# Tightness and weak convergence for jump processes

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**Abstract**

The aim of this note is to extend tightness criteria for random measures and simple point processes to processes with general jump distributions.

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## 1 Introduction

Consider random elements in the function space $D[0,1]$ endowed with the Skorohod $J_1$-topology. As is well known, weak convergence of a sequence $\{X_n,\, n \geq 1\}$ (to $X \in D[0,1]$) follows if the finite-dimensional distributions of $X_n$ converge (appropriately to those of $X$) and the sequence $\{X_n\}$ is tight. For background information see Billingsley (1968). Much attention has been devoted to finding sufficient conditions for tightness. A particular setup has been random measures and point processes; see Jagers (1974) and Kallenberg (1997) and further references given there. Motivated by Gut and Hüsler (1999) where extreme shock models are investigated, the aim of the paper is to prove some tightness criteria for pure jump processes. A particular case would be when the jumps only take integer values, in which case the processes reduce to (simple) point processes, which are more easily handled.

## 2 Results

Throughout we thus suppose that the random elements of the sequence $\{X_n,\, n \geq 1\}$ in $D[0,1]$ are pure jump processes.

A crucial object is $w'(\,\cdot\,)$, a kind of modulus of continuity; the proofs below amount to showing that it is suitably small. For an element $x \in D[0,1]$ and a set $T \subset [0,1]$ we set

$$w_x(T) = \sup_{s,t \in T} |x(s) - x(t)|, \qquad \text{and} \qquad w'_x(\delta) = \inf_{\{s_i\}} \max_{0<i\leq r} w_x[s_{i-1}, s_i), \tag{2.1}$$

where the infimum extends over the finite sets of points $\{s_i\}$ satisfying

$$0 = s_0 < s_1 < \ldots < s_r = 1 \qquad \text{and} \qquad s_i - s_{i-1} > \delta, \quad \text{for} \quad i = 1, 2, \ldots, r \tag{2.2}$$

(cf. Billingsley (1968), pp. 109-110). For brevity, $w'_{X_n}(\,\cdot\,)$ will be denoted $w'_n(\,\cdot\,)$.

We are now ready to state our results.

**Theorem 2.1.** *The sequence* $\{X_n,\, n \geq 1\}$ *of jump processes is tight if*

(i) *For each positive* $\eta$ *there exists* $b$*, such that*

$$P(\sup_{0\leq s\leq 1} |X_n(s)| > b) \leq \eta \quad \textit{for all} \quad n \geq 1. \tag{2.3}$$

(ii) *For each* $a > 0$

$$\delta^{-1} \limsup_{n\to\infty} \sup_{a\leq s\leq 1-a} P(\text{at least two } X_n\text{-jumps in } [s, s+\delta)) \to 0 \quad \textit{as } \delta \to 0. \tag{2.4}$$

(iii) *For every* $\varepsilon > 0$*,*

$$\limsup_{n\to\infty} P(\sup_{0\leq s\leq a} |X_n(s) - X_n(0)| > \varepsilon) \to 0 \quad \textit{as} \quad a \to 0. \tag{2.5}$$

(iv)

$$\limsup_{n\to\infty} P(\sup_{1-a\leq s\leq 1} |X_n(s) - X_n(1)| > \varepsilon) \to 0 \quad \textit{as} \quad a \to 0. \tag{2.6}$$

**Remark 2.2.** The conditions of the theorem resemble those of Billingsley (1968), Theorem 15.3. To see this, note that

$$\sup_{0\leq s\leq a} |X_n(s) - X_n(0)| \leq w_n([0,a]) \leq 2 \sup_{0\leq s\leq a} |X_n(s) - X_n(0)|,$$

that is, we may replace (2.5) by the assumption that

$$\limsup_{n\to\infty} P(w_n([0,a]) > \varepsilon) \to 0 \quad \text{as} \quad a \to 0; \tag{2.7}$$

cf. condition (15.8) in the cited result. Similarly, our (2.6) is equivalent to

$$\limsup_{n\to\infty} P(w_n([1-a,1]) > \varepsilon) \to 0 \quad \text{as} \quad a \to 0; \tag{2.8}$$

cf. (15.9) there.

