SUPERREPLICATION OF OPTIONS ON SEVERAL UNDERLYING ASSETS

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ABSTRACT. We investigate when a hedger who over-estimates the volatility will superreplicate a convex claim on several underlying assets. It is shown that the classical Black and Scholes model is the only model, within a large class, for which over-estimation of the volatility yields the desired superreplication property. This is in contrast to the one-dimensional case, in which it is known that over-estimation of the volatility with any model guarantees superreplication of convex claims.

1. INTRODUCTION

For options written on one underlying asset it is well-known that convexity of the contract function ensures certain monotonicity properties of the option price with respect to the volatility. For example, if the contract function is convex, then the option price increases with the volatility, see Bergman, Grundy and Wiener (1996), El Karoui, Jeanblanc-Picque and Shreve (1998), Hobson (1998) or Janson and Tysk (2003a). It is also known that a hedger who over-estimates the volatility with any model will superreplicate a given convex claim, see El Karoui et al (1998) or Hobson (1998). Crucial both for the monotonicity result and for the superreplication property is the fact that the price of a convex claim is convex (as a function of the current stock price) at any time before maturity. This fact, however, is one-dimensional in nature; it is easy to find examples of an option with a convex pay-off of two underlying assets which has a non-convex price, see Janson and Tysk (2003a).

Janson and Tysk (2003b) consider second order parabolic differential equations of the form

$$\frac{\partial G}{\partial t} = \mathcal{L}G, \tag{1}$$

where the differential operator

$$\mathcal{L} = \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x,t)\frac{\partial}{\partial x_i} + c(x,t)$$

is elliptic. The authors find a necessary and sufficient local condition on the operator $\mathcal{L}$ that guarantees that the unique (satisfying appropriate growth conditions) solution to the equation (1) remains convex at every time $t > 0$ provided the initial condition is convex. If $\mathcal{L}$ satisfies the condition it is said to be *locally convexity preserving* (LCP). In the present paper we apply the results of Janson and Tysk (2003b) to the problem of monotonicity in the

Key words and phrases. Parabolic equations; superreplication; convexity; options.

volatility; in particular we study the problem of superreplication of options written on several underlying assets.

Consider a market consisting of a bank account with price process

$$B(t) = B(0)\exp\Big\{\int_0^t r(u)\,du\Big\},$$

where the interest rate r is a deterministic function, and n risky assets, with the price X_i of the ith asset satisfying the stochastic differential equation

$$dX_i = \mu_i(X,t)\,dt + \sum_{j=1}^{n} \beta_{ij}(X,t)\,dW_j. \tag{2}$$

In this equation the drift μ_i is some deterministic function of the current stock prices and time, W is an n-dimensional Brownian motion and the diffusion matrix $\beta = (\beta_{ij}(x,t))_{i,j=1}^n$ is assumed to be non-singular for all x with positive components. Note that the only source of randomness in the diffusion matrix β is in the dependence on the current stock prices. In finance it is natural to consider stock prices that cannot become negative. Therefore we let 0 be an absorbing barrier, i.e. if X_i is 0 at some time, then X_i remains 0 forever.

Given a finite time horizon $T > 0$, there exists a unique probability measure $\tilde{\mathbb{P}}$ equivalent with the original measure $\mathbb{P}$ such that

$$dX_i = r(t)X_i\,dt + \sum_{j=1}^{n} \beta_{ij}(X,t)\,d\tilde{W}_j$$

for some $\tilde{\mathbb{P}}$-Brownian motion $\tilde{W}$, i.e. $B_t^{-1}X_t$ is a local martingale under $\tilde{\mathbb{P}}$. Let $g : \mathbb{R}_+^n \to \mathbb{R}$ be continuous and of at most polynomial growth. Standard arbitrage theory yields that the price at time t of the option which at time $T_0 \le T$ pays $g(X(T_0))$ is $F(X(t),t)$, where

$$F(x,t) = \exp\Big\{-\int_t^{T_0} r(u)\,du\Big\}\tilde{E}_{x,t}\,g(X(T_0)). \tag{3}$$

Here $\tilde{E}$ denotes expected value with respect to the measure $\tilde{P}$ and the indices indicate that $X_t = x$. Moreover, this pricing function F solves the Black-Scholes parabolic differential equation

$$\frac{\partial F}{\partial t} + \mathcal{L}F = 0, \tag{4}$$

where

$$\mathcal{L} = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} r x_i \frac{\partial}{\partial x_i} - r, \tag{5}$$

with terminal condition

$$F(x,T_0) = g(x).$$

In this equation the coefficients $a_{ij} = a_{ij}(x,t)$ are the entries of the $n \times n$-matrix $\beta\beta^*$. Note that the invertibility of β guarantees parabolicity of the equation (4) (since the direction of the time variable is opposite to the customary one).

