PRESERVATION OF CONVEXITY OF SOLUTIONS TO PARABOLIC EQUATIONS

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ABSTRACT. In the present paper we find necessary and sufficient conditions on the coefficients of a parabolic equation for convexity to be preserved. A parabolic equation is said to preserve convexity if given a convex initial condition, any solution of moderate growth remains a convex function of the spatial variables for each fixed time.

1. INTRODUCTION

We consider the Cauchy problem for second order parabolic operators on $\Omega_T = \mathbb{R}^n \times (0,T]$. Necessary and sufficient conditions on the operator are found that guarantee that the solutions to the equation remain convex, for each fixed time t, if the inital condition is a convex function, under appropriate growth conditions. Some type of restriction of the growth of the solution is necessary due to the well-known fact that the solutions to parabolic equations, in general, are not unique. There are classical examples of solutions to the associated heat equation with zero initial condition that grow faster than $\exp(a|x|^2)$ for any a. Of course, subtracting such a function from a solution with a given convex initial condition would typically change the convexity properties for $|x|$ large and convexity might be lost instantaneously at infinity.

In [7], compare also [1] and [6], we study this problem, in the case of one spatial variable, in connection with applications to finance and we show that convexity is indeed preserved for solutions, given by the stochastic representation formula, to an equation of the form

$$F_t = a^2(x,t)F_{xx} \tag{1.1}$$

where $a(x,t)$ is measurable on $\mathbb{R} \times [0,\infty)$, locally Hölder(1/2) in the x-variable, and satisfies the growth condition $|a(x,t)| \le C(1+|x|)$ for some constant C.

In the present paper, we study the case of several spatial variables. In this case preservation of convexity is a rather rare property, in contrast to the case of one spatial dimension. Note that convexity is always preserved (for the solution of moderate growth) in the case of operators with constant coefficients since a solution is obtained by integrating the initial condition against a translation invariant positive kernel.

The question of preservation of convexity is formulated more precisely and some basic definitions are introduced in Section 2. In Section 3 we consider the case of regular coefficients, see (3.1). We give a necessary and sufficient condition for the infinitesimal preservation of convexity at some point. We call this condition LCP, an abbreviation for locally convexity preserving, see Definition 2.2. We find a

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characterization of LCP in terms of a differential inequality on the coefficients of the operator, see Lemma 3.12. We then show that LCP holds if and only if convexity is preserved for solutions to the equation that are of polynomial growth, see Theorem 3.1.

The convexity inequality of Lemma 3.12 is a pointwise condition in the leading coefficients and their first two spatial derivatives. The algebraic properties of this inequality are discussed in Section 4.

In Section 5 we give some explicit examples of convexity preserving operators and in Section 6 we study non-homogeneous equations.

In Section 7 we show the perhaps surprising result that for operators with bounded coefficients, it is only the operators with coefficients only depending on time that preserve convexity, see Theorem 7.2.

In Section 8 we relax the regularity conditions for the coefficients of the operator to a customary Hölder condition. However, in this case we only show that the obtained conditions for the preservation of convexity are sufficient. We conjecture that these conditions also are necessary. The LCP condition, which applies to operators with regular coefficients is linear in the operator. Thus convex combinations of LCP operators are also LCP, which easily extends to suitably defined infinite sums and integrals. Thus convexity preserving operators with regular coefficients form a positive convex cone, a property not apparent form the definition of preservation of convexity. We have not been able to show the corresponding result for operators with Hölder coefficients.

In the following section, see Theorem 9.1 we apply the property of convexity preservation to study monotonicity properties of solutions to different parabolic equations. In Section 10 we consider preservation of convexity for the Dirichlet problem on bounded domains in $\mathbb{R}^n$. In the section thereafter, we discuss extension of our results to nonlinear equations.

In the appendix we have, for the benefit of the reader, collected results on second order parabolic equations that we needed to obtain our results. These results, or at least versions of them, are well-known, but for many of these statements we have not found references for exactly the version that we have used.

Finally, some words about notation. D_u denotes differentiation in the direction of the vector $u \in \mathbb{R}^n$ and ∇_x the gradient in the spatial directions. For typographical convenience, we will write D_i for D_{e_i}, where e_i is the i:th coordinate vector. We will use several spaces of functions on Ω_T or $\mathbb{R}^n$; for convenience we have collected the definitions in the appendix, and urge the reader to check there when necessary.

When we say that a function on Ω_T is convex, we always mean convex in the spatial variables for every fixed t. If $F \in C^{2,0}(\Omega_T)$, we similarly say that F is convex at a point if the spatial Hessian matrix $\nabla_x^2 F \geq 0$ there, where $A \geq 0$ for a square matrix means positive semidefinite. We will several times use the obvious fact that the pointwise limit of a sequence of convex functions is convex. Recall also that $\mathrm{Tr}(AB) \geq 0$ if $A \geq 0$ and $B \geq 0$ are matrices of the same size.

2. Problem formulation and basic definitions

As stated in the introduction, we consider parabolic differential operators

$$\mathcal{M} = \frac{\partial}{\partial t} - \mathcal{L}$$

where

$$\mathcal{L}F = \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 F}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x,t)\frac{\partial F}{\partial x_i} + c(x,t)F. \tag{2.1}$$

We assume that all coefficients a_{ij}, b_i, c are real-valued functions defined on the domain $\Omega_T = \mathbb{R}^n \times (0,T]$ for some given (finite) $T > 0$.

Our minimal assumptions on the operator, which hold throughout this article, are the following.

(A1) $\mathcal{M}$ is parabolic ($\mathcal{L}$ is elliptic) everywhere, i.e. the matrix $\big(a_{ij}(x,t)\big)_{ij}$ is positive definite for every $(x,t) \in \Omega_T$. Explicitly, $\sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j > 0$ for every $(x,t) \in \Omega_T$ and every $\xi \in \mathbb{R}^n \setminus \{0\}$.

(C0) The coefficients a_{ij}, b_i, c are continuous functions in Ω_T.

Some stronger versions of these are defined in the appendix.

We study the Cauchy problem for $\mathcal{M}$ in the following form:

Given a continuous function f on $\mathbb{R}^n$ and a continuous function h on Ω_T, find a function $u \in C(\overline{\Omega_T}) \cap C^{2,1}(\Omega_T)$ such that

$$\begin{aligned} \mathcal{M}F(x,t) &= h(x,t), \qquad && (x,t) \in \Omega_T, \\ F(x,0) &= f(x), && x \in \mathbb{R}^n. \end{aligned} \tag{2.2}$$

We would like to find conditions on the operator $\mathcal{M}$ and the function h that guarantee that solutions to the equation remain convex as a function of x for any fixed t if the initial condition is a convex function. Since we allow the coefficients of the operator $\mathcal{M}$ to depend on time, it is natural to allow that the initial condition can be given at any τ with $0 \le \tau < T$ and ask that the solution remains convex for any fixed t with $\tau \le t \le T$. We therefore define $\Omega_{(\tau,T]} = \mathbb{R}^n \times (\tau, T]$ and its closure $\Omega_{[\tau,T]} = \mathbb{R}^n \times [\tau, T]$, and introduce the following variation of Problem (2.2):

Given a continuous function f on $\mathbb{R}^n$ and a number $\tau \in [0,T)$, find a function $F \in C(\Omega_{[\tau,T]}) \cap C^{2,1}(\Omega_{(\tau,T]})$ such that

$$\begin{aligned} \mathcal{M}F(x,t) &= h(x,t), \qquad && (x,t) \in \Omega_{(\tau,T]}, \\ F(x,\tau) &= f(x), && x \in \mathbb{R}^n. \end{aligned} \tag{2.3}$$

Usually we take $h = 0$ and thus consider the homogeneous equation

$$\begin{aligned} \mathcal{M}F(x,t) &= 0, \qquad && (x,t) \in \Omega_{(\tau,T]}, \\ F(x,\tau) &= f(x), && x \in \mathbb{R}^n. \end{aligned} \tag{2.4}$$

We can now state our definition of convexity preserving operators.

Definition 2.1. Let $\mathcal{F}$ be a space of continuous functions on Ω_T. Further, let $\mathcal{F}_\tau$ for $0 \le \tau < T$ denote the space of obtained by restricting functions in $\mathcal{F}$ to $\mathbb{R}^n \times \{\tau\}$ and let $\mathcal{F}_{[\tau,T]}$ denote the space obtained by restricting function in $\mathcal{F}$ to $\Omega_{[\tau,T]}$. We say that the operator $\mathcal{M}$ is *convexity preserving* in the function space $\mathcal{F}$ if for every $\tau \in [0,T)$ and every convex $f \in \mathcal{F}_\tau$ there is unique solution to (2.4) in $\mathcal{F}_{[\tau,T]}$ and this solution is convex in the spatial variables for each $t \in (\tau, T]$.

More generally, we say that the pair $(\mathcal{M},\, h)$ is *convexity preserving*, where $h \in C(\Omega_T)$, if this holds for (2.3).

We note that the condition of unicity of the solution is redundant in the sense that if there is more than one solution then convexity is easily seen not to be preserved.

However, it seems natural to explicitly state that we consider spaces where there is a unique solution.

A typical space $\mathcal{F}$ of functions is the class of functions of polynomial growth:

$$C_{\mathrm{pol}}(\overline{\Omega_T}) = \bigcup_{C>0,\,p>0} \{F \in C(\overline{\Omega_T}) : |F(x,t)| \le C(|x|^p + 1),\ (x,t) \in \overline{\Omega_T}\}.$$

The corresponding initial conditions f then form the set

$$C_{\mathrm{pol}}(\mathbb{R}^n) = \bigcup_{C>0,\,p>0} \{f \in C(\mathbb{R}^n) : |f(x)| \le C(|x|^p + 1),\ x \in \mathbb{R}^n\}.$$

We will, assuming that the coefficients are sufficiently smooth, connect the property of convexity preserving with a certain local property. Intuitively, convexity could be lost after some point in time at a point where the solution F has vanishing second derivate in some spatial direction u. To ensure that convexity is not lost, the infinitesimal change of F in time, i.e. $\mathcal{L}F$ in the case of $h = 0$, needs to be convex in the direction of u. We formulate this idea in the following definition.

Definition 2.2. Assume that the operator $\mathcal{M} = \frac{\partial}{\partial t} - \mathcal{L}$ has coefficients that are in $C^{2,0}(\overline{\Omega_T})$. Then $\mathcal{M}$ (or $\mathcal{L}$) is *locally convexity preserving*, abbreviated *LCP*, at (x,t), if

$$D_{uu}(\mathcal{L}f)(x,t) \ge 0 \tag{2.5}$$

whenever $u \in \mathbb{R}^n$, $f \in C^\infty(\mathbb{R}^n)$ is convex in a neighborhood of x, and $D_{uu}f(x) = 0$.

Remark 2.3. The LCP condition involves spatial derivatives but no derivatives with respect to t. Hence the condition can be regarded as a condition for elliptic differential operators $\mathcal{L}$ on $\mathbb{R}^n$, without any time parameter at all. A time dependent operator $\mathcal{L}$, as above, then is LCP if and only if it is LCP for every fixed t.

Remark 2.4. It is easy to show, using Lemma 3.8, that every parabolic (elliptic) differential operator with constant coefficients is LCP.

The LCP condition will enable us to characterize convexity preserving operators (under some mild technical conditions) using a certain differential inequality which we simply refer to as the convexity inequality.

Definition 2.5. Let $A(x,t)$ denote the $n \times n$ matrix $(a_{ij}(x,t))$, $n \ge 2$, and also the corresponding linear operator in $\mathbb{R}^n$, and let A_u denote the operator Q_uAQ_u restricted to $u^\perp$, where Q_u is the orthogonal projection of $\mathbb{R}^n$ onto $u^\perp$. Then the correspondning operator $\mathcal{M}$ is said to satisfy the convexity inequality in Ω_T if

$$\langle M, D_{uu}A_u\rangle + 2\langle N, D_uA_u\rangle + 2\langle P, A_u\rangle \ge 0 \tag{2.6}$$

in Ω_T, for every unit vector $u \in \mathbb{R}^n$ and all symmetric linear operators M, N and P in $u^\perp$ such that

$$\begin{pmatrix} M & N \\ N & P \end{pmatrix} \ge 0. \tag{2.7}$$

Unless the coefficients of $\mathcal{M}$ are C^2, (2.6) is interpreted in the sense of distributions.

Remark 2.6. For the local conditions LCP and the convexity inequality, it is obvious that the set of operators that satisfy them are convex cones. In other words, for example, convex combinations of several LCP operators are LCP. This extends to suitably convergent infinite sums and integrals. We thus note that if we convolve the entries of the matrix A with the same non-negative approximate identity, the inequality (2.6) holds for the then obtained matrix if it holds for A.

3. The case of regular coefficients

In this section we assume the regularity condition

$$a_{ij}, b_i, c \in C^{2,1}_\alpha(\Omega_T), \quad \text{for some } \alpha > 0, \tag{3.1}$$

where $C^{2,1}_\alpha(\Omega_T)$ is defined in the appendix, and the boundedness condition

(B2) The coefficients a_{ij}, b_i, c satisfy the bounds

$$|a_{ij}(x,t)| \le B(|x|^2+1), \quad |b_i(x,t)| \le B(|x|+1), \quad |c(x,t)| \le B,$$

for some constant B and all $(x,t) \in \Omega_T$.

We further assume that the the right hand side h vanishes.

The main result of this section is the following theorem.

Theorem 3.1. *Assume that the operator $\mathcal{M}$ is such that* (A1), (B2) *and* (3.1) *hold. Then the following are equivalent.*

(i) *$\mathcal{M}$ is convexity preserving in $C_{\mathrm{pol}}(\overline{\Omega_T})$.*
(ii) *$\mathcal{M}$ is LCP in Ω_T.*
(iii) (a) *$n = 1$ or $\mathcal{M}$ satisfies the convexity inequality* (2.6) *in Ω_T, and*
(b) *for each fixed t, each b_i is an affine function of x and c is constant.*

Proof. We will prove a series of lemmas, beginning with the following extension lemma for convex functions.

Lemma 3.2. *If $f \in C^m(\mathbb{R}^n)$ is convex in a neighborhood of $x_0 \in \mathbb{R}^n$, where $2 \le m \le \infty$, then there exists a convex function $g \in C^m(\mathbb{R}^n)$ such that $g(x) = f(x)$ in a neighborhood of x_0, $g(x) = O(|x|)$ as $|x| \to \infty$ and each derivative $D^k g$ with $1 \le |k| \le m$ is bounded on $\mathbb{R}^n$.*

Proof. For notational convenience we assume that $x_0 = 0$. From the assumptions of the theorem we see that f is convex in $U = \{x : |x| < \delta\}$ for some $\delta > 0$. Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ have support in U, with $\varphi(x) = 1$ when $|x| \le \delta/2$. Let $\psi \in C^\infty(R)$ be an increasing function with $\psi(t) = 0$ for $t \le \delta/4$, $\psi(t) = 2t$ for $\delta/2 \le t \le 2\delta$ and $\psi(t) = 5\delta$, for $t \ge 3\delta$. Let $\Psi(t) = \int_0^t \psi(s)\,ds$, which is increasing and convex. It is then easily verified that if K is large enough, then

$$g(x) = \varphi(x)f(x) + K\Psi(|x|) \tag{3.2}$$

satisfies the requirements. □

The following lemma shows that the condition LCP of Definition 2.2 is necessary for convexity to be preserved.

Lemma 3.3. *Suppose that $\mathcal{M}$ satisfies* (A2), (B2) *and* (3.1), *and that for every convex $f \in C^\infty(\mathbb{R}^n)$ with $f(x) = O(1+|x|)$ there exists a convex solution $F(x,t) \in C_{\mathrm{pol}}(\overline{\Omega_T})$ to $\mathcal{M}F = f$ with $F(x,0) = f(x)$. Then $\mathcal{M}$ is LCP at $(x,0)$ for all $x \in \mathbb{R}^n$.*

Proof. Suppose that $u \in \mathbb{R}^n$ and that f is C^∞ and convex in a neighborhood of $x_0 \in \mathbb{R}^n$, with $D_{uu}f(x_0) = 0$. By Lemma 3.2, there exists a convex function $g \in C^\infty(\mathbb{R}^n)$ with $g(x) = O(|x|+1)$ such that $g = f$ in a neighborhood U of x_0. By assumption, there exists a convex solution $F(x,t) \in C_{\mathrm{pol}}(\overline{\Omega_T}) \cap C^{2,1}(\Omega_T)$ to $\mathcal{M}F = 0$ with $F(x,0) = g(x)$, $x \in \mathbb{R}^n$. In particular, for $0 < t < T$, $D_{uu}F(x_0,t) \ge 0$. Further,

$$D_{uu}F(x_0,0) = D_{uu}g(x_0) = D_{uu}f(x_0) = 0. \tag{3.3}$$

By Theorem A.18, $F \in C^{4,1}(\overline{\Omega_T})$. Thus $D_t D_{uu} F(x_0, t)$ exists and we have by the above $D_t D_{uu} F(x_0, 0) \geq 0$. Consequently, using (2.2),

$$D_{uu}(\mathcal{L}f)(x_0) = D_{uu}(\mathcal{L}F)(x_0, 0) = D_{uu}(D_t F)(x_0, 0) = D_t D_{uu} F(x_0, 0) \geq 0. \qquad \square$$

The implication (i) $\Longrightarrow$ (ii) in Theorem 3.1 follows immediately from this lemma by considering $\Omega_{[\tau,T]}$. To prove the converse, we first impose extra regularity conditions on both the coefficients and the solution.

