# RADEMACHER CHAOS: TAIL ESTIMATES VS LIMIT THEOREMS

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ABSTRACT. We study Rademacher chaos indexed by a sparse set which has a fractional combinatorial dimension. We obtain tail estimates for finite sums and a normal limit theorem as the size tends to infinity. The tails for finite sums may be much larger that the tails of the limit.

## 1. INTRODUCTION AND RESULTS

A (homogeneous) Rademacher chaos is a random variable of the type

$$S = \sum_{i_1 < \cdots < i_d} a_{i_1 \cdots i_d} r_{i_1} \cdots r_{i_d}, \qquad (1.1)$$

where $d \geq 1$, $a_{i_1 \cdots i_d}$ are real or complex numbers and $r_1, r_2, \ldots$ is a sequence of independent random variables with the symmetric two-point distribution $\mathbb{P}(r_i = 1) = \mathbb{P}(r_i = -1) = 1/2$. (For example, $r_i$ could be the classical Rademacher functions [19], defined on $[0, 1]$ (with the usual Lebesgue measure) by $r_i(x) = 1 - 2b_i$ when $x \in [0, 1]$ has the binary expansion $0.b_1 b_2 \ldots$, but it is often more convenient to let $r_i$ be defined on the Cantor group $\mathbb{Z}_2^\infty$. For our purposes, the choice of $r_i$ does not matter.) Equivalently, $S$ is a linear combination of the Walsh functions of the type $r_{i_1} \cdots r_{i_d}$.

We will consider only finite sums (1.1), so there is no problem of convergence, and all moments of $S$ are finite.

We are interested in two related properties of the random variables $S$: the tail behaviour, i.e. the size of the probabilities $\mathbb{P}(|S| > x)$ for large $x$, and the size of the $L^q$ norms $\|S\|_q = \left(\mathbb{E}\,|S|^q\right)^{1/q}$ for large $q$. For convenience, we define $\widetilde{S} = S/\|S\|_2$; thus $\mathbb{E}\,\widetilde{S} = 0$ and $\mathrm{Var}\,\widetilde{S} = \mathbb{E}\,\widetilde{S}^2 = 1$.

Bonami's hypercontractive inequality [4] implies that every $S$ in (1.1) satisfies

$$\|S\|_q \leq (q-1)^{d/2}\|S\|_2 = (q-1)^{d/2}\Big( \sum_{i_1 < \cdots < i_d} |a_{i_1 \cdots i_d}|^2 \Big)^{1/2}, \qquad q \geq 2, \quad (1.2)$$

or, equivalently, $\|\widetilde{S}\|_q \leq (q-1)^{d/2}$, $q \geq 2$. (See also [1, 2, 11, 13].)

In general, this estimate is best possible, up to a constant depending on $d$ but not on $q$. For example, it is easily seen that $a_{i_1 \cdots i_d} = 1$ for $1 \leq i_1 < \cdots < i_d \leq n$ (and 0 otherwise) yields an $S = S_n$ that after suitable normalization converges, as $n \to \infty$, to a (Hermite) $d$-degree polynomial in a Gaussian random variable; see Example 3.2. It thus follows that for some $c(d) > 0$ and every $q \geq 2$,

$\|S\|_q \geq c(d)(q-1)^{d/2}\|S\|_2$ provided $n$ is large enough. (See e.g. [11, Chapter XIII].)

In this paper we study Rademacher chaos (1.1) where most coefficients $a_{i_1 \cdots i_d} = 0$ so that we really only sum over an indexing set which is combinatorially sparse in the sense of [2, Chapters XII and XIII]. In this case, Bonami's hypercontractive inequality (1.2) can be improved, precisely reflecting the sparsity of the indexing set.

We first recall some definitions [2, Chapters XIII], which we modify and adapt to our purposes in this paper.

For $F \subseteq \mathbb{N}^d$ and $\alpha > 0$, define

$$\Psi_F(s) = \max\{|F \cap (A_1 \times \cdots \times A_d)| : A_j \subset \mathbb{N}, |A_j| \leq s, j = 1, \ldots, m\},$$

and

$$d_F(\alpha) = \sup_{s \geq 1} \Psi_F(s)/s^\alpha = \sup_{A_1, \ldots, A_d} |F \cap (A_1 \times \cdots \times A_d)| \big/ \big(\max_{j \leq d} |A_j|\big)^\alpha.$$

In [2, §XIII.4], the combinatorial dimension of a set $F \subseteq \mathbb{N}^d$ is defined to be

$$\dim(F) = \limsup_{s \to \infty} \big(\log \Psi_F(s)/\log s\big)$$
$$= \sup\{\alpha : d_F(\alpha) = \infty\} = \inf\{\alpha : d_F(\alpha) < \infty\}. \tag{1.3}$$

In this paper, we consider sequences of index sets $F_N \subseteq [N]^d$, where $[N] = \{1, \ldots, N\}$, and we adopt the definition below. Because we want to consider only non-empty index sets, we consider sequences starting at some index $N_0 \geq 1$; this allows for some empty $F_N$ for smaller $N$ that we ignore.

**Definition.** A sequence $F_N \subseteq [N]^d$, $N = N_0, N_0 + 1, \ldots$, has combinatorial dimension $\alpha$ if there exist positive constants $C_1$, $C_2$ such that for all $N \geq N_0$,

$$d_{F_N}(\alpha) \leq C_1,$$

(i.e. $|F_N \cap (A_1 \times \cdots \times A_d)| \leq C_1\big(\max_{j \leq d} |A_j|\big)^\alpha)$, and

$$|F_N| \geq C_2 N^\alpha.$$

We write $\dim\{F_N\} = \alpha$.

Given a set $F \subseteq \mathbb{N}^d$, we define $F_N = F \cap [N]^d$. In the present paper, we define $\dim(F) = \dim\{F_N\}$ when the latter exists (and leave the dimension undefined otherwise).

**Remark 1.1.** Note that this is a stricter definition than (1.3); there are sets $F$ with no dimension in the present sense, but it is easily seen that when the dimension exists in the present sense, it coincides with (1.3).