We will say that a model for the stock price vector X (or the diffusion matrix β) is *convexity preserving* if, for any $T_0 \le T$, the price of an option with a convex pay-off $g(X(T_0))$ at T_0 is convex in $X(t)$ at all times t prior to T_0. In Section 2 we introduce the LCP-condition. Then we show that, for $n \geq 2$, the only convexity preserving model which is standard, see Definition 2.5, is geometric Brownian motion. In Section 3 we show that if the hedger uses a convexity preserving model, then over-estimation of the diffusion matrix (in the sense of quadratic forms, see below) guarantees superreplication of convex claims.

2. Convexity preserving models

Assume that the diffusion coefficients β_{ij} are linear in x_i and independent of x_l, $l \neq i$. Then, under the measure $\tilde{\mathbb{P}}$, the stock price vector X satisfies

$$dX_i = r(t)X_i\,dt + X_i \sum_{j=1}^{n} \sigma_{ij}(t)\,d\tilde{W}_j \tag{6}$$

for some deterministic functions σ_{ij}. Such a process X is called n-dimensional geometric Brownian motion with time-dependent volatility, or simply geometric Brownian motion. Thus, to show that a process X defined by (2) is geometric Brownian motion one has to show that $\beta_{ij}(x,t) = x_i\sigma_{ij}(t)$ for all i and j.

Theorem 2.1. *Geometric Brownian motion is convexity preserving.*

Proof. Let $T_0 \leq T$. It is well-known that if X is geometric Brownian motion as in (6), then

$$X_i(T_0) = x_i \exp\Big\{ \int_t^{T_0} \big(r(u) - \frac{1}{2}\sum_{j=1}^{n} \sigma_{ij}^2(u)\big)\,du + \sum_{j=1}^{n} \int_t^{T_0} \sigma_{ij}(u)\,d\tilde{W}_j \Big\},$$

where $x_i = X_i(t)$. Let $g : \mathbb{R}_+^n \to \mathbb{R}$ be convex. The price $F(X(t),t)$ at time t of an option which at time T_0 pays $g(X(T_0))$ is given by

$$F(x,t) = \exp\big\{ -\int_t^{T_0} r(u)\,du \big\} \tilde{E}_{x,t}\, g(X(T_0)),$$

which is convex in x since $X_i(T_0)$ is linear in x_i and g is convex. Thus geometric Brownian motion is convexity preserving. □

Next we investigate which models are convexity preserving. It turns out that if one imposes some conditions on the diffusion matrix, then geometric Brownian motion is the only model which is convexity preserving. We first introduce the LCP-condition:

Definition 2.2. *Assume that the coefficients of the differential operator $\mathcal{L}$ are in $C^3(\mathbb{R}_+^n \times [0,T])$. Let $x \in \mathbb{R}_+^n$ be an interior point, and let $t \in [0,T]$. Then $\mathcal{L}$ is said to be locally convexity preserving (LCP) at (x,t) if*

$$D_{uu}(\mathcal{L}f)(x,t) \geq 0$$

whenever $u \in \mathbb{R}^n \setminus \{0\}$, $f \in C^\infty(\mathbb{R}_+^n)$ is convex in a neighborhood of x and $D_{uu}f(x) = 0$.

If G is a solution to (1), then the infinitesimal change of G during a short time interval Δt is approximately $\Delta t(\mathcal{L}G)$. Thus the LCP-condition is intuitively the right condition to preserve convexity: if, at some instant, convexity is almost lost in some direction u, then the infinitesimal change of G is convex in that direction.

For simplicity we will in this paper work under the following assumption.