Lemma 3.4. *Suppose that $\mathcal{M}$ is such that* (A1), (B2^{2,1}) *and* (3.1) *hold. Assume that $\mathcal{M}$ is LCP everywhere in Ω_T. If $F(x,t) \in C^{4,0}(\overline{\Omega_T}) \cap C^{2,0}_{\mathrm{pol}}(\overline{\Omega_T})$ is a solution to $\mathcal{M}F = 0$ and $F(x, 0)$ is convex, then F is convex in Ω_T.*

Proof. Let m be an even integer chosen so large that $|F(x,t)|, |\nabla_x F(x,t)|, |\nabla_x^2 F(x,t)| = O(|x|^m + 1)$, $(x,t) \in \overline{\Omega_T}$. Let

$$g(x) = |x|^2 + |x|^{m+4}. \tag{3.4}$$

Clearly,

$$\mathcal{L}g(x,t) = \sum_{i,j=1}^{n} a_{ij}(x,t)\, p_{ij}^{(2)}(x) + \sum_{i=1}^{n} b_i(x,t)\, p_i^{(1)}(x) + c(x,t)\, p^{(0)}(x), \tag{3.5}$$

where $p_{\dots}^{(l)}$ are polynomials of degree $m+4-l$, and thus, uniformly for all unit vectors $u \in \mathbb{R}^n$, by (B2^{2,1}),

$$D_{uu}(\mathcal{L}g)(x,t) = O(1 + |x|^{m+2}).$$

Moreover,

$$\begin{aligned} D_{uu} g(x) &= 2 + (m+4)|x|^{m+2} + (m+4)(m+2)\langle x, u\rangle^2 |x|^m \\ &\geq 2 + (m+4)|x|^{m+2}, \end{aligned} \tag{3.6}$$

and thus there exists a constant C such that

$$1 + |D_{uu}(\mathcal{L}g)(x,t)| \leq C\, D_{uu} g(x), \tag{3.7}$$

for all x, t and unit vectors u. Let $\varepsilon > 0$ and define

$$F_\varepsilon(x,t) = F(x,t) + \varepsilon e^{Ct} g(x).$$

Let $E = \{(x,t) \in \mathbb{R}^n \times [0,T] : F_\varepsilon \text{ is not convex at } (x,t)\}$ and suppose that $E \neq \emptyset$. For all unit vectors $u \in \mathbb{R}^n$, by (3.6),

$$D_{uu} F_\varepsilon(x,t) = D_{uu} F(x,t) + \varepsilon e^{Ct} D_{uu} g(x) \geq \varepsilon |x|^{m+2} + O(1 + |x|)^m.$$

Hence, for some $\rho < \infty$, $D_{uu} F_\varepsilon(x,t) \geq 0$ for all unit vectors u and $|x| \geq \rho$. In other words, $E \subseteq B(0,\rho) \times [0,T]$. Thus E is bounded and $\overline{E}$ is compact. Let

$$t_0 = \inf\{t \geq 0 : (x,t) \in \overline{E} \text{ for some } x \in \mathbb{R}^n\}. \tag{3.8}$$

This infimum is attained and thus $(x_0, t_0) \in \overline{E}$ for some $x_0 \in \mathbb{R}^n$. If $\nabla_x^2 F_\varepsilon$ is strictly positive definite at (x_0, t_0), then by continuity, it is positive in a neighborhood and thus F_ε is strictly convex there, which contradicts the fact that $(x_0, t_0) \in \overline{E}$.

For $t = 0$ we have, by assumption, $\nabla_x^2 F(x, 0) \geq 0$, and thus $\nabla_x^2 F_\varepsilon \geq \varepsilon \nabla_x^2 g$, which by (3.6) is strictly positive definite. Hence $t_0 > 0$.

Since $(x,t) \notin E$ for $0 < t < t_0$, F_ε is convex there and by continuity, $F_\varepsilon(x, t_0)$ is convex. Consequently, $f(x) = F_\varepsilon(x, t_0)$ is convex, but $D_{uu} f(x_0) = 0$ for some unit vector $u \in \mathbb{R}^n$. By the LCP property,

$$D_{uu}(\mathcal{L}F_\varepsilon)(x_0, t_0) \geq 0. \tag{3.9}$$

Moreover, $D_{uu}F_\varepsilon(x_0,t) \geq 0$ for $0 < t < t_0$, and thus

$$D_{uu}D_tF_\varepsilon(x_0,t_0) = D_tD_{uu}F_\varepsilon(x_0,t_0) \leq 0. \tag{3.10}$$

However, since F solves (2.2), by (3.7),

$$\begin{aligned} D_{uu}(D_tF_\varepsilon - \mathcal{L}F_\varepsilon) &= D_{uu}(\varepsilon Ce^{Ct}g - \varepsilon e^{Ct}\mathcal{L}g) \\ &= \varepsilon e^{Ct}(CD_{uu}g - D_{uu}(\mathcal{L}g)) \\ &\geq \varepsilon e^{Ct} > 0, \end{aligned}$$

for all (x,t), which contradicts (3.9) and (3.10). Consequently, $E = \emptyset$, and thus F_ε is convex everywhere. Letting $\varepsilon \to 0$, we find that F is convex. □

To extend this result, we first note that the property of convexity preserving is preserved under suitable pointwise limits.

Lemma 3.5. *Suppose that $\mathcal{M}^{(m)}$, $m = 1,2,\dots$ is a sequence of parabolic differential operators in Ω_T that are convexity preserving in $C_{\mathrm{pol}}(\overline{\Omega_T})$ and such that:*

(i) (A1) *holds for $\mathcal{M}^{(m)}$ uniformly in m, i.e. for every compact $K \subset \Omega_T$, all operators $\mathcal{M}^{(m)}$ satisfy* (A.5) *with some $\lambda_K > 0$ not depending on m.*
(ii) (B2) *holds uniformly in m, i.e.* (A.3) *holds for every $\mathcal{M}^{(m)}$, for some B not depending on m.*
(iii) (C1) *holds uniformly in m, i.e., for every compact $K \subset \Omega_T$, there exists C independent of m such that $\|a_{ij}^{(m)}\|_{H_\alpha(K)} \leq C$, $\|b_i^{(m)}\|_{H_\alpha(K)} \leq C$, $\|c^{(m)}\|_{H_\alpha(K)} \leq C$.*
(iv) *$\mathcal{M}^{(m)} \to \mathcal{M}$ as $m \to \infty$, in the sense that $a_{ij}^{(m)} \to a_{ij}$, $b_i^{(m)} \to b_i$, $c^{(m)} \to c$ pointwise in Ω_T.*

Then $\mathcal{M}$ is convexity preserving in $C_{\mathrm{pol}}(\overline{\Omega_T})$.

Proof. Let $f \in C_{\mathrm{pol}}(\mathbb{R}^n)$ be convex, and let $F^{(m)} \in C_{\mathrm{pol}}(\overline{\Omega_T}) \cap C^{2,1}(\Omega_T)$ solve the Cauchy problem $\mathcal{M}^{(m)}F^{(m)} = 0$ in Ω_T with $F^{(m)} = f$ on $\mathbb{R}^n$. By Theorem A.12, $F^{(m)}$ converges in $C^{2,1}(\Omega_T)$, as $m \to \infty$, to the solution $F \in C_{\mathrm{pol}}(\overline{\Omega_T})$ of $\mathcal{M}F = 0$ in Ω_T with $F = f$ on $\mathbb{R}^n$.

By assumption, each $F^{(m)}$ is convex, and thus their pointwise limit F is too. □

Let $G = O(n,\mathbb{R}) \times \mathbb{R}_+ \times \mathbb{R}^n$ be the group of similarity mappings of $\mathbb{R}^n$; G acts on $\mathbb{R}^n$ by $(\sigma,t,y)x = t\sigma(x) + y$. G acts naturally on functions on $\mathbb{R}^n$ and Ω_T; we write both actions as $f \mapsto f \circ \gamma$ with $f \circ \gamma(x) = f(x)$ and $F \circ \gamma(x,t) = F(\gamma(x),t)$, respectively. This induces an action on differential operators, and we define

$$\mathcal{M}_\gamma = \mathcal{M}(f \circ \gamma^{-1}) \circ \gamma.$$

Clearly $\mathcal{M}_\gamma$ is a parabolic differential operator and $\mathcal{M}_\gamma$ satisfies LCP if and only if $\mathcal{M}$ does (we have just changed the coordinate system). We write $G = G_1 \times G_2$ where $G_1 = O(n,\mathbb{R}) \times \mathbb{R}_+$ is the subgroup that fixes 0 and $G_2 = \mathbb{R}^n$ is the subgroup of translations. If $\gamma \in G$, we write $\gamma = (\gamma_1,\gamma_2)$ with the components $\gamma_i \in G_i$. Fix Haar measures $d\gamma_i$ on G_i. Let $\varphi_i \in C_0^\infty(G_i)$ be non-negative with $\int_{G_i} \varphi_i(\gamma_i)\,d\gamma_i = 1$. Define $\varphi \in C_0^\infty(G)$ by $\varphi(\gamma) = \varphi_1(\gamma_1)\varphi_2(\gamma_2)$ and let $d\gamma = d\gamma_1\,d\gamma_2$, a right-invariant Haar measure on G_2. Then $\varphi \geq 0$ and $\int_G \varphi(\gamma)\,d\gamma = 1$. Finally, define

$$\widetilde{\mathcal{M}} = \int_G \varphi(\gamma)\mathcal{M}_\gamma\,d\gamma.$$

$\widetilde{\mathcal{M}}$ is a parabolic differential operator which satisfies LCP if $\mathcal{M}$ does, cf. Remark 2.6.

Lemma 3.6. *If $\mathcal{M}$ satisfies* (B2)*, then $\widetilde{\mathcal{M}}$ satisfies* (B2^{p,0}) *for every $p \geq 0$.*

Proof. We consider only the second order coefficients a_{ij}; the lower order coefficients b_i and c are treated in the same way.

For $\gamma \in G$, let γ_1 denote the component of γ in G_1, i.e. $\gamma = (\gamma_1, y)$ for some $y \in G_2 = \mathbb{R}^n$. We may regard γ_1 as a matrix acting on $\mathbb{R}^n$. Let A be the matrix (a_{ij}). Then the corresponding matrix for $\mathcal{M}_\gamma$ is $\gamma_1^{-1}(A \circ \gamma)\gamma_1^{-1}$. Thus the corresponding matrix $\widetilde{A}$ for $\widetilde{\mathcal{M}}$ is given by

$$\widetilde{A}(x,t) = \int_G \gamma_1^{-1} A(\gamma x, t) \gamma_1^{-1} \varphi(\gamma)\, d\gamma. \tag{3.11}$$

Hence, for $g \in G$, using the right-invariance of the measure,

$$\begin{aligned}
\widetilde{A}(gx,t) &= \int_G \gamma_1^{-1} A(\gamma g x, t) \gamma_1^{-1} \varphi(\gamma)\, d\gamma \\
&= \int_G (\gamma g^{-1})_1^{-1} A(\gamma x, t) (\gamma g^{-1})_1^{-1} \varphi(\gamma g^{-1})\, d\gamma \\
&= \int_G g_1 \gamma_1^{-1} A(\gamma x, t) g_1 \gamma_1^{-1} \varphi(\gamma g^{-1})\, d\gamma.
\end{aligned}$$

The final integral is an infinitely differentiable function of $g \in G$, since we may differentiate under the integral sign. Hence $\widetilde{A}(z,t)$ is an infinitely differentiable function of z. (For this, it suffices to use the translation subgroup G_2.)

To obtain the required bounds for the derivatives of A, it suffices to consider $|x| > 1$. We now use the subgroup G_1. We obtain from (3.11) by Fubini's theorem

$$\widetilde{A}(x,t) = \int_{G_1} \gamma_1^{-1} A^*(\gamma_1 x, t) \gamma_1^{-1} \varphi_1(\gamma_1)\, d\gamma,$$

where

$$A^*(x,t) = \int_{G_2} A(x+y,t) \varphi_2(y)\, dy.$$

Arguing as above, we find for $g_1 \in G_1$

$$\widetilde{A}(g_1 x, t) = \int_{G_1} g_1 \gamma_1^{-1} A^*(\gamma_1 x, t) g_1 \gamma_1^{-1} \varphi_1(\gamma_1 g_1^{-1})\, d\gamma_1.$$

Again, the final integral is an infinitely differentiable function of g. For fixed x_0, the mapping $g \mapsto g x_0 : G_1 \to \mathbb{R}^n$ has surjective differential at the unit $e \in G_1$. Hence this mapping has a right inverse in a neighborhood U of x_0, and it follows that $\widetilde{A}$ is infinitely differentiable in U (as we already know), with estimates of the type

$$|D^k \widetilde{A}(x,t)| \leq C_k \sup\{|A^*(y,t)| : c_1|x_0| \leq |y| \leq c_2|x_0|\}$$

in a smaller compact neighborhood U_1 of x_0. Here the constants may depend on x_0, but by covering the unit sphere by such neigborhood U_1 and applying this inequality to $\widetilde{A}(rx,t)$ and $A^*(rx,t)$ for $r > 0$, it follows that for all $x \neq 0$

$$|x|^{|k|} D^k \widetilde{A}(x,t) \leq C_k' \sup\{|A^*(y,t)| : c_1'|x| \leq |y| \leq c_2'|x|\}.$$

For $|x| > 1$, the right hand side is $O(|x|^2)$ by (B2), and hence $|D^k \widetilde{A}(x,t)| \leq C_k''|x|^{2-|k|}$. $\square$

Assume that $\mathcal{M}$ satisfies (A1), (B2), (3.1) and LCP. We have so far regularized in the x directions. The t direction is much simpler; we define, for a given $\delta > 0$,

$$\widetilde{\mathcal{M}}^*(x,t) = \delta^{-1} \int_t^{t+\delta} \widetilde{\mathcal{M}}(x, u \wedge T)\, du.$$

It is easy to see from Lemma 3.6 that $\widetilde{\mathcal{M}}^*$ satisfies (B2^{2,1}). Moreover, $\widetilde{\mathcal{M}}^*$ is LCP.

Finally, let $\varepsilon > 0$ and define

$$\widetilde{\mathcal{M}}^{**} = \widetilde{\mathcal{M}}^* + \varepsilon(1+|x|^2)\Delta.$$

Then $\widetilde{\mathcal{M}}^{**}$ satisfies both (A4) and (B2^{2,1}). Moreover, $\widetilde{\mathcal{M}}^{**}$ is LCP by Remark 2.6, because $(1+|x|^2)\Delta$ is. The latter fact follows easily from Lemma 3.12 below, see also Example 5.4; we omit the details.

Lemma 3.7. *If $\mathcal{M}$ satisfies* (B2) *and LCP, then $\widetilde{\mathcal{M}}^{**}$ is convexity preserving in $C_{\mathrm{pol}}(\overline{\Omega_T})$, for each choice of φ_1, φ_2, δ, ε.*

Proof. Suppose that $f \in C_{\mathrm{pol}}(\mathbb{R}^n)$ and let F be the unique solution in C_{pol} to $\widetilde{\mathcal{M}}^{**}F = 0$ with $F(x,0) = f(x)$.

First assume that $f \in C^5_{\mathrm{pol}}(\mathbb{R}^n)$. By Theorems A.18 and A.20, $F \in C^{4,1}(\overline{\Omega_T}) \cap C^{2,1}_{\mathrm{pol}}(\overline{\Omega_T})$. Thus F is convex by Lemma 3.4.

In general, we first regularize f in the usual way by convolution with a smooth approximation of the identity. This gives us a sequence of functions $f^{(m)} \in C^5_{\mathrm{pol}}(\mathbb{R}^n)$ that converge to f uniformly on compact sets. We have just shown that the corresponding solutions are convex, and by Theorem A.12 they converge pointwise to F. Thus F is convex.

The same holds in every $\Omega_{(\tau,T]}$. □

We can now let first $\varepsilon \to 0$, then $\delta \to 0$ and finally let the supports of φ_1 and φ_2 shrink to $\{0\}$. It is easy to check that the conditions of Lemma 3.5 are satisfied each time; hence $\widetilde{\mathcal{M}}^*$, $\widetilde{\mathcal{M}}$ and finally $\mathcal{M}$ are convexity preserving in $C_{\mathrm{pol}}(\overline{\Omega_T})$. This completes the proof of (ii) $\Longrightarrow$ (i) in Theorem 3.1.

Next we analyze the condition LCP. By translation invariance, it suffices to consider $(x,t) = (0,0)$. We first take care of the lower order terms.

