

Spread-out percolation in \mathbb{R}^d

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Abstract

Fix $d \geq 2$, and let X be either \mathbb{Z}^d or the points of a Poisson process in \mathbb{R}^d of intensity 1. Given parameters r and p , join each pair of points of X within distance r independently with probability p . This is the simplest case of a ‘spread-out’ percolation model studied by Penrose [8], who showed that, as $r \rightarrow \infty$, the average degree of the corresponding random graph at the percolation threshold tends to 1, i.e., the percolation threshold and the threshold for criticality of the naturally associated branching process approach one another. Here we show that this result follows immediately from a general result of [3] on inhomogeneous random graphs.

1 Introduction and results

The study of percolation and the study of the emergence of the giant component in a random graph are closely related topics. In both cases, one can phrase the key question as follows: ‘As $n \rightarrow \infty$, for what parameters does a certain n -vertex random graph have with high probability a component of order $\Theta(n)$?’ The key difference is that in percolation there is some global geometric structure: for example, the graph might be a random subgraph of \mathbb{Z}^d , or of a finite portion of \mathbb{Z}^d , or it might be the graph formed by the points of a Poisson process in \mathbb{R}^d , joined if they are within a certain distance r . In the classical theory of random graphs, the graph model, $G(n, p)$, is totally structureless: each pair of vertices is joined independently with probability p , with $p = c/n$, c constant, being the appropriate normalization for the very simplest results on the giant component. Of course, each instance of $G(n, p)$ has a rich structure, but the model does not. Many authors have studied inhomogeneous random graphs in which the model does have some structure, but the behaviour of these random graphs is still much closer to the behaviour of $G(n, p)$ than to percolation. In particular, it was

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shown in [3] that, for a very general inhomogeneous model with independence between the edges, the threshold for the emergence of a giant component is given by the point at which a certain (multi-type) branching process becomes critical, generalizing the classical result for $G(n, p)$. In contrast, in percolation there are only a few models where the exact threshold can be determined: unless one of a few very special things happens, it seems to be impossible to give a simple formula for the critical point.

Penrose [8] determined the asymptotic behaviour of the critical parameters for a certain natural family of two-parameter percolation models with global geometric structure. A special case of this result, as in the present abstract but with $X = \mathbb{Z}^d$ and distance replaced by ℓ_∞ -distance, was proved independently by Bollobás and Kohayakawa [4]. Our aim in this paper is to show that Penrose's result is a simple consequence of the main (and very general) results of [3].

Writing μ for the Lebesgue measure on \mathbb{R}^d , we say that a set $A \subset \mathbb{R}^d$ is a μ -continuity set if A is μ -measurable and $\mu(\partial A) = 0$, where ∂A is the boundary of A . Note that we take the d -dimensional measure of the boundary.

Throughout, the vertex set of our (infinite) random graph will be a deterministic or random set $X \subset \mathbb{R}^d$, of 'density' one. The dimension $d \geq 2$ will be fixed throughout. The natural examples are \mathbb{Z}^d (perhaps with each point displaced by a small random amount), any other (suitably scaled) lattice, or a Poisson process of intensity 1. Formally, we require two assumptions on the distribution of the random discrete set X : the *density assumption* that if $A \subset \mathbb{R}^d$ is a bounded μ -continuity set, then for any sequence $\mathbf{v}_r \in \mathbb{R}^d$ we have

$$\frac{1}{r^d} |X \cap (rA + \mathbf{v}_r)| \xrightarrow{\mathbb{P}} \mu(A)$$

as $r \rightarrow \infty$, where $\xrightarrow{\mathbb{P}}$ denotes convergence in probability, $|\cdot|$ denotes the cardinality of a set, and $rA + \mathbf{v}_r = \{r\mathbf{x} + \mathbf{v}_r : \mathbf{x} \in A\}$. The second assumption we require is that well separated regions are independent: there is a constant D such that whenever $A, B \subset \mathbb{R}^d$ are measurable sets at Euclidean distance at least D from each other, then the set-valued random variables $X \cap A$ and $X \cap B$ are independent.