Hypothesis 2.3. *The diffusion matrix β is in $C(\overline{\mathbb{R}^n_+} \times [0,T]) \cap C^3(\mathbb{R}^n_+ \times [0,T])$ and is such that, for any vector of non-negative initial values of the stocks, there exists a unique strong solution X to (2) with absorption at 0 of the i:th component X_i for all i.*

We also assume that the diffusion matrix β is such that, for any smooth terminal value g, the function F defined by (3) has continuous derivatives $D^k_x D^m_t F$, $m \in \{0,1\}$, $0 \leq |k| + 2m \leq 4$ up to time T_0.

Remark The assumption that β is C^3 is unnecessarily strong, see section 3 in Janson and Tysk (2003b). To clarify the presentation, however, we keep this assumption.

Theorem 2.4. *Let the diffusion matrix β satisfy Hypothesis 2.3, and let $\mathcal{L}$ be the corresponding differential operator as in (5). Now, if β is convexity preserving, then $\mathcal{L}$ is LCP at all points (x_0, T_0) such that $x_0 \in \mathbb{R}^n_+$ is interior and $T_0 \in (0,T)$.*

Proof. Suppose that $f \in C^\infty(\mathbb{R}^n_+)$ is convex in a neighborhood of some point x_0 in the interior of $\mathbb{R}^n_+$, and suppose that $D_{uu}f(x_0) = 0$ for some direction $u \neq 0$. Then there exists a smooth convex function $g : \mathbb{R}^n_+ \to \mathbb{R}$ which equals f in a neighborhood of x_0, for details see Lemma 3.2 in Janson and Tysk (2003b). Since β is convexity preserving there exists a solution $F(x,t)$ (defined by (3)), which is convex in x for all $t \leq T_0$, to equation (4) with terminal condition $F(x,T_0) = g(x)$. By assumption, $D^k_x D^m_t F$, $m \in \{0,1\}$, $0 \leq |k| + 2m \leq 4$, exist and are continuous up to time T_0. Since $D_{uu}F(x_0,t) \geq 0$ for all $t \leq T_0$ and $D_{uu}F(x_0,T_0) = 0$, we have that $D_t D_{uu} F(x_0,T_0) \leq 0$. Therefore, using equation (4),

$$\begin{aligned} D_{uu}(\mathcal{L}f)(x_0,T_0) &= D_{uu}(\mathcal{L}F)(x_0,T_0) = -D_{uu}(D_tF)(x_0,T_0) \\ &= -D_t D_{uu} F(x_0,T_0) \geq 0. \end{aligned}$$

□

Remark Janson and Tysk (2003b) show that the LCP-condition is both necessary and sufficient to guarantee that an operator $\mathcal{L}$ defined on $\mathbb{R}^n \times [0,T]$ is convexity preserving. To get sufficiency in the present setting we would have to add some conditions on the boundary of $\mathbb{R}^n_+$ in the definition of LCP. In our analysis, however, we only need that LCP is a necessary condition, and we therefore leave the rather technical considerations about the appropriate LCP-condition for boundary points.

We now present the class of models under consideration.

Definition 2.5. *A model with a diffusion matrix β that satisfies Hypothesis 2.3 is* standard *if the following conditions are satisfied:*

(i) *the diffusion coefficient β_{ij} is a function only of x_i and t;*

(ii) *the diffusion coefficient* $\beta_{ij} = 0$ *for* $x_i = 0$*;*
(iii) *for all* $i = 1, .., n$ *and fixed times* t*, the volatility*

$$\sqrt{a_{ii}(x_i, t)}/x_i = \sqrt{\beta_{i1}^2(x_i, t) + .. + \beta_{in}^2(x_i, t)}/x_i$$

of the i:th asset is not an increasing function of x_i*, unless it is constant.*

Remark Note that (i) does not exclude dependence between the assets. Instead it merely says that the volatility of the ith asset depends only on the value of that asset and time. Note further that condition (ii) allows volatilities tending to infinity for asset values close to zero. Condition (iii) seems to be satisfied for virtually all models for option pricing; in fact many models have larger volatilities at zero than at infinity.

We can now state our main theorem.

Theorem 2.6. *Let* $n \geq 2$*. Then geometric Brownian motion (with time-dependent volatility) is the only standard model that is convexity preserving.*

Remark It follows, for example, that the constant elasticity of variance model (which is a standard model) in which the price of the ith asset is given by

$$dX_i = r(t)X_i\, dt + \sigma_i X_i^\gamma\, d\tilde{W}_i$$

with $0 < \gamma < 1$, is not convexity preserving.

We start with a lemma.