If the cardinalities of $F_N$ are uniformly bounded, then $\dim\{F_N\} = 0$; otherwise $1 \leq \dim\{F_N\} \leq d$ (if $\dim\{F_N\}$ exists at all). We are mainly interested in the case $1 < \dim\{F_N\} < d$.

Let $\Delta^d = \{(i_1, \ldots, i_d) : 1 \leq i_1 < \cdots < i_d < \infty\}$ and $\Delta_N^d = \Delta^d \cap [N]^d$. We will in the sequel consider only $F \subseteq \Delta^d$ and $F_N \subseteq \Delta_N^d$; This is not essential, but restriction to ordered sets of indices is convenient when we study sums (1.6).

It is proved in [2, Chapter XIII] that for every $\alpha \in [1, d]$, there exist sets $F \subset \Delta^d$ of combinatorial dimension $\alpha$ (also in the stricter sense used here). Such sets can always be constructed by a random procedure; for rational $\alpha \geq 1$ and $d$ such that $d\alpha$ is an integer, it is also possible to use the following deterministic construction.

**Example 1.2.** (*Minimal fractional Cartesian products* [2, §XIII.1 and p. 493].) Fix arbitrary integers $d \geq 3$ and $1 \leq m \leq d$, and let $\{S_1, \ldots, S_d\}$ be a cover of $[d]$ consisting of $m$-subsets of $[d]$, such that every $i \in [d]$ appears in exactly $m$ elements of $S_1, \ldots, S_d$; i.e., $\bigcup_{j=1}^d S_j = [d]$, $|S_j| = m$, and for every $i \in [d]$, $|\{j : i \in S_j\}| = m$.

We employ the following notation: if $X$ is a set, $\mathbf{y} = (y_1, \ldots, y_d) \in X^d$ and $S \subseteq [d]$, then

$$\pi_S \mathbf{y} = (y_i : i \in S).$$

For an integer $N \geq d^m$, let $n$ be the greatest integer such that $n \leq N^{1/m}$. Fix a one-one map $\varphi$ from $[n]^m$ into $[N]$, and consider

$$F_N^* = \left\{ \left( \varphi(\pi_{S_1} \mathbf{k}), \ldots, \varphi(\pi_{S_d} \mathbf{k}) \right) : \mathbf{k} \in [n]^d \right\}. \tag{1.4}$$

In order to obtain a subset of $\Delta_N^d$, for the purposes of this paper, we modify this set to

$$F_N = \left\{ (i_1, \ldots, i_d) \in \Delta_N^d : (i_{\rho 1}, \ldots, i_{\rho d}) \in F_N^* \quad \text{for some permutation } \rho \right\}. \tag{1.5}$$

We call the sequence $\{F_N\}$ a *fractional Cartesian product*.

We further say the the fractional Cartesian product is *disconnected* if $[d]$ can be partitioned into two disjoint nonempty subsets $T_1$ and $T_2$ such that each $S_j$ is a subset of either $T_1$ or $T_2$, and *connected* otherwise.

The archetypal case is $d = 3$, $m = 2$, $S_1 = \{1, 2\}$, $S_2 = \{1, 3\}$ and $S_3 = \{2, 3\}$. This gives a connected fractional Cartesian product.

*Claim.* $\dim\{F_N\} = \dim\{F_N^*\} = d/m$.

We verify the claim in the archetypal case $d = 3$, $m = 2$ only. The general case is similar; see [2, Corollary XIII.16].

Let $1 \leq s \leq N$ be an integer, and let $A$, $B$, $C$ be arbitrary subsets of $[N]$. Then,

$$|F_N^* \cap (A \times B \times C)| = \sum_{k_1, k_2, k_3 \in [n]} \mathbf{1}_A(\varphi(k_1, k_2)) \mathbf{1}_B(\varphi(k_1, k_3)) \mathbf{1}_C(\varphi(k_2, k_3)).$$

A three-fold application of the Cauchy–Schwarz inequality yields

$$|F_N^* \cap (A \times B \times C)| \leq \left( \sum_{k_1, k_2 \in [n]} \mathbf{1}_A(\varphi(k_1, k_2)) \right)^{1/2} \left( \sum_{k_1, k_3 \in [n]} \mathbf{1}_B(\varphi(k_1, k_3)) \right)^{1/2}$$

$$\cdot \left( \sum_{k_2, k_3 \in [n]} \mathbf{1}_C(\varphi(k_2, k_3)) \right)^{1/2}$$

$$\leq |A|^{1/2} |B|^{1/2} |C|^{1/2},$$

which implies $\Psi_{F_N^*}(s) \leq s^{3/2}$ and thus $\Psi_{F_N}(s) \leq 6s^{3/2}$. In the opposite direction,

$$|F_N| \geq |\Delta_n^3| = \binom{n}{3} \geq c_1 n^3 \geq c_2 N^{3/2}.$$

**Remark 1.3.** Again, the definition differs slightly from [2]; there the fractional Cartesian product is defined on an infinite set ($n = \infty$ in (1.4)).

**Remark 1.4.** Note that the function $\varphi$ appears in the definition of a fractional Cartesian product, only because we let the indices be integers in this paper. We might avoid $\varphi$ by changing the notation slightly; for example, for the case $d = 3$, $m = 2$, we could equivalently write (1.9) below as $S_N = \sum_{i<j<k\leq n} r_{ij}r_{ik}r_{jk}$, where $r_{ij}$, $i < j$ are independent Rademacher variables.

It is shown in [2] (e.g., Corollary XIII.29, see Remark 1.9 below) that if $F \subseteq \Delta^d$ (finite or infinite), and $S$ is a Rademacher chaos

$$S = \sum_{(i_1,\ldots,i_d)\in F} a_{i_1\cdots i_d} r_{i_1} \cdots r_{i_d}, \tag{1.6}$$

then

$$\|S\|_q \leq K d_F(\alpha)^{1/2} q^{\alpha/2} \|S\|_2, \qquad q \geq 1, \tag{1.7}$$

where $K < \infty$ depends only on the ambient dimension $d$. In particular, if $\dim\{F_N\} < d$, the exponent in (1.2) can be improved, with $d$ replaced by the combinatorial dimension.