To state our form of Penrose's result, we consider a function $\phi : \mathbb{R}^d \rightarrow [0, \infty)$ satisfying the following assumptions: ϕ is *symmetric*, in that $\phi(\mathbf{x}) = \phi(-\mathbf{x})$, bounded, continuous almost everywhere, and satisfies $\int_{\mathbb{R}^d} \phi(\mathbf{x}) d\mu(\mathbf{x}) = 1$. In addition, for convenience we shall assume that ϕ is strictly positive in a neighbourhood of the origin. To obtain the example described in the abstract, we choose for ϕ the function that is 1 on a ball of volume 1 centred at the origin, and 0 otherwise.

Given a 'scale' r and a 'degree parameter' $\lambda > 0$, we form the random graph $G = G(X) = G_{r, \lambda}(X)$ with vertex set X as follows: given X , for each pair $\mathbf{x}, \mathbf{y} \in X$ join \mathbf{x} and \mathbf{y} with probability

$$\min \left\{ \lambda r^{-d} \phi \left(\frac{\mathbf{x} - \mathbf{y}}{r} \right), 1 \right\},$$

independently of all other pairs. Note that $\int_{\mathbb{R}^d} r^{-d} \phi(\mathbf{x}/r) d\mu(\mathbf{x}) = 1$, so, at least in the Poisson case, the average degree of a vertex of $G_{r,\lambda}(X)$ tends to λ as $r \rightarrow \infty$ with λ fixed.

Theorem 1. *Let X and ϕ satisfy the conditions above with $d \geq 2$, and let $\lambda > 1$ be fixed. If r is large enough, then with probability 1 the graph $G_{r,\lambda}(X)$ has an infinite component.*

The basic idea is to show that the neighbourhood of a vertex of $G(X)$ is ‘tree-like’, and can be approximated by a Galton-Watson branching process where each particle has a Poisson number of children with mean λ . We shall approximate the local structure of $G_{r,\lambda}(X)$ using the results of [3], and then deal with the global structure using the concept of k -independent percolation.

2 Proofs

Throughout this section, $d \geq 2$, D , ϕ , and the distribution of X will be fixed. Much of the time, we shall rescale the model in the following natural way: let $X/r = \{\mathbf{x}/r : \mathbf{x} \in X\}$. The (rescaled) graph $G'(X) = G'_{r,\lambda}(X)$ has vertex set X/r , and each pair $\mathbf{x}, \mathbf{y} \in X/r$ is joined independently with probability $\min\{\lambda r^{-d} \phi(\mathbf{x} - \mathbf{y}), 1\}$.

To prove Theorem 1 we shall need two results. The first is a simple observation concerning locally dependent percolation.

A *bond percolation measure* on \mathbb{Z}^d is a measure on the set of assignments of a *state, open or closed*, to each edge of \mathbb{Z}^d , the graph with vertex set \mathbb{Z}^d in which vertices at Euclidean distance 1 are adjacent. Such a measure is *k-independent* if, for every pair S, T of sets of edges of \mathbb{Z}^d at graph distance at least k , the states of the edges in S are independent of the states of the edges in T . When $k = 1$, the separation condition is exactly that no edge of S shares a vertex with an edge of T .

Measures of this type arise very naturally in static renormalization arguments, and comparisons between k -independent measures and product measures (or arguments amounting to such comparisons) have been considered by many people; see Liggett, Schonmann and Stacey [6] and the references therein.

Lemma 2. *Let $d \geq 2$ and $k \geq 1$ be fixed. There is a $p_0 = p_0(k) < 1$ such that in any k -independent bond percolation measure on \mathbb{Z}^d satisfying the additional condition that each edge is open with probability at least p_0 , with probability 1 there is an infinite path consisting of open edges.*

This result is a special case of the very general main result of [6]; in the form above, it is essentially trivial. Note that without loss of generality we may take $d = 2$, as \mathbb{Z}^d contains \mathbb{Z}^2 as a subgraph. Here the value of p_0 is irrelevant, but in many contexts this value is very important. For $k = 1$, the best bound known is due to Balister, Bollobás and Walters [2], who showed that one can take $p_0(1) = 0.8639$.