Lemma 2.7. *Let* β *be a* C^2 *diffusion matrix such that* β_{ij} *is a function of* x_i *alone, and such that the property (ii) in Definition 2.5 is satisfied. Assume that there exist constants* B_{ij} *such that*

$$\sum_{k=1}^{n} \beta_{ik}(x_i)\beta_{jk}(x_j) = B_{ij}x_i x_j \tag{7}$$

for all i, j *with* $i \neq j$*. Then there exists an index* $l \in \{1, ..., n\}$ *such that* $(\beta_{l1}, ..., \beta_{ln}) = f(x_l)v$ *for some constant vector* $v \in \mathbb{R}^n$ *and some function* f*.*

If, in addition, (7) also holds for $i = j$*, then* β *is the diffusion matrix of a geometric Brownian motion.*

Proof. Let $\beta_i = (\beta_{i1}, ..., \beta_{in})$ and $\beta_i'' = (\beta_{i1}'', ..., \beta_{in}'')$ for $i = 1, ..., n$. Here the double primes refer to differentiation with respect to the x_i-variables. If $\beta_i'' \equiv 0$ for some i, then all components of β_i are affine functions of x_i. From (ii) it follows that the components in fact are linear, and thus the lemma is true in this particular case.

Now, assume that we can choose coordinates $x_1, ..., x_n > 0$ such that $\beta_i''(x_i) \neq 0$. From (7) we have

$$\beta_i''(x_i) \cdot \beta_j(x_j) = 0 \tag{8}$$

for $i \neq j$. Since $\beta_1(x_1), ..., \beta_n(x_n)$ are linearly independent (recall that the diffusion matrix is assumed to be non-singular), it follows from (8) that $\beta_1''(x_1), ..., \beta_n''(x_n)$ are linearly independent. Now, still keeping $x_1, ..., x_{n-1}$ fixed it follows from (8) that, for any x_n, $\beta_n(x_n)$ is in the orthogonal (1-dimensional) complement of $\beta_1''(x_1), ..., \beta_{n-1}''(x_{n-1})$. Thus $\beta_n(x_n) = f(x_n)v$ for some constant vector $v \in \mathbb{R}^n$, which finishes the first part of the lemma.

Next, assume in addition that (7) also holds for $i=j$, i.e. that $a_{ii}(x_i)=B_{ii}x_i^2$ for some positive constants B_{ii}. Fix $x_2,...,x_n$ and let $v_i=\frac{1}{x_i}\beta_i(x_i)$ for $i=2,...,n$. Then

$$\frac{1}{x_1}\beta_1(x_1)\cdot v_i=B_{1i}$$

for all x_1. Since v_i, $i=2,...,n$ are linearly independent it follows that

$$\frac{1}{x_1}\beta_1(x_1)\in\{\beta_1(1)+tw;t\in\mathbb{R}\}$$

for some vector $w\in\mathbb{R}^n$. But since the inner product

$$\big(\frac{1}{x_1}\beta_1(x_1)\big)\cdot\big(\frac{1}{x_1}\beta_1(x_1)\big)=B_{11}$$

is constant we find that $\frac{1}{x_1}\beta_1(x_1)$ is a vector of constant length. By continuity it follows that $\frac{1}{x_1}\beta_1(x_1)$ is a constant vector. Similarly we deduce that all entries in the ith row of β are linear in x_i, and thus the lemma follows. □

Proof of Theorem 2.6. Assume that β is a standard diffusion matrix which is convexity preserving, and let $\mathcal{L}$ be the corresponding differential operator appearing in the Black-Scholes equation (4). We suppress the time variable t in the calculations below. First choose two components x_i and x_j. For fixed s, let $f(x_i,x_j)=\frac{1}{2}(sx_i-x_j)^2$. Then f is constant along lines $x_j=sx_i+x_{i,0}$, where $x_{i,0}$ is a constant. If $u=e_i+se_j$ then, since $\mathcal{L}$ is convexity preserving, it follows from Theorem 2.4 that

$$\begin{aligned}
0 &\leq 2D_{uu}(\mathcal{L}f)\\
&= (\partial^2_{x_i}+2s\partial_{x_i}\partial_{x_j}+s^2\partial^2_{x_j})(\sum_{i,j=1}^n a_{ij}\frac{\partial^2 f}{\partial x_i\partial x_j}+2r\sum_{i=1}^n x_i\frac{\partial f}{\partial x_i}-2rf)\\
&= (\partial^2_{x_i}+2s\partial_{x_i}\partial_{x_j}+s^2\partial^2_{x_j})(s^2a_{ii}-2sa_{ij}+a_{jj}+2rx_if_{x_i}+2rx_jf_{x_j}-2rf)\\
&= -2s^3\partial^2_{x_j}a_{ij}+s^2(\partial^2_{x_i}a_{ii}-4\partial_{x_i}\partial_{x_j}a_{ij}+\partial^2_{x_j}a_{jj})-2s\partial^2_{x_i}a_{ij}.
\end{aligned}$$