These norm estimates lead to tail estimates by the customary procedure: If (1.7) holds and $d_F(\alpha) < \infty$, then for any $x > 0$ and $q \geq 1$, by Markov's inequality,

$$\mathbb{P}(|\widetilde{S}| \geq x) \leq x^{-q} \, \mathbb{E} \, |\widetilde{S}|^q = x^{-q}\|\widetilde{S}\|_q^q \leq \left(x^{-1}Cq^{\alpha/2}\right)^q,$$

where $C = K d_F(\alpha)^{1/2}$. Taking $q = (x/C)^{2/\alpha} e^{-1}$ (if $x \geq Ce^{\alpha/2}$), we obtain

$$\mathbb{P}(|\widetilde{S}| \geq x) \leq e^{-\alpha q/2} = \exp\left(-cx^{2/\alpha}\right), \tag{1.8}$$

for a constant $c > 0$ depending on $d$, $\alpha$ and $d_F(\alpha)$ only.

The norm and tail estimates above are in fact sharp, in a sense made precise below. (Cf. [2, Corollary XIII.29].) For simplicity, we will consider only the case where $a_{i_1\cdots i_d} = 1$ or $0$. Specifically, we consider a sequence of non-empty sets $F_N \subseteq \Delta_N^d$ and Rademacher chaos

$$S_N = \sum_{(i_1,\ldots,i_d)\in F_N} r_{i_1} \cdots r_{i_d}. \tag{1.9}$$

Clearly, $\|S_N\|_2 = |F_N|^{1/2}$, and thus $\widetilde{S}_N = |F_N|^{-1/2} S_N$.

**Theorem 1.5.** *Suppose* $\dim\{F_N\} = \alpha \geq 1$, *where* $F_N \subseteq \Delta_N^d$. *Let* $S_N$ *be given by* (1.9). *Then there exist positive constants* $c_1$, $c_2$, $c_3$, $c_4$ *(depending only on* $d$, $\alpha$, $C_1$, $C_2$, $N_0$ *above) such that for every* $q \geq 1$,

$$c_1 q^{\alpha/2} \leq \sup_N \|\widetilde{S}_N\|_q \leq c_2 q^{\alpha/2}, \tag{1.10}$$

*and for all $x \geq 2$,*

$$\exp(-c_3 x^{2/\alpha}) \leq \sup_N \mathbb{P}(|\widetilde{S}_N| > x) \leq \exp(-c_4 x^{2/\alpha}). \qquad (1.11)$$

A natural question arises: can $\sup_N$ in (1.10) and (1.11) be replaced by $\lim_{N\to\infty}$? (See Remark ii in [2, p.524].) In the standard integer-dimensional case $F_N = \Delta^d$, the answer is affirmative (by a $d$-fold application of the usual Central Limit theorem). But in many fractional-dimensional cases, the answer is negative: the precise relation between tail estimates and combinatorial dimension, as per (1.11), is completely wiped out in the limit. We illustrate this in two important cases.

**Theorem 1.6.** *Let $F_N \subseteq \Delta_N^d$, $N = 1, \ldots$, and let $S_N$ be given by (1.9). Suppose either (i) $d = 2$ and $1 < \dim\{F_N\} < 2$, or (ii) $F_N$ is a connected fractional Cartesian product as in Example 1.2. Then $\widetilde{S}_N \xrightarrow{\mathrm{d}} N(0,1)$ with convergence of all moments. In particular, if $\xi \sim N(0,1)$, then, for all $q \geq 1$,*

$$\lim_{N\to\infty} \|\widetilde{S}_N\|_q = \|\xi\|_q \leq q^{1/2}$$

*and for all $x \geq 2$,*

$$\lim_{N\to\infty} \mathbb{P}(|\widetilde{S}_N| > x) = \mathbb{P}(|\xi| > x) \leq \exp(-x^2/2).$$

Case (ii) with $m = 2$ can be translated (using Remark 1.4) into a result for random graphs, which is a special case (with $p = 1/2$) of [9, Theorem 1]; see also [10] and [11, Chapter XI].

Theorems 1.5 and 1.6 complement one another in the following (heuristic) sense. Let us agree that tail probabilities of sums of uncorrelated symmetric variables provide a gauge of interdependence between the variables: larger tail probabilities (smaller likelihood of cancellations) convey higher degree of interdependence, and conversely. In this light, Theorem 1.5 provides a precise assessment of interdependence of the random variables $r_{i_1} \cdots r_{i_d}$, $(i_1, \ldots, i_d) \in F_N$. As a counterpoint, reflecting increasing sparsity of $F_N$ relative to the full product set $\Delta_N^d$, Theorem 1.6 asserts that $F_N$ *in the limit*, as $N \to \infty$, is asymptotically independent.

Theorems 1.5 and 1.6 show that, for large $q$ or $x$, the limits as $N \to \infty$ are much smaller than the largest values for finite $N$. If we fix a large $q$ and study $\|\widetilde{S}_N\|_q$ as $N$ grows, we begin with rather small values (at most $|F_N|^{1/2}$) that grow to a maximum of the order $q^{\alpha/2}$ (when $N$ is about $q$, see Section 2), but then the norms decrease again towards a limit of the order $q^{1/2}$. (We do not know whether the increase and decrease are monotone; there might be several local maxima.) A similar story holds for $\mathbb{P}(|\widetilde{S}_N| > x)$ for a fixed large $x$. Consequently, the limit results in Theorem 1.6 are misleading when we consider $\widetilde{S}_N$ for finite $N$.

A central limit theorem in fact holds generally under a condition of sparsity in $F_N$ that is milder than the sparsity implied by non-integer combinatorial dimension. The condition is in effect that $F_N$ is not "too close" to a product

set. To express this precisely we use the following terminology. For $j \in [N]$, define

$$F_{Nj}^* = \big\{(i_1, \ldots, i_d) \in F_N : j \in \{i_1, \ldots, i_d\}\big\}.$$

Further, let $F_N^{\#}$ be the subset of $F_N \times F_N$ defined as follows: a pair of $d$-tuples $\big((i_1, \ldots, i_d), (j_1, \ldots, j_d)\big) \in F_N^{\#}$ if $\{i_1, \ldots, i_d\} \cap \{j_1, \ldots, j_d\} = \emptyset$ and there exist $(k_1, \ldots, k_d) \in F_N$, $(l_1, \ldots, l_d) \in F_N$ such that $\{k_1, \ldots, k_d, l_1, \ldots, l_d\} = \{i_1, \ldots, i_d, j_1, \ldots, j_d\}$ but $(k_1, \ldots, k_d)$ does not equal $(i_1, \ldots, i_d)$ or $(j_1, \ldots, j_d)$. (In other words, the $2d$ indices $i_1, \ldots, i_d, j_1, \ldots, j_d$ can be partitioned in at least two ways into elements of $F_N$.)