The meaning of (ii) is that the jumps are well separated from each other as long as one stays away from the endpoints; on the other hand, no assumption is made on the size of these jumps apart from the overall boundedness (i). In contrast, we allow many, even infinitely many, jumps close to 0 or 1, but, by (iii) and (iv), they all have to be small.

In the formulation of the theorem the points 0 and 1 are taken care of in a "symmetric" fashion. In cases, such as the one in Gut and Hüsler (1999), where the jumps cluster at one endpoint only, the following set of conditions is more convenient.

**Theorem 2.3.** *The sequence* $\{X_n,\, n \geq 1\}$ *of jump processes is tight if* (i), (ii), (iii) *hold, together with*

(iv′)

$$\liminf_{n\to\infty} P(\text{no } X_n\text{-jump in } [1-\delta, 1)) \to 1 \quad \textit{as } \delta \to 0. \tag{2.9}$$

One case of particular importance is when the jumps can only take positive values. In this case oscillation over an interval reduces to an increase over that same interval, in which case conditions (iii) and (iv) reduce to

(iii$^+$) For every $\varepsilon > 0$,

$$\limsup_{n\to\infty} P(X_n(a) - X_n(0) > \varepsilon) \to 0 \quad \text{as} \quad a \to 0. \tag{2.10}$$

and (iv$^+$) For every $\varepsilon > 0$,

$$\limsup_{n\to\infty} P(X_n(1) - X_n(a) > \varepsilon) \to 0 \quad \text{as} \quad a \to 1. \tag{2.11}$$

respectively, which leads to the following immediate corollary.

**Theorem 2.4.** *Suppose that the jumps of the elements $X_n$ are positive a.s. The sequence $\{X_n,\, n \geq 1\}$ is tight if conditions* (i), (ii), (iii$^+$), and (iv$^+$) *are satisfied.*

**Remark 2.5.** Note that condition (iii$^+$) corresponds to $X_n(a)-X_n(0) \to 0$ in probability, or stochastic continuity at 0. Compare the stronger condition (iii) and Remark 2.2. A similar remark applies to condition (iv$^+$).

**Remark 2.6.** As an immediate corollary (cf. Billingsley (1968)) it follows that, if in addition to the tightness conditions above, the finite-dimensional distributions of $X_n$ converge to those of $X$, then $X_n \Rightarrow X$ in $D[0,1]$.

In the case of only positive jumps an alternative formulation runs as follows.

**Theorem 2.7.** *Suppose that $\{X_n,\, n \geq 1\}$ is a sequence of pure jump processes with only positive jumps. Suppose further that the finite-dimensional distributions converge to those of $X$. If $X$ has no jump at 1 and condition* (ii) *in Theorem 2.1 holds, then $X_n \Rightarrow X$ in $D[0,1]$.*

# 3 Proofs

## 3.1 Proof of Theorem 2.1

The idea is to show that the conditions in Billingsley (1968), Theorem 15.2 are satisfied. In fact, as mentioned above, our conditions resemble those of his Theorem 15.3, where a variation $w''(\,\cdot\,)$ of $w'(\,\cdot\,)$ is involved. However, since the latter always dominates the former we actually verify the conditions of both those theorems.

Let $\varepsilon > 0$ and $\eta > 0$, arbitrary, be given. It follows from (iii), (iv) and Remark 2.2 that we may choose $a > 0$ so that

$$P\{w_n[0,2a] > \varepsilon\} < \eta, \tag{3.1}$$

and

$$P(w_n[1-2a,1)) < \eta \tag{3.2}$$

for $n \geq n_0$, and then $\delta$, $0 < \delta < a$, in (ii) such that

$$\delta^{-1} P(\text{at least two } X_n\text{-jumps in } [s, s+\delta)) < \eta, \tag{3.3}$$

for $s \in [a, 1-a]$ and $n \geq n_1$.