Lemma 3.8. *If $f \in C^4(\mathbb{R}^n)$ is convex in a neighborhood of* 0 *and $D_{uu}f(0) = 0$, then for any $v \in \mathbb{R}^n$, $D_{uv}f(0) = 0$, $D_{uuv}f(0) = 0$, $D_{uuvv}f(0) \geq 0$.*

Proof. The first claim follows because $\nabla^2 f(0) \geq 0$. The second and third follow because $D_{uu}f \geq 0$ and thus 0 is a minimum point of $D_{uu}f$. □

Lemma 3.9. *Let $\mathcal{M}$ have C^2 coefficients in a domain D. Then $\mathcal{M}$ is LCP at a point $Q \in D$ if and only if the principal part $\mathcal{M}_0$ is LCP at Q and, at Q,*

$$\nabla_x^2(b_i + x_i c) = 0, \qquad i = 1, \dots, n, \tag{3.12}$$

$$\nabla_x^2 c = 0. \tag{3.13}$$

Proof. Suppose first that $\mathcal{M}$ is LCP at Q. If f is affine, the f is convex and $D_{uu}f = 0$ for every u, so $D_{uu}(\mathcal{L}f)(Q) \geq 0$ by the LCP condition. Since $-f$ too is affine, this yields $D_{uu}(\mathcal{L}f)(Q) = 0$, and thus $\nabla_x^2(\mathcal{L}f)(Q) = 0$, Taking $f = 1$ and $f = x_i$, we obtain (3.13) and (3.12).

Next, suppose that (3.13) and (3.12) hold and that f is convex with $D_{uu}f(Q)=0$. Then, by Lemma 3.8, at Q,

$$\begin{aligned}
D_{uu}(\mathcal{L}f-\mathcal{L}_0f) &= D_{uu}\Bigl(\sum_i b_iD_if+cf\Bigr)\\
&=\sum_i\bigl(D_{uu}b_i\cdot D_if+2D_ub_i\cdot D_{ui}f+b_iD_{uui}f\bigr)+D_{uu}c\cdot f+2D_uc\cdot D_uf+cD_{uu}f\\
&=\sum_i D_{uu}b_i\cdot D_if+D_{uu}c\cdot f+2D_uc\cdot D_uf\\
&=\sum_i D_{uu}(b_i+x_ic)\cdot D_if=0
\end{aligned}$$

The lemma follows. □

Note that (3.12) and (3.13) hold in Ω_T (or in another cylindrical domain) if and only if, for each fixed t, c and b_i+x_ic are affine, i.e.

$$\begin{aligned}
c(x,t)&=c_0(t)+\langle c_1(t),x\rangle\\
b_i(x,t)&=b_{i0}(t)+\langle b_{i1}(t),x\rangle-x_i\langle c_1(t),x\rangle,
\end{aligned}$$

for some $c_0(t),b_{i0}(t)\in\mathbb{R}$ and $c_1(t),b_{i1}(t)\in\mathbb{R}^n$. In our case, the growth condition $c(x,t)=O(1)$ forces $c_1(t)=0$, and we obtain c constant and b_i affine as asserted in (iii) in Theorem 3.1.

To complete the proof of the theorem, it is thus sufficient to show that (ii) $\iff$ (iii) for the principal part $\mathcal{M}_0$. In other words, we may and will in the remainder of this section assume $b_i=c=0$.

We next narrow down the class of test functions in the definition.of LCP. We begin by considering the function f defined by

$$f(x)=\langle x,y\rangle^4, \tag{3.14}$$

for a vector $y\in\mathbb{R}^n$. Clearly, f is smooth and convex and

$$D_{uu}f(x)=12\langle u,y\rangle^2\langle x,y\rangle^2, \tag{3.15}$$

so $D_{uu}f(0)=0$. Moreover,

$$\mathcal{L}f(x,t)=\sum_{i,j=1}^n a_{ij}(x,t)12y_iy_j\langle x,y\rangle^2, \tag{3.16}$$

and thus

$$D_{uu}(\mathcal{L}f)(0,0)=24\sum_{i,j=1}^n a_{ij}(0,0)y_iy_j\langle u,y\rangle^2\ge 0, \tag{3.17}$$

because $\mathcal{L}$ is assumed to be elliptic, see (A1).

Remark 3.10. We have assumed that $\mathcal{L}$ is elliptic in order to guarantee the solvability of the initial value problem (2.2) with $F(x,0)=f(x)$. The calculation leading to (3.17) shows that even if we took care of this problem in another way, we would at least have the weak ellipticity condition

$$\sum_{i,j=1}^n a_{ij}(x,t)y_iy_j\ge 0,\qquad y\in\mathbb{R}^n, \tag{3.18}$$

as a necessary condition.

We note that for $n = 1$, the condition LCP is automatically satisfied (for the principal part). To see this let $\mathcal{L} = a(x,t)D_{xx}$ with $a \geq 0$ and let f be as in Definition 2.2. It suffices to consider $u = e_1$, so $D_u = D_x$. The condition $D_{uu}f(x) = 0$ thus says that $f''(x) = 0$. Furthermore, by Lemma 3.8, $f'''(x) = 0$ and $f''''(x) \geq 0$ and we have

$$\begin{aligned} D_{uu}\big(\mathcal{L}f\big)(x,t) &= D_{xx}(a(x,t)f'')(x,t) \\ &= a_{xx}(x,t)f''(x) + 2a_x(x,t)f'''(x) + a(x,t)f''''(x) \\ &= a(x,t)f''''(x) \geq 0, \end{aligned}$$

i.e. the LCP condition holds.

Hovever, for $n \geq 2$, LCP is no longer automatic. To study higher dimension we let $n \geq 2$ and for $u \in \mathbb{R}^n \setminus \{0\}$, we define $u^\perp$ as the $(n-1)$-dimensional space $\{v \in \mathbb{R}^n : \langle u, v\rangle = 0\}$. We introduce an arbitrary ON-basis in $u^\perp$, and can thus identify linear operators on $u^\perp$ with $(n-1) \times (n-1)$ matrices.

Lemma 3.11. *Let $\mathcal{M} = \mathcal{M}_0$ with coefficients in $C^{2,0}$. Then $\mathcal{M}$ is LCP at $(0,t)$ if and only if*

$$D_{uu}(\mathcal{L}g)(0,t) \geq 0, \tag{3.19}$$

for every $u \in \mathbb{R}^n \setminus \{0\}$, where $n \geq 2$ and every function g on $\mathbb{R}^n$ of the form

$$g(su + v) = \langle Mv, v\rangle + s\langle Nv, v\rangle + s^2\langle Pv, v\rangle, \tag{3.20}$$

where $s \in \mathbb{R}$, $v \in u^\perp$ and M, N and P, are symmetric linear operators on $u^\perp$ satisfying

$$\langle Mw, w\rangle + \langle Pv, v\rangle + 2\langle Nv, w\rangle \geq 0, \tag{3.21}$$

for all $v, w \in u^\perp$; equivalently, the $(2n-2) \times (2n-2)$ matrix

$$\begin{pmatrix} M & N \\ N & P \end{pmatrix}$$

is positive semidefinite.

Proof. If g is given by (3.20), its Hessian matrix at the point $su + v$, in the basis given by u and the chosen basis for $u^\perp$, has the block form

$$\operatorname{Hess}(g) = \begin{pmatrix} 2\langle Pv, v\rangle & 2Nv + 4sPv \\ 2Nv + 4sPv & 2M + 2sN + 2s^2P \end{pmatrix}. \tag{3.22}$$

Suppose that (3.20) and (3.21) hold. Let

$$g_\varepsilon(su + v) = g(su + v) + \varepsilon k(su + v), \tag{3.23}$$

where $0 < \varepsilon < 1$ and $k(su + v) = |v|^2 + s^2|v|^2$, i.e. g_ε is given by (3.20) with M and P replaced by $M + \varepsilon I$ and $P + \varepsilon I$, respectively. By (3.22) we find, for $r \in R$ and $w \in u^\perp$ and with $*$ denoting transpose,

$$\begin{aligned} (r, w)\operatorname{Hess}(g_\varepsilon)(r, w)^* &= 2\langle (P + \varepsilon I)v, v\rangle r^2 + 4r\langle Nv, w\rangle \\ &\quad + 8rs\langle (P + \varepsilon I)v, w\rangle + 2\langle (M + \varepsilon I)w, w\rangle \\ &\quad + 2s\langle Nw, w\rangle + 2s^2\langle (P + \varepsilon I)w, w\rangle \\ &= 2\big(\langle P(rv), rv\rangle + \langle Mw, w\rangle + 2\langle N(rv), w\rangle\big) \\ &\quad + 2\varepsilon(r^2|v|^2 + |w|^2) + O(|s||rv||w| + |s||w|^2 + s^2|w|^2), \end{aligned}$$

which by (3.21) is nonnegative for sufficiently small $|s|$ and all r, v and w. Hence $\operatorname{Hess}(g_\varepsilon) \geq 0$ in a neighborhood of 0, i.e. g_ε is convex in a neighborhood of 0.

Moreover, at the point 0, for which $s = v = 0$, $D_{uu}(g_\varepsilon) = 2\langle (P+\varepsilon)v, v\rangle = 0$. If $\mathcal{M}$ is LCP at $(0,t)$ we thus have

$$0 \le D_{uu}(\mathcal{L}g_\varepsilon)(0,t) = D_{uu}(\mathcal{L}g)(0,t) + \varepsilon D_{uu}(\mathcal{L}k)(0,t),$$

and letting $\varepsilon \to 0$, we obtain $D_{uu}(Lg)(0,t) \ge 0$.

Conversely, suppose that $f \in C^4$ is convex in a neighborhood U of 0 and that $D_{uu}f(0) = 0$. The Taylor expansion of f to order 4 can be written

$$f(x) = p_0 + p_1(x) + p_2(x) + p_3(x) + p_4(x) + R(x),$$

where p_k, $k = 0, 1, \dots, 4$, is a homogeneous polynomial of order k and $|\nabla^j R| = o(|x|^{4-j})$ as $x \to 0$, for $0 \le j \le 4$. By Lemma 3.8, several of the Taylor coefficients vanish, and thus

$$\begin{aligned}
&p_2(su+v) = \alpha_2(v),\\
&p_3(su+v) = s\beta_2(v) + \beta_3(v),\\
&p_4(su+v) = as^4 + s^3\langle b, v\rangle + s^2\gamma_2(v) + s\gamma_3(v) + \gamma_4(v),
\end{aligned}$$

for some $a \in \mathbb{R}$, $b \in u^\perp$ and homogeneous polynomials α_k, β_k and γ_k of degree k; moreover, $a = \frac{1}{24}D_{uuuu} \ge 0$. Suppose first that $a > 0$. Then, with $\tilde b = b/4a$,

$$\begin{aligned}
p_4(su+v) &= a(s+\langle \tilde b, v\rangle)^4 + s^2\tilde\gamma_2(v) + s\tilde\gamma_3(v) + \tilde\gamma_4(v)\\
&= ah(su+v) + s^2\tilde\gamma_2(v) + s\tilde\gamma_3(v) + \tilde\gamma_4(v),
\end{aligned}$$

where $h(x) = \langle x, u+\tilde b\rangle^4$ and $\tilde\gamma_k$, $k = 2, 3$ and 4, have the same homogeneity properties as γ_k and are defined so that the equation above holds. Consequently,

$$\begin{aligned}
f(su+v) = p_0 + p_1(su+v) + g(su+v) + \beta_3(v) + ah(su+v)&\\
+ s\tilde\gamma_3(v) + \tilde\gamma_4(v) + R(su+v),&\quad (3.24)
\end{aligned}$$

where g is as in (3.20) for some M, N, P. Letting $r \in \mathbb{R}$ and $w \in u^\perp$, we have for $|su+v| \le 1$, say, using (3.22),

$$\begin{aligned}
0 \le D^2_{ru+w}f(su+v) &= D^2_{ru+w}g(su+v) + 12a(s+\langle\tilde b, v\rangle)^2(r+\langle\tilde b, w\rangle)^2\\
&\quad + O(|v||w|^2 + |r||v|^2|w|) + o(|su+v|^2|ru+w|^2)\\
&= 2r^2\langle Pv, v\rangle + 4r\langle Nv, w\rangle + 2\langle Mw, w\rangle\\
&\quad + 12a(s+\langle\tilde b, v\rangle)^2)(r+\langle\tilde b, w\rangle)^2 + O(|s||rv||w|)\\
&\quad + O(|s||w|^2 + |v||w|^2 + |r||v|^2|w|) + o(|su+v|^2|ru+w|^2).
\end{aligned}$$

Take $s = -\langle\tilde b, v\rangle$ and $v = \overline{v}/r$, for some fixed vector $\overline{v}$, and let $r \to \infty$ and thus $v \to 0$ and $s \to 0$. In the limit we obtain,

$$2\langle P\overline{v}, \overline{v}\rangle + 4\langle N\overline{v}, w\rangle + 2\langle Mw, w\rangle \ge 0$$

and thus (3.21) holds for g. Assuming the condition in the lemma, we thus have (3.19), i.e. $D_{uu}(\mathcal{L}g)(0,t) \ge 0$.

$D_{uu}\mathcal{L}$ is a partial differential operator with continuous coefficients containing derivatives of order 2, 3 and 4, but no term with more than two derivatives in directions orthogonal to u. Hence at $(0,t)$,

$$D_{uu}\mathcal{L}\big(p_0 + p_1(su+v) + \beta_3(v) + s\tilde\gamma_3(v) + \tilde\gamma_4(v) + R(su+v)\big) = 0.$$

and consequently, by (3.24),

$$D_{uu}(\mathcal{L}f)(0,t) = D_{uu}(\mathcal{L}g)(0,t) + aD_{uu}(\mathcal{L}h)(0,t) \ge 0$$

by (3.19) and (3.17). In the case $a = 0$, we obtain the same conclusion by considering $f_\varepsilon = f(x) + \varepsilon\langle x, u\rangle^4$ and letting $\varepsilon \to 0$. $\square$

Let $A(x,t)$ denote the $n \times n$ matrix $(a_{ij}(x,t))$, and the corresponding linear operator in $\mathbb{R}^n$ and let A_u denote the operator $Q_u A Q_u$ restricted to $u^\perp$, where Q_u is the orthogonal projection of $\mathbb{R}^n$ onto $u^\perp$. Now, let g be as in (3.20). If we rotate the system so that $u = e_1$, we have

$$\begin{aligned}\mathcal{L}g(x,t) = &\sum_{i,j=2}^{n} 2a_{ij}(x,t)(M_{ij} + x_1 N_{ij} + x_1^2 P_{ij})\\ &+ \sum_{i=2}^{n} 2a_{1i}(x,t) \sum_{j=2}^{n} (2N_{ij}x_j + 4x_1 P_{ij} x_j)\\ &+ 2a_{11}(x,t) \sum_{i,j=2}^{n} P_{ij} x_i x_j\end{aligned}$$

and thus

$$D_{uu}(\mathcal{L}g)(0,0) = 2\sum_{i,j=2}^{n} \big(D_{uu}a_{ij}(0,0)M_{ij} + 2D_u a_{ij}(0,0)N_{ij} + 2a_{ij}(0,0)P_{ij}\big).$$

Lemma 3.11 thus implies the following, which completes the proof of Theorem 3.1.

Lemma 3.12. *Let $\mathcal{M} = \mathcal{M}_0$ with coefficients in $C^{2,0}$, and let the spatial dimension $n \geq 2$. Then $\mathcal{M}$ is LCP at (x_0, t) if and only if*

$$\langle M, D_{uu}A_u\rangle + 2\langle N, D_u A_u\rangle + 2\langle P, A_u\rangle \geq 0 \tag{3.25}$$

at (x_0, t) for every unit vector $u \in \mathbb{R}^n$ and every symmetric linear operators M, N and P in $u^\perp$ such that

$$\begin{pmatrix} M & N \\ N & P \end{pmatrix} \geq 0. \tag{3.26}$$

$\square$

4. An algebraic digression

The convexity inequality in Definition 2.5 and Lemma 3.12 is a pointwise algebraic condition on the coefficients a_{ij} and their first two x-derivatives. As we will see in the next section, this makes it easy to investigate some examples. Nevertheless, the algebraic condition is somewhat implicit, and it would be nice to simplify it. We have not been able to do so, in general, but in order to stimulate the reader to further research, we describe the algebraic situation in some detail. (Note that n below corresponds to $n-1$ elsewhere.)

Let $\mathcal{M}(n)$ be the set of (real) $n \times n$ matrices, and let $\mathcal{S}(n)$ and $\mathcal{P}(n)$ be the subsets of symmetric and positive semidefinite matrices, respectively, and let $\mathcal{Q}(2n) \subseteq \mathcal{S}(2n)$ be the set of $2n \times 2n$ matrices of the form $\left(\begin{smallmatrix} A & B \\ B & C \end{smallmatrix}\right)$ with $A, B, C \in \mathcal{S}(n)$.

Let $\mathcal{PQ}(2n) = \mathcal{P}(2n) \cap \mathcal{Q}(2n)$ and let

$$\mathcal{PQ}^*(2n) = \{T \in \mathcal{M}(2n) : \operatorname{Tr}(TU) \geq 0 \text{ when } U \in \mathcal{PQ}(2n)\}.$$

Thus the convexity inequality in Definition 2.5 says

$$\begin{pmatrix} D_{uu}A_u & D_u A_u \\ D_u A_u & 2A_u \end{pmatrix} \in \mathcal{PQ}^*(2(n-1)), \tag{4.1}$$

so the problem is to find a simple characterization of $\mathcal{PQ}^*(2n)$, or at least $\mathcal{PQ}^*(2n)\cap Q(2n)$ (since the matrix in (4.1) evidently belongs to $\mathcal{Q}(2n)$).

For $n=1$, this is simple. $\mathcal{Q}(2)=S(2)\supset\mathcal{P}(2)$ and thus $\mathcal{PQ}(2)=\mathcal{P}(2)$, and it is easy to see (and well known) that $\mathcal{PQ}^*(2)\cap\mathcal{Q}(2)=\mathcal{P}^*(2)\cap\mathcal{S}(2)=\mathcal{P}(2)$.

