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THE DENSITY OF THE ISE AND LOCAL LIMIT LAWS FOR EMBEDDED TREES

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It has been known for a few years that the occupation measure of several models of embedded trees converges, after a suitable normalization, to the random measure called ISE (Integrated SuperBrownian Excursion). Here, we prove a local version of this result: ISE has a (random) Hölder continuous density, and the *vertical profile* of embedded trees converges to this density, at least for some such trees.

As a consequence, we derive a formula for the distribution of the density of ISE at a given point. This follows from earlier results by Bousquet-Mélou on convergence of the vertical profile at a fixed point.

We also provide a recurrence relation defining the moments of the (random) moments of ISE.

1. Introduction. We consider some families of random labelled trees; the labels will be integers (positive or negative). Our main case is *binary trees*, in which each node has a right and/or a left child, or no child at all (Figure 1). The label of a node is the difference between the number of right steps and the number of left steps occurring in the path from the root to the node. In particular, the root has label 0, and the labels of two adjacent nodes differ by ± 1. Note that the label of each node is simply its abscissa, if we draw the tree in the plane in such a way that the right [left] child of a node lies one unit to the right [left] of its parent. We call this the *natural labelling* of a binary tree.

Given a labelled tree T, let $X(j;T)$ be the number of nodes in T with label j; the sequence $(X(j;T))_{j=-\infty}^{\infty}$ is the *vertical profile* of the tree (Figure 1).

Let T_n be a random binary tree with n nodes with the uniform distribution, and let $X_n(j) := X(j;T_n)$ be its vertical profile. It was shown by Marckert [23, Theorem 5] that the (random) distribution of the labels in the tree converges, after appropriate normalization, to the ISE (integrated superbrownian excursion) introduced by Aldous [4], see also [7]. The ISE is a random probability measure; to emphasize this we will usually write it as μ_{ISE}. (Actually, the result in [23] is stated for *complete binary trees*, i.e. binary trees where each node has either 0 or 2 children, but the result transfers immediately by considering internal nodes only; see at the end of Section 8

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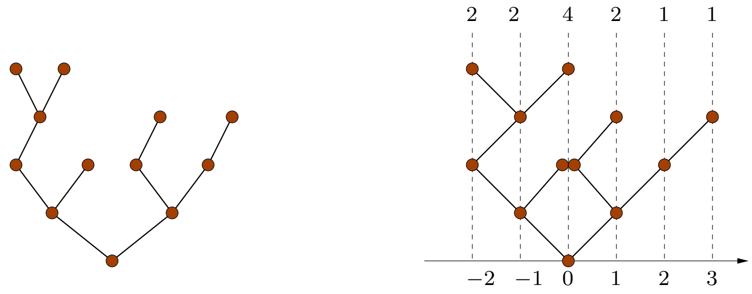

FIG 1. *A binary tree having vertical profile* $[2, 2, 4; 2, 1, 1]$.

for details.) Marckert's result can be stated as follows, where $\gamma := 2^{-1/4}$, $\ell(v)$ denotes the label of v, and δ_x is the Dirac measure at x,

$$\frac{1}{n}\sum_{j=-\infty}^{\infty} X_n(j)\delta_{\gamma n^{-1/4} j} = \frac{1}{n}\sum_{v\in T_n} \delta_{\gamma n^{-1/4}\ell(v)} \xrightarrow{\mathrm{d}} \mu_{\mathrm{ISE}}, \qquad n\to\infty, \tag{1.1}$$

with convergence in the space of probability measures on $\mathbb{R}$. For complete binary trees, the result is the same, except that now $\gamma = 1$ (and n has to be odd).

Our first main result is a local version of (1.1), showing that the vertical profile of random binary trees, properly normalized, converges to the density f_{ISE} of μ_{ISE}; see Section 3 for details. Our second result consists of a recurrence relation that characterizes the joint law of the moments of the ISE.

REMARK 1.1. Different normalizations of μ_{ISE} are used in the literature. We use the normalization of [4], also used by e.g. [23]. The normalization in [7, 24] differs by a scale factor $2^{1/4}$.