**Theorem 1.7.** *Suppose*

$$\lim_{N \to \infty} \max_j |F_{Nj}^*|/|F_N| = 0 \tag{1.12}$$

*and*

$$\lim_{N \to \infty} \big(|F_N^{\#}|/|F_N|^2\big) = 0. \tag{1.13}$$

*Then $\widetilde{S}_N \xrightarrow{d} N(0,1)$, with convergence of all moments.*

We have the following partial converse. (The trivial example $F = \{(1, j) : j \geq 2\}$ shows that (1.12) is not necessary; we do not know whether it is needed at all in Theorem 1.7.)

**Theorem 1.8.** *Suppose that $\widetilde{S}_N \xrightarrow{d} N(\mu, \sigma^2)$ for some $\mu$ and $\sigma^2 > 0$. Then $\mu = 0$, $\sigma^2 = 1$ and (1.13) holds.*

The proof of Theorem 1.5 is given in Section 2, and the proofs of Theorems 1.6, 1.7, 1.8 are given in Section 3. Some simple examples of nonnormal limits when (1.13) is not satisfied are given also in Section 3. Further remarks and open problems are presented in Section 4.

**Remark 1.9.** The results in the present paper use Corollary XIII.29 in [2]. A correction to an argument in the proof of that theorem is included in the preprint version of the present paper [3]. The referee has pointed out that (1.7) also follows from [2, Corollary XIII.28] together with the decoupling inequality, see e.g. [6, Theorem 3.1.1].

## 2. Proof of Theorem 1.5

The upper bounds follow by (1.7) and (1.8) (for $x \geq x_0$, say; the case $2 \leq x \leq x_0$ follows by Chebyshev's inequality if $c_4$ is small enough).

To verify the lower bounds, let $\mathcal{E}_N$ be the event $r_1 = \cdots = r_N = 1$; thus $\mathbb{P}(\mathcal{E}_N) = 2^{-N}$. On $\mathcal{E}_N$, we have $S_N = |F_N|$ and thus $\widetilde{S}_N = |F_N|^{1/2}$. Hence, for every $q \geq 1$,

$$\|\widetilde{S}_N\|_q \geq |F_N|^{1/2} \, \mathbb{P}(\mathcal{E}_N)^{1/q} \geq cN^{\alpha/2} 2^{-N/q}.$$

To verify the left inequality in (1.10), in the line above choose $N = \max(N_0, \lfloor q \rfloor)$.

Similarly, given $x$ (large enough), let $N = \lceil Cx^{2/\alpha} \rceil$ for a constant $C > C_2^{-1/\alpha}$, where $C_2$ is as in the definition above. Then, on $\mathcal{E}_N$

$$\widetilde{S}_N = |F_N|^{1/2} \geq C_2^{1/2} N^{\alpha/2} > x,$$

and thus
$$\mathbb{P}(\widetilde{S}_N > x) \geq \mathbb{P}(\mathcal{E}_N) = 2^{-N} \geq \exp(-c_3 x^{2/\alpha}). \qquad \square$$

## 3. Asymptotic normality

**Lemma 3.1.** *If $d = 2$, then for any $\alpha$ and any finite subsets $A$ and $B$ of $\mathbb{N}$ such that $|A| \leq |B|$,*

$$|F \cap (A \times B)| \leq 2d_F(\alpha)|A|^{\alpha-1}|B|,$$
$$|F \cap (B \times A)| \leq 2d_F(\alpha)|A|^{\alpha-1}|B|.$$

*Proof.* We may assume $|A| \geq 1$. Partition $B$ into $\lceil |B|/|A| \rceil \leq 2|B|/|A|$ subsets $B_j$ with $|B_j| \leq |A|$. For each $j$, $|F \cap (A \times B_j)| \leq \Psi_F(|A|) \leq d_F(\alpha)|A|^\alpha$, and similarly for $|F \cap (B_j \times A)|$. The result follows by summing over $j$. $\square$

*Proof of Theorem 1.6.* We verify the conditions of Theorem 1.7. Let $\alpha = \dim\{F_N\}$.

First consider case (i), i.e., suppose that $d = 2$ and $1 < \alpha < 2$. Then, $|F_{Nj}^*| \leq N = o(|F_N|)$, which verifies (1.12).

Next, choose $\varepsilon_N > 0$ such that $\varepsilon_N \to 0$ and $N\varepsilon_N^{1/(2-\alpha)} \to \infty$. (For example, $\varepsilon_N = N^{-\delta}$ for $0 < \delta < 2 - \alpha$.) Let $A = A_N = \{i : |F_{Ni}^*| \geq \varepsilon_N N\}$. Then,

$$|F_N \cap (A \times [N])| + |F_N \cap ([N] \times A)| = \sum_{i \in A} |F_{Ni}^*| \geq \varepsilon_N N|A|,$$

and thus, by Lemma 3.1

$$\varepsilon_N N|A| \leq 4d_{F_N}(\alpha)|A|^{\alpha-1}N,$$

which implies

$$|A| \leq \left(4d_{F_N}(\alpha)/\varepsilon_N\right)^{1/(2-\alpha)} = o(N). \tag{3.1}$$

By definition, $F_N^\#$ is the set of all $\big((i,j),(k,l)\big) \in F_N \times F_N$, all of which entries are distinct, such that either $\big(\{i,k\},\{j,l\}\big) \in F_N \times F_N$ or $\big(\{i,l\},\{j,k\}\big) \in F_N \times F_N$ (or both), where $\{i,k\} = (i,k)$ when $i < k$ and $(k,i)$ when $i > k$. We let $F_{N1}^\#$ be the subset of $F_N^\#$ where $i \in A$, and $F_{N2}^\#$ the subset where $i \notin A$.