The second result we shall need is a special case of the main result of [3]; to state this we recall some definitions from [3].

Let \mathcal{S} be a separable metric space and μ a Borel measure on \mathcal{S} with $0 < \mu(\mathcal{S}) < \infty$. In this paper, \mathcal{S} will be either a cube in \mathbb{R}^d of side-length L , or the union of two such cubes sharing a face, and μ will be Lebesgue measure.

Let $\rho > 0$ be a ‘density parameter’ that will tend to infinity. Here we shall fix \mathcal{S} and take $\rho = r^d$ with $r \rightarrow \infty$. For each ρ , let V_ρ be a deterministic or random finite subset of \mathcal{S} . The triple $(\mathcal{S}, \mu, (V_\rho))$ forms a *generalized vertex space* if \mathcal{S} and μ satisfy the conditions above, and

$$\frac{1}{\rho} |V_\rho \cap A| \xrightarrow{P} \mu(A) \tag{1}$$

as $\rho \rightarrow \infty$, for each μ -continuity set $A \subset \mathcal{S}$. Here, we shall take $V_\rho = (X/r) \cap \mathcal{S}$, so the condition above is simply that

$$r^{-d} |(X/r) \cap A| \xrightarrow{P} \mu(A),$$

which follows from our density assumption on X .

A *kernel* on (\mathcal{S}, μ) is a symmetric, non-negative, Borel-measurable function on $\mathcal{S} \times \mathcal{S}$; we shall consider kernels $\kappa = \kappa_{\phi, \mathcal{S}, \lambda}$ given by $\kappa(\mathbf{x}, \mathbf{y}) = \lambda \phi(\mathbf{x} - \mathbf{y})$ for $\mathbf{x}, \mathbf{y} \in \mathcal{S}$. Note that as ϕ is continuous almost everywhere on \mathbb{R}^d , the kernel κ is continuous almost everywhere on $\mathcal{S} \times \mathcal{S}$.

A kernel κ is *irreducible* if, whenever A is a measurable subset of \mathcal{S} with $\kappa = 0$ a.e. on $A \times (\mathcal{S} \setminus A)$, then $\mu(A) = 0$ or $\mu(\mathcal{S} \setminus A) = 0$. The assumption that ϕ is strictly positive in a neighbourhood of the origin ensures that the kernel $\kappa = \kappa_{\phi, \mathcal{S}, \lambda}$ is irreducible. (To see this, pick a $\delta > 0$ such that $\phi(\mathbf{x}) > 0$ on $B_{2\delta}(0) = \{\mathbf{x} : \|\mathbf{x}\| < 2\delta\}$. If $A \subset \mathcal{S}$ with $0 < \mu(A) < \mu(\mathcal{S})$, then, since \mathcal{S} is connected, there is some $\mathbf{x} \in \mathcal{S}$ such that both A and $\mathcal{S} \setminus A$ meet $B_\delta(\mathbf{x})$ in sets of positive measure. Since κ is positive on $B_\delta(\mathbf{x}) \times B_\delta(\mathbf{x})$ it follows that κ is irreducible.)

Given a kernel κ on (\mathcal{S}, μ) , let T_κ be the integral operator on (\mathcal{S}, μ) with kernel κ , defined by

$$(T_\kappa f)(x) = \int_{\mathcal{S}} \kappa(x, y) f(y) d\mu(y),$$

for any (measurable) function $f : \mathcal{S} \rightarrow \mathbb{R}$ such that the integral is defined (finite or $+\infty$) for a.e. x . For the bounded κ we consider, $T_\kappa f$ is defined for every $f \in L^2 = L^2(\mathcal{S}, \mu)$, and the operator T_κ maps L^2 into itself. Let $\|T_\kappa\|$ be the operator norm of $T_\kappa : L^2 \rightarrow L^2$.