For $n\geq 2$, it is still true that $\mathcal{P}^*(2n)\cap\mathcal{S}(2n)=\mathcal{P}(2n)$, and thus $\mathcal{PQ}^*(2n)\supseteq\mathcal{P}^*(2n)\supseteq\mathcal{P}(2n)$. However, the following example (for $n=2$ but easily extended to $n>2$) shows that $\mathcal{PQ}^*(2n)\cap\mathcal{Q}(2n)\supsetneq\mathcal{PQ}(2n)$ when $n\geq 2$.

We leave it as an open problem to find a simple characterization of $\mathcal{PQ}^*(2n)\cap\mathcal{Q}(2n)$, $n\geq 2$. A related problem is to find the extremal rays in $\mathcal{PQ}(2n)$.

Example 4.1. Let a and b be real with $a>0$ and define

$$A=\begin{pmatrix}1&0&0&b\\0&a&b&0\\0&b&a&0\\b&0&0&1\end{pmatrix}\in\mathcal{Q}(4). \tag{4.2}$$

If $\left(\begin{smallmatrix}M&N\\N&P\end{smallmatrix}\right)\in\mathcal{PQ}(4)$, then $\left(\begin{smallmatrix}M_{22}&N_{12}\\N_{12}&P_{11}\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}M_{11}&N_{12}\\N_{12}&P_{22}\end{smallmatrix}\right)\in\mathcal{P}(2)$. Conversely, if M_{11}, M_{22}, P_{11}, P_{22}, N_{12} are given with these two matrices in $\mathcal{P}(2)$, we can take $M_{12}=N_{11}=N_{22}=P_{12}=0$ and obtain $\left(\begin{smallmatrix}M&N\\N&P\end{smallmatrix}\right)\in\mathcal{PQ}(4)$. Hence $A\in\mathcal{PQ}^*(4)$ if and only if

$$M_{11}+aM_{22}+aP_{11}+P_{22}+4bN_{12}\geq 0 \tag{4.3}$$

whenever $M_{11},M_{22},P_{11},P_{22}\geq 0$, and $N_{12}^2\leq M_{22}P_{11}$, $N_{12}^2\leq M_{11}P_{22}$. Fixing N_{12}, the left side of (4.3) is minimized, under these restrictions, by $M_{11}=P_{22}=M_{22}=P_{11}=|N_{12}|$. Consequently,

$$A\in\mathcal{PQ}^*(4)\iff 2+2a\pm 4b\geq 0\iff 2|b|\leq 1+a.$$

For example, $b=2$ and $a=3$ gives $A\in\mathcal{PQ}^*(4)\cap\mathcal{Q}(4)$, but $A\notin\mathcal{P}(4)$.

5. Examples

We use the results in Section 3 to give explicit characterizations of the convexity preserving differential operators in some cases. More precisely, we assume that the coefficients are in $C^{2,0}$ and characterize the operators that are LCP (using Lemma 3.12). Assuming the slightly stronger regularity hypotheses in Theorem 3.1, this characterizes the operators that are convexity preserving.

Since the condition LCP is expressed only in the spatial variables, see Remark 2.3, it is enough to study equations without explicit time dependence; we could let the coefficients depend on t too without changing anything else in the examples. Moreover, we will mainly consider equations without lower order terms, since equations with such terms then have a simple characterization by Lemma 3.9 (locally) and Theorem 3.1 (globally).

Example 5.1. An operator in divergence form

$$\frac{\partial F}{\partial t}=\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial F}{\partial x_j}\Big)$$

is LCP if and only if (a_{ij}) satisfies the convexity inequality and $\sum_{i=1}^n\frac{\partial}{\partial x_i}a_{ij}(x)$ is affine for each $j=1,\dots,n$.

If $n=1$, this holds only if a is a quadratic polynomial in x.

Recall first that if $n = 1$, the LCP condition holds for any operator $a(x)D_{xx}$.

For $n = 2$, A_u is 1-dimensional, i.e. real valued, and the discussion in Section 4 shows that the convexity inequality holds if and only if

$$\begin{pmatrix} D_{uu}A_u & D_uA_u \\ D_uA_u & 2A_u \end{pmatrix} \geq 0, \tag{5.1}$$

or, equivalently, since $A_u > 0$ by the parabolicity,

$$2A_uD_{uu}A_u \geq (D_uA_u)^2. \tag{5.2}$$

Moreover, if $u = (u_1, u_2)$, then $u^\perp$ is spanned by $(u_2, -u_1)$, and the restriction of Q_uAQ_u to $u^\perp$ equals multiplication by $(u_2, -u_1)A(u_2, -u_1)'$. Consequently, we have a simple characterization.

Corollary 5.2. *Let $n = 2$. The parabolic equation $D_tF = a_{11}D_{11}F + 2a_{12}D_{12}F + a_{22}D_{22}F$ with coefficients in $C^{2,0}$ is LCP if and only if, for every (unit) vector $u \in \mathbb{R}^2$, (5.2) or, equivalently,*

$$D_{uu}(A_u^{1/2}) \geq 0 \tag{5.3}$$

holds, where $A_u = a_{11}u_2^2 - 2a_{12}u_1u_2 + a_{22}u_1^2$. □

Example 5.3. Consider an equation of the form

$$D_tF = g^2(x)D_{xx}F + h^2(y)D_{yy}F$$

where $F = F(x, y, t)$ and g and h are twice continuously differentiable and non-zero. In other words, the matrix A is given, in the xy-coordinates by

$$A = \begin{pmatrix} g^2(x) & 0 \\ 0 & h^2(y) \end{pmatrix}.$$

Let a direction u in (5.3) be given in the same coordinates by $u = (a, b)$. Condition (5.3) is thus given by

$$(a^2\partial_x^2 + 2ab\partial_x\partial_y + b^2\partial_y^2)\sqrt{b^2g^2(x) + a^2h^2(y)} \geq 0. \tag{5.4}$$

By direct computation we see that the left-hand-side of (5.4) equals

$$(b^2g^2(x) + a^2h^2(x))^{-3/2} \cdot \Big(\big(a^2bg'(x)h(y) - ab^2g(x)h'(y)\big)^2 \\ + a^2b^2\big(b^2g^2(x) + a^2h^2(y)\big)\big(g(x)g''(x) + h(y)h''(y)\big)\Big)$$

When $ab \neq 0$, this is equivalent to

$$\big(ag'(x)h(y) - bg(x)h'(y)\big)^2 + \big(b^2g^2(x) + a^2h^2(y)\big)\big(g(x)g''(x) + h(y)h''(y)\big) \geq 0,$$

which by continuity must hold for $ab = 0$ too. Since we always may choose a and b such that the first parenthesis vanishes, the operator is LCP if and only if

$$g(x)g''(x) + h(y)h''(y) \geq 0. \tag{5.5}$$

Thus, for example, the operator $x\frac{\partial^2}{\partial x^2} + y\frac{\partial^2}{\partial y^2}$ defined on the first quadrant is not LCP, but the operator $x^2\frac{\partial^2}{\partial x^2} + y^2\frac{\partial^2}{\partial y^2}$ is.

For $n > 2$, on the other hand, the condition (3.25) does not reduce to (5.1), see Section 4. Hence we can only say that (5.1) is sufficient for LCP (when there are no lower order terms), and we have to use Definition 2.5.

Example 5.4. In this example we consider an equation of the form

$$D_t F = g^2(x)\Delta F, \tag{5.6}$$

where Δ is the Laplacian on $\mathbb{R}^n, n \geq 2$ and $x = (x_1, \dots, x_n)$. We assume that $g > 0$ and $g \in C^2$. Using the notation of Lemma 3.12 we see that in this case $A = g^2(x)I$, where I is the identity matrix on $\mathbb{R}^n$. Thus $A_u = g^2 I_{u^\perp}$, with $I_{u^\perp}$ denoting the identity matrix on $u^\perp$. Thus (3.25) reduces to

$$(\mathrm{Tr}M)D_{uu}(g^2) + 2(\mathrm{Tr}N)D_u(g^2) + 2(\mathrm{Tr}P)g^2 \geq 0. \tag{5.7}$$

If M, N, P satisfy (3.26), then, for each i, the submatrix $\begin{pmatrix} M_{ii} & N_{ii} \\ N_{ii} & P_{ii} \end{pmatrix} \geq 0$ and thus $N_{ii}^2 \leq M_{ii}P_{ii}$. Consequently, by the Cauchy–Schwarz inequality,

$$|\operatorname{Tr} N| \leq \sum_i |N_{ii}| \leq \sum_i |M_{ii}|^{1/2}|P_{ii}|^{1/2} \leq (\operatorname{Tr} M)(\operatorname{Tr} P).$$

Conversely, given any real numbers μ, ν and π with $\mu, \pi \geq 0$ and $\nu^2 \leq \mu\pi$, let M, N and P be suitable multiples of the identity such that $\operatorname{Tr} M = \mu$, $\operatorname{Tr} N = \nu$, $\operatorname{Tr} P = \pi$. It then follows from the Cauchy–Schwarz inequality that (3.26) holds.

Consequently, (5.7) holds for all M, N, P satisfying (3.26) if and only if

$$\mu D_{uu}(g^2) + 2\nu D_u(g^2) + 2\pi g^2 \geq 0$$

for all μ, ν, π with $\mu, \pi \geq 0$ and $\nu^2 \leq \mu\pi$. This is equivalent to

$$2g^2 D_{uu}(g^2) \geq \big(D_u(g^2)\big)^2,$$

which by simple calculations can be written

$$gD_{uu}g \geq 0. \tag{5.8}$$

(The last steps are the same as the ones leading to (5.2) and (5.3) for $n = 2$.) Consequently, when $n \geq 2$, the equation (5.6) is LCP if and only if g is convex.

The stochastic representation of a solution to (5.6) is given by

$$F(x, t) = \mathbb{E}\, F(X(t), 0),$$

where X is a vector valued stochastic process satisfying $X(0) = x$ and

$$dX = \sqrt{2}g(X)dW,$$

where W is a Wiener process. From this point of view it is thus rather natural to have convexity conditions on g rather than g^2.

6. Non-homogeneous equations

In this section we show that the case of a non-homogeneous equation (2.3) easily is reduced to the homogeneous case treated above.

Theorem 6.1. *Assume* (A1), (B2), (C1) *and let* $h \in C_{\mathrm{pol}}(\Omega_T)$ *be locally Hölder*(α) *in* Ω_T. *Then* $(\mathcal{M}, h)$ *is convexity preserving on* $C_p(\overline{\Omega_T})$ *if and only if* $\mathcal{M}$ *is convexity preserving on* $C_p(\overline{\Omega_T})$ *and* h *is convex (for each* $t \in (0, T]$*).*

Proof. We use the formula (and notation) in Theorem A.15.

If $\mathcal{M}$ is convexity preserving and $h(x, \tau)$ is convex for every τ, then, for each convex $f \in C_{\mathrm{pol}}(\mathbb{R}^n)$, $F_0(x, t)$ and $H_\tau(x, t)$ are convex, so $F(x, t)$ is convex by (A.11). Thus, translating to domains $\Omega_{(\tau,T]}$, $(\mathcal{M}, h)$ is convexity preserving.

Conversely, assume that $(\mathcal{M}, h)$ is convexity preserving. Let $f \in C_{\mathrm{pol}}(\mathbb{R}^n)$ be convex and let $r > 0$. The solution with boundary values rf is by (A.11) $rF_0 + \int_0^t H_\tau$.

By assumption this is convex. We divide by r and let $r \to \infty$; thus F_0 is convex. Consequently, $\mathcal{M}$ is convexity preserving.

Now consider (2.3) with $0 < \tau < T$ and $f = 0$. By Theorem A.15, the solution is

$$F(x,t) = \int_\tau^t H_s(x,t)\,ds.$$

It follows from Theorem A.9 and (C1) that for given x, $\tau > 0$ and $\varepsilon > 0$, there exists $\eta > 0$ such that for $\tau \le s \le t < \tau + \eta$,

$$|H_s(x,t) - h(x,\tau)| \le |H_s(x,t) - h(x,s)| + |h(x,s) - h(x,\tau)| < 3\varepsilon.$$

Consequently, $(t-\tau)^{-1}F(x,t) \to h(x,\tau)$ as $t \downarrow \tau$, and $h(x,\tau)$ is convex. □

7. Bounded coefficients

It is common in studies of parabolic partial differential equations to assume that the coefficients are bounded. The following, perhaps surprising, result shows that when $n \ge 2$, the only such operators that are LCP are the operators with coefficients depending on t only. (Recall that this does not hold for $n = 1$, when a can be any bounded sufficiently smooth function.)

Theorem 7.1. *Assume that $n \ge 2$ and that $\mathcal{M}$ has coefficients in $C^{2,0}$ and satisfies* (A1) *and* (B3)*. Then $\mathcal{M}$ is LCP if and only if a_{ij} and c depend on t only, and $b_i(x,t) = b_{i0}(t) + \langle b_{i1}(t), x\rangle$ for some functions $b_{i0}(t) \in \mathbb{R}$ and $b_{i1}(t) \in \mathbb{R}^n$.*

In particular, if $\mathcal{M}$ has bounded coefficients, it is LCP if and only if the coefficients do not depend on x.

Proof. The claims for b_i and c follow from Lemma 3.9 and the assumed bounds.

For a, we for convenience assume that $n = 2$. The case $n > 2$ is similar, or follows by considering two-dimensional subspaces. Moreover, we fix t.

Using the function $f(x_1, x_2) = x_1^2$ in the definition of LCP, we conclude that a_{11} is convex as a function of x_2. However, by assumption a_{11} is bounded. Thus a_{11} can only depend on x_1. Similarly, a_{22} can only depend on x_2. Now, consider instead the function $f(x_1, x_2) = (sx_1 - x_2)^2$ for some fixed s. We compute

$$\sum_{i,j=1}^{2} a_{ij} D_{ij} f = 2(s^2 a_{11} - 2s a_{12} + a_{22}) \tag{7.1}$$

On lines of the form $x_2 = x_{2,0} + sx_1$ where $x_{2,0}$ is a constant, f is constant. Thus using again the LCP condition, we conclude that the expression in equation (7.1) is convex along such lines. However, this expression is, by assumption on the coefficients a_{ij}, bounded and hence constant along these lines. Recalling that a_{11} only depends on x_1 and a_{22} only depends on $x_2 = x_{2,0} + sx_1$ whereas $a_{12} = a_{12}(x_1, x_{2,0} + sx_1)$, we take the partial derivative of the right hand side of (7.1) with respect to x_1 arriving at

$$s^2 a_{11}'(x_1) + s a_{22}'(x_2) - 2s\frac{\partial a_{12}}{\partial x_1}(x_1,x_2) - 2s^2\frac{\partial a_{12}}{\partial x_2}(x_1,x_2) = 0. \tag{7.2}$$

Now, this holds for any s. Identifying the coefficients we thus conclude that

$$a_{11}'(x_1) = 2\frac{\partial a_{12}}{\partial x_2}(x_1,x_2), \tag{7.3}$$

$$a_{22}'(x_2) = 2\frac{\partial a_{12}}{\partial x_1}(x_1,x_2). \tag{7.4}$$

From equation (7.3) we conclude that $a_{12} = f_1(x_1) + x_2 f_2(x_1)$ for some functions f_1 and f_2. Inserting this into equation (7.4) we conclude that f_1 and f_2 are affine. However, since a_{12} is bounded we deduce that f_1 is a constant and that f_2 vanishes, so a_{12} is a constant. Finally, (7.3) and (7.4) yield that a_{11} and a_{22} are constants.

The converse is easy, see Remark 2.4. □

If we require the entries of the matrix (a_{ij}) to be bounded, or more generally that (B3) holds, we can enlarge the space of functions we allow as initial conditions. We thus define

$$C_{\exp}(\overline{\Omega_T}) = \{F \in C(\overline{\Omega_T}) : |F(x,t)| = \exp(o(|x|^2)),\ (x,t) \in \overline{\Omega_T}\}.$$

We then have the following result, showing that only very special operators of the studied class preserve convexity when the spatial dimension n is at least two. (The case $n = 1$ is, again, different.)

Theorem 7.2. *Assume that $n \geq 2$ and that the operator $\mathcal{M}$ is such that* (A1)*,* (B3)*,* (C1) *hold and $a_{ij}, b_i, c \in C^{2,0}_\alpha$. Then the following are equivalent.*

(i) *$\mathcal{M}$ is convexity preserving in $C_{\exp}(\overline{\Omega_T})$.*
(ii) *$\mathcal{M}$ is LCP in Ω_T.*
(iii) *for each fixed t, a and c are constant and each b_i is an affine function of x.*

Proof. (i) $\Longrightarrow$ (ii). Let $FC^{2,1} \cap C_{\exp}(\overline{\Omega_T})$ solve $\mathcal{M}F = 0$ with $F(x,0) = f(x)$, where $f \in C^\infty(\mathbb{R}^n) \cap C_{\exp}$ is convex. By Theorem A.19, $F \in C^{4,1}_\alpha(\overline{\Omega_T})$ and the proof of Lemma 3.3 applies.

(ii) $\Longrightarrow$ (iii). By Theorem 7.1.

(iii) $\Longrightarrow$ (i). We could argue as in Lemma 3.4 and so on, but since the operators are so special, we use an alternative.