The number of possible $(i,j) \in F_N$ with $i \in A$ is $|F_N \cap (A \times [N])|$, and thus, by Lemma 3.1 and (3.1),

$$|F_{N1}^\#| \leq |F_N \cap (A \times [N])| \cdot |F_N| \leq 2d_{F_N}(\alpha)|A|^{\alpha-1}N|F_N|$$
$$= o\big(N^\alpha|F_N|\big) = o(|F_N|^2). \tag{3.2}$$

On the other hand, let $F_{Ni}^{**} = \{k : (i,k) \in F_{Ni}^* \text{ or } (k,i) \in F_{Ni}^*\}$. Thus $|F_{Ni}^{**}| = |F_{Ni}^*|$. If $\big((i,j),(k,l)\big) \in F_N^\#$, then either $k$ or $l$ is in $F_{Ni}^{**}$, and thus the number of possible $(k,l) \in F_N$ for a given $i \notin A$ is at most, again by Lemma 3.1,

$$|F_N \cap (F_{Ni}^{**} \times [N])| + |F_N \cap ([N] \times F_{Ni}^{**})| \leq 4d_F(\alpha)|F_{Ni}^{**}|^{\alpha-1}N \leq 4d_F(\alpha)\varepsilon_N^{\alpha-1}N^\alpha.$$

Summing over all possible $(i,j)$ we find

$$|F_{N2}^\#| \leq 4d_F(\alpha)\varepsilon_N^{\alpha-1}N^\alpha|F_N| = o(|F_N|^2). \tag{3.3}$$

Combining (3.2) and (3.3), we obtain (1.13) and the result follows in this case.

In case (ii), we first observe that fixing an index $j$ in $F_N$ means that the corresponding $\mathbf{k}$ in (1.4) is such that $\pi_{S_i}\mathbf{k} = \varphi^{-1}(j)$ for some $i$; for each $i$ this means that $m$ of the $d$ coordinates of $\mathbf{k}$ have given values, so the number of choices of $\mathbf{k}$ is at most $dn^{d-m}$. Consequently, $|F_{Nj}^*| \le dn^{d-m} = o(|F_N|)$, proving (1.12).

Next, suppose that $\big((i_1,\ldots,i_d),(j_1,\ldots,j_d)\big) \in F_N^\#$, and that $(i_1,\ldots,i_d)$ and $(j_1,\ldots,j_d)$ are generated by (1.4) and (1.5) by some vectors $\mathbf{i}$ and $\mathbf{j}$ in $[n]^d$, respectively. By the definition of $F_N^\#$, there exists also $(k_1,\ldots,k_d) \in F_N$, generated in the same way by, say, $\mathbf{k} \in [n]^d$, such that $\{k_1,\ldots,k_d\} \subseteq \{i_1,\ldots,i_d,j_1,\ldots,j_d\}$ but $(k_1,\ldots,k_d)$ does not equal $(i_1,\ldots,i_d)$ or $(j_1,\ldots,j_d)$.

Hence, each $\pi_{S_\nu}\mathbf{k}$, $1 \le \nu \le d$, coincides with some $\pi_{S_\mu}\mathbf{i}$ or $\pi_{S_\mu}\mathbf{j}$, $1 \le \mu \le d$. Define

$$J_1 = \{\nu \in [d] : \pi_{S_\nu}\mathbf{k} = \pi_{S_\mu}\mathbf{i} \text{ for some } \mu\};$$
$$J_2 = \{\nu \in [d] : \pi_{S_\nu}\mathbf{k} = \pi_{S_\mu}\mathbf{j} \text{ for some } \mu\};$$
$$T_s = \bigcup_{\nu \in J_s} S_\nu, \qquad s = 1, 2.$$

Then $J_1 \cup J_2 = [d]$ and $T_1 \cap T_2 \ne \emptyset$, because otherwise the fractional Cartesian product would be disconnected.

If $q \in T_1$, then $q \in S_\nu$ for some $\nu \in J_1$, and thus $\pi_{S_\nu}\mathbf{k} = \pi_{S_\mu}\mathbf{i}$ for some $\mu$. In particular, the $q$th coordinate of $\mathbf{k}$ is one of the coordinates of $\mathbf{i}$. Similarly, if $q \in T_2$, then the $q$th coordinate of $\mathbf{k}$ is one of the coordinates of $\mathbf{j}$.

Because $T_1 \cap T_2 \ne \emptyset$, it follows that $\mathbf{i}$ and $\mathbf{j}$ have at least one coordinate in common (not necessarily in the same position). Consequently, the number of possible pairs $(\mathbf{i}, \mathbf{j})$ is $O(n^{2d-1})$, and

$$|F_N^\#| = O\big(n^{2d-1}\big) = O\big(N^{2(d/m)-1}\big) = o\big(N^{2\alpha}\big) = o\big(|F_N|^2\big),$$

verifying (1.13). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

*Proof of Theorem 1.7.* All limits in the proof are as $N \to \infty$. We begin by observing that the assumption (1.12) implies

$$\sum_{j=1}^N |F_{Nj}^*|^2 \le \max_j |F_{Nj}^*| \sum_{j=1}^N |F_{Nj}^*| \le \max_j |F_{Nj}^*| \cdot d|F_N| = o\big(|F_N|^2\big). \qquad (3.4)$$

We use the martingale central limit theorem, as stated in [16, Corollary (2.13)]. We let

$$F_{Nj} = \big\{(i_1,\ldots,i_d) \in F_N : i_d = j\big\} \subseteq F_{Nj}^*,$$

and let

$$X_{Nj} = \sum_{(i_1,\ldots,i_d)\in F_{Nj}} r_{i_1}\cdots r_{i_d} = r_j \sum_{(i_1,\ldots,i_d)\in F_{Nj}} r_{i_1}\cdots r_{i_{d-1}}.$$

Then

$$S_N = \sum_{j=1}^N X_{Nj}$$

and, with $\widetilde{X}_{Nj} = |F_N|^{-1/2}X_{Nj}$,

$$\widetilde{S}_N = \sum_{j=1}^{N} \widetilde{X}_{Nj}.$$

Evidently, $(\widetilde{X}_{Nj})_{j=1}^{N}$ is a martingale difference sequence for the filtration $\mathcal{F}_j = \mathcal{F}(r_1, \ldots, r_j)$, and we have $\mathbb{E}\,\widetilde{S}_N^2 = \sum \mathbb{E}\,\widetilde{X}_{Nj}^2 = 1$.