This provides an upper bound for *one* interval of length $\delta$. Since there are at most $1/\delta$ intervals in $[0,1]$ of this length (and all the more so in the interval $[a, 1-a]$) it follows that the probability of having at least two $X_n$-jumps in any of the intervals $[(j-1)\delta, j\delta) \subset [a, 1-a]$ is at most equal to $\eta$ for $n \geq n_1$.

By considering the $\delta$-intervals $[(j-1/2)\delta, (j+1/2)\delta) \subset [a, 1-a]$ it follows that the probability of having two $X_n$-jumps within $\delta/2$ from each other in adjacent $\delta$-intervals from the first set of $\delta$-intervals is also bounded by $\eta$ for $n \geq n_1$. It thus follows that the probability of having two or more $X_n$-jumps in the interval $[2a, 1-2a]$ of distance at most $\delta/2$ from each other is bounded by $2\eta$ for $n \geq n_1$.

We now choose a partition as described in (2.2), however, with $\delta$ replaced by $\delta/2$. For a given $\omega$ we then distinguish between the following three cases depending on the sample path behaviour in the interval $[2a, 1-2a]$:

**(i)** There exist jumps in $[2a, 1-2a]$ closer than $\delta/2$ from each other. For $n$ large we know that the probability for this event is at most equal to $2\eta$.

**(ii)** There are no jumps in $[2a, 1-2a]$ closer than $\delta/2$ from each other. In this case we choose $s_1$ as the time of the first jump after $a$, after which the remaining points, $s_2, s_3, \ldots, s_{r-1}$, are assigned to the further jump points of $X_n$ in $[2a, 1-2a]$ (recall that $s_r = 1$). It follows that

$$\begin{aligned} w_n'(\delta) &\leq \max_{0<i\leq r} w_n[s_{i-1}, s_i) = \max\{w_n[0, s_1), w_n[s_{r-1}, 1)\} \\ &\leq \max\{w_n[0, 2a], w_n[1-2a, 1)\}, \end{aligned} \tag{3.4}$$

since $w_n[s_{i-1}, s_i) = 0$ for $2 \leq i \leq r-1$ and $X_n$ is constant on $[2a, s_1)$ and $[s_{r-1}, 1-2a]$.

**(iii)** There are no jumps at all in $[2a, 1-2a]$. In this case we let $r = 2$ with $s_1 = 1/2$ (say), and (3.4) still holds.

Combining the above estimates we find that, given $\varepsilon$ and $\eta$ positive as above, we can choose $a$ and $\delta$, such that

$$P(w_n'(\delta) > \varepsilon) \leq 2\eta + P(w_n[0, 2a] > \varepsilon) + P(w_n[1-2a, 1) > \varepsilon) \leq 4\eta \tag{3.5}$$

uniformly for large values of $n$.

The proof is complete.

**Remark 3.1.** An inspection of the proof above shows that we may replace $X(1)$ by $X(1-)$ in (2.6). Similarly, $w_n[1-a, 1]$ may be replaced by $w_n[1-a, 1)$ in (2.8), that is, a possible jump at 1 does, in fact, not matter.

### 3.2 Proof of Theorem 2.3

Since $w_n[1-\delta, 1) = 0$ unless $X_n$ has at least one jump in $[1-\delta, 1)$, (iv′) implies that (2.8) holds (with $w_n[1-a, 1)$). The conclusion thus follows from Theorem 2.1 and Remark 3.1.

### 3.3 Proof of Theorem 2.7

Since $X_n(0) \to_d X(0)$ and $X_n(1) \to_d X(1)$ as $n \to \infty$, it follows that $\{X_n(0),\, n \geq 1\}$ and $\{X_n(1),\, n \geq 1\}$ are tight. Moreover, $\sup_{0\leq s\leq 1} |X_n(s)| = \max\{|X_n(0)|, |X_n(1)|\}$, which shows that (i) holds.