Since s is arbitrary we find that

$$\partial^2_{x_j}a_{ij}=\partial^2_{x_i}a_{ij}=0,$$

so $a_{ij}=B_{ij}x_ix_j+B_1x_i+B_2x_j+B_3$. The condition (ii) guarantees that $a_{ij}(x_i,x_j)$ vanishes for $x_i=0$ and for $x_j=0$. It follows that $B_1=B_2=B_3=0$. Hence $a_{ij}=B_{ij}x_ix_j$ for $i\neq j$. It remains to show the same for the diagonal elements a_{ii}. Using the first part of Lemma 2.7 we may assume that say $\beta_n(x_n)=f(x_n)v$ for some constant vector $v\in\mathbb{R}^n$.

Now, choose a row β_i, $i\in\{1,..,n-1\}$, in the diffusion matrix, and consider the matrix block

$$A_{in}=\begin{pmatrix} a_{ii} & a_{in}\\ a_{ni} & a_{nn}\end{pmatrix}=\begin{pmatrix} a_{ii}(x_i) & B_{in}x_ix_n\\ B_{in}x_ix_n & a_{nn}(x_n)\end{pmatrix}.$$

If A is convexity preserving we must have that A_{in} is convexity preserving for equations in two spatial variables x_i and x_n. Two different cases can occur. First, assume that $B_{in}=0$. Then

$$A_{in}=\begin{pmatrix} g^2(x_i) & 0\\ 0 & h^2(x_n)\end{pmatrix}, \tag{9}$$

where g and h are some positive functions. From Example 5.3 in Janson and Tysk (2003b) we see that in order to have a convexity preserving operator we must have

$$g(x_i)g''(x_i) + h(x_n)h''(x_n) \geq 0$$

for all x_i and x_n. Hence at least one of g and h is convex. Let this function be g. By (ii) in Definition 2.5 we conclude that $g(0) = 0$. From (iii) the function g is seen to be linear using the fact that $g(x_i)/x_i = \sqrt{a_{ii}(x_i,t)}/x_i$ is increasing since g is a convex function vanishing at 0. Then the above inequality has only one term, so h is convex, and thus also linear by the same argument. Therefore $a_{ii}(x_i) = C_1^2 x_i^2$ and $a_{nn}(x_n) = C_2^2 x_n^2$ for some constants C_1 and C_2.

Next, if $B_{in} \neq 0$, then

$$a_{in}(x_i, x_n) = B_{in} x_i x_n = \beta_i(x_i) \cdot \beta_n(x_n) = f(x_n)(\beta_i(x_i) \cdot v)$$

for some constant vextor v. It follows that $\beta_n(x_n) = x_n u$ for some constant vector $u \in \mathbb{R}^n$. Therefore

$$A_{in} = \begin{pmatrix} g^2(x_i) + D^2 x_i^2 & DEx_i x_n \\ DEx_i x_n & E^2 x_n^2 \end{pmatrix}$$

for some non-zero constants D and E and some function g which is strictly positive for $x_i > 0$. From Corollary 5.2 by Janson and Tysk (2003b) it follows that

$$D_{uu}\sqrt{b^2 a_{ii}(x_i) - 2abDEx_i x_n + a^2E^2x_n^2} \geq 0 \tag{10}$$

for all directions $u = ae_i + be_n$. Direct calculations show that

$$\begin{aligned} & D_{uu}\sqrt{b^2 a_{ii}(x_i) - 2abDEx_i x_n + a^2E^2x_n^2} \\ & = a^2b^2(b^2a_{ii}(x_i) - 2abDEx_ix_n + a^2E^2x_n^2)^{-3/2} \cdot \\ & \Big(\big(b(D-E)g(x_i) - (bDx_i - aEx_n)g'(x_i)\big)^2 \\ & + \big(b^2g^2(x_i) + (bDx_i - aEx_n)^2\big)g(x_i)g''(x_i)\Big). \end{aligned}$$

