VOLATILITY TIME AND PROPERTIES OF OPTION PRICES: A SUMMARY

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Abstract. We use a notion of stochastic time, here called volatility time, to show convexity of option prices in the underlying asset if the contract function is convex as well as continuity and monotonicity of the option price in the volatility.

1. Introduction

In this article we will announce some of the results of [J-T]. Consider the spot price S of some asset following the risk neutral process

$$dS = S\sigma(S,t)\,dB, \tag{1}$$

with initial condition $S_t = s$, where B is a Brownian motion and σ is the volatility of S. We here compute the price with respect to some suitable numeraire process, for instance the price of a zero coupon bond, maturing at some future time T, to avoid the drift in the process for S associated with interest rates. We are interested in general properties of prices of simple contingent claims maturing at T. The arbitrage free price of a simple claim with contract function Φ is given by

$$F(s,t) = E_{s,t}[\Phi(S_T)], \tag{2}$$

according to [B-S]. In [B-G-W] it is shown that the price $F(s,t)$ is a convex function of s if Φ is a convex function. From the Black–Scholes equation

$$F_t + \frac{1}{2}s^2\sigma^2(s,t)F_{ss} = 0, \tag{3}$$

corresponding to (2) through the Feynman–Kac stochastic representation formula, it follows that the convexity of F in s is equivalent to F decaying with time when the price of the underlying asset is constant. In [B-G-W] it is also shown that the price F is monotonic in the volatility. However, the arguments presented in [B-G-W] for these results require the volatility to be a differentiable function of the underlying asset price.

Below we state the generalization of the results above to volatilities that are not even continuous in time and only satisfy a local Hölder(1/2) condition in s. We believe that these conditions, especially the lack of continuity assumption in t, are natural in applications as well as mathematically satisfying. To obtain these results, we use the standard fact that a local martingale can be represented as a time-change of a Brownian motion. In our context, this entails defining a notion of stochastic time for risky assets which we refer to as *volatility time*. Its aim is to reduce the study of price processes

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modelled by local martingales in the form (1) to the study of Brownian motion. This stochastic time has also been used by D. Hobson in [Ho] in the same context. However, in [Ho], existence and uniqueness of volatility time, which is needed in these applications, is assumed to hold without further discussing conditions for this. In [J-T] we find very general conditions under which the volatility time exists uniquely. The methods used to prove these results are also very general in nature, and we believe they can be of independent interest in the study of stochastic differential equations. From these arguments we are also able to establish the continuity of the option price in the volatility, a result which we have not seen elsewhere. The question of the continuity of option prices under perturbations of the volatility is of obvious interest in applications. A related reference, where the relation between options prices and volatility is treated, especially in the context of hedging, is [K-J-S].

2. Applications to option pricing

Assume that $\sigma(s,t)$ is measurable on $(0,\infty)\times[t_0,\infty)$ and that for some constant C and all $s\in(0,\infty)$ and $t\geq t_0$,

$$\sigma(s,t)\leq C(1+s^{-1}). \tag{4}$$

We also assume that the following local Hölder(1/2) condition is satisfied: for every $K>0$, there exists a constant C_K such that

$$|\sigma(s_1,t)-\sigma(s_2,t)|\leq C_K|s_1-s_2|^{1/2} \tag{5}$$

when $s_1,s_2\in[K^{-1},K]$ and $|t|\leq K$.

In our first application we compare option prices at two different times.

Theorem 1. *Let σ be as above and let $t_0\leq t_1\leq t_2\leq T$. Let $S_t^{(1)}$ and $S_t^{(2)}$ be solutions to*

$$dS_t=S\sigma(S_t,t)\,dB_t,$$

where B is a Brownian motion, with $S_{t_1}^{(1)}=s_0=S_{t_2}^{(2)}$. Assume that 0 is an absorbing state for $S_t^{(1)}$ and $S_t^{(2)}$. Finally, let Φ be a convex function. Then

$$E\Phi(S_T^{(1)})\geq E\Phi(S_T^{(2)}).$$

We note that interpreting S as a price process with volatility $\sigma(S,t)$ we have, in view of equation (2), that option prices with convex contract functions decay with time, or in view of equation (3), that the price is convex as a function of the price of underlying asset. As noted above, this has earlier been proved by [B-G-W] and [Ho] under somewhat different conditions.

We also have the following result concerning the continuity of option prices under perturbation of the volatility.

Theorem 2. *Suppose that σ and $\sigma_1,\sigma_2,\dots,$ satisfy the conditions above uniformly and suppose that*

$$\sigma_n(s,t)\to\sigma(s,t),$$

as $n\to\infty$ for all $s>0$ and t. Let S_t and $S_t^{(n)}$ be solutions to

$$dS_t=S_t\sigma(S_t,t)\,dB_t,\quad dS_t^{(n)}=S_t^{(n)}\sigma_n(S_t^{(n)},t)\,dB_t$$

with $S_{t_0} = S_{t_0}^{(n)} = s_0$. *Assume that* 0 *is an absorbing state for* S_t *and* $S_t^{(n)}$. *Let* $T \geq t_0$. *Then* $S_T^{(n)}$ *converges in distribution to* S_T *as* n *tends to infinity. Further, if* Φ *is a continuous function with* $|\Phi(s)| \leq C_1(1+s)^k$ *for some* C_1 *and* $k < \infty$, *then*

$$E\Phi(S_T^{(n)}) \to E\Phi(S_T).$$

Using the results on volatility time we can also show the following monotonicity result for option prices in the volatility.

Theorem 3. *Suppose that* σ *and* $\tilde{\sigma}$ *satisfy the conditions above and that* $\sigma(s,t) \leq \tilde{\sigma}(s,t)$ *for all* $s > 0$ *and* t. *Let* S_t *and* $\tilde{S}_t$ *be solutions to*

$$dS_t = S_t\sigma(S_t,t)\,dB_t, \quad d\tilde{S}_t = \tilde{S}\tilde{\sigma}(\tilde{S}_t,t)\,dB_t$$

with $S_0 = \tilde{S}_0 = s_0$. *If* $T \geq t_0$ *and* Φ *is a convex function, then*

$$E\Phi(S_T) \leq E\Phi(\tilde{S}_T).$$

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