Our local limit result actually holds for other families of labelled (or: embedded) trees too. Indeed, the random measure μ_{ISE} arises naturally as a limit for embedded trees in the following way [4]. Let T_n be a random conditioned Galton–Watson tree with n nodes, i.e. a random tree obtained as the family tree of a Galton–Watson process conditioned on a given total population of n. (See e.g. [2, 9] for details, and recall that this includes e.g. binary trees, complete binary trees, plane trees and labelled (=Cayley) trees. These random trees are also known as simply generated trees.) The Galton–Watson process is defined using an offspring distribution; let ξ denote a random variable with this distribution. We assume, as usual, $\mathbb{E}\,\xi = 1$ (the Galton–Watson process is critical) and $0 < \sigma_\xi^2 := \operatorname{Var}\xi < \infty$. Assign i.i.d.

random variables η_e to the edges of T_n. We regard η_e as the displacement from one endpoint of the edge e to the other, in the direction from parent to child; this gives a labelling of the nodes such that the root has label 0 and each other node v has label $\ell(v) := \ell(v') + \eta_{vv'}$, where v' is the parent of v. For the purposes of this paper, we assume η_e to be integer valued. We further assume $\mathbb{E}\,\eta_e = 0$ and $0 < \sigma_\eta^2 := \operatorname{Var}\eta_e < \infty$. Define X and X_n as was done above for binary trees with their natural labelling. Then (1.1) holds, with $\gamma := \sigma_\eta^{-1}\sigma_\xi^{1/2}$ [4], see also [17].

We conjecture that a local version of (1.1) holds in this generality, provided η_e is not supported on a subgroup $d\mathbb{Z}$ of the integers with $d \geq 2$, but we will only prove this for two special cases, viz. random plane trees with η_e uniformly distributed on either $\{\pm 1\}$ or $\{-1, 0, 1\}$, see Theorem 3.6.

We state in Section 2 some properties of the (random) density function f_{ISE}, in particular that it exists. The proofs are given in Section 6 after some preliminaries on the Brownian snake and the Brownian CRT (continuum random tree) in Section 5. Our results on the local limit law are stated in Section 3 and proved in Sections 7–10. Some further computations of (mixed) moments of the density $f_{\mathrm{ISE}}(\lambda)$ are given in Section 11. Our results on moments of μ_{ISE} are stated in Section 4 and proved in Section 12.

All unspecified limits below are as $n \to \infty$. We will use C and c with various subscripts to denote various positive constants, not depending on n or other variables; for constants depending on a parameter we use $C(a)$ and so on.

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2. The density of the ISE. It is no surprise that the random measure μ_{ISE} is absolutely continuous; the following theorem may well be known to experts, but we have not found an explicit reference. (Related results for super-Brownian motion have been given by [19, 26, 30]. It seems to be possible but non-trivial to derive the existence of a density for ISE from these results.) We give a proof in Section 6.

Theorem 2.1. ISE *has a Hölder continuous density. In other words, there exists a continuous stochastic process* $f_{\mathrm{ISE}}(x)$, $-\infty < x < \infty$, *such that*

$d\mu_{\mathrm{ISE}}(x) = f_{\mathrm{ISE}}(x)\,dx$. *Moreover, the random function* $f_{\mathrm{ISE}}(x)$ *has a.s. the following properties:*

(i) f_{ISE} *has compact support:* $\sup\{|x| : f_{\mathrm{ISE}}(x) \neq 0\} < \infty$*;*
(ii) f_{ISE} *is Hölder(α)-continuous for every* $\alpha < 1$*;*
(iii) f_{ISE} *has a derivative* $f'_{\mathrm{ISE}}(x)$ *a.e. and in the distribution sense, and* $f'_{\mathrm{ISE}} \in L^p(\,dx)$ *for every* p *with* $2 \leq p < \infty$.

Of course, the support of f_{ISE} is random; (i) says that there exists a random $M < \infty$ such that $f_{\mathrm{ISE}}(x) = 0$ for $|x| > M$, but no deterministic M will do.

REMARK 2.2. More precisely, the proof in Section 6 shows that f_{ISE} belongs to the generalized Sobolev space $L^{2,\alpha}$ for any $\alpha < 3/2$. Loosely speaking, f_{ISE} thus has "α derivatives in L^2" for every real $\alpha < 3/2$.