Theorem 3 below is a special case of parts of the main results, Theorem 3.1 (part (iii)) and Theorem 3.6, of [3]. In Theorem 3, $G_\rho(\kappa)$ denotes the random graph with vertex set V_ρ where, given V_ρ , each pair $\{\mathbf{x}, \mathbf{y}\}$ of vertices is joined independently with probability $\min\{\kappa(\mathbf{x}, \mathbf{y})/\rho, 1\}$. We write $C_i(G)$ for the number of vertices in the i th largest component of a graph G .

Theorem 3. *Let $(\mathcal{S}, \mu, (V_\rho))$ be a generalized vertex space, and let κ be an irreducible, bounded, almost everywhere continuous kernel on (\mathcal{S}, μ) . If $\|T_\kappa\| > 1$, then $C_1(G_\rho(\kappa))/\rho \xrightarrow{\mathbb{P}} a$ as $\rho \rightarrow \infty$, for some constant $a = a(\mathcal{S}, \mu, \kappa) > 0$, while $C_2(G_\rho(\kappa))/\rho \xrightarrow{\mathbb{P}} 0$.*

For comparison with the statement in [3], note that the additional condition there, that κ be ‘graphical’ on $(\mathcal{S}, \mu, (V_\rho))$, is not needed here. Indeed, taking $A = \mathcal{S}$ in (1), we have $\rho^{-1}|V_\rho| \xrightarrow{\mathbb{P}} \mu(\mathcal{S}) < \infty$. In particular, $\rho^{-1}|V_\rho| \leq 2\mu(\mathcal{S})$ with probability $1 - o(1)$. Redefining V_ρ to be empty if this inequality is not satisfied, which changes $G_\rho(\kappa)$ on a set of measure $o(1)$ and hence does not affect the conclusion of Theorem 3, we obtain a new vertex space with $\rho^{-1}|V_\rho|$ bounded. But now convergence in probability in (1) implies convergence of all moments. As noted in [3] (Remarks 2.8 and 8.2), under this condition any bounded, almost everywhere continuous kernel is graphical. [This argument, which applies to all ‘with probability $1 - o(1)$ ’ results in [3], shows that the definition of graphical there should perhaps be modified not to refer to expectation; this is purely a matter of convenience, since one can always modify the model on events with probability $o(1)$ as here.]

Proof of Theorem 1. Fix $\lambda > 1$ throughout. Let us rescale the vertex set X as above, considering the graph $G'(X)$ with vertex set X/r .

Let L be a (large) constant to be chosen later. Let $\mathcal{S}_1 = [0, L]^d$ and $\mathcal{S}_2 = [0, 2L] \times [0, L]^{d-1}$. With $\mathcal{S} = \mathcal{S}_1$ or $\mathcal{S} = \mathcal{S}_2$, let $\kappa = \kappa_{\phi, \mathcal{S}, \lambda}$ be defined as above, and let f be the constant function on \mathcal{S} taking value 1. Then

$$\begin{aligned} (T_\kappa f)(\mathbf{x}) &= \int_{\mathbf{y} \in \mathcal{S}} \lambda \phi(\mathbf{y} - \mathbf{x}) d\mu(\mathbf{y}) \\ &= \lambda \int_{\mathbf{y} \in \mathcal{S} - \mathbf{x}} \phi(\mathbf{y}) d\mu(\mathbf{y}). \end{aligned}$$

Since $\int_{\|\mathbf{y}\| \leq K} \phi(\mathbf{y}) \rightarrow \int_{\mathbb{R}^d} \phi(\mathbf{y}) = 1$ as $K \rightarrow \infty$, there is a constant K such that $(T_\kappa f)(\mathbf{x}) \geq (1 + \lambda)/2 > 1$ if $\mathbf{x} \in \mathcal{S}$ is at distance at least K from the boundary of \mathcal{S} . It follows that if L is large enough, then $\|T_\kappa f\|_2 > \|f\|_2$, so $\|T_\kappa\| > 1$. From now on we choose L large enough that $\|T_\kappa\| > 1$ for $\mathcal{S} = \mathcal{S}_1, \mathcal{S}_2$.