Next, given $\varepsilon$, $\eta > 0$, we choose $a > 0$ such that $P(|X(a) - X(0)| \geq \varepsilon) < \eta$, and since $(X_n(0), X_n(a)) \to_d (X(0), X(a))$ as $n \to \infty$ it follows that $P(|X_n(a) - X_n(0)| \geq \varepsilon) < \eta$ for $n \geq n_0$, that is, (iii$^+$) holds. A similar argument applies to (iv$^+$). An application of Theorem 2.4 concludes the proof.

# 4 Remarks and comments

We begin this section by describing the problem in Gut and Hüsler (1999) that motivated the present investigation, after which we provide some additional remarks and comments.

As mentioned in the introduction, the present paper was motivated by the paper Gut and Hüsler (1999) which deals with extreme shock models. Shock models describe systems that at random times are subject to shocks of random magnitudes. One distinguishes between two major types; cumulative shock models and extreme shock models. Systems governed by the former kind break down when the cumulative shock magnitude exceeds some given level, whereas systems modeled by the latter kind break down as soon as an individual shock exceeds some given level.

The general setup in extreme shock models is a family $\{(X_k, Y_k),\ k \geq 1\}$ of i.i.d. two-dimensional random vectors, where $X_k$ represents the magnitude of the $k$th shock and where $Y_k$ represents the time between the $(k-1)$st and the $k$th shock. We further set $T_n = \sum_{k \leq n} Y_k,\ n \geq 1$, and define

$$\tau(t) = \min\{n : X_n > t\}, \quad t \geq 0. \tag{4.1}$$

The main object of investigation is the failure time $T_{\tau(t)}$.

In order to prove weak convergence for the whole process one introduces the process

$$\{Z_t(s),\ 0 \leq s \leq 1\} = \{T_{\tau(st)}(1 - F(t)),\ 0 \leq s \leq 1\}, \tag{4.2}$$

with $Z_t(0) = 0$. We first observe that the realizations of the process are nondecreasing step functions in the function space $D[0,1]$. To see that the process fits into the present discussion we observe that the process starts at 0 and stays constant as $s$ moves from 0 to 1 until a new (shock-)record occurs, at which time point the process jumps, and then stays constant until the next shock occurs, and so on. Note also that the process differs from a simple point process in that the jumps are not of size one. Under the additional assumption that the tail of the distribution of the shock magnitudes is regularly varying at infinity it is shown in Gut and Hüsler (1999) that the finite-dimensional distributions converge, and that the sequence $\{Z_t(s),\ 0 \leq s \leq 1\}$ is tight, that is, that the process converges weakly. The limit process is also described.

**Remark 4.1.** In Theorem 2.4 we consider the special case when the jumps are positive, replacing conditions (iii) and (iv) with conditions (iii$^+$) and (iv$^+$), respectively. A completely analogous argument can of course be made if the jumps only assume negative values, in which case conditions (iii) and (iv) are replaced by the obvious conditions (iii$^-$) and (iv$^-$), respectively, the exact formulation of which leave to the interested reader(s).

**Remark 4.2.** The condition (trouble) at the endpoint 1 can be dispensed with (avoided) by considering jump processes defined on $[0, \infty)$ endowed with the Skorohod $J_1$-topology on $D[0, \infty)$ (see Lindvall (1973)).

In this case conditions (i) and (ii) become

($\mathrm{i}^\infty$) For each $T, \eta > 0$ there exists $b$, such that

$$P(\sup_{0\le s\le T} |X_n(s)| > b) \le \eta \quad \text{for all} \quad n \ge 1. \tag{4.3}$$

($\mathrm{ii}^\infty$) For each $a_1, a_2 > 0$

$$\delta^{-1} \limsup_{n\to\infty} \sup_{a_1\le s\le a_2} P(\text{at least two } X_n\text{-jumps in } [s, s+\delta)) \to 0 \quad \text{as } \delta \to 0. \tag{4.4}$$

A sequence $\{X_n,\, n \ge 1\}$ of jump processes on $[0,\infty)$ is tight in $D[0,\infty)$ if ($\mathrm{i}^\infty$), ($\mathrm{ii}^\infty$), and (iii) hold. In particular, if the jumps are positive, the finite-dimensional distributions of $X_n$ converge to those of $X$ and ($\mathrm{ii}^\infty$) holds, then $X_n \to_d X$ as $n \to \infty$; cf. Theorem 2.7.