Parts (ii) and (iii) of Theorem 2.1 come close to showing that f_{ISE} has a continuous derivative, but we have not been able to prove it. Indeed, it seems likely that the (fractional) derivatives in L^2 asserted by Remark 2.2 are continuous. Hence we make the following conjecture.

CONJECTURE 2.3. *The density* f_{ISE} *has a.s. a continuous derivative, but not a second derivative.*

The marginal distributions of f_{ISE}, i.e. the distributions of $f_{\mathrm{ISE}}(\lambda)$ for fixed λ, will be described in Corollaries 3.3 and 3.4. Moments and mixed moments of $f_{\mathrm{ISE}}(\lambda)$ will be computed in Section 11.

3. Local limit results. Our main result is the following local limit result for naturally embedded random binary trees, conjectured in [7].

We let $\bar{X}_n(x)$ denote the function obtained by extending $X_n(j)$ to arbitrary real arguments by linear interpolation; thus $\bar{X}_n(j) = X_n(j)$ for every integer j, and $\bar{X}_n$ is linear on each interval $[j, j+1]$.

$C_0(\mathbb{R})$ denotes, as usual, the Banach space of continuous functions on $\mathbb{R}$ that tend to 0 at $\pm\infty$. We equip $C_0(\mathbb{R})$ with the usual uniform topology defined by the supremum norm.

Recall that we have defined the constant γ as $2^{-1/4}$ for binary trees and 1 for complete binary trees.

THEOREM 3.1. *Consider random binary trees or random complete binary trees with their natural labelling. Then, as* $n \to \infty$,

$$\frac{1}{n}\gamma^{-1} n^{1/4} \bar{X}_n(\gamma^{-1} n^{1/4} x) \xrightarrow{\mathrm{d}} f_{\mathrm{ISE}}(x), \tag{3.1}$$

in the space $C_0(\mathbb{R})$ with the usual uniform topology. Equivalently,

$$n^{-3/4}\bar{X}_n(n^{1/4}x) \xrightarrow{\mathrm{d}} \gamma f_{\mathrm{ISE}}(\gamma x). \tag{3.2}$$

Note that the functions on the left-hand sides of (3.1) and (3.2) are density functions, i.e. non-negative functions with integral 1. Proofs will be given in Sections 7–10.

COROLLARY 3.2. *For random binary trees or random complete binary trees with their natural labelling, if $n \to \infty$ and $j_n/n^{1/4} \to x$, where $-\infty < x < \infty$, then $n^{-3/4}X_n(j_n) \xrightarrow{\mathrm{d}} \gamma f_{\mathrm{ISE}}(\gamma x)$.*

It follows by combining this with results in Bousquet-Mélou [7] that the marginal distributions of f_{ISE} are as conjectured there.

COROLLARY 3.3. *For every real x, the distribution of $f_{\mathrm{ISE}}(x)$ is given by the moment generating function, with a real or complex,*

$$\mathbb{E}\, e^{af_{\mathrm{ISE}}(x)} = L(2^{-1/4}|x|, 2^{-1/4}a), \qquad |a| < 2^{2+1/4}3^{-1/2},$$

where

$$L(x,a) := 1 + \frac{48}{\mathrm{i}\sqrt{\pi}} \int_\Gamma \frac{A(a/v^3)e^{-2xv}}{(1+A(a/v^3)e^{-2xv})^2} v^5 e^{v^4}\, dv, \qquad x \ge 0,$$

$A(y) \equiv A$ is the unique solution of

$$A = \frac{y}{24}\frac{(1+A)^3}{1-A}$$

satisfying $A(0) = 0$, and the integral is taken over

$$\Gamma = \{1 - te^{-\mathrm{i}\pi/4}, t \in (-\infty, 0]\} \cup \{1 + te^{\mathrm{i}\pi/4}, t \in [0,\infty)\}.$$

In particular, the density at $x = 0$ has a simple law. (See again [7].)