Setting $\rho = r^d$, let $V_\rho = (X/r) \cap \mathcal{S}$ be the set of vertices of $G'(X)$ lying in \mathcal{S} . The graph $G_\rho(\kappa)$ considered in Theorem 3 has exactly the distribution of $G'[\mathcal{S}]$, the subgraph of $G'(X)$ induced by vertices in \mathcal{S} . Hence, taking $\mathcal{S} = \mathcal{S}_1$ and applying Theorem 3, there is a constant $a = a(\mathcal{S}_1, \mu, \kappa) > 0$ such that, as $r \rightarrow \infty$,

$$\mathbb{P}(C_1(G'[\mathcal{S}_1]) \leq a\rho) \rightarrow 0. \quad (2)$$

Taking $\mathcal{S} = \mathcal{S}_2$ and applying Theorem 3 again, we have

$$\mathbb{P}(C_2(G'[\mathcal{S}_2]) \geq a\rho) \rightarrow 0.$$

For each $\mathbf{v} = (v_1, \dots, v_d) \in \mathbb{Z}^d$, let $\mathcal{S}_{\mathbf{v}} = \prod_{i=1}^d [v_i L, v_i L + L)$. Also, for each edge $e = \{\mathbf{v}, \mathbf{w}\}$ of \mathbb{Z}^d , let $\mathcal{S}_e = \mathcal{S}_{\mathbf{v}} \cup \mathcal{S}_{\mathbf{w}}$. We claim that $\mathbb{P}(C_1(G'[\mathcal{S}_{\mathbf{v}}]) \leq a\rho) \rightarrow 0$

uniformly in \mathbf{v} . In the Poisson case, this is immediate from (2), since the translation invariance of the model implies that the relevant probability is independent of \mathbf{v} . In the general case, a little technical argument is needed: taking \mathbf{v}_r as any sequence of points of \mathbb{Z}^d , and defining V_ρ , $\rho = r^d$, by translating $(X/r) \cap \mathcal{S}_{\mathbf{v}_r}$ through $-L\mathbf{v}_r$, our density assumption on X implies that $(\mathcal{S}_1, \mu, (V_\rho))$ is a vertex space. Hence, $\mathbb{P}(C_1(G'[\mathcal{S}_{\mathbf{v}_r}]) \leq a\rho) \rightarrow 0$. As the sequence \mathbf{v}_r is arbitrary, this is the same as uniform convergence. Similarly, $\mathbb{P}(C_2(G'[\mathcal{S}_e]) \geq a\rho) \rightarrow 0$ uniformly in the edges e of \mathbb{Z}^d .

For each edge $e = \{\mathbf{v}, \mathbf{w}\}$ of the graph \mathbb{Z}^d , let $Y(e)$ be the event that $C_1(G'[\mathcal{S}_{\mathbf{v}}]), C_1(G'[\mathcal{S}_{\mathbf{w}}]) > a\rho$, while $C_2(G'[\mathcal{S}_e]) < a\rho$. We have shown that $\mathbb{P}(Y(e)) \rightarrow 1$ uniformly in e as $r \rightarrow \infty$. Let $p_0 = p_0(d+1)$ be the constant in Lemma 2, taking $k = d+1$, and choose r large enough that $\mathbb{P}(Y(e)) \geq p_0$ for every edge e of \mathbb{Z}^d . We shall also assume that $r > D/L$, where D is the constant appearing in our independence assumption on X .