If the coefficients are functions of t only, there exists a fundamental solution $\Gamma(x,t,\xi,\tau) = \Gamma(x-\xi,t,\tau)$ that depends only on $x - \xi$ [4, Theorems 9.1], and thus the Cauchy problem (2.2) with $h = 0$ and $f \in C_{\exp}(\mathbb{R}^n)$, has a solution that, for fixed t, is a convolution on f with a certain kernel, see [4, 1.12].

By the maximum principle, Theorem A.5, the solution is non-negative when f is, so this kernel is non-negative. Consequently, the solution is convex whenever f is.

The case when each b_i is affine can be reduced to the previous case by a change of coordinates: if $G(x,t) = F(V(t)x,t)$ for a suitable matrix valued function $V(t)$, then $\widetilde{\mathcal{M}}G = 0$ for a parabolic operator $\widetilde{\mathcal{M}}$ with coefficients depending on t only. Hence G is convex, and thus F is convex. We omit the details. □

8. The case of Hölder coefficients

In Theorem 3.1 we assumed the smoothness condition (3.1) for the coefficients. Assuming instead only Hölder continuity, (C1), we can give a partial result.

Theorem 8.1. *Assume that $\mathcal{M}$ is such that* (A1)*,* (B2) *and* (C1) *hold. If $\mathcal{M}$ satisfies the convexity inequality, see Definition 2.5, in Ω_T, and, for fixed t, each b_i is an affine function of x and c is constant, then $\mathcal{M}$ is convexity preserving on $C_{\mathrm{pol}}(\overline{\Omega_T})$.*

Proof. We regularize the coefficients of $\mathcal{M}$. Thus, let $\varphi \in C_0^\infty$ be nonnegative with $\int \varphi = 1$, and let $\widetilde{\mathcal{M}}$ be the parabolic differential operator with coefficients $\tilde{a}_{ij} = \varphi * a_{ij}$, $\tilde{b}_i = b_i$, $\tilde{c} = c$. Then $\widetilde{\mathcal{M}}$ satisfies the convexity inequality, cf. Remark 2.6, so

$\widetilde{\mathcal{M}}$ is convexity preserving by Theorem 3.1. The result follows from Lemma 3.5 by taking a sequence φ_m with supports shrinking to $\{0\}$. □

We conjecture that the converse holds too, but we have been unable to show it. (The major technical problem is to show that if $\mathcal{M}$ is convexity preserving, then so are suitable regularizations of it. For smooth coefficients, this follows by Theorem 3.1 and Remark 2.6.)

9. Monotonicity of solutions

For one spatial dimension, it was shown in [7] that a certain monotonicity property follow from the convexity preserving. We give here a corresponding result for arbitrary dimensions.

Theorem 9.1. *Assume that the first and zeroth order terms of the operators $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ are identical, and that their respective second order terms satisfy, using natural notation,*

$$(a_{ij}) \geq (\widetilde{a}_{ij}) \tag{9.1}$$

as quadratic forms. Assume also in that these operators are such that the conditions (A1), (B2) *and* (C1) *hold for both operators and that at least one of the operators is convexity preserving on $C_{\text{pol}}(\overline{\Omega_T})$. Now, consider the Cauchy problem* (2.2) *for these two operators where the corresponding non-homogeneous terms h and $\widetilde{h}$ are in $C_{\text{pol}}(\Omega_T)$, locally Hölder(α) in Ω_T, and satisfy the inequality $h \geq \widetilde{h}$. Let the corresponding initial conditions f and $\widetilde{f}$ be convex functions in $C_{\text{pol}}(\mathbb{R}^n)$ satisfying $f(x) \geq \widetilde{f}(x)$. Let $F(x,t)$ and $\widetilde{F}(x,t)$ be the corresponding solutions to* (2.2) *in $C_{\text{pol}}(\overline{\Omega_T})$. Then*

$$F(x,t) \geq \widetilde{F}(x,t) \tag{9.2}$$

for all (x,t) in $\overline{\Omega_T}$.

Proof. Let m be an even integer so large that $|F(x,t)| + |\widetilde{F}(x,t)| = O(|x|^m + 1)$. Define

$$F_\varepsilon(x,t) = F(x,t) + \varepsilon e^{Ct} g(x), \tag{9.3}$$

where $g(x) = 1 + |x|^{m+2}$. We can choose C so large that

$$Cg(x) > \mathcal{L}g(x) + 1, \tag{9.4}$$

due to the assumed growth rate of the coefficients of $\mathcal{L}$. Let $E = \{(x,t) \in \mathbb{R}^n \times [0,T] : F_\varepsilon(x,t) < \widetilde{F}(x,t)\}$ and suppose that $E \neq \emptyset$. By the construction of g, for some $\rho < \infty$, $E \subseteq B(0,\rho) \times [0,T]$. Thus E is bounded and $\overline{E}$ is compact. By continuity, $F_\varepsilon \leq \widetilde{F}$ in $\overline{E}$. Let

$$t_0 = \inf\{t \geq 0 : (x,t) \in \overline{E} \text{ for some } x \in \mathbb{R}^n\}.$$

This infimum is attained and thus $(x_0, t_0) \in \overline{E}$ for some $x_0 \in \mathbb{R}^n$.

For $t = 0$ we have $F_\varepsilon(x,0) = f(x) + \varepsilon g(x) > \widetilde{f}(x) = \widetilde{F}(x,0)$. Hence $(x,0) \notin \overline{E}$ and thus $t_0 > 0$. Since $(x,t) \notin E$ for $0 < t < t_0$, by continuity we have $F_\varepsilon(x,t_0) \geq \widetilde{F}(x,t_0)$. Hence $F_\varepsilon(x_0,t_0) = \widetilde{F}(x_0,t_0)$, and x_0 has to be a local minimum point for $F_\varepsilon(\cdot,t_0) - \widetilde{F}(\cdot,t_0)$, so the first order derivatives agree at this point and the Hessian matrix of $F_\varepsilon - \widetilde{F}$ is positive semi-definite.

Since $(x,t) \notin E$ for $0 < t < t_0$, we further must have

$$D_t(F_\varepsilon - \widetilde{F})(x_0,t_0) \leq 0. \tag{9.5}$$

On the other hand, since F and $\widetilde{F}$ solve (2.2) for the operators $\mathcal{M}$ and $\widetilde{\mathcal{M}}$, we have at (x_0, t_0), using (9.4) and $h \geq \widetilde{h}$,

$$\begin{aligned} D_t(F_\varepsilon - \tilde{F}) &= \sum_{i,j=1}^n \Bigl(a_{ij} D_{ij}(F_\varepsilon) - \widetilde{a}_{ij} D_{ij}(\widetilde{F})\Bigr) + \varepsilon C e^{Ct} g - \varepsilon e^{Ct} \mathcal{L} g + h - \widetilde{h} \\ &\geq \sum_{i,j=1}^n (a_{ij} - \widetilde{a}_{ij}) D_{ij}(F_\varepsilon) + \sum_{i,j=1}^n \widetilde{a}_{ij} D_{ij}(F_\varepsilon - \widetilde{F}) + \varepsilon e^{Ct}. \end{aligned} \tag{9.6}$$

Let us first assume that the operator $\mathcal{M}$ is convexity preserving. Since f and g are convex, then F and F_ε are convex. Hence the Hessian matrix $(D_{ij} F_\varepsilon)_{ij} \geq 0$. By the assumption (9.1), also $(a_{ij} - \widetilde{a}_{ij})_{ij} \geq 0$, and thus the first sum in the last line is non-negative. The same is true for the second sum because $(\widetilde{a}_{ij})_{ij} \geq 0$ by (A1) and, as shown above, $\bigl(D_{ij}(F_\varepsilon - \widetilde{F})\bigr)_{ij} \geq 0$ at (x_0, t_0). Consequently, $D_t(F_\varepsilon - \tilde{F})(x_0, t_0) \geq \varepsilon e^{Ct_0} > 0$ which contradicts (9.5). Consequently, $E = \emptyset$, and thus $F_\varepsilon \geq \widetilde{F}$ everywhere. Letting $\varepsilon \to 0$, we find that $F \geq \tilde{F}$.

If instead we assume that $\widetilde{\mathcal{M}}$ is convexity preserving we change the second line of (9.6) to $\sum_{i,j=1}^n (a_{ij} - \widetilde{a}_{ij}) D_{ij}(\widetilde{F}) + \sum_{i,j=1}^n a_{ij} D_{ij}(F_\varepsilon - \widetilde{F}) + \varepsilon e^{Ct}$ and argue similarly. $\square$

10. Bounded domains

In this section we give some comments on the case of a bounded domain. We study a boundary value problem for $\mathcal{M}$ in the following form: Let B be a bounded convex domain in $\mathbb{R}^n$ and let $D = B \times (0, T]$.

Given a continuous function f on the closure $\overline{B}$ of B with $f = 0$ on the boundary ∂B, find a function $F \in C(\overline{D}) \cap C^{2,1}(D)$ such that

$$\begin{aligned} \mathcal{M}F(x,t) &= 0, && (x,t) \in D, \\ F(x,0) &= f(x), && x \in D, \\ F(x,t) &= 0, && x \in \partial B,\ 0 \leq t \leq T. \end{aligned} \tag{10.1}$$

We assume that the coeffients of $\mathcal{M}$ are defined and continuous on $\overline{D}$ and that the parabolicity condition (A1) holds on $\overline{D}$ as well as condition (C1). Since we assume that D is bounded we thus in fact have uniform parabolicity and uniform Hölder estimates. We now define preservation of convexity for these problems in complete analogy with Definition 2.1.

In problem (10.1) we only consider convex domains B and zero boundary conditions. The motivation for this restriction is as follows. Let B be a bounded domain, not necessarily convex, in $\mathbb{R}^n$ and and consider the problem above, with for simplicity all lower order terms of $\mathcal{M}$ vanishing, but with a more general time-independent boundary condition. Let us assume that convexity is preserved for all t. Then, with a denoting the matrix (a_{ij}), $D_t F = \mathcal{L} F = \operatorname{Tr}\bigl(a \nabla_x^2 F\bigr) \geq 0$ and thus $F(x,t)$ is increasing in time, but on the other hand F is bounded by the maximum of the prescribed values on ∂B. Thus we can form the function

$$\varphi(x) = \sup_t F(x,t) = \lim_{t \to \infty} F(x,t).$$

Then φ is convex since it is the supremum of convex functions. Hovever, φ also solves $\mathcal{L}\varphi = 0$, being the steady-state solution of our equation. The only possibility is then that $\nabla_x^2 \varphi = 0$ so φ is affine. Thus the boundary conditions have to be affine. Since the function f assuming these boundary values is convex, the domain B has to be convex. Further, since the boundary conditions are affine we can subtract the corresponding affine function and instead assume that we have zero boundary values. Thus convex domains and zero boundary conditions is a natural class to consider. The next example shows that even for these problems, convexity is often not preserved.

Example 10.1. Let B be a ball in Euclidean space and the operator $\mathcal{L} = \Delta$ the standard Laplacian where the spatial dimension n is at least two. We assume that the initial condition f is a given convex function with vanishing boundary values. In this particular example, convexity is lost for any initial condition. To see this let λ_i and φ_i be the eigenvalues and eigenfunctions of $-\Delta$ on B (with Dirichlet condition). Then the solution $F(x,t)$ to the heat equation can be written

$$F(x,t) = \sum_{i=1}^{\infty} c_i e^{-\lambda_i t} \varphi_i(x),$$

where the constants c_i are chosen so that $F(x,0) = f(x)$. We then note that $e^{\lambda_1 t} F(x,t)$ converges uniformly to $c_1 \varphi_1(x)$ as t tends to infinity. But the first eigenfunction of the Laplacian for a ball in $\mathbb{R}^n$, $n \geq 2$, is not convex (or concave if one makes the standard choice with a positive first eigenfunction). Indeed, for the unit ball, $\varphi_1(x) = \psi(|x|)$, where $\psi''(r) + \frac{n-1}{r}\psi'(r) + \lambda_1 \psi(r) = 0$. ψ is smooth also at $r = 1$, and there $\psi(1) = 0$ and $\psi'(1) > 0$; hence $\psi''(1) = (1-n)\psi'(1) < 0$, so φ is not convex close to the boundary.

Thus the convexity of f is not preserved for all t, regardless of the choice of the convex initial condition.

In contrast to the example above we have a positive result in the case of one spatial dimension. In this case the operator $\mathcal{L}$ reduces to

$$\mathcal{L}F = a(x,t)F_{xx} + b(x,t)F_x + c(x,t)F.$$

We study problem (10.1) for $\mathcal{M} = \frac{\partial}{\partial t} - \mathcal{L}$ on a bounded interval on the real axis. For notational convenience, we let the interval be $[0,1]$. In this case we need also conditions at the boundary.

Theorem 10.2. *Consider problem* (10.1) *with* $n = 1$ *and* $B = (0,1)$. *Assume that the operator* $\mathcal{L}$ *has continuous coefficients that are Hölder*(α) *on* $[0,1] \times [0,T]$. *Assume that*

(i) *The function* b *satisfies* $b(0,t) \geq 0$ *and* $b(1,t) \leq 0$.
(ii) $b \in C^{1,0}(D)$ *and* $2c + b_x$ *is a function of* t *only.*
(iii) *As a function of* x, c *is concave for each fixed* t.

Then $\mathcal{M}$ *is convexity preserving in the set of continuous functions on* $[0,1] \times [0,T]$ *that vanish on* $\{0\} \times [0,T]$ *and* $\{1\} \times [0,T]$.

Proof. We begin by reducing to the case of smooth initial conditions f with the property that $\mathcal{L}f(x)$ vanishes at 0 and 1. (Smooth means infinitely differentiable in this proof.) Let f be convex on $[0,1]$. We first replace the graph of f close to 0 by a secant connecting the origin with the graph of f. Then we perform the analogous construction at 1. The thus obtained function can be approximated arbitrarily well with a smooth convex function $\tilde{f}$ which still is affine near the endpoints and vanishes there. To $\tilde{f}$ we then add a convex function, vanishing at the endpoints, having slope -1 at 0 and 1 at 1 but having second derivates at the endpoints chosen such that the resulting sum φ has the property that $\mathcal{L}\varphi(x)$ vanishes at 0 and 1. (This uses

$\widetilde{f}'(0) \le 0$, $\widetilde{f}'(1) \ge 0$ and (i).) This construction can be made so that the supremum norm of $f - \varphi$ is bounded by any given $\varepsilon > 0$. Let Φ be the solution to problem (10.1) with initial value given by φ. By the maximum principle Theorem A.3

$$|F(x,t) - \Phi(x,t)| \le \varepsilon e^{\mu t}$$

where μ equals supremum norm of c. Since ε is arbitrary, we thus see that if every such Φ is convex, then F is convex too.

For initial conditions φ with $\mathcal{L}\varphi(x)$ vanishing at the boundary at 0 and 1, problem (10.1) is uniquely solvable and $F \in C_\alpha^{2,1}(\overline{D})$, see [4, Theorem 3.7]. Moreover, [4, Theorem 3.6] shows that the norm of F in $C_\alpha^{2,1}(\overline{D})$ can be bounded by a constant depending only on φ, α, the norms of a, b and c in $C_\alpha(\overline{D})$, and the parabolicity constant. We can then argue as in the proof of Theorem A.12 (with the simplification that no special argument is required to guarantee the boundary conditions) to show that the solution depends continuously on the coefficients a, b, c. Thus, if we can show that convexity is preserved in the case of smooth coefficents and initial condition f with $\mathcal{L}f$ vanishing at the endpoints, the general case is obtained by taking uniform limits of such solutions, and are thus also convex.

Let therefore $F(x,t) \in C_\alpha^{2,1}(\overline{D})$ be a solution to equation (10.1) where the coefficients and f are smooth, f is convex, and $f(0) = f(1) = \mathcal{L}f(0) = \mathcal{L}f(1) = 0$. Since then $f \le 0$, the maximum principle Theorem A.2 yields $F \le 0$.

The function F_{xx} is continuous on $\overline{D}$ and smooth in D, see [4, Theorem 3.11, Corollary 2] or Theorem A.11. For $t = 0$, $F_{xx} = f_{xx} \ge 0$. Further, for $0 \le t \le T$, $F(0,t) = 0$ and since $F(x,t) \le 0$ for all x, $F_x(0,t) \le 0$. Since $\mathcal{L}F(0,t) = D_tF(0,t) = 0$, assumption (i) yields $a(0,t)F_{xx}(0,t) = -b(0,t)F_x(0,t) \ge 0$. Thus $F_{xx}(0,t) \ge 0$. Similarly $F_{xx}(1,t) \ge 0$. Moreover,

$$\begin{aligned} D_tF_{xx} &= D_{xx}D_tF = D_{xx}(\mathcal{L}F) \\ &= a_{xx}F_{xx} + 2a_xF_{xxx} + aF_{xxxx} + b_{xx}F_x + 2b_xF_{xx} + bF_{xxx} \\ &\qquad + c_{xx}F + 2c_xF_x + cF_{xx} \\ &= aF_{xxxx} + (2a_x + b)F_{xxx} + (a_{xx} + 2b_x + c)F_{xx} + c_{xx}F, \end{aligned}$$

where we used assumption (ii). Consequently, F_{xx} satisfies the parabolic differential equation

$$\Big(D_t - aD_{xx} - (2a_x + b)D_{xx} - (a_{xx} + 2b_x + c)\Big)F_{xx} = c_{xx}F.$$

We apply the maximum principle Theorem A.2 to this equation. Since $F_{xx} \ge 0$ on $\partial_0 D = B \times \{0\} \cup \{0,1\} \times [0,T]$ as shown above, and $c_{xx}F \ge 0$ by (iii) and $F \le 0$, this shows $F_{xx} \ge 0$. Thus F is convex in x for each t and the proof is complete. $\square$

Remark 10.3. We note that the conditions of Theorem 10.2 are necessary too, at least assuming sufficient smoothness. If convexity is preserved and $F \in C^{2,1}(\overline{D})$, then $F = 0$ and $\mathcal{L}F = D_tF = 0$ on $\{0,1\} \times [0,T]$ and we see that condition (i) has to hold since F is convex with negative x-derivate at 0 and positive x-derivative at 1 (unless F vanishes identically). Furthermore, formulating the LCP condition for this operator, which amounts to the same calculation as we did before Lemma 3.11 with added lower order terms (and also, for several variables, in the proof of Lemma 3.9), we see that the additional terms are $f'(x)(b_{xx} + 2c_x) + c_{xx}f$, using the notation of that calculation. Since f is non-positive and $f'(x)$ can be chosen arbitrarily, we arrive at conditions (ii) and (iii) of the statement of the theorem.