COROLLARY 3.4. *$f_{\mathrm{ISE}}(0)$ has the same distribution as $2^{1/4}3^{-1}T^{-1/2}$, where T is a positive $2/3$-stable variable with Laplace transform $\mathbb{E}\, e^{-tT} = e^{-t^{2/3}}$. Hence $f_{\mathrm{ISE}}(0)$ has the moments*

$$\mathbb{E}\, f_{\mathrm{ISE}}(0)^r = 2^{r/4}3^{-r}\frac{\Gamma(3r/4+1)}{\Gamma(r/2+1)}, \qquad -4/3 < r < \infty.$$

As said in the introduction, we conjecture that the local limit results hold also for conditioned Galton–Watson trees with random labellings defined by i.i.d. random increments η_e along the edges; a precise formulation is as follows. Recall that the *span* of η_e is the largest integer $d \geq 1$ such that η_e a.s. is a multiple of d.

CONJECTURE 3.5. *Consider a random conditioned Galton–Watson tree T_n with a random labelling defined as above by integer valued random variables η_e with mean* 0*, finite variance $\sigma_\eta^2 > 0$ and span* 1*. Then, the conclusions* (3.1) *and* (3.2) *of Theorem* 3.1 *hold, with $\gamma := \sigma_\eta^{-1}\sigma_\xi^{1/2}$.*

If this conjecture holds, the conclusion of Corollary 3.2 holds too.

As said in the introduction, we can prove the conjecture in two special cases, both considered in [7].

THEOREM 3.6. *Conjecture* 3.5 *holds if T_n is a random plane tree and η_e is uniformly chosen at random from $\{\pm 1\}$ or from $\{-1, 0, 1\}$.*

For these two cases, $\sigma_\xi^2 = 2$ and $\operatorname{Var}\eta_e = 1$ and $2/3$; hence $\gamma = 2^{1/4}$ and $\gamma = 2^{-1/4}3^{1/2}$, respectively.

REMARK 3.7. It follows from the proof of Theorem 3.1 in Section 7 that to prove Conjecture 3.5 in further cases, it suffices to prove the estimate in Lemma 7.3.

4. The moments of ISE. Let T_n be a random binary tree with n nodes, and let μ_n be the following (random) probability distribution:

$$\mu_n = \frac{1}{n} \sum_{v \in T_n} \delta_{(2n)^{-1/4}\ell(v)}. \tag{4.1}$$

As recalled in the introduction, μ_n converges to μ_{ISE}. The ith moment of μ_n, denoted $m_{i,n}$, is itself a random variable:

$$m_{i,n} = \frac{1}{n} \sum_{v \in T_n} (2n)^{-i/4} \ell(v)^i = 2^{-i/4} n^{-1-i/4} \sum_{v \in T_n} \ell(v)^i.$$

We shall prove that the sequence $m_{1,n}, m_{2,n}, \dots$ converges in distribution to the sequence $m_1, m_2, \dots$ of moments of ISE, and compute the joint moments of the m_i:

$$\mathbb{E}(m_1^{p_1} m_2^{p_2} \cdots m_r^{p_r}),$$

for all (fixed) values of $p_1, p_2, \dots, p_r$. The moments of the m_i, being the moments of the moments (of μ_{ISE}) should probably be called the *grand-moments* of μ_{ISE}. Note that the grand-moments of a random probability measure, provided they do not grow too quickly, determine the distribution of the sequence of moments of the measure, and thus the distribution of the random measure.

In order to state our result, we introduce some notation. A *partition* λ of an integer k is a nondecreasing sequence $(\lambda_1, \dots, \lambda_p)$ of positive integers summing to k. The value k is called the *weight* of λ, also denoted $k = |\lambda|$. For instance, $\lambda = (1, 1, 3, 4)$ is a partition of $k = 9$. The λ_i are called the parts of λ. We shall also use *extended partitions*, in which the positivity condition on the parts is relaxed by simply requiring that λ_i is non-negative. Hence $\lambda = (0, 0, 1, 1, 3, 4)$ is an extended partition of 9. The *union* $\sigma \cup \tau$ of two extended partitions $\sigma = (\sigma_1, \dots, \sigma_p)$ and $\tau = (\tau_1, \dots, \tau_q)$ is obtained by reordering the sequence $(\sigma_1, \dots, \sigma_p, \tau_1, \dots, \tau_q)$. For any p-tuple $(\sigma_1, \dots, \sigma_p)$ of non-negative integers, we denote by $\bar{\sigma}$ the extended partition obtained by reordering the σ_i. Given two p-tuples σ and λ, we write $\sigma \le \lambda$ if $\sigma_i \le \lambda_i$ for $1 \le i \le p$.