Define a bond percolation measure on \mathbb{Z}^d by declaring the edge e to be open if $Y(e)$ holds. For edges e, f at graph distance at least $d+1$, the sets \mathcal{S}_e and \mathcal{S}_f are separated by a Euclidean distance of at least $L > D/r$. Hence, from our assumptions on X and the independence of edges in the graph, the graphs $G'[\mathcal{S}_e]$ and $G'[\mathcal{S}_f]$ are independent, and so are the events $Y(e)$ and $Y(f)$. This observation also holds for sets of edges at graph distance at least $d+1$, so the bond percolation measure we have defined is $(d+1)$ -independent. Hence, by Lemma 2, with probability 1 there is an infinite open path, i.e., an infinite sequence $\mathbf{v}_1, \mathbf{v}_2, \dots$ such that $Y(\{\mathbf{v}_i, \mathbf{v}_{i+1}\})$ holds for each $i \geq 1$.

Let \mathcal{C}_i be the largest component of $G'[\mathcal{S}_{\mathbf{v}_i}]$, chosen arbitrarily if there is a tie. As $Y(e)$ holds for $e = \{\mathbf{v}_i, \mathbf{v}_{i+1}\}$, \mathcal{C}_i and \mathcal{C}_{i+1} have more than $a\rho$ vertices. As $G'[\mathcal{S}_{\mathbf{v}_j}]$ is a subgraph of $G'[\mathcal{S}_e]$ for $j = i, i+1$, each of $\mathcal{C}_i, \mathcal{C}_{i+1}$ is contained entirely within some component of $G'[\mathcal{S}_e]$. But as $Y(e)$ holds, $G'[\mathcal{S}_e]$ has at most one component with more than $a\rho$ vertices. Hence \mathcal{C}_i and \mathcal{C}_{i+1} are connected in $G'[\mathcal{S}_e]$, and thus in $G'(X)$. It follows that, with probability 1, $G'(X)$ and hence $G(X)$ contains an infinite path, completing the proof of Theorem 1. \square

The idea of combining local information (here the events $Y(e)$) to deduce global information, in particular via comparison with a product measure, is natural and has been used many times. Here the events we use are as in Balister, Bollobás and Walters [2]. For an earlier application of related ideas in a more complicated context, see Pisztor [9].

3 Discussion

Theorem 1 is the main part of the related results of Penrose [8], showing that the threshold $\lambda(r)$ for percolation to occur in $G(X) = G_{r,\lambda}(X)$ approaches 1 as $r \rightarrow \infty$. Note that for $X = \mathbb{Z}^d$ or X Poisson, the cases considered in [8], the existence of a threshold for each r follows from monotonicity of the model in λ and Kolmogorov's 0-1 law: constructing X and then $G(X)$ from appropriate independent random variables, the event that $G(X)$ has an infinite component

is a tail event, and so has probability 0 or 1 for any fixed λ . Hence there is a (perhaps infinite) $\lambda(r)$ such that this probability is 0 for $\lambda < \lambda(r)$ and 1 for $\lambda > \lambda(r)$.

The condition of Theorem 1 that ϕ be positive in a neighbourhood of the origin is not essential: it was imposed here for convenience, to avoid the complication of dealing with reducible kernels in the proof. This condition is not imposed in [8]. On the other hand, the stronger conditions on ϕ in [8] (or at least some stronger conditions) are needed for the ‘easy’ part of the result, that percolation does not occur if $\lambda < 1$. This result is trivial in ‘nice’ cases (see below), but fails under the conditions of Theorem 1, for example if $X = \mathbb{Z}^d$ and ϕ is large at all points with integer coordinates and small elsewhere.

For completeness, we give a short proof of the reverse bound in simple cases; for proofs under slightly different assumptions see Penrose [8] and Meester, Penrose and Sarkar [7].