By [16, Corollary (2.13)], to prove $\widetilde{S}_N \xrightarrow{d} N(0,1)$ it suffices to verify the Lindeberg condition

$$\sum_{j=1}^{N} \mathbb{E}\big(\widetilde{X}_{Nj}^2 \mathbf{1}[|\widetilde{X}_{Nj}| > \varepsilon]\big) \to 0 \qquad \text{for every } \varepsilon > 0, \tag{3.5}$$

together with

$$\limsup_{N \to \infty} \sum_{i \neq j} \mathbb{E}\big(\widetilde{X}_{Ni}^2 \widetilde{X}_{Nj}^2\big) \leq 1. \tag{3.6}$$

Because every moment of $\widetilde{S}_N$ stays bounded by (1.2), moment convergence will follow as well.

To prove (3.5) it suffices to show

$$\sum_{j=1}^{N} \mathbb{E}\,\widetilde{X}_{Nj}^4 \to 0. \tag{3.7}$$

In our case, by (1.2) we note $\|\widetilde{X}_{Nj}\|_4 \leq 3^{d/2}\|\widetilde{X}_{Nj}\|_2$ and therefore

$$\sum_{j=1}^{N} \mathbb{E}\,\widetilde{X}_{Nj}^4 \leq 3^{2d}\sum_{j=1}^{N}\|\widetilde{X}_{Nj}\|_2^4 = 3^{2d}\sum_{j=1}^{N}\frac{|F_{Nj}|^2}{|F_N|^2} \leq 3^{2d}|F_N|^{-2}\sum_{j=1}^{N}|F_{Nj}^*|^2,$$

which by (3.4) implies (3.7).

It remains to verify (3.6). For simplicity we treat first the case $d = 2$, and will describe later the modifications needed in the general case. If $d = 2$, then

$$\mathbb{E}\big(X_{Ni}^2 X_{Nj}^2\big) = \sum_{k,l,m,n} \mathbb{E}\,\mathbf{1}_{F_N}(k,i)\mathbf{1}_{F_N}(l,i)\mathbf{1}_{F_N}(m,j)\mathbf{1}_{F_N}(n,j)r_k r_l r_m r_n.$$

We have, $\mathbb{E}\,r_k r_l r_m r_n = 0$ unless the indices $k, l, m, n$ coincide in pairs, and obtain (overcounting the case when all four indices coincide)

$$\mathbb{E}\big(X_{Ni}^2 X_{Nj}^2\big) \leq \sum_{k,m} \mathbf{1}_{F_N}(k,i)\mathbf{1}_{F_N}(k,i)\mathbf{1}_{F_N}(m,j)\mathbf{1}_{F_N}(m,j)$$

$$+ 2\sum_{k,l} \mathbf{1}_{F_N}(k,i)\mathbf{1}_{F_N}(l,i)\mathbf{1}_{F_N}(k,j)\mathbf{1}_{F_N}(l,j).$$

Summing the first term on the right over all $i$ and $j$, we obtain $|F_N|^2$. Therefore, to show (3.6), it suffices to verify

$$\sum_{i \neq j}\sum_{k,l} \mathbf{1}_{F_N}(k,i)\mathbf{1}_{F_N}(l,i)\mathbf{1}_{F_N}(k,j)\mathbf{1}_{F_N}(l,j) = o(|F_N|^2). \tag{3.8}$$

The sum above equals the number of pairs $\big((k,i),(l,j)\big) \in F_N \times F_N$ such that also $\big((l,i),(k,j)\big) \in F_N \times F_N$. The number of such pairs with distinct $i, j, k, l$ is at most $|F_N^\#|$. Further, the number of pairs $\big((k,i),(l,j)\big) \in F_N \times F_N$ where two indices are equal to some $r$ is at most $|F_{Nr}^*|^2$. Consequently, the sum in (3.8) is at most

$$|F_N^\#| + \sum_{r=1}^N |F_{Nr}^*|^2,$$

and (3.8) follows by (1.13) and (3.4).

In the case $d \geq 2$, we similarly find that $\mathbb{E}\, X_{Ni}^2 X_{Nj}^2$ equals the number of quadruples $I_1, I_2, I_3, I_4$ of $d$-tuples in $F_N$ wherein the $4d$ indices coincide in pairs, and the last index is $i$ in $I_1$ and $I_2$ and $j$ in $I_3$ and $I_4$. We group such quadruples according to the positions of the pairs of coinciding elements (again overcounting in the cases with less than $2d$ distinct indices, when there are several possibilities of pairing).

To do this precisely, let $\hat{I}_k = \{1, \ldots, d\} \times \{k\}$, $k = 1, 2, 3, 4$; thus, $\hat{I}_1, \hat{I}_2, \hat{I}_3, \hat{I}_4$ are four disjoint copies of $\{1, \ldots, d\}$. We define a *pattern* to be a complete matching in $\hat{I}_1 \cup \hat{I}_2 \cup \hat{I}_3 \cup \hat{I}_4$, i.e. a partition of the $4d$ points into $2d$ pairs, which are regarded as the edges of a graph.

For a pattern $\pi$, any assignment of indices in $\{1, \ldots, N\}$ to the $2d$ edges defines 4 $d$-tuples $I_1, I_2, I_3, I_4$ in the obvious way. Let $T_N(\pi)$ be the number of quadruples $(I_1, I_2, I_3, I_4) \in F_N^4$ generated this way, i.e., the number of all assignments such that $I_1 \in F_N$, $I_2 \in F_N$, $I_3 \in F_N$, $I_4 \in F_N$.

Finally, let $\Pi'$ denote the set of all patterns that contain the two edges $\{(d,1),(d,2)\}$ and $\{(d,3),(d,4)\}$.