Here

$$b^2 a_{ii} - 2abDEx_ix_n + a^2E^2x_n^2 = b^2g^2(x_i) + (bDx_i - aEx_n)^2 > 0$$

if $b \neq 0$ and $x_i > 0$. We claim that

$$g''(x_i) \geq 0 \tag{11}$$

for all x_i. Indeed, for given x_i and x_n, let $K_1 = Ex_n g'(x_i)$ and $K_2 = (D-E)g(x_i) - Dx_i g'(x_i)$. Then we know that

$$(K_1 a + K_2 b)^2 + (b^2g^2(x_i) + (bDx_i - aEx_n)^2)g(x_i)g''(x_i) \geq 0$$

for all $a, b \neq 0$. If K_1 and K_2 both are 0, then (11) follows immediately. If K_1 and K_2 both are non-zero, then $a, b \neq 0$ can be chosen so that $K_1a + K_2b = 0$, and thus (11) follows. If $K_1 = 0$, $K_2 \neq 0$ and $g(x_i)g''(x_i) < 0$, then $b = 1$ and a very big yields a contradiction, so (11) follows. The case $K_1 \neq 0 = K_2$ is similar. Considering the inequality (11) together with the assumptions (ii) and (iii) of Definition 2.5 we find that $a_{ii}(x_i) = C^2x_i^2$ for some constant C. □

Remark The theorem tells us that to be sure to preserve convexity, regardless of the (convex) contract function, geometric Brownian motion should be used to model the asset prices. However, given a *particular* convex claim, it can of course be the case that the class of models that guarantee convex option prices is bigger. For example, if the contract function $g(x_1, x_2) = g_1(x_1) + g_2(x_2)$, where g_1 and g_2 both are convex, then all models with diagonal diffusion matrices satisfying (i) in Definition 2.5 are convexity preserving. The reason is that such a claim is the sum of two one-dimensional claims, both of which have convex prices.

3. Superreplication of convex claims

Assume that an option writer believes that the diffusion matrix is β as in (2), whereas the true stock price vector $\tilde{X}$ evolves according to

$$d\tilde{X}_i = \tilde{\mu}_i(\tilde{X}, t)\, dt + \sum_{j=1}^{n} \tilde{\beta}_{ij}(\tilde{X}, t)\, dW_j \tag{12}$$

for some functions $\tilde{\mu}_i$ and $\tilde{\beta}_{ij}$. He will then (incorrectly) price an option on the stocks according to (3), where X is a diffusion with diffusion matrix β. Moreover, if he tries to replicate the option with the hedging strategy suggested by his model, then he will form a self-financing portfolio which has initial value $F(X(0), 0)$ and is such that it at each instant t contains $\frac{\partial F}{\partial x_i}(\tilde{X}(t), t)$ numbers of shares of the ith asset (and the remaining amount invested in the bank account). In this section we provide conditions under which the terminal value of the hedger's portfolio exceeds the option pay-off $g(X(T_0))$ almost surely.

Given two diffusion matrices β and $\tilde{\beta}$ we say that β dominates $\tilde{\beta}$ if $A(x,t) = \beta(x,t)\beta^*(x,t) \geq \tilde{\beta}(x,t)\tilde{\beta}^*(x,t) = \tilde{A}(x,t)$ as quadratic forms for all x and t.

Theorem 3.1. *Assume that a hedger overestimates (underestimates) the volatility, i.e he uses a diffusion matrix β which dominates (is dominated by) the true diffusion matrix $\tilde{\beta}$. Moreover, assume that β is convexity preserving. Then the hedger will superreplicate (subreplicate) any convex claim written on X.*

Proof. Let $g : \mathbb{R}_+^n \to \mathbb{R}$ be a convex contract function, and define F as in (3). Since β is convexity preserving, $F(x,t)$ is convex in x for all $t \in [0, T_0]$. The value $V(t)$ of the self-financing portfolio with $\frac{\partial F}{\partial x_i}(\tilde{X}_i(t), t)$ shares of the ith asset and initial value $V(0) = F(\tilde{X}(0), 0)$ has the dynamics

$$dV = r\big(V(t) - \sum_{i=1}^{n} \tilde{X}_i(t) \frac{\partial F}{\partial x_i}(\tilde{X}(t), t)\big)\, dt + \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(\tilde{X}(t), t)\, d\tilde{X}_i(t).$$