Lemma 4. *Let X be either \mathbb{Z}^d or a Poisson process in \mathbb{R}^d of intensity 1, and let ϕ satisfy the conditions of Theorem 1. If $X = \mathbb{Z}^d$, suppose in addition that ϕ has bounded support. If $\lambda < 1$ is fixed and r is large enough, then with probability 1 every component of $G(X)$ is finite.*

Proof. We start with the Poisson case, working with the rescaled graph $G'(X)$. Let U be a fixed unit cube in \mathbb{R}^d . From elementary properties of Poisson processes, the expected number E_n of paths $(\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n)$ in $G'(X)$ with $\mathbf{x}_0 \in U$ is given by

$$\int_{(\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_n) \in U \times (\mathbb{R}^d)^n} r^{d(n+1)} \prod_{i=1}^n \lambda r^{-d} \phi(\mathbf{x}_i - \mathbf{x}_{i-1}),$$

where the integral is with respect to $(d(n+1))$ -dimensional Lebesgue measure, the factor $r^{d(n+1)}$ is due to the density r^d of the (rescaled) Poisson process on \mathbb{R}^d , and each factor $\lambda r^{-d} \phi(\mathbf{x}_i - \mathbf{x}_{i-1})$ is an edge probability. As $\int_{\mathbb{R}^d} \phi(\mathbf{x}) d\mu(\mathbf{x}) = 1$, we thus have

$$E_n = r^{d(n+1)} \mu(U) \left(\lambda r^{-d} \int_{\mathbb{R}^d} \phi \right)^n = r^d \lambda^n.$$

By assumption, $\lambda < 1$, so $E_n \rightarrow 0$ as $n \rightarrow \infty$ with r fixed. Since a Poisson process has (with probability 1 or by definition) no accumulation points, every vertex of $G'(X)$ has finite degree. Thus the probability that $G'(X)$ has an infinite component meeting U is at most the probability that $G'(X)$ contains a path of length n starting in U , and hence at most E_n . So with probability 1 all components of $G'(X)$ meeting U are finite. Considering countably many choices for U , the result follows.

For the case $X = \mathbb{Z}^d$, as ϕ is bounded, almost everywhere continuous, and has bounded support, we have $r^{-d} \sum_{\mathbf{x} \in \mathbb{Z}^d} \phi(\mathbf{x}/r) \rightarrow \int_{\mathbb{R}^d} \phi(x) = 1$. (In fact, all we need is that ϕ is directly Riemann integrable, as assumed in [8].) Hence, choosing r large enough, we may assume that $\lambda r^{-d} \sum_{\mathbf{x} \in \mathbb{Z}^d} \phi(\mathbf{x}/r) < 1$. Thus $G(X)$, the unrescaled graph, is a random graph (on \mathbb{Z}^d) where edges are

independent, and the expected degree of every vertex is at most $c < 1$. Hence the expected number of paths of length n starting at a given vertex is at most c^n , and it follows as above that with probability 1 every component of $G(X)$ is finite. \square

It follows from Theorem 1, Lemma 4 and the remarks on the existence of a threshold that, under the assumptions of Lemma 4, the critical value $\lambda(r)$ for percolation to occur tends to 1 as $r \rightarrow \infty$. A slightly more general version of this result is the main result of [8].

Penrose [8] also shows that with $\lambda > 1$ fixed and $r \rightarrow \infty$, the probability that the origin (added as an extra point if X is Poisson) is in an infinite component tends to $\psi(\lambda)$, the survival probability of a Galton-Watson branching process in which each particle has a Poisson number of children with mean λ . The lower bound in this result also follows from the results of [3], which relate the size of the giant component in an inhomogeneous random graph to a branching process. Again, the upper bound requires stronger conditions.

We close by noting that the results of Penrose considered here are similar to, but distinct from, results for ‘annulus percolation’ due to Balister, Bollobás and Walters [1] and Franceschetti, Booth, Cook, Meester and Bruck [5]. In both cases, the planar percolation process locally looks like a tree with constant average degree, and the result is that the average degree at the threshold approaches 1, but the methods needed to show this are rather different in the two cases. (A similar comment applies to the results of Meester, Penrose and Sarkar [7] and of [1] concerning a related model whose dimension tends to infinity.) There may well be a common generalization of these results.

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