11. Nonlinear equations

It is challenging to try to extend our results to a suitable class of non-linear operators. The following theorem shows that a large class of quasilinear operators automatically are LCP. We therefore conjecture that, under suitable boundedness and regularity conditions, they are convexity preserving too. However, our proof for the linear case that LCP implies convexity preserving does not apply. (At least not without modification; there are several technical problems.) Hence we leave this as an open problem.

Theorem 11.1. *Consider the operator $\mathcal{M} = \frac{\partial}{\partial t} - \mathcal{L}$ where $\mathcal{L}$ is a a quasilinear elliptic operator of the form*

$$\mathcal{L}F = \sum_{i,j=1}^{n} a_{ij}(\nabla F, t)\frac{\partial^2 F}{\partial x_i \partial x_j} + G(\nabla F, t) + H(F, t),$$

where we assume that the functions a_{ij}, G and H are twice differentiable and the matrix a_{ij} is positive definite. If further H is a convex function of the first variable, then $\mathcal{M}$ is LCP.

Proof. Take a smooth convex function f satisfying $D_{uu}f(x_0) = 0$ at some point x_0 and evaluate $D_{uu}(\mathcal{L}f)$ at this point. Lemma 3.9 implies that $D_u a_{ij}(\nabla f, t)$ and $D_{uu}a_{ij}(\nabla f, t)$ vanish at x_0, and similarly for G. Hence, at x_0 and fixing t,

$$D_{uu}(\mathcal{L}f) = \sum_{i,j=1}^{n} a_{ij}(\nabla f, t)D_{ijuu}f + H'(f, t)D_{uu}f + H''(f, t)(D_u f)^2.$$

Now, again by Lemma 3.9, the matrix $\big(D_{ijuu}f(x_0)\big)_{ij} \geq 0$. Since further $(a_{ij})_{ij} \geq 0$, and $D_{uu}f = 0$ and $H'' \geq 0$, it follows that $D_{uu}(\mathcal{L}f) \geq 0$. □

Example 11.2. In [9, Proposition 2.1] the equation for the mean curvature flow is given as

$$D_t F = \sum_{i,j=1}^{n} g^{ij} D_{ij} F,$$

where $(g^{ij}) = (g_{ij})^{-1}$ and $g_{ij} = \delta_{ij} + D_i F D_j F$. Theorem 11.1 says that this equation is LCP. It is therefore natural to conjecture that it is convexity preserving, i.e. if we consider this flow on $\mathbb{R}^n$ and the initial surface is convex, then each surface is convex.

Example 11.3. Various models in physics, chemistry and biology lead to a non-linear parabolic equation of the form

$$F_t = \delta \triangle F + H(F)$$

where for instance in biology H can be related to the rate of reproduction of a certain species. The theorem above says that this equation is LCP, and thus presumably preserves convexity, if H is convex.

12. Higher order operators

Finally we note that the problem of preservation of convexity is not natural to pose for parabolic operators of an order greater than two. To see this consider the mth power Δ^m of the Laplace operator. We note that the function $f(x) = x_1^{2m} - x_1^{2m+2} + cx_1^{2m+4}$ is convex for a suitably large constant c, and that

$$D_{11}(\Delta^m f)(0) = -(2n+2)! \tag{12.1}$$

For $n \geq 2$, $D_{11}f(0) = 0$ and thus for those n equation (12.1) shows that the LCP condition is not satisfied. Moreover, the Cauchy problem is solved by $F(x,t) = f(x) + t\Delta^m f(x) + \frac{1}{2}t^2\Delta^{2m}f(x)$, which is not convex at $x = 0$ for any $t > 0$ because $D_{11}F(0,t) = tD_{11}(\Delta^m f)(0) = -t(2n+2)!$ by (12.1). Hence convexity is not preserved even for these simple operators.

Appendix A. Some general facts for partial differential equations

We consider a parabolic differential operator

$$\mathcal{M} = \frac{\partial}{\partial t} - \mathcal{L}$$

where

$$\mathcal{L}F = \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 F}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x,t)\frac{\partial F}{\partial x_i} + c(x,t)F. \tag{A.1}$$

We assume that all coefficients a_{ij}, b_i, c are real-valued functions defined in the domain $\Omega_T = \mathbb{R}^n \times (0,T]$ for some given (finite) $T > 0$. We will use the notations $D_t = \partial/\partial t$, $D_i = \partial/\partial x_i$; we let D_x^k, where k is a multiindex, denote multiple x-derivatives.

Our minimal assumptions on the operator are the following, which we always assume.

(A0) $\mathcal{M}$ is weakly parabolic ($\mathcal{L}$ is weakly elliptic) everywhere, i.e. the matrix $\big(a_{ij}(x,t)\big)_{ij}$ is positive semidefinite for every $(x,t) \in \Omega_T$. Explicitly, $\sum_{i,j=1}^{n} a_{ij}(x,t)u_iu_j \geq 0$ for every $(x,t) \in \Omega_T$ and every $u \in \mathbb{R}^n$.

(C0) The coefficients a_{ij}, b_i, c are continuous functions in Ω_T.

We will later require both stronger regularity conditions and boundedness conditions; we list below various conditions that will be used in different combinations.

We first need some definitions. We assume that $0 < \alpha < 1$. We consider only real-valued functions.

- For any metric (or topological) space E, $C(E)$ is the space of all continuous functions on E. If E is compact, this is a Banach space with the norm
$$\|f\|_{C(E)} = \sup_{x \in E} |f(x)|.$$
- A function f on a metric space E is (uniformly) Hölder(α) if $|f(x) - f(y)| \leq Cd(x,y)^\alpha$ for some constant C and all $x, y \in E$. We define
$$\|f\|_{H_\alpha(E)} = \sup_{x \in E} |f(x)| + \sup_{x,y \in E} \frac{|f(x) - f(y)|}{d(x,y)^\alpha}$$
and observe that $\|f\|_{H_\alpha(E)} < \infty$ if and only if f is bounded and uniformly Hölder(α) on E. (If E is bounded, this holds if and only if f is uniformly Hölder(α).)
- We say that f is locally Hölder(α) on E if it is Hölder(α) on every compact subset of E. We let $C_\alpha(E)$ denote the space of all locally Hölder(α) functions on E. If E is compact, this equals the space of all uniformly Hölder(α) functions; in this case we let $\|f\|_{C_\alpha(E)} = \|f\|_{H_\alpha(E)}$.

We will only consider sets $E \subseteq \mathbb{R}^n$, with usual Euclidean metric, or $E \subseteq \mathbb{R}^{n+1}$, with the parabolic metric $d\big((x,t),(y,s)\big) = (|x-y|^2 + |t-s|)^{1/2}$. In these cases we further define the following spaces, for $p, q \in \{0,1,2,\dots\}$ and $0 < \alpha < 1$.

- $C_{\mathrm{pol}}(E)$, where $E \subseteq \mathbb{R}^n$ or $E \subseteq \mathbb{R}^{n+1}$, is the space of $f \in C(E)$ of at most polynomial growth (in x). Thus (for $E \subseteq \mathbb{R}^{n+1}$), $C_{\mathrm{pol}}(E) = \bigcup_{p>0}\{f \in C(E) : |f(x,t)| = O(|x|^p + 1),\ x \in E\}$.
- $C_{\mathrm{Exp}}(E)$, where $E \subseteq \mathbb{R}^n$ or $E \subseteq \mathbb{R}^{n+1}$, is the space of $f \in C(E)$ that are bounded by $\exp\big(O(|x|^2)\big)$. Thus (for $E \subseteq \mathbb{R}^{n+1}$), $C_{\mathrm{Exp}}(E) = \bigcup_{\beta>0}\{f \in C(E) : |f(x,t)| = O(e^{\beta|x|^2}),\ x \in E\}$.
- $C_{\mathrm{exp}}(E)$, where $E \subseteq \mathbb{R}^n$ or $E \subseteq \mathbb{R}^{n+1}$, is the space of $f \in C(E)$ that are bounded by $\exp\big(o(|x|^2)\big)$. Thus (for $E \subseteq \mathbb{R}^{n+1}$), $C_{\mathrm{exp}}(E) = \bigcap_{\beta>0}\{f \in C(E) : |f(x,t)| = O(e^{\beta|x|^2}),\ x \in E\}$.
- $C^p(E)$, where $E \subseteq \mathbb{R}^n$, is the space of $f \in C(E)$ such that all derivatives $D^k f$ with $0 \le |k| \le p$ exist in the interior E° and have extensions to continuous functions on E.
- $C^p_\alpha(E)$, where $E \subseteq \mathbb{R}^n$, is the space of $f \in C^p(E)$ such that $D^k f \in C_\alpha(E)$ when $0 \le |k| \le p$.
- $C^p_{\alpha,\mathrm{b}}(E)$, where $E \subseteq \mathbb{R}^n$, is the space of all $f \in C^p_\alpha(E)$ such that $D^k f$ are bounded and uniformly Hölder(α) for $0 \le |k| \le p$.
- $C^p_{\mathrm{pol}}(E)$, where $E \subseteq \mathbb{R}^n$, is the space of all $f \in C^p(E)$ such that $D^k f \in C_{\mathrm{pol}}(E)$ for $0 \le |k| \le p$.
- $C^{p,q}(E)$where $E \subseteq \mathbb{R}^{n+1}$, is the space of all functions f in $C(E)$ such that all derivatives $D^k_x D^l_t f$ with $|k| + 2l \le p$ and $0 \le l \le q$ exist in the interior E° and have extensions to continuous functions on E.
- $C^{p,q}_\alpha(E)$, where $E \subseteq \mathbb{R}^{n+1}$, is the space of all $f \in C^{p,q}(E)$ such that $D^k_x D^l_t f \in C_\alpha(E)$ for $|k| + 2l \le p$ and $0 \le l \le q$.
- $C^{p,q}_{\alpha,\mathrm{b}}(E)$, where $E \subseteq \mathbb{R}^{n+1}$, is the space of all $f \in C^{p,q}_\alpha(E)$ such that $D^k_x D^l_t f$ are bounded and uniformly Hölder(α) for $|k| + 2l \le p$ and $0 \le l \le q$.
- $C^{p,q}_{\mathrm{pol}}(E)$, where $E \subseteq \mathbb{R}^{n+1}$, is the space of all $f \in C^{p,q}(E)$ such that $D^k_x D^l_t f \in C_{\mathrm{pol}}(E)$ for $|k| + 2l \le p$ and $0 \le l \le q$.

We give $C^{p,q}(E)$ the topology of uniform convergence on compact sets of the function together with the derivatives in the definition. In other words, the topology is defined by the seminorms $\sup_K |D^k_x D^l_t F|$ where $|k| + l \le p$, $l \le q$, and K ranges over the compact subsets of E. It is easy to see $C^{p,q}(E)$ is metrizable and complete, and thus a Fréchet space. (Argue as in [8, Example 10.I].) When E is compact, $C^{p,q}_\alpha$ is a Banach space with the norm

$$\|f\|_{C^{p,q}_\alpha(E)} = \sum_{|k|+l\le p,\ l\le q} \|D^k_x D^l_t\|_{H_\alpha(E)}$$

The other spaces defined above may be given topologies defined by similar seminorms or norms (and inductive limits for the polynomially or exponentially bounded cases), but we do not need them.

Our conditions are the following. In the sequel, α is a fixed number with $0 < \alpha < 1$.

(A1) $\mathcal{M}$ is parabolic ($\mathcal{L}$ is elliptic) everywhere, i.e. the matrix $\big(a_{ij}(x,t)\big)_{ij}$ is positive definite for every $(x,t) \in \Omega_T$. Explicitly, $\sum_{i,j=1}^n a_{ij}(x,t)\xi_i\xi_j > 0$ for every $(x,t) \in \Omega_T$ and every $\xi \in \mathbb{R}^n \setminus \{0\}$.

(A2) $\mathcal{M}$ is uniformly parabolic ($\mathcal{L}$ is uniformly elliptic) in each bounded subset of Ω_T, i.e. the matrix $\big(a_{ij}(x,t)\big)_{ij}$ is uniformly positive definite for $|x| \le R$ and $0 < t \le T$ for each R.

(A3) $\mathcal{M}$ is uniformly parabolic ($\mathcal{L}$ is uniformly elliptic) in Ω_T, i.e. the matrix $\big(a_{ij}(x,t)\big)_{ij}$ is uniformly positive definite for $(x,t) \in \Omega_T$. Explicitly, for some $\lambda > 0$,

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \ge \lambda|\xi|^2 \tag{A.2}$$

for every $(x,t) \in \Omega_T$ and every $\xi \in \mathbb{R}^n$.

(A4) $(1+|x|^2)^{-1}\mathcal{M}$ is uniformly parabolic ($(1+|x|^2)^{-1}\mathcal{L}$ is uniformly elliptic) in Ω_T.

(B1) The coefficients a_{ij}, b_i, c are bounded functions in Ω_T.

(B2) The coefficients a_{ij}, b_i, c satisfy the bounds

$$|a_{ij}(x,t)| \le B(|x|^2+1), \quad |b_i(x,t)| \le B(|x|+1), \quad |c(x,t)| \le B, \tag{A.3}$$

for some constant B and all $(x,t) \in \Omega_T$.

(B2^{p,q}) The coefficients $a_{ij}, b_i, c \in C^{p,q}(\Omega_T)$ and satisfy the bounds

$$\begin{aligned} |D_x^k D_t^l a_{ij}(x,t)| &\le B(|x|+1)^{2-|k|}, \\ |D_x^k D_t^l b_i(x,t)| &\le B(|x|+1)^{1-|k|}, \\ |D_x^k D_t^l c(x,t)| &\le B(|x|+1)^{-|k|}, \end{aligned}$$

for some constant B and all $(x,t) \in \Omega_T$. and all k and l with $|k| + 2l \le p$ and $0 \le l \le q$.

(B3) The coefficients a_{ij}, b_i, c satisfy the bounds

$$|a_{ij}(x,t)| \le B, \quad |b_i(x,t)| \le B(1+|x|), \quad |c(x,t)| \le B(1+|x|^2), \tag{A.4}$$

for some constant B and all $(x,t) \in \Omega_T$.

(C1) The coefficients a_{ij}, b_i, c are locally Hölder(α) in Ω_T.

(C2) The coefficients a_{ij}, b_i, c are uniformly Hölder(α) in Ω_T.

Note that if (A1) holds, then for every compact $K \subset \Omega_T$, there exists a constant $\lambda_K > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \ge \lambda_K|\xi|^2, \qquad x \in K. \tag{A.5}$$

(By continuity and compactness, first considering $|\xi| = 1$ only.) In other words, (A.2) holds locally.

We study the Cauchy problem for $\mathcal{M}$ in the following form:

Given a continuous function f on $\mathbb{R}^n$ and a continuous function h on Ω_T, find a function $F \in C(\overline{\Omega_T}) \cap C^{2,1}(\Omega_T)$ such that

$$\begin{aligned} \mathcal{M}F(x,t) &= h(x,t), \quad (x,t) \in \Omega_T, \\ F(x,0) &= f(x), \qquad x \in \mathbb{R}^n. \end{aligned} \tag{A.6}$$

A.1. **Maximum principle.** Recall the weak maximum principle for bounded domains. For simplicity we consider only domains of the type $D = B \times (0,T]$ with $B \subset \mathbb{R}^n$ open. We let $S = \partial B \times (0,T]$ and $\partial_0 D = B \cup S$ (this subset of ∂D is known as the parabolic boundary). A standard form of the (weak) maximum principle is as follows, see [5, Theorem 6.3.1] or [4, Theorem 2.6] (with a different sign of $\mathcal{M}$).

Theorem A.1. *Assume that* (A0) *and* (C0) *hold in a bounded domain* $D = B \times (0,T]$, *and that* $c(x,t) \le 0$ *in* D. *If* $F \in C(\overline{D}) \cap C^{2,1}(D)$ *satisfies* $\mathcal{M}F \le 0$ *in* D, *then*

$$\sup_D F \le \max\Bigl(0, \sup_{\partial_0 D} F\Bigr). \qquad \square$$

As a simple consequence we have the following, where no condition on c is needed.