We shall denote

$$m_{1,n}^{p_1} m_{2,n}^{p_2} \cdots m_{r,n}^{p_r} := m_{\lambda,n} \quad \text{and} \quad m_1^{p_1} m_2^{p_2} \cdots m_r^{p_r} := m_\lambda \tag{4.2}$$

where $\lambda = 1^{p_1} 2^{p_2} \cdots$ is the partition having p_1 parts equal to 1, p_2 parts equal to 2 and so on. The value of $\mathbb{E}(m_\lambda)$ will be expressed in terms of a rational number c_λ, which we actually define for any *extended* partition $\lambda = (\lambda_1, \dots, \lambda_p)$. The definition works by induction on $p + |\lambda|$ as follows:

- $c_\emptyset = -2$,
- $c_\lambda = 0$ if $|\lambda|$ is odd,
- $c_\lambda = (p + |\lambda|/4 - 3/2) c_{\lambda'}$ if $\lambda_1 = 0$, with $\lambda' = (\lambda_2, \dots, \lambda_p)$,
- if $\lambda_1 > 0$ and $|\lambda|$ is even,

$$c_\lambda = \frac{1}{4} \sum_{\emptyset \ne I \subsetneq [p]} c_{\lambda_I} c_{\lambda_J} + \sum_{\sigma \le \lambda, |\sigma| = |\lambda| - 2} \binom{\lambda}{\sigma} c_{\bar{\sigma}} \tag{4.3}$$

where $[p] := \{1, 2, \dots, p\}$, $J = [p] \setminus I$, $\lambda_I = (\lambda_{i_1}, \lambda_{i_2}, \dots, \lambda_{i_r})$ if $I = \{i_1, \dots, i_r\}$ with $1 \le i_1 < \dots < i_r \le p$, the second sum runs over all non-negative p-tuples σ (not necessarily partitions) satisfying the two required conditions, and $\binom{\lambda}{\sigma} = \prod_{i=1}^p \binom{\lambda_i}{\sigma_i}$. (Trivially, (4.3) holds for $|\lambda|$ odd too.)

THEOREM 4.1. *As $n \to \infty$, the moments $m_{1,n}, m_{2,n}, \dots$ of the occupation measure of binary trees converge jointly in distribution to the moments $m_1, m_2, \dots$ of ISE. The convergence of moments holds as well, and for all partitions λ, the joint λ-moment of the random variables $m_{i,n}$, defined by* (4.2)*, satisfies*

$$\mathbb{E}(m_{\lambda,n}) = \mathbb{E}(m_\lambda) = 0 \quad \text{if } |\lambda| \text{ is odd,}$$

and otherwise

$$\mathbb{E}(m_{\lambda,n}) \to \mathbb{E}(m_\lambda) = \frac{2^{-|\lambda|/4}\, c_\lambda\, \Gamma(1/2)}{\Gamma(p + |\lambda|/4 - 1/2)}, \tag{4.4}$$

where the number c_λ is defined just above.

The vanishing of $\mathbb{E}(m_{\lambda,n})$ when $|\lambda|$ is odd, is a straightforward consequence of the symmetry of T_n. The proof is given in Section 12.

EXAMPLE 4.2 (**The average moments of ISE**). When λ has a single part, equal to $2k$ for $k \geq 1$, the above recurrence relation gives $c_{(2k)} = k(2k-1)c_{(2k-2)}$, together with the initial condition $c_{(0)} = (1 - 3/2)c_\emptyset = 1$. Hence the mean of the $2k$th moment of the random probability μ_n satisfies

$$\mathbb{E}(m_{2k,n}) \to \mathbb{E}(m_{2k}) = \frac{(2k)!\, \Gamma(1/2)}{2^{3k/2}\, \Gamma((k+1)/2)}. \tag{4.5}$$

