PROPERTIES OF OPTIONS ON SEVERAL UNDERLYING ASSETS

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Abstract. It is well-known that prices of options on one underlying asset decay with time and are convex in the underlying asset if the contract function is convex. Here, options on several underlying assets are studied and we prove that if the volatility matrix is independent of time, then the option prices decay with time if the contract function is convex. However, the option prices are no longer necessarily convex in the underlying assets. If a time dependent volatility is allowed we note that the option prices do not necessarily decay with time. Moreover, we show that even if the price processes are independent, convexity is preserved only for very special volatilities including price processes driven by a Geometric Brownian motion.

1. Introduction

Let the assets S_i have risk neutral processes given by

$$dS_i = S_i(t) \sum_{j=1}^{n} \sigma_{ij}(S(t), t)\, dB_j$$

for $i = 1, \ldots, n$, where B_j are independent Brownian motions and $S(t) = (S_1(t), \ldots, S_n(t))$. The matrix σ with entries σ_{ij} is called the *volatility matrix*. In this model we get rid of the effect of interest rates by using a bond as a numeraire. The pricing function of a contingent claim with the contract $\Phi(S(T))$ is given by

$$F(s, t) = E_{s,t}[\Phi(S(T)],$$

compare [B-S]. Alternatively, one has that the pricing function is a solution of the partial differential equation

$$F_t + \frac{1}{2} \sum_{i,j=1}^{n} s_i s_j F_{ij} C_{ij} = 0$$

with the boundary condition $F(s, T) = \Phi(s)$, where $C_{ij} = [\sigma\sigma^*]_{ij}$. Here, we will only consider the solutions that come from the stochastic representation.

Prices of options on one underlying asset decay with time and are convex in the underlying asset if the contract function is convex, compare [B-G-W], [H] and [J-T1]. In the present paper we study the corresponding properties for options on several underlying assets. We show that if the volatility matrix is independent of time, and the contract function is convex, then

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indeed the option prices decay with time. However, many examples show that the case of several underlying assets is essentially different from the case of one underlying asset. For instance, even though the contract function is convex and the solutions decay with time, the option price for a fixed time t, need not be a convex function of the underlying assets, in contrast to the one dimensional case. On the other hand, if the price remains a convex function of the asset values we see directly from the differential equation that the prices will decay with time. Furthermore, if a time dependent volatility matrix is allowed, it is easy to find examples where the option prices do not decrease with time. In the last section we show that even if the price processes are independent, convexity is preserved only for very special volatilities including price processes driven by Geometric Brownian motion.

2. Time-decay of option prices

As noted above prices of options on one underlying asset decay with time if the contract function is convex. This generalizes to options on several underlying assets when the volatility matrix is independent of time.

Theorem 1. *If the volatility matrix is independent of time, and the contract function is convex, then the option price $F(s,t)$ given by*

$$F(s,t) = E_{s,t}(\Phi(S(T)).$$

decreases with time.

Proof. Since

$$F(s,t) = E_{s,t}(\Phi(S(T)),$$

we know that if $F(s,T) = \Phi(S(T)) \geq 0$, then $F(s,t) \geq 0$ for all $t \leq T$. We also note that if the contract function is an affine function then $F(s,t) = F(s,T)$ for all $t \leq T$. First, we will show that if the contract function is convex then $F(s,t) \geq F(s,T)$ for all $t \leq T$ and all s. To show this for some particular s_0 we compare the solution F with a solution U having a supporting hyperplane at s_0 as contract function. We then have $F(s,T) - U(s,T) \geq 0$, because F is convex, and thus $F(s,t) - U(s,t) \geq 0$ for every $t \leq T$. Moreover, $F(s_0,T) = U(s_0,T)$ and $U(s_0,t) = U(s_0,T)$ for all $t \leq T$ because U is affine. Hence,