Consider the process $Y(t) := V(t) - F(\tilde{X}(t), t)$. From Ito's formula it follows that

$$\begin{aligned} dY &= dV - \frac{\partial F}{\partial t}(\tilde{X}(t), t)\, dt - \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(\tilde{X}(t), t)\, d\tilde{X}_i \\ &\quad - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \tilde{a}_{ij}(\tilde{X}(t), t) \frac{\partial^2 F}{\partial x_i \partial x_j}(\tilde{X}(t), t)\, dt \\ &= rV(t)\, dt - \frac{\partial F}{\partial t}(\tilde{X}(t), t)\, dt - r\sum_{i=1}^{n} \tilde{X}_i(t) \frac{\partial F}{\partial x_i}(\tilde{X}(t), t)\, dt \\ &\quad - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \tilde{a}_{ij}(\tilde{X}(t), t) \frac{\partial^2 F}{\partial x_i \partial x_j}(\tilde{X}(t), t)\, dt \\ &= r(V(t) - F(\tilde{X}(t), t))\, dt \\ &\quad + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \big(a_{ij}(\tilde{X}(t), t) - \tilde{a}_{ij}(\tilde{X}(t), t)\big) \frac{\partial^2 F}{\partial x_i \partial x_j}(\tilde{X}(t), t)\, dt \\ &= rY(t)\, dt + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \big(a_{ij}(\tilde{X}(t), t) - \tilde{a}_{ij}(\tilde{X}(t), t)\big) \frac{\partial^2 F}{\partial x_i \partial x_j}(\tilde{X}(t), t)\, dt, \end{aligned}$$

where we have used that F solves the Black-Scholes equation (4) with diffusion matrix β. Now, assume that the hedger overestimates the volatility, i.e. that $\beta(x,t)$ dominates $\tilde{\beta}(x,t)$ for all x and t. Then, since F is convex, the last double sum is non-negative. Thus $Y(0) = 0$ and $Y(T_0) \geq 0$. It follows that the final value $V(T_0)$ of the hedger's portfolio satisfies $V(T_0) \geq F(\tilde{X}(T_0), T_0) = g(\tilde{X}(T_0))$, so the hedger superreplicates. The case of underestimation of the volatility is similar. □

Theorem 2.6 together with Theorem 3.1 shows that if a hedger wants to be sure to superreplicate a convex claim on several underlying assets, then he should overestimate the true diffusion matrix $\tilde{\beta}$ with a diffusion matrix β such that $\beta_{ij}(x,t) = x_i \sigma_{ij}(t)$ for some functions $\sigma_{ij}(t)$. Note that there is no assumption on the true diffusion matrix to be LCP; what matters is that the hedger should use a model which is LCP. Actually, it is not essential that the true diffusion matrix $\tilde{\beta}$ is a function of time and the current stock prices. In fact, the theorem is true also in the case when $\tilde{\beta}$ is some adapted process dominated by $\beta(\tilde{X}(t), t)$ for all t almost surely. This is of course in analogy with the one-dimensional case, see El Karoui, Jeanblanc-Picque and Shreve (1998).

Remark By considering an American option as the limit of a sequence of European options (for the one-dimensional case, see El Karoui, Jeanblanc-Picque and Shreve (1998) or Ekström (2003)) it is clear that a model which gives convex European option prices also gives convex American option prices. For American claims, an option writer needs to be sure that the value V of his hedging portfolio at each instant t satisfies $V_t \geq g(\tilde{X}(t))$. It can be shown that this is indeed the case if the option writer overestimates the diffusion matrix with a convexity preserving model.

Remark Janson and Tysk (2003b) show that if $A \geq \tilde{A}$ as quadratic forms, and if either A or $\tilde{A}$ is convexity preserving, then the corresponding solutions F and $\tilde{F}$ satisfy the inequality $F(x,t) \geq \tilde{F}(x,t)$ for all x and t. Thus, if one knows that the true diffusion matrix can be bounded both above and below by some diffusion matrices corresponding to geometric Brownian motions, then one also has upper and lower bounds for the price of an option. This can also be seen as a consequence of Theorem 3.1.

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