Theorem A.2. *Assume that* (A0) *and* (C0) *hold in a bounded domain* $D = B \times (0,T]$. *If* $F \in C(\overline{D}) \cap C^{2,1}(D)$ *satisfies* $\mathcal{M}F \ge 0$ *in* D *and* $F \ge 0$ *on* $\partial_0 D$, *then* $F \ge 0$ *in* $\overline{D}$.

Proof. Let $\mathcal{M}'F = \mathcal{M}F + KF$ with $K = \sup_D |c(x,t)|$. $\mathcal{M}'$ is a partial differential operator of the type in Theorem A.1; note that the coefficient of F is $c'(x,t) = c(x,t) - K \le 0$. We have $\mathcal{M}'\bigl(e^{-Kt}F(x,t)\bigr) = e^{-Kt}\mathcal{M}F(x,t) \ge 0$, and the result follows from Theorem A.1 applied to $\mathcal{M}'$ and and $-e^{-Kt}F(x,t)$. $\square$

Other simple consequences are estimates such as the following, cf. [4, Section 2.3].

Theorem A.3. *Assume that* (A0) *and* (C0) *hold in a bounded domain* $D = B \times (0,T]$, *and let* $K = \sup_D |c(x,t)|$. *If* $F \in C(\overline{D}) \cap C^{2,1}(D)$ *satisfies* $\mathcal{M}F = h$ *in* D, *then*

$$\sup_D |F| \le e^{KT} \sup_{\partial_0 D} |F| + Te^{KT} \sup_D |h|.$$

Proof. Let again $\mathcal{M}'F = \mathcal{M}F + KF$, so that $\mathcal{M}'$ is a partial differential operator with zeroth order coefficient $c'(x,t) = c(x,t) - K \le 0$. We have $\mathcal{M}'t = 1 - c'(x,t)t \ge 1$ and thus, with $A = \sup_D |h|$,

$$\begin{aligned}\mathcal{M}'\bigl(e^{-Kt}F(x,t) - At\bigr) &\le \mathcal{M}'\bigl(e^{-Kt}F(x,t)\bigr) - A = e^{-Kt}\mathcal{M}F(x,t) - A \\ &= e^{-Kt}h(x,t) - A \le 0.\end{aligned}$$

Hence Theorem A.1 applies and yields, for any $(x,t) \in D$,

$$e^{-Kt}F(x,t) - At \le \max\Bigl(0, \sup_{\partial_0 D}\bigl(e^{-Kt}F(x,t) - At\bigr)\Bigr) \le \sup_{\partial_0 D} |F|,$$

which yields $F(x,t) \le e^{KT} \sup_{\partial_0 D} |F| + Te^{KT}A$. Considering also $-F$, we get the result. $\square$

Further simple consequences are maximum principles for Ω_T under suitable growth conditions, for example the two following, see e.g. [5, Theorems 6.4.3 and 6.4.1].

Theorem A.4. *Assume* (A0), (B2), (C0). *If* $F \in C_{\mathrm{pol}}(\overline{\Omega_T}) \cap C^{2,1}(\Omega_T)$ *satisfies* $\mathcal{M}F \ge 0$ *in* Ω_T *and* $F \ge 0$ *on* $\mathbb{R}^n$, *then* $F \ge 0$ *in* $\overline{\Omega_T}$. $\square$

Theorem A.5. *Assume* (A0), (B3), (C0). *If* $F \in C_{\mathrm{Exp}}(\overline{\Omega_T}) \cap C^{2,1}(\Omega_T)$ *satisfies* $\mathcal{M}F \ge 0$ *in* Ω_T *and* $F \ge 0$ *on* $\mathbb{R}^n$, *then* $F \ge 0$ *in* $\overline{\Omega_T}$. $\square$

Corollary A.6. (i) *Assume* (A0), (B2), (C0). *If f and h are given, the Cauchy problem* (A.6) *has at most one solution $F \in C_{\mathrm{pol}}(\overline{\Omega_T})$.*

(ii) *Assume* (A0), (B3), (C0). *If f and h are given, the Cauchy problem* (A.6) *has at most one solution $F \in C_{\mathrm{Exp}}(\overline{\Omega_T})$.* □

The same method easily yields a bound for the solution (if it exists).

Theorem A.7. *Assume* (A0), (B2), (C0). *If $F \in C_{\mathrm{pol}}(\overline{\Omega_T}) \cap C^{2,1}(\Omega_T)$ solves the Cauchy problem* (A.6), *and*

$$|f(x)| \le A(|x|^p + 1), \qquad x \in \mathbb{R}^n, \tag{A.7}$$

$$|h(x,t)| \le A(|x|^p + 1), \qquad (x,t) \in \Omega_T, \tag{A.8}$$

for some constants A and $p \ge 0$, then

$$|F(x,t)| \le CA(|x|^p + 1), \qquad (x,t) \in \Omega_T,$$

where C is a constant depending on n, T, p and B in (B2).

Proof. Let $w(x,t) = (1+|x|^2)^{p/2} e^{\gamma t}$, where γ will be chosen later. Then, using (B2),

$$\begin{aligned} -\mathcal{M}w &= p(p-2)\sum_{i,j=1}^n a_{ij}x_ix_j(1+|x|^2)^{-2}w + p\sum_{i=1}^n a_{ii}(1+|x|^2)^{-1}w \\ &\quad + p\sum_{i=1}^n b_ix_i(1+|x|^2)^{-1}w + cw - \gamma w \\ &\le C_1p(p+2)Bw + C_2pBw + Bw - \gamma w \qquad \text{in } \Omega_T, \end{aligned} \tag{A.9}$$

for some constants C_1 and C_2 depending on n only. If $\gamma = 1 + \bigl(C_1p(p+2) + C_2p + 1\bigr)B$, we thus have $\mathcal{M}w \ge w$. Hence, with $G = 2Aw + F$, the assumptions (A.7), (A.8) imply

$$\mathcal{M}G = 2A\mathcal{M}w + \mathcal{M}F \ge 2Aw + h \ge 0 \qquad \text{in } \Omega_T,$$

and

$$G(x,0) = 2A(1+|x|^2)^{p/2} + f(x) \ge 0, \qquad x \in \mathbb{R}^n.$$

Consequently, Theorem A.4 yields $G \ge 0$ in $\overline{\Omega_T}$, i.e. $F \ge -2Aw$. The same argument applied to $-F$ yields $-F \ge -2Aw$. Hence $|F| \le 2Aw \le 2Ae^{\gamma T}(1+|x|^2)^{p/2}$. □

Theorem A.8. *Assume* (A0), (B3), (C0). *If $F \in C_{\mathrm{Exp}}(\overline{\Omega_T}) \cap C^{2,1}(\Omega_T)$ solves the Cauchy problem* (A.6), *and*

$$|f(x)| \le Ae^{\beta|x|^2}, \qquad x \in \mathbb{R}^n,$$

$$|h(x,t)| \le Ae^{\beta|x|^2}, \qquad (x,t) \in \Omega_T,$$

for some constants A and $\beta \ge 0$, then

$$|F(x,t)| \le 2Ae^{2\beta|x|^2}, \qquad 0 \le t \le t_0,$$

where $t_0 > 0$ is a constant depending on n, T, β and B in (B3).

Proof. Similar to Theorem A.7, using $G = Aw \pm F$ and $w(x,t) = \exp\bigl(\frac{\beta|x|^2}{1-\mu t} + \nu t\bigr)$, where μ and ν are large constants depending on B and β ($CB(\beta+1)+1$ will do), and $t_0 = \min(1/2\mu, \ln 2/\nu)$. □

We also need an estimate for the modulus of continuity at the boundary.

Theorem A.9. *Assume* (A0), (B2), (C0). *Suppose that* $F \in C_{\mathrm{pol}}(\overline{\Omega_T}) \cap C^{2,1}(\Omega_T)$ *solves the Cauchy problem* (A.6), *and that* (A.7)–(A.8) *hold. If* $x_0 \in \mathbb{R}^n$, $\varepsilon > 0$ *and* $\delta > 0$ *are such that* $|f(x) - f(x_0)| < \varepsilon$ *for* $|x - x_0| < \delta$, *then there exists* $\eta > 0$, *depending on* x_0, ε, δ, p, A *and* B *in* (B2), *such that*

$$|F(x,t) - f(x_0)| < 2\varepsilon \qquad \text{for } |x - x_0| < \eta \text{ and } 0 \le t < \eta.$$

Proof. Translating everything by x_0, which preserves the conditions if we multiply A and B by factors $2^p(|x_0|^p + 1)$ and $2(|x_0|^2 + 1)$, we may assume $x_0 = 0$.

This time we use two auxiliary functions, $w_1(x,t) = (1 + |x|^2)^{p/2}(e^{\gamma t} - 1)$ and $w_2(x,t) = (|x|^2 + Ct)^{p/2} e^{\gamma t}$. By calculations similar to (A.9), it is seen that if we choose first C and then γ large enough (depending on n, p and B), then $\mathcal{M}w_1 \ge (1 + |x|^2)^{p/2}$ and $\mathcal{M}w_2 \ge 0$ in Ω_T.

Let $C_1 = 2A + AB + \varepsilon B$ and $C_2 = 3A(1 + \delta^{-2})^{p/2}$, and define

$$G(x,t) = -F(x,t) + f(0) + \varepsilon + C_1 w_1(x,t) + C_2 w_2(x,t).$$

Then

$$\begin{aligned}\mathcal{M}G &= -h - cf(0) - c\varepsilon + C_1 \mathcal{M}w_1 + C_2 \mathcal{M}w_2 \\ &\ge -A(|x|^p + 1) - BA - B\varepsilon + C_1(1 + |x|^2)^{p/2} \ge 0\end{aligned}$$

in Ω_T. On $\mathbb{R}^n$ we observe first that if $|x| < \delta$, then $|F(x,0) - f(0)| = |f(x) - f(0)| < \varepsilon$, and thus $G(x,0) > 0$. If $|x| \ge \delta$, then

$$C_2 w_2(x,0) = 3A(1 + \delta^{-2})^{p/2}|x|^p \ge 3A(1 + |x|^{-2})^{p/2}|x|^p = 3A(1 + |x|^2)^{p/2},$$

and $G(x,0) > 0$ follows in this case too. Consequently, Theorem A.4 applies to G, and shows that $G \ge 0$ in $\overline{\Omega_T}$, and thus $F(x,t) \le f(0) + \varepsilon + C_1 w_1 + C_2 w_2$. Since $w_1(0,0) = w_2(0,0) = 0$, there exists $\eta < \delta$ such that if $|x| < \eta$ and $0 \le t < \eta$, then $C_1 w_1(x,t) + C_2 w_2(x,t) < \varepsilon$ and thus $F(x,t) < f(0) + 2\varepsilon$. The same argument applied to $-F$ (and $-f$, $-h$) completes the proof. □

A.2. **Regularity.** We will use the following regularity theorem, see e.g. [3, Sections II.1.7 and II.1.3] or [4, Theorems 3.10 and 3.5]. (But note that the proof in [4] of Theorem 3.10 contains a gap, since it uses a consequence of Theorem 3.9, which is incorrect as stated). We state the theorem for Ω_T only, although it is valid for any domain $\Omega \subseteq \mathbb{R}^{n+1}$ with obvious extensions of the definitions. (Actually, the theorems in [3] and [4] yields more precise information close to the boundary of the domain; our version is an immediate corollary.)

Theorem A.10. *Assume* (A1) *and* (C1). *If* $F \in C^{2,1}(\Omega_T)$ *satisfies* $\mathcal{M}F = h$ *and* h *is locally Hölder*(α) *in* Ω_T, *then* $F \in C^{2,1}_\alpha(\Omega_T)$. *Moreover, for any compact* $K \subset \Omega_T$ *and any relatively compact domain* $U \subset \Omega_T$ *with* $K \subset U$, *there exists a constant* C *depending on* α, K, U, $\|a_{ij}\|_{H_\alpha(U)}$, $\|b_i\|_{H_\alpha(U)}$, $\|c\|_{H_\alpha(U)}$, *and* λ_U *in* (A.5), *such that*

$$\|F\|_{C^{2,1}_\alpha(K)} \le C \sup_U |F| + C\|h\|_{H_\alpha(U)}. \qquad \square$$

With higher differentiability of the coefficients and h, we get corresponding higher differentiability of the solution F. The following theorem (also valid for any domain Ω) is [4, Theorem 3.10]; see also [3, Theorem 3.2, p. 206].

Theorem A.11. *Let* $p \ge 0$. *Assume* (A1) *and* (C1), *and assume further that the coefficients* a_{ij}, b_i, c *and* h *belong to* $C^{p,0}_\alpha(\Omega_T)$. *If* $F \in C^{2,1}(\Omega_T)$ *satisfies* $\mathcal{M}F = h$, *then* $F \in C^{p+2,1}_\alpha(\Omega_T)$. □

A.3. **Approximation.** The next theorem shows that the solution to the Cauchy problem behaves continuously if we change the coefficients of the equation or the data f and h in an appropriate way.

Theorem A.12. *Suppose that $\mathcal{M}^{(m)}$, $m = 1, 2, \dots$ is a sequence of parabolic differential operators in Ω_T such that:*

(i) (A1) *holds for $\mathcal{M}^{(m)}$ uniformly in m, i.e. for every compact $K \subset \Omega_T$, all operators $\mathcal{M}^{(m)}$ satisfy* (A.5) *with some $\lambda_K > 0$ not depending on m.*
(ii) (B2) *holds uniformly in m, i.e.* (A.3) *holds for every $\mathcal{M}^{(m)}$, for some B not depending on m.*
(iii) (C1) *holds uniformly in m, i.e., for every compact $K \subset \Omega_T$, there exists C independent of m such that $\|a_{ij}^{(m)}\|_{H_\alpha(K)} \le C$, $\|b_i^{(m)}\|_{H_\alpha(K)} \le C$, $\|c^{(m)}\|_{H_\alpha(K)} \le C$.*
(iv) *$\mathcal{M}^{(m)} \to \mathcal{M}$ as $m \to \infty$, in the sense that $a_{ij}^{(m)} \to a_{ij}$, $b_i^{(m)} \to b_i$, $c^{(m)} \to c$ pointwise in Ω_T.*

Suppose further that $f^{(m)} \in C(\mathbb{R}^n)$ and $h^{(m)} \in C(\Omega_T)$ with $h^{(m)}$ locally Hölder(α), and that for some A and p not depending on m

$$|f^{(m)}(x)| \le A(|x|^p + 1), \qquad x \in \mathbb{R}^n,$$
$$|h^{(m)}(x,t)| \le A(|x|^p + 1), \qquad (x,t) \in \Omega_T,$$

and that for every compact $K \subset \Omega_T$

$$\|h^{(m)}\|_{H_\alpha(K)} \le A_K$$

for some A_K, independent of m. Finally, suppose that $f^{(m)} \to f$ uniformly on compact sets and $h^{(m)} \to h$ pointwise, for some functions $f \in C(\mathbb{R}^n)$ and $h \in C(\Omega_T)$.

If $F^{(m)} \in C_{\mathrm{pol}}(\overline{\Omega_T}) \cap C^{2,1}(\Omega_T)$ solves the Cauchy problem $\mathcal{M}^{(m)} F^{(m)} = h^{(m)}$ in Ω_T with $F^{(m)} = f^{(m)}$ on $\mathbb{R}^n$, then $F^{(m)}$ converges in $C^{2,1}(\Omega_T)$, as $m \to \infty$, to a solution $F \in C_{\mathrm{pol}}(\overline{\Omega_T})$ of (A.6).

Proof. Let K be a compact subset of Ω_T, and let U be a relatively compact domain with $K \subset U \subset \Omega_T$. By Theorem A.7,

$$|F^{(m)}(x,t)| \le CA(|x|^p + 1), \tag{A.10}$$

and thus $\sup_m \sup_U |F^{(m)}| < \infty$. It follows by Theorem A.10 that

$$\|F^{(m)}\|_{C_\alpha^{2,1}(K)} \le C_K$$

for some constant C_K. It is an easy consequence of the Arzela–Ascoli theorem [8, Theorem 14.1] that this implies that the sequence $(F^{(m)})$ is relatively compact in $C^{2,1}(\Omega_T)$, i.e. every subsequence has a subsequence that converges in $C^{2,1}(\Omega_T)$.

Let $F \in C^{2,1}(\Omega_T)$ be the limit of such a convergent subsequence. Taking the pointwise limit in $\mathcal{M}^{(m)} F^{(m)} = h^{(m)}$, we see that $\mathcal{M} F = h$ in Ω_T.

Let $x_0 \in \mathbb{R}^n$ and $\varepsilon > 0$. Since $f^{(m)} \to f$ uniformly on compact sets, and f is continuous, there exists a $\delta > 0$ such that $|f^{(m)}(x) - f^{(m)}(x_0)| < \varepsilon$ for $|x - x_0| < \delta$ and every m. By Theorem A.9, there exists $\eta > 0$ such that $|F^{(m)}(x,t) - f^{(m)}(x_0)| < 2\varepsilon$ for $|x - x_0| < \eta$, $t < \eta$ and every m. Letting $m \to \infty$, it follows that $|F(x,t) - f(x_0)| \le 2\varepsilon$ for $|x - x_0| < \eta$, $t < \eta$.

Since ε was arbitrary, this shows that if we define $F(x,0) = f(x)$, then F is continuous at x_0. Thus $F \in C(\overline{\Omega_T})$ with $F = f$ on $\mathbb{R}^n$. Consequently, F solves the Cauchy problem.

