Remarks to Random Dyadic Tilings of the Unit Square

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This note contains some further remarks to *Random Dyadic Tilings of the Unit Square* by Svante Janson, Dana Randall and Joel Spencer, that were not deemed suitable for inclusion in the paper. (Although only one of the authors is reponsible for this note, the help of the other two is acknowledged.)

The note is informal, and not intended for publication.

1 Number of edges in the graph T_n

Let e_n be the number of edges in the graph T_n (see Section 2 and Problem 2.8). Thus $e_0 = 0$, $e_1 = 1$, $e_2 = 8$, ...

If $n \geq 2$, then the subgraph consisting of tilings with a vertical cut is isomorphic to $T_{n-1} \times T_{n-1}$ and has $2A_{n-1}e_{n-1}$ edges, the same holds for the subgraph consisting of tilings with a horizontal cut, every edge belongs to at least one of these two subgraphs by Theorem 2.3 and the number of edges belonging to both subgraphs equals the number of edges in $T_{n-2} \times T_{n-2} \times T_{n-2} \times T_{n-2}$, which is $4e_{n-2}A^3_{n-2}$. Hence, we have the recursion formula

$$e_n = 4A_{n-1}e_{n-1} - 4A^3_{n-2}e_{n-2}, \qquad n \geq 2. \tag{1}$$

This gives for small n the values in Table 1.

e_1	1
e_2	8
e_3	192
e_4	52000
e_5	1874325376
e_6	1210503319261219968
e_7	251888520933684030206341784482360832
e_8	5448250438458315814543618408036661448863619847306233051620928065384960

Table 1: number of edges in $\mathcal{T}_n$, $n = 1, \dots, 8$

Let $d_n := e_n/A_n$ be (half) the average degree. Then the recursion formula (1) can be rewritten

$$e_n = 4d_{n-1}A^2_{n-1} - 4d_{n-2}A^4_{n-2}$$

and thus, see (4.1) and (1.1),

$$d_n = 4d_{n-1}p_n - 4d_{n-2}(2p_n - 1), \qquad n \geq 2.$$

Let $x_n := 2^{-n}d_n$. Then $x_0 = 0$, $x_1 = 1/4$, $x_2 = 2/7$, $x_3 = 12/41$, and, by the relations above,

$$x_n = 2p_nx_{n-1} - (2p_n - 1)x_{n-2}, \qquad n \geq 2.$$

Hence,

$$x_n - x_{n-1} = (2p_n - 1)(x_{n-1} - x_{n-2}), \qquad n \geq 2,$$

and thus

$$x_n - x_{n-1} = \tfrac{1}{4}\prod_{k=2}^{n}(2p_k - 1), \qquad n \geq 1,$$

and finally

$$x_n = \tfrac{1}{4}\sum_{j=1}^{n}\prod_{i=2}^{j}(2p_i - 1) \to \gamma := \tfrac{1}{4}\sum_{j=1}^{\infty}\prod_{i=2}^{j}(2p_i - 1).$$

In other words,

$$d_n = 2^{n-2}\sum_{j=1}^{n}\prod_{i=2}^{j}(2p_i - 1) \sim \gamma 2^n,$$

and hence, see (1.2), we have the asymptotical expression

$$e_n \sim \gamma 2^n A_n \sim \gamma\phi^{-1}2^n\rho^{2^n}.$$

Since $2p_i - 1 \to 2\phi^{-1} - 1 = 2\phi - 3 = \sqrt{5} - 2 = 0.236\cdots$, the sum defining γ converges rapidly. Numerically, $\gamma = 0.2946462157\cdots$.

2 More lattice theory

By a theorem of Birkhoff's, every finite distributive lattice is isomorphic to the family of all hereditary subsets (down-sets) in some finite partially ordered set, which is unique up to isomporphism. (The converse holds too, and this defines a bijection between sets of isomorphism classes of finite distributive lattices and finite posets.) The height function (in our case $H(T)/2$) is the cardinality of the corresponding hereditary subset, and thus the diameter of the lattice equals the cardinality of the poset.

It follows by induction and the recursive construction of $\mathcal{T}_n$ as the union of the two (overlapping) subsets of tilings with a horizontal or vertical cut, respectively, that the partially ordered set for the lattice $\mathcal{T}_n$ looks like a n-dimensional truss. It has $n2^{n-1}$ elements, cf. Corollary 2.7, and can be realized as follows: Take the product set $\{1,\ldots,n\} \times \{0,1\}^{n-1}$, and define $(i+1, x_1, \ldots, x_{n-1}) > (i, y_1, \ldots, y_{n-1})$ if $x_j = y_j$ for every $j \neq i$ (but x_i and y_i are arbitrary). Take the transitive closure.

(Thanks to Anders Björner for helpful remarks.)

3 Path coupling, a remark

Actually, the first ceiling in the conclusion of Theorem 5.4 is not necessary, but [2] has it, so its simplest to keep it.

4 Comparison method

Theorem 5.8 is a slight modification of a theorem in [3]. Here is a detailed proof.