$$F(s_0,t) \geq U(s_0,t) = U(s_0,T) = F(s_0,T).$$

Thus we have for arbitrary s and $t_1 \geq 0$ that

$$F(s, T - t_1) \geq F(s,T).$$

Now, let us consider both sides of this inequality as contract functions and consider the corresponding solutions at some time $T - t_2$ where $t_2 \geq 0$. The corresponding solutions satisfy the same inequality by the argument above. However, the time-independence of the equation yield that these solutions are simply given by translates in time of $F(s,t)$ and we obtain

$$F(s, T - t_1 - t_2) \geq F(s, T - t_2)$$

which is the desired monotonicity in t. ∎

Remark on convexity and time-decay of option prices and monotonicity in volatility. Consider a market with two underlying assets S_1 and S_2. Let this market have a diffusion matrix which is independent of time in accordance with the theorem above and with a convex contract function. Then the theorem yields that the option price decays with time. However, let the contract function be that of a call option, with strike price K on one of the assets, say S_1, but let the volatility of S_1 depend on S_2 in such a way that the volatility has a strict local maximum for some value $s_{2,0}$ of S_2. It is then easy to see that the solution to the pricing equation is not convex near the point $(K, s_{2,0})$ in the $S_1 - S_2$-plane. Thus convexity is lost but not time-decay of the prices. Another property which is lost with the convexity is the monotonicity in volatility. In the present example we see that if the volatility in S_2 is made larger then the value of the option at $(K, s_{2,0})$ decreases.

In the time-dependent case there are no general results corresponding to the theorem above. One can, for instance, easily modify the example above by letting the volatility for the second asset be time-dependent such that it is very large at some time and then decreases to a very small value until the time of expiration. Then the option price at $(K, s_{2,0})$ will increase with time during some interval. This corresponds directly to examples in [B-G-W] of bloating option prices when the volatility is stochastic.

3. Convexity of option prices

The results of the previous section show that time-decay for option prices with convex contract functions only holds for certain classes of volatility matrices. We also noted that time-decay can hold without preservation of convexity holding, but as we remarked in the introduction, if the convexity is preserved then the option prices do decay with time.

The main result of this section indicates that preservation of convexity is indeed quite rare. The counterexample above against preservation of convexity in the case of time independent processes involved dependent processes. Let us therefore limit ourselves to independent price processes with, naturally, a time independent volatility matrix. For the sake of notational convenience we then choose the Brownian motions so that S_i has a risk neutral processes given by

$$dS_i = S_i(t)\sigma_{ii}(S_i(t))\, dB_i$$

for $i = 1, \ldots, n$, where, as above, B_i are independent Brownian motions. Let us further consider the constant elasticity of variance model studied in [C-R], meaning that

$$\sigma_{ii} = c_i S_i(t)^{-\alpha_i}$$

where c_i is a constant and $0 \leq \alpha_i \leq 1$ for each i. Note that $\alpha_i = 0$ corresponds to Geometric Brownian motion and $\alpha_i = 1$ corresponds to Brownian motion. We then have the following result.

Theorem 2. *Let the stock prices be independent processes with*

$$\sigma_{ii} = c_i S_i(t)^{-\alpha_i}$$

where c_i is a constant and $0 \leq \alpha_i \leq 1$ for each i. Then the corresponding option prices are convex in the underlying assets for any convex contract *function, if and only if, for each i, $\alpha_i = 0$ or 1, i.e. if each price process is described by a Geometric Brownian motion or a Brownian motion.*

Proof. (Outline) In the case of Geometric Brownian motion and Brownian motion there are explicit solution formulas for the stochastic differential equation describing the price formation. Using these formulas one can show that convexity is preserved in these cases. Choosing contract functions of the form $(s_1 - s_2)^2 s_2^{-q}$, where $0 \leq q \leq 1$, one can show that for other processes included in the statement of the theorem, convexity is not necessarily preserved. $\square$

Remark. In [J-T2] we find necessary and sufficient conditions for the preservation of convexity of solutions to second order parabolic equations. The theorem above is a special case of the results in this paper.

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