**Remark 4.3.** It is also possible to consider processes defined on the open interval $(0,1)$. One can then show as above that if $\{X_n,\, n \ge 1\}$ is a sequence of jump processes such that (i) and (ii) hold, then $\{X_n,\, n \ge 1\}$ is tight. Here we may also replace (i) by

($\mathrm{i}'$) For each $0 < a_1 < a_2 < 1$

$$P(\sup_{a_1\le s\le a_2} |X_n(s)| > b) \le \eta \quad \text{for all} \quad n \ge 1. \tag{4.5}$$

Similarly, if (i), (ii), and (iii) hold, then $\{X_n,\, n \ge 1\}$ is tight in $D[0,1)$, and with (i), (ii), and (iv) it is tight in $D(0,1]$, and so on. For further remarks in this direction, see Janson (1994), pp. 5-6, in particular, Proposition 2.4.

By obvious modifications it is also possible to consider processes on arbitrary intervals $[S,T]$, or $[S,T)$, etc, $0 \le S \le T \le \infty$.

**Remark 4.4.** Condition (ii) is used as a convenient tool in order to verify the weaker

($\mathrm{ii}'$) For each $a > 0$

$$\limsup_{n\to\infty} P(\text{there exist two } X_n\text{-jumps in } [a, 1-a] \text{ closer than } \delta) \to 0 \quad \text{as } \delta \to 0. \tag{4.6}$$

The purpose of the condition is to prevent asymptotic double points; to ensure that the limiting process is simple. The reason for this is that if $X_n$ has two points "close" to each other in such a way that they collaps in the limit, the $J_1$-topology is too strong to allow this; whereas $X_n$ are simple for all $n$ this is not necessarily the case for $X$, that is, being simple is not a $J_1$-continuous property.

As an example, suppose that

$$X_n(t) = \begin{cases} 0 & \text{for } 0 \le t < \frac{1}{2}, \\ Z_1 & \text{for } \frac{1}{2} \le t < \frac{1}{2} + \frac{1}{n}, \\ Z_2 & \text{for } \frac{1}{2} + \frac{1}{n} \le t \le 1, \end{cases}$$

where $Z_1, Z_2$ assume the values 1 and 2 with probability $1/2$, say, then every $X_n$ is clearly simple, whereas $X$ is not.

The last remark leads us, in turn, to the following two remarks.

**Remark 4.5.** Consider, for example, the special case of simple non-decreasing point processes, that is the jumps are all equal to $+1$. Suppose further that the finite-dimensional distributions of $X_n$ converge to those of $X$, and assume, for simplicity, that $X(0) = 0$ (the typical case) and that $X$ has no jump at 1. Then $X$ is a point process too and it is easily seen that (ii′) is equivalent to $X$ being simple as well. Theorem 2.7 now implies that $X_n \Rightarrow X$ in $D[0,1]$ as $n \to \infty$. This has been proved by other means in Jagers (1974), Section 3, see also Kallenberg (1997), Chapter 14.

**Remark 4.6.** In Remark 4.4 we found that the $J_1$-topology is too strong for handling the case when the limit process may have double points. The weaker Skorohod $M_1$-topology introduced in Skorohod (1956) can cope with this problem. Namely, in this topology convergence of elements in $D[0,1]$ amounts to uniform convergence of continuous parametric representations of the elements, that is, convergence of parametric representations that move continuously along the graphs. For the example considered in Remark 4.5 it follows that weak convergence in $D[0,1]$ (or $D[0,\infty)$) endowed with the $M_1$-topology holds without condition (ii′), that is, the limit process may have double points in this case.

**Remark 4.7.** The above results remain true if the "time" parameter is continuous, that is, we may alternatively consider a family $\{X_t,\, t \geq 0\}$ instead of the sequence $\{X_n,\, n \geq 1\}$; cf. Gut and Hüsler (1999) (and the introduction of this section) for an example.

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