In this framework, we then observe

$$\sum_{i,j} \mathbb{E}\, X_{Ni}^2 X_{Nj}^2 \leq \sum_{\pi \in \Pi'} T_N(\pi).$$

We classify the patterns in $\Pi'$ into three types: a pattern is of type I if all its edges are inside $\hat{I}_1 \cup \hat{I}_2$ or $\hat{I}_3 \cup \hat{I}_4$; it is of type II if it is not of type I and there are no edges connecting $\hat{I}_2$ and $\hat{I}_3$, and type III otherwise.

First, consider a pattern $\pi$ of type I. Since the $d$-tuples in $F_N$ are ordered, it follows that $T_N(\pi) = 0$ unless $\pi$ is the pattern with edges $\{(i,1),(i,2)\}$ and $\{(i,3),(i,4)\}$, $i = 1, \ldots, d$. In this case, $I_1 = I_2$ and $I_3 = I_4$, which are arbitrary elements of $F_N$, and thus $T_N(\pi) = |F_N|^2$.

Because the set of patterns is finite, it suffices to show that $T_N(\pi) = o(|F_N|^2)$ for every pattern $\pi$ of type II or III.

If $\pi$ is of type II, then $I_1$ and $I_4$ together determine $I_2$ and $I_3$. As in the case $d = 2$, the number of allowed pairs $(I_1, I_4)$ with distinct indices is at most $|F_N^\#|$, and the number of pairs $(I_1, I_4)$ with at least one common index is at most $\sum_{r=1}^N |F_{Nr}^*|^2$. Therefore $T_N(\pi) = o(|F_N|^2)$ by (1.13) and (3.4).

Finally, suppose that $\pi$ is of type III. Let $\hat{I}_L = \hat{I}_1 \cup \hat{I}_2$ and $\hat{I}_R = \hat{I}_3 \cup \hat{I}_4$, and call these the left and right sides of the pattern. We further say that the points $(i, k) \in \hat{I}_L$ and $(i, k+2) \in \hat{I}_R$ are the *mirror images* of one another.

Suppose that there are $r$ edges between $\hat{I}_L$ and $\hat{I}_R$; call these $r$ edges crossing, and order them (in some way). Let $t_N^L(k_1, \ldots, k_r)$ be the number of ways to assign indices to the edges inside $\hat{I}_L$ such that, with $k_1, \ldots, k_r$ assigned to the crossing edges, $I_1, I_2 \in F_N$. Similarly, let $t_N^R(k_1, \ldots, k_r)$ be the corresponding number of ways to assign indices in $\hat{I}_R$ such that $I_3, I_4 \in F_N$. Then,

$$T_N(\pi) = \sum_{k_1, \ldots, k_r = 1}^{N} t_N^L(k_1, \ldots, k_r) t_N^R(k_1, \ldots, k_r).$$

Further, let $\pi'$ be the pattern obtained by taking the edges inside $\hat{I}_L$ in $\pi$ together with their mirror images in $\hat{I}_R$ and the edges connecting each remaining point to its mirror image. Define $\pi''$ similarly, starting with the edges inside $\hat{I}_R$ in $\pi$. Note that both $\pi'$ and $\pi''$ are patterns of type II. Then, by the Cauchy–Schwarz inequality,

$$\begin{aligned}
T_N(\pi) &= \sum_{k_1, \ldots, k_r = 1}^{N} t_N^L(k_1, \ldots, k_r) t_N^R(k_1, \ldots, k_r) \\
&\leq \Big( \sum_{k_1, \ldots, k_r = 1}^{N} t_N^L(k_1, \ldots, k_r)^2 \Big)^{1/2} \Big( \sum_{k_1, \ldots, k_r = 1}^{N} t_N^R(k_1, \ldots, k_r)^2 \Big)^{1/2} \\
&= T_N(\pi')^{1/2} T_N(\pi'')^{1/2} \\
&= o(|F_N|)^2,
\end{aligned}$$

where the final estimate holds because $\pi'$ and $\pi''$ are of type II.

This completes the proof of (3.6) and thus of the theorem. $\square$

*Proof of Theorem 1.8.* If $\widetilde{S}_N$ converges in distribution, then (1.2) implies that all moments converge (as remarked in the proof of Theorem 1.7). In particular, $\mu = \lim \mathbb{E}\, \widetilde{S}_N = 0$ and $\sigma^2 = \lim \mathbb{E}\, \widetilde{S}_N^2 = 1$; further,

$$\mathbb{E}\, \widetilde{S}_N^4 \to \mathbb{E}\, \xi^4 = 3. \tag{3.9}$$

Similarly, as in the proof above, $ES_N^4$ equals the number of quadruples $(I_1, I_2, I_3, I_4)$ of $d$-tuples in $F_N$ such that the $4d$ indices in them coincide in pairs. To estimate this number from above, we note that the number of possibilities that $I_1, I_2, I_3, I_4$ can coincide in two different pairs is $3|F_N|(|F_N| - 1)$, and that each element in $F_N^{\#}$ contributes (at least) one more to the count. Hence,

$$|F_N|^2 \mathbb{E}\, \widetilde{S}_N^4 = \mathbb{E}\, S_N^4 \geq 3|F_N|^2 - 3|F_N| + |F_N^{\#}|. \tag{3.10}$$

Obviously, $|F_N| \to \infty$ if $\widetilde{S}_N \xrightarrow{\mathrm{d}} N(0, 1)$. Hence, (3.9) and (3.10) imply (1.13). $\square$

We end this section with some counterexamples where the set $F_N$ is close to a product set and asymptotic normality does not hold.

**Example 3.2.** Take $F_N = \Delta_N^d$ with $d \geq 2$. It is easily seen that (1.13) does not hold, so asymptotic normality fails by Theorem 1.8. Actually, it is easy to see that in this case, $\widetilde{S}_N$ converges to a Hermite polynomial of degree $d$ in

a standard normal variable [20], see also [11, Section XI.1] and [2, Theorem X.26].

In particular, with $d = 2$, this example shows that Theorem 1.6(i) does not extend to $\dim\{F_N\} = 2$.

**Example 3.3.** Fix an integer $\ell \geq 1$ and let, for $N > \ell$, $F_N$ be the product set $\{1, \ldots, \ell\} \times \{\ell + 1, \ldots, N\}$.