Theorem (5.8). *Let (P,π,Ω) and $(\widetilde{P},\pi,\Omega)$ be two reversible Markov chains such that $\widetilde{P}(x,y) \neq 0$ implies $P(x,y) \neq 0$ for all $x,y \in \Omega$. Let $\pi_* = \min_{x\in\Omega} \pi(x)$. Then, for $0 < \epsilon < 1/2$,*

$$\tau(\epsilon) \leq 4\ln(1/(\epsilon\pi_*))A\max\Big(\frac{\widetilde{\tau}(\epsilon)}{\ln(1/2\epsilon)},1\Big), \tag{2}$$

where

$$A = \max_{x\neq y,\widetilde{P}(x,y)>0} \frac{\widetilde{P}(x,y)}{P(x,y)}. \qquad \square$$

Proof. The argument in [3] yields that $\lambda_1 < 1/2$ or

$$1-\lambda_1 \geq \frac{1}{4\widetilde{\tau}(\epsilon)}\log(1/2\epsilon)$$

and thus always

$$1-\lambda_1 \geq \min\Big(\frac{1}{4\widetilde{\tau}(\epsilon)}\log(1/2\epsilon),\frac{1}{2}\Big).$$

Moreover, since we only use paths of length 1, which is odd, the same argument but using Theorem 2.2 in [1] shows that the same estimate holds for $1+\lambda_{|\Omega|-1}$. Thus the result holds by [3, Theorem 1(i)]. $\square$

Remark. Although we do not need it, the constant 4 in (2) can be replaced by 2 by the following sharpening of [3, Theorem 1(ii)].

Theorem 1. *Let $\lambda_* = \max(|\lambda_1|,|\lambda_{\Omega-1}|)$. For $0<\epsilon<1$, we have*

$$\max_x \tau_x(\epsilon) \geq \frac{\lambda_*}{1-\lambda_*}\log\Big(\frac{1}{2\epsilon}\Big).$$

Proof. Let L^p denote $L^p(\Omega,\pi)$, let P denote the operator on these spaces (which coincide as vector spaces but have different norms) defined by $Pf(x) = \sum_y P(x,y)f(y)$ and let Q denote the operator defined by $Qf(x) = \sum_y \pi(y)f(y)$; then P is a self-adjoint operator on L^2 and Q is the orthogonal projection onto the space of constant functions.

Since $P-Q$ is self-adjoint, the operator norm in L^2 of $P^t - Q = (P-Q)^t$ is given by

$$\|P^t - Q\|_{B(L^2)} = \|(P-Q)^t\|_{B(L^2)} = \lambda_*^t. \tag{3}$$

On the other hand, it is easily seen that the operator norm in L^1 and in L^∞ both are given by

$$\|P^t - Q\|_{B(L^1)} = \|P^t - Q\|_{B(L^\infty)} = \max_x \sum_y |P^t(x,y) - \pi(y)| = 2 \max_x \Delta_x(t).$$

(That these two norms are equal follows also because the operator is self-adjoint.)

Hence, if $t = \max_x \tau_x(\epsilon)$, we have

$$\|P^t - Q\|_{B(L^1)} = \|P^t - Q\|_{B(L^\infty)} \leq 2\epsilon.$$

By interpolation (in this case special case due to Schur (1911) and known as Schur's lemma), this implies

$$\|P^t - Q\|_{B(L^2)} \leq 2\epsilon$$

and the result follows by (3) and $\log(1/\lambda_*) \leq 1/\lambda_* - 1$. $\square$

5 A non-uniform random tiling

We have shown that Algorithm 4.1 generates uniformly distributed random tilings in $\mathcal{T}_n$. Evidently, one can also produce random tilings in $\mathcal{T}_n$ by the following simpler algorithm: *Make a vertical or horizontal cut, with probability* $1/2$ *each, and continue recursively in each half (independently)* n *levels.* This method, however, does *not* give a uniformly distributed tiling when $n \geq 2$. For example, the probability of obtaining the all horizontal tiling $\widetilde{T}_0$ is $2^{-(2^n-1)} \ll A_n^{-1}$.

This simpler method is equivalent to choosing a random labeling of the complete binary tree with H and V (or A and D) uniformly among all 2^{2^n-1} possibilities without any restrictions, and constructing the corresponding tiling as in Section 3.

It might be interesting to study properties of such non-uniform random tilings too. We give only a few simple remarks.

A branching process argument, similar to the one in Section 6.2 but now with a critical branching process, shows that for the random tiling generated by this procedure, $\mathbf{P}$(there is a vertical cut) $\to 1$ as $n \to \infty$, in contrast to (4.1). By symmetry, $\mathbf{P}$(there is a horizontal cut) $\to 1$ too, and hence $\mathbf{P}$(there is a strut) $\to 0$.

For the total height we now find easily from (6.4) that, with the same normalization (6.1), $\tilde{H}_n \xrightarrow{d} \tilde{H}'_\infty$, where

$$\tilde{H}'_\infty = \sum_{k=1}^{\infty} \sum_{j=1}^{2^{k-1}} 2^{-k} Y_{kj}$$

with Y_{kj} independent random variables with $\mathbf{P}(Y_{kj} = \pm 1) = \frac{1}{2}$. Hence, $\mathbf{Var}\, \tilde{H}'_\infty = \frac{1}{2}$ and the moment generating function is given by

$$\mathbf{E}\, e^{z\tilde{H}'_\infty} = \prod_{k=1}^{\infty} (\cosh(2^{-k}z))^{2^{k-1}}.$$

Problems. Recalling the minimum height $h_{\min}$ from Problem 6.6, it is not difficult to show that for the non-uniform model $h_{\min} \xrightarrow{p} \infty$. How fast? What is $\mathbf{E}\, h_{\min}$? Does $h_{\min}$ have an asymptotic distribution after normalization? If so, what is it?

References

[1] Diaconis, P. and Saloff-Coste, L. Comparison theorems for reversible Markov chains. *Ann. Appl. Probability* **3**:696–730, 1993.

[2] Dyer, M. and Greenhill, C. A more rapidly mixing Markov chain for graph colorings. *Random Structures and Algorithms* **13**:285–317, 1998.

[3] Randall, D. and Tetali, P. Analyzing Glauber dynamics by comparison of Markov chains. *J. Mathematical Physics* **41**:1598–1615, 2000.