Since the assumptions (i)–(iv) imply that (A1), (B2) and (C1) hold, and (A.10) implies $|F(x,t)| \le CA(|x|^p + 1)$ and thus $F \in C_{\mathrm{pol}}(\overline{\Omega_T})$, Corollary A.6 shows that the solution F is unique. Hence every convergent subsequence of $(F^{(m)})$ has the same limit F; since the sequence is relatively compact, this implies that $F^{(m)} \to F$ in $C^{2,1}(\Omega_T)$. □

A.4. **Existence.** We begin with a standard result.

Theorem A.13. *Assume* (A3), (B1), (C2). *If $f \in C(\mathbb{R}^n)$ and $h \in C(\overline{\Omega_T})$ are bounded with h locally Hölder(α) in $\overline{\Omega_T}$, then the Cauchy problem* (A.6) *has a unique bounded solution.*

Proof. The existence is an immediate consequence of [4, Theorem 1.12 and (1.6.12)]. The uniqueness follows by Corollary A.6. □

Theorem A.14. *Assume* (A1), (B2), (C1). *Then the Cauchy problem* (A.6) *has a unique solution $F \in C_{\mathrm{pol}}(\overline{\Omega_T}) \cap C^{2,1}(\Omega_T)$ for every $f \in C_{\mathrm{pol}}(\mathbb{R}^n)$ and $h \in C_{\mathrm{pol}}(\Omega_T)$ with h locally Hölder(α).*

Proof. Let $\psi \in C_c^\infty(\mathbb{R})$ be a test function with compact support with $\psi(x) = 1$ for $|x| \le 1$. Let $\psi^{(m)}(x) = \psi(x/m)$ and $\Psi^{(m)}(x,t) = \psi(x/m)\psi(1/mt)$, and define $\mathcal{L}^{(m)} = \Psi^{(m)}\mathcal{L} + (1 - \Psi^{(m)})\Delta$, $\mathcal{M}^{(m)} = \partial/\partial t - \mathcal{L}^{(m)}$, $f^{(m)} = \psi^{(m)} f$, $h^{(m)} = \Psi^{(m)} h$.

The conditions of Theorem A.13 are satisfied with $\mathcal{M}^{(m)}$, $f^{(m)}$, $h^{(m)}$ for each m, and thus the Cauchy problem $\mathcal{M}^{(m)} F^{(m)} = h^{(m)}$ in Ω_T with $F^{(m)} = f^{(m)}$ on $\mathbb{R}^n$ has a (unique) bounded solution. Theorem A.12 now applies, and shows that $F^{(m)}$ converges to a solution $F \in C_{\mathrm{pol}}(\overline{\Omega_T}) \cap C^{2,1}(\Omega_T)$ of (A.6).

Uniqueness follows by Corollary A.6. □

The general solution can be expressed in solutions to the homogeneous equation as follows.

Theorem A.15. *Assume* (A1), (B2), (C1) *and let $f \in C_{\mathrm{pol}}(\mathbb{R}^n)$ and $h \in C_{\mathrm{pol}}(\Omega_T)$ with h locally Hölder(α). Let $F_0 \in C_{\mathrm{pol}}(\overline{\Omega_T}) \cap C^{2,1}(\Omega_T)$ solve $\mathcal{M}F_0 = 0$ with $F_0(x,0) = f(x)$ and let, for $0 < \tau < T$, $H_\tau \in C_{\mathrm{pol}}(\Omega_{[\tau,T]}) \cap C^{2,1}(\Omega_{(\tau,T]})$ solve $\mathcal{M}H_\tau = 0$ in $\Omega_{(\tau,T]}$ with $H_\tau(x,\tau) = h(x,\tau)$. Then the unique solution $F \in C_{\mathrm{pol}}(\overline{\Omega_T}) \cap C^{2,1}(\Omega_T)$ to the Cauchy problem* (A.6) *is*

$$F(x,t) = F_0(x,t) + \int_0^t H_\tau(x,t)\, d\tau. \tag{A.11}$$

Proof. This is well known if the stronger conditions (A3), (B1), (C2) hold, by the formula for the solution in terms of a fundamental solution, see [4, Theorem 1.12]. In general, we define $\mathcal{M}^{(m)}$, $f^{(m)}$, $h^{(m)}$ as in Theorem A.14 and denote the corresponding solutions by $F^{(m)}$, $F_0^{(m)}$, $H_\tau^{(m)}$. Thus

$$F^{(m)}(x,t) = F_0^{(m)}(x,t) + \int_0^t H_\tau^{(m)}(x,t)\, d\tau. \tag{A.12}$$

By Theorem A.12, $F^{(m)} \to F$, $F_0^{(m)} \to F_0$ and $H_\tau^{(m)} \to H_\tau$ pointwise, and by Theorem A.7 (applied to $\Omega_{(\tau,T]}$), $|H_\tau^{(m)}(x,t)| \le C(|x|^p + 1)$ for some C and p and all m and τ. Hence (A.11) follows from (A.12) by dominated convergence. □

A.5. **Regularity at the boundary.** We begin with an a priori estimate for bounded domains. Let $B_r = \{x \in \mathbb{R}^n : |x| < r\}$ and $D_r = B_r \times (0, T]$.

Lemma A.16. *Let $0 < r_0 < r$ and let $p \geq 0$. Suppose that the coefficients a_{ij}, b_i, c belong to $C^{p,0}_\alpha(\overline{D}_r)$ and that $\mathcal{M}$ is uniformly parabolic in D_r, i.e.* (A.2) *holds for some $\lambda > 0$ and all $(x,t) \in D_r$.*

If $F \in C^{p+2,1}_\alpha(\overline{D}_r)$ satisfies $\mathcal{M}F = h$ in D_r and $F = f$ on B_r, where $h \in C^{p,0}_\alpha(\overline{D}_r)$ and $f \in C^{p+2}_\alpha(\overline{B}_r)$, then

$$\|F\|_{C^{p+2,1}_\alpha(\overline{D}_{r_0})} \leq K\Big(\|F\|_{C(\overline{D}_r)} + \|f\|_{C^{p+2}_\alpha(\overline{B}_r)} + \|h\|_{C^{p,0}_\alpha(\overline{D}_r)}\Big), \tag{A.13}$$

where K is a constant depending only on n, r, r_0, α, p, the norms of the coefficients in $C^{p,0}_\alpha(\overline{D}_r)$, and the parabolicity constant λ.

Proof. We use induction on p. The case $p = 0$ follows from [4, Theorem 4.4], taking there $D = D_r$, $R_0 = B_{r_0}$ and $R = B_{r_1}$, where $r_1 = (r + r_0)/2$, say. More precisely, this theorem as stated in [4] yields the estimate (A.13) for the norm in $C^{2,0}_\alpha$; since $D_t F = \mathcal{L}F + h$, the required estimates for $D_t F$ follow too.

If $p \geq 1$, fix $l \in \{1, \dots, n\}$. Then

$$\begin{aligned}\mathcal{M}(D_l F) &= \frac{\partial}{\partial t} D_l F - \mathcal{L}(D_l F) = D_l \frac{\partial}{\partial t} F - \mathcal{L}(D_l F) = D_l(\mathcal{L}F) - \mathcal{L}(D_l F) + D_l h \\ &= \sum_{i,j=1}^n (D_l a_{ij}) D_{ij} F + \sum_{i=1}^n (D_l b_i) D_i F + (D_l c) F + D_l h. \end{aligned} \tag{A.14}$$

Denoting the right hand side by H_l, it follow by Leibniz' rule that $H_l \in C^{p-1,0}_\alpha(\overline{D}_r)$ and, again with $r_1 = (r + r_0)/2$,

$$\|H_l\|_{C^{p-1,0}_\alpha(\overline{D}_{r_1})} \leq K_1 \|F\|_{C^{p+1,0}_\alpha(\overline{D}_{r_1})} + \|h\|_{C^{p,0}_\alpha(\overline{D}_{r_1})}. \tag{A.15}$$

The induction hypothesis applied to $D_l F$ yields, using (A.15),

$$\begin{aligned}\|D_l F\|_{C^{p+1,1}_\alpha(\overline{D}_{r_0})} &\leq K_2 \Big(\|D_l F\|_{C(\overline{D}_{r_1})} + \|D_l f\|_{C^{p+1}_\alpha(\overline{B}_{r_1})} + \|H_l\|_{C^{p-1,0}_\alpha(\overline{D}_{r_1})}\Big) \\ &\leq K_3 \Big(\|F\|_{C^{p+1,0}_\alpha(\overline{D}_{r_1})} + \|f\|_{C^{p+2}_\alpha(\overline{B}_r)} + \|h\|_{C^{p,0}_\alpha(\overline{D}_{r_1})}\Big). \end{aligned}$$

The induction hypothesis again shows that this is dominated by the right hand side of (A.13). The result follows. □

Lemma A.17. *Let $p \geq 0$. Assume* (A3), (B1), (C2), *and assume furthermore that the coefficients a_{ij}, b_i, c and h belong to $C^{p,0}_{\alpha,\mathrm{b}}(\overline{\Omega_T})$ and that $f \in C^{p+2}_{\alpha,\mathrm{b}}(\mathbb{R}^n)$. Suppose that F is the bounded solution of the Cauchy problem* (A.6). *Then $F \in C^{p+2,1}_{\alpha,\mathrm{b}}(\overline{\Omega_T})$.*

Proof. For $p = 0$, [3, Theorem 5.3, p. 283] yields $F \in C^{2,0}_{\alpha,\mathrm{b}}(\overline{\Omega_T})$, and the conclusion follows because $D_t F = \mathcal{L}F + h$.

For $p \geq 1$ we use induction. By Theorem A.11, $F \in C^{p+2,1}_\alpha(\overline{\Omega_T})$. Hence (A.14) holds as above. The induction hypothesis implies $F \in C^{p+1,1}_{\alpha,\mathrm{b}}(\overline{\Omega_T})$, and thus $H_l \in C^{p-1,1}_{\alpha,\mathrm{b}}(\overline{\Omega_T})$. Since $D_l F$ is bounded and in $C(\overline{\Omega_T})$ by the case $p = 0$, the induction hypothesis and (A.14) show that $D_l F \in C^{p+1,1}_\alpha(\overline{\Omega_T})$ for every l, and the result follows. □

Theorem A.18. *Assume* (A2), (B2), (C1), *and that $h \in C_{\mathrm{pol}}(\Omega_T)$. Let $p \geq 0$ and assume furthermore that the coefficients a_{ij}, b_i, c and h belong to $C^{p,0}_\alpha(\overline{\Omega_T})$ and that*

$f \in C_{\text{pol}}(\mathbb{R}^n) \cap C_\alpha^{p+2}(\mathbb{R}^n)$. *If* $F \in C_{\text{pol}}(\overline{\Omega_T}) \cap C^{2,1}(\Omega_T)$ *solves the Cauchy problem* (A.6)*, then* $F \in C_\alpha^{p+2,1}(\overline{\Omega_T})$.

Proof. Let again $\psi \in C_c^\infty(\mathbb{R})$ be a test function with compact support with $\psi(x) = 1$ for $|x| \le 1$. This time, let $\Psi^{(m)}(x,t) = \psi^{(m)}(x) = \psi(x/m)$, and define as before $\mathcal{L}^{(m)} = \Psi^{(m)} L + (1 - \Psi^{(m)})\Delta$, $\mathcal{M}^{(m)} = \partial/\partial t - \mathcal{L}^{(m)}$, $f^{(m)} = \psi^{(m)} f$, $h^{(m)} = \Psi^{(m)} h$.

For each m, the Cauchy problem $\mathcal{M}^{(m)} F^{(m)} = h^{(m)}$ in Ω_T with $F^{(m)} = f^{(m)}$ on $\mathbb{R}^n$ has a (unique) bounded solution by Theorem A.13, and Lemma A.17 shows that $F^{(m)} \in C_\alpha^{p+2,1}(\overline{\Omega_T})$.

Moreover, Lemma A.16 shows, together with Theorem A.7, that for every fixed r_0, $\|F^{(m)}\|_{C_\alpha^{p+2,1}(\overline{D}_{r_0})}$ stays bounded as $m \to \infty$. By the Arzela–Ascoli theorem we may thus select a subsequence such that every derivative $D_x^k D_t^l F^{(m)}$ with $|k| + 2l \le p + 2$ and $l \le 1$ converges uniformly on $\overline{D}_{r_0}$. Since $F^{(m)} \to F$ by Theorem A.12, this implies that $F \in C_\alpha^{p+2,1}(\overline{D}_{r_0})$. Since r_0 is arbitrary, this completes the proof. □

Theorem A.19. *Assume* (A2), (B3), (C1)*, and that* $h \in C_{\text{Exp}}(\Omega_T)$. *Let* $p \ge 0$ *and assume furthermore that the coefficients* a_{ij}, b_i, c *and* h *belong to* $C_\alpha^{p,0}(\overline{\Omega_T})$ *and that* $f \in C_{\text{Exp}}(\mathbb{R}^n) \cap C_\alpha^{p+2}(\mathbb{R}^n)$. *If* $F \in C_{\text{Exp}}(\overline{\Omega_T}) \cap C^{2,1}(\Omega_T)$ *solves the Cauchy problem* (A.6)*, then* $F \in C_\alpha^{p+2,1}(\overline{\Omega_T})$.

Proof. First note that for any real κ, $\widetilde{F}(x,t) = e^{-\kappa|x|^2} F(x,t)$ satisfies a similar differential equation $\widetilde{\mathcal{M}}\widetilde{F} = e^{-\kappa|x|^2} h(x,t)$, where $\widetilde{\mathcal{M}}$ too satisfies (A2), (B3), (C1). (This is an advantage of (B3).) Consequently we may assume that f, F and h are bounded.

If, further, c is bounded, also (B2) holds, and the result follows by Theorem A.18. In general, we replace c by $c^{(m)} = c\psi(x/m)$, with ψ as in the previous proof, and find a solution $F^{(m)} \in C_\alpha^{p+2,1}(\overline{\Omega_T})$. Using Lemma A.16 and the maximum principle Theorem A.8 to obtain norm estimates not depending on m, reducing T to t_0 if necessary, we see by a limit argument as in Theorem A.12 that $F^{(m)} \to F$ in $\mathbb{R}^n \times [0, t_0]$ and that $F \in C_\alpha^{p+2,1}(\mathbb{R}^n \times [0, t_0])$. The extension to the whole strip Ω_T follows by Theorem A.11. □

Theorem A.20. *Assume* (A4) *and* (B2^{2,1})*, and that* $f \in C^3_{\text{pol}}(\mathbb{R}^n)$ *and* $h \in C^{1,1}_{\text{pol}}(\Omega_T)$. *Then the Cauchy problem* (A.6) *has a unique solution* $F \in C_{\text{pol}}(\overline{\Omega_T}) \cap C^{2,1}(\Omega_T)$ *and* $F \in C^{2,1}_{\text{pol}}(\overline{\Omega_T})$.

Proof. Let $0 < \alpha < 1$. Note that (B2^{2,1}) entails both (B2) and (C1). Clearly, $f \in C_\alpha^2(\mathbb{R}^n)$ and $a_{ij}, b_i, c, h \in C_\alpha(\overline{\Omega_T})$. Theorem A.14 thus shows that F exists and is unique, and Theorem A.18 (with $p = 0$) shows that $F \in C_\alpha^{2,1}(\overline{\Omega_T})$.

Fix x_0 with $R = |x_0| \ge 1$. Define $\widetilde{f}(y) = f(x_0 + Ry)$, $\widetilde{F}(y,t) = F(x_0 + Ry, t)$, $\widetilde{h}(y,t) = h(x_0 + Ry, t)$, $\tilde{a}_{ij}(y,t) = R^{-2} a_{ij}(x_0 + Ry, t)$, $\widetilde{b}_i(y,t) = R^{-1} b_i(x_0 + Ry, t)$, $\tilde{c}(y,t) = c(x_0 + Ry, t)$. The corresponding operator $\widetilde{\mathcal{M}}$ satisfies $\widetilde{\mathcal{M}}\widetilde{F} = \widetilde{h}$ in Ω_T and $\widetilde{F}(y, 0) = \widetilde{f}(y)$.

We apply Lemma A.16 with $p = 0$, $r_0 = 1/4$ and $r = 1/2$, and note that the norms of $\tilde{a}_{ij}$ etc. that enter in the constant are bounded uniformly in x_0 and R by (B2^{2,1}). Hence, with K, K_1 independent of x_0, if $0 \le k + 2l \le 2$,

$$\begin{aligned} R^k |D_x^k D_t^l F(x_0, t)| &\le \|\widetilde{F}\|_{C_\alpha^{2,1}(\overline{D}_{1/4})} \le K \left(\|\widetilde{F}\|_{C(\overline{D}_{1/2})} + \|\widetilde{f}\|_{C_\alpha^2(\overline{B}_{1/2})} + \|\widetilde{h}\|_{C_\alpha(\overline{D}_{1/2})} \right) \\ &\le K_1 \left(\|F\|_{C(\overline{D}_{2R})} + R^3 \|f\|_{C_\alpha^3(\overline{B}_{2R})} + R\|h\|_{C^{1,1}(\overline{D}_{2R})} \right). \end{aligned}$$

The right hand side is bounded by a polynomial in $R = |x_0|$, and it follows that $F \in C^{2,1}_{\mathrm{pol}}(\overline{\Omega_T})$. $\square$

Remark A.21. We did not use the full strength of ($\mathrm{B2}^{2,1}$).

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