Clearly, $S_N = \sum_1^\ell r_i \cdot \sum_{\ell+1}^N r_j$ and it follows from the central limit theorem that

$$\widetilde{S}_N \xrightarrow{\mathrm{d}} Y\xi,$$

where $Y$ and $\xi$ are independent, $\xi \sim N(0,1)$ and $Y = \ell^{-1/2} \sum_1^\ell r_i$.

Hence, if $\ell = 1$, the limit is normal, but not if $\ell \geq 2$. For example, if $\ell = 2$, the limit variable is 0 with probability $1/2$. (The limit can be regarded as a mixture of normal distributions with different variances.)

In particular, this example shows that Theorem 1.6(i) does not extend to $\dim\{F_N\} = 1$.

**Example 3.4.** Consider a *disconnected* fractional Cartesian product. For example, take $d = 6$, $m = 2$ and let $S_1, \ldots, S_6$ be the sets $\{1,2\}$, $\{1,3\}$, $\{2,3\}$, $\{4,5\}$, $\{4,6\}$, $\{5,6\}$. It is easily seen that (1.13) does not hold, so asymptotic normality fails by Theorem 1.8.

This case is related to the case of disconnected $G$ or $H$, respectively, in [9, Theorem 1] or [10, Theorem 1]. We expect that, as in those results, $\widetilde{S}_N$ converges to a polynomial in normal variables, but we have not checked the details.

## 4. Further remarks and open problems

**Remark 4.1.** It would be interesting to know more about $\|\widetilde{S}_N\|_q$ and $\mathbb{P}(|\widetilde{S}_N| > x)$ as functions of $N$. For example, how fast is the transition from the maxima in Theorem 1.5 to the limits in Theorem 1.6 as $N$ grows?

**Remark 4.2.** We considered for simplicity only $a_{i_1 \cdots i_d} = 1$ in Theorems 1.5 and 1.6. The upper bounds in (1.7) and (1.8) are given for arbitrary $a_{i_1 \cdots i_d}$, and, in particular, $a_{i_1 \cdots i_d} = \pm 1$, but the proof of the lower bounds uses the fact that all coefficients have the same sign. In general, there will be cancellations among the terms in (1.6), for any values of $r_1, \ldots, r_N$, and it seems likely that the lower bounds in Theorem 1.5 do not extend to general $a_{i_1 \cdots i_d}$. What is the correct result? Give an extension of Theorem 1.5 to arbitrary $a_{i_1 \cdots i_d}$.

Certainly, the central limit theorems 1.6 and 1.7 extend to sums (1.6) with suitable conditions on $a_{i_1 \cdots i_d}$, but we have not worked out the details of such extensions.

See also [8] and [17] for some related bounds.

**Remark 4.3.** Note that if $S$ is given by (1.6) and $d_F(\alpha) < \infty$, then

$$\|S\|_\infty \geq c\|\{a_{i_1 \cdots i_d}\}\|_{\ell^{2\alpha/(\alpha+1)}},$$

where the exponent $2\alpha/(\alpha+1)$ is the best possible; see [2, Section XIII.7]. This generalizes a result for $F = \Delta^d$ (i.e. sums (1.1), with $\alpha = d$) proved by Littlewood [15] for $d = 2$ and for general $d$ by [5] and [12]. It would be interesting to obtain lower bounds to the probability $\mathbb{P}\big(S \geq c\|\{a_{i_1\cdots i_d}\}\|_{\ell^{2\alpha/(\alpha+1)}}\big)$.

**Remark 4.4.** The proofs above show that the tail estimates in Theorem 1.5 hold for the upper tails $\mathbb{P}(\widetilde{S}_N > x)$ too. If $d$ is odd, we obtain the same results for $\mathbb{P}(\widetilde{S}_N < -x)$ by symmetry, but if $d$ is even this fails. It seems likely that $\sup_N \mathbb{P}(\widetilde{S}_N < -x)$ is smaller than $\exp(-cx^{2/\alpha})$ for even $d$, for example for $d = 2$. How small is it?

We can also replace the Rademacher system by other orthogonal systems. (See e.g. [13, Chapter 6] for a general background.)

**Remark 4.5.** If we replace the Rademacher variables $r_i$ by Steinhaus functions $\chi_i$, i.e. independent complex random variables that are uniformly distributed on the unit circle, then Theorem 1.5 still holds.

Indeed, (1.7) is still valid [2, Corollary XIII.29], and thus (1.8) holds by the same proof, so the upper bounds in Theorem 1.5 hold. For the lower bounds, we use the same proof as above, now taking $\mathcal{E}_N = \{\mathrm{Re}\,\chi_k \geq 1/2,\ k = 1, \ldots, N\}$.

For the upper bound in (1.7), we can alternatively introduce a Rademacher system $\{r_i\}$ independent of $\{\chi_i\}$, replace $\chi_i$ by $\chi_i r_i$, which has the same distribution, and use the Rademacher version above conditioning on $\{\chi_i\}$. This standard trick works for all i.i.d. sequences of bounded symmetric random variables.

Are the central limit theorems 1.6 and 1.7 true for the Steinhaus system too, now with complex Gaussian limits? (We believe so, but we have not checked the details.)

**Remark 4.6.** Let us instead consider a Gaussian chaos, obtained by replacing $r_i$ by independent Gaussian variables $\xi_i \sim N(0,1)$.

The hypercontractive inequality (1.2) holds in this case too [18], see also [1, 11, 13], but the combinatorial dimension version (1.7) fails in the Gaussian case, as is seen by taking $F$ to be a set with a single element.

Hence Theorem 1.5 is not true in the Gaussian case. What is true? There is no problem with the lower bounds in Theorem 1.5; the proof in Section 2 works if we take $\mathcal{E}_N = \{\xi_i > 1,\ i = 1, \ldots, N\}$.

We believe that Theorems 1.6 and 1.7 hold for the Gaussian case too, but we have not checked the details.

The estimates in [7] and [14] for $d = 2$ might be useful.

**Remark 4.7.** Are the results true if we replace $r_k$ by a lacunary sequence $\exp(2\pi i n_k t)$, where $\inf n_{k+1}/n_k > 1$?

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