A NOTE ON THE VARIANCE CALCULATION FOR GENERALIZED POLYA URNS

SVANTE JANSON

1. Introduction

Limit theorems for generalized Pólya urns are given in [3]. In particular, a central limit theorem for the composition is shown under very general conditions, with an explicit but rather complicated formula for the covariance matrix of the asymptotic multi-dimensional normal distribution.

When computing the covariance matrix numerically in some applications (see [2] and [1]), Cecilia Holmgren and Axel Heimbürger found a minor simplification. The purpose of this note is to explain why this simplification works, in a general setting.

We use the assumptions and notation of [3]. In particular, all vectors are column vectors. Furthermore,

- $\xi_i = (\xi_{ij})_j$ is the (possibly random) replacement vector when a ball of type (colour) i is drawn;
- $a = (a_i)_i$ is the vector of activities of the different types;
- $A := (a_j \mathbb{E}\,\xi_{ji})_{i,j}$; $\lambda_1, \dots$ are the eigenvalues of A, ordered with $\lambda_1 \geqslant \operatorname{Re}\lambda_2 \geqslant \operatorname{Re}\lambda_3 \dots$;
- u_1' and v_1 are left and right eigenvectors corresponding to the largest eigenvalue λ_1; these are normalized by $a \cdot v_1 = a' v_1 = 1$ and $u_1 \cdot v_1 = u_1' v_1 = 1$ and are then uniquely defined under the assumptions in [3].

2. Results

Lemma 1. *Suppose that $a \cdot \xi_i = m$ deterministically for some $m > 0$ and every i. (In other words, the activity increases deterministically by a fixed amount every time a ball is drawn.) Then*

$$Bu_1 = m^2 v_1. \tag{1}$$

Proof. Note first that the condition implies $m = \lambda_1$ and $u_1 = a$, see [3, Lemma 5.4]. By [3, (2.13)], $B_i := \mathbb{E}(\xi_i \xi_i')$, and thus, since $\xi_i' a = \xi \cdot a = m$,

$$B_i u_1 = B_i a = \mathbb{E}\big(\xi_i \xi_i' a\big) = m\,\mathbb{E}\,\xi_i. \tag{2}$$

Date: October 1, 2014.

Partly supported by the Knut and Alice Wallenberg Foundation.

Furthermore, the matrix B is by [3, (2.14)] defined by $B := \sum_i v_{1i}a_iB_i$, and thus (2) yields

$$Bu_1 = \sum_i v_{1i}a_i m\,\mathbb{E}\,\xi_i, \tag{3}$$

where the j:th component is

$$(Bu_1)_j = \sum_i v_{1i}a_i m\,\mathbb{E}\,\xi_{ij} = m\sum_i v_{1i}A_{ji} = m(Av_1)_j = m\lambda_1(v_1)_j. \tag{4}$$

Hence,

$$Bu_1 = m\lambda_1 v_1 = m^2 v_1. \qquad \square$$

Lemma 2. *Suppose that $a\cdot\xi_i = m$ deterministically for some $m > 0$ and every i. Suppose further that $\operatorname{Re}\lambda_2 < \frac12\lambda_1$. Then*

$$P_I B P_I' = P_I B = B P_I' = B - m^2 v_1 v_1'. \tag{5}$$

Proof. When $\operatorname{Re}\lambda_2 < \frac12\lambda_1$, we have by definition and [3, (2.7)]

$$P_I = I - P_{\lambda_1} = I - v_1 u_1'.$$

Consequently, $P_I' = I - u_1 v_1'$ and, by Lemma 1,

$$B - BP_I' = B(I - P_I') = Bu_1 v_1' = m^2 v_1 v_1', \tag{6}$$

which yields

$$P_I B - P_I B P_I' = m^2 P_I v_1 v_1' = 0,$$

since $P_I v_1 = v_1 - v_1 u_1' v_1 = 0$ by the construction of P_I. Thus $P_I B = P_I B P_I'$, and taking the transpose yields $BP_I' = P_I B P_I'$.

The final equation in (5) follows by (6). $\square$

Under the conditions in Lemma 2, the asymptotic covariance matrix Σ in [3, Theorem 3.22] equals by [3, Lemma 5.4] $m\Sigma_1$, where by [3, (2.15)] and the fact that P_I commutes with A,

$$\Sigma_I := \int_0^\infty P_I e^{sA} B e^{sA'} P_I' e^{-\lambda_1 s}\,\mathrm{d}s = \int_0^\infty e^{sA} P_I B P_I' e^{sA'} e^{-\lambda_1 s}\,\mathrm{d}s. \tag{7}$$

Theorem 3. *Suppose that $a\cdot\xi_i = m$ deterministically for some $m > 0$ and every i. Suppose further that $\operatorname{Re}\lambda_2 < \frac12\lambda_1$. Then we may drop either P_I or P_I' (but not both) from (7). Furthermore,*

$$\Sigma_I = \int_0^\infty \Bigl(e^{-\lambda_1 s} e^{sA} B e^{sA'} - m^2 e^{\lambda_1 s} v_1 v_1'\Bigr)\,\mathrm{d}s. \tag{8}$$

Proof. That we can drop P_I or P_I' is an immediate consequence of (5) in Lemma 2. To see that we cannot drop both P_I and P_I', note that by $A'u_1 = \lambda_1 u_1$ and thus $e^{sA'}u_1 = e^{s\lambda_1}u_1$, which by transposing also yields $u_1' e^{sA} = e^{s\lambda_1} u_1'$. Hence, by Lemma 1,

$$\begin{aligned} u_1'\bigl(e^{sA} B e^{sA'} e^{-\lambda_1 s}\bigr)u_1 e^{-\lambda_1 s} e^{\lambda_1 s} u_1' B\bigl(e^{\lambda_1 s} u_1\bigr) &= e^{\lambda_1 s} u_1' B u_1 \\ &= m^2 e^{\lambda_1 s} u_1' v_1 = m^2 e^{\lambda_1 s}. \end{aligned} \tag{9}$$

Thus the integral (7) diverges without P_I or P_I'.

Finally, (8) follows from (7) and (5), recalling that $e^{sA}v_1 = e^{s\lambda_1}v_1$ and $v_1'e^{sA'} = e^{s\lambda_1}v_1'$. □

Remark 4. It is easily seen that, for some $q \geqslant 0$, $P_I e^{sA} = O\bigl((1+s^q)e^{\operatorname{Re}\lambda_2 s}\bigr)$, and thus the integrand in (7) is $O\bigl((1+s^{2q})e^{(2\operatorname{Re}\lambda_2-\lambda_1)s}\bigr)$, which is integrable because $2\operatorname{Re}\lambda_2 - \lambda_1 < 0$. The same holds for (8), since its integrand is the same, by the proof above.

Acknowledgement. I thank Cecilia Holmgren for noting this property and discussing it with me.

References

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Department of Mathematics, Uppsala University, PO Box 480, SE-751 06 Uppsala, Sweden

E-mail address: `svante.janson@math.uu.se`

URL: `http://www2.math.uu.se/~svante/`