EXAMPLE 4.3 (**The moments of the average of ISE**). Let $m_{1,n}$ denote the mean of μ_n. Then $\mathbb{E}(m_{1,n}^p) = \mathbb{E}(m_1^p) = 0$ if p is odd, while

$$\mathbb{E}\left(m_{1,n}^{2k}\right) \to \mathbb{E}\left(m_1^{2k}\right) = \frac{a_k\, \Gamma(1/2)}{2^{k/2}\, \Gamma((5k-1)/2)},$$

where $a_0 = -2$ and for $k > 0$,

$$4\, a_k = \sum_{i=1}^{k-1} \binom{2k}{2i} a_i a_{k-i} + k(2k-1)(5k-4)(5k-6)a_{k-1}.$$

Indeed, $a_k = c_\lambda$, where $\lambda = 1^{2k}$, and the recurrence relation (4.3) translates into the above recursive definition of a_k. Note that each $2k$-tuple σ occurring in (4.3) contains 2 coefficients equal to zero, so that we also use the part of the definition of c_λ that deals with the case $\lambda_1 = 0$. This value of $\mathbb{E}(m_1^{2k})$ was already obtained in [16].

EXAMPLE 4.4 (**The first two moments of ISE**). Let us finally work out the joint distribution of m_1 and m_2. We have $\mathbb{E}(m_{1,n}^{2k+1} m_{2,n}^{\ell}) = \mathbb{E}(m_1^{2k+1} m_2^{\ell}) = 0$, and

$$\mathbb{E}(m_{1,n}^{2k} m_{2,n}^{\ell}) \to \mathbb{E}(m_1^{2k} m_2^{\ell}) = \frac{a_{k,\ell}\,\Gamma(1/2)}{2^{(k+\ell)/2}\,\Gamma((5k+3\ell-1)/2)},$$

where $a_{0,0} = -2$ and the $a_{k,\ell}$ are determined by induction on $k+\ell$:

$$a_{k,\ell} = \frac{1}{4} \sum_{(0,0)<(i,j)<(k,\ell)} \binom{2k}{2i}\binom{\ell}{j} a_{i,j} a_{k-i,\ell-j} + 2\ell(\ell-1) a_{k+1,\ell-2}$$
$$+ \frac{1}{4} k(2k-1)(5k+3\ell-4)(5k+3\ell-6) a_{k-1,\ell} + \frac{1}{2}(4k+1)\ell(5k+3\ell-4) a_{k,\ell-1}.$$

Here, $a_{k,\ell} = c_\lambda$ with $\lambda = 1^{2k}2^{\ell}$. In the right-hand side of the equation, the second (resp. third, fourth) term corresponds to the case $\bar{\sigma} = 1^{2k+2}2^{\ell-2}$ (resp. $\bar{\sigma} = 0^2 1^{2k-2} 2^{\ell}$, $\bar{\sigma} = 01^{2k}2^{\ell-1}$). The last case occurs both when we replace a part of λ equal to 2 by a zero part, and when we decrease by 1 a part equal to 1 *and* a part equal to 2. Of course, this generalizes Example 4.3 (which corresponds to $\ell = 0$).

It seems likely that Theorem 4.1 extends to randomly labelled conditioned Galton–Watson trees as in Conjecture 3.5, at least under some moment conditions on ξ and η_e, where the measure μ_n is defined by the left-hand side of (1.1) and, as usual, $\gamma := \sigma_\eta^{-1}\sigma_\xi^{1/2}$. We show this for the special case in Theorem 3.6.

THEOREM 4.5. *If T_n is a random plane tree and η_e is uniformly chosen at random from $\{\pm 1\}$ or from $\{-1,0,1\}$, then the conclusions of Theorem* 4.1 *hold, where the measure μ_n is defined by* (1.1), *with $\gamma = 2^{1/4}$ and $\gamma = 2^{-1/4}3^{1/2}$, respectively.*

To conclude this section, we want to underline briefly a similarity between the density of ISE and the local time of the (normalized) Brownian excursion. In fact, Theorem 2.1 shows that the vertical profile of a random binary tree converges, after suitable rescaling, to the density of the ISE. Similarly, as shown by Drmota and Gittenberger [10], the horizontal profile converges to the local time of the Brownian excursion.

We can develop this analogy to grand-moments as follows. Consider again the random binary tree T_n. Let $d(v)$ denote the depth (the distance from the root) of vertex v and define the probability measure

$$\nu_n := \frac{1}{n}\sum_{v\in T_n} \delta_{2^{-3/2}n^{-1/2}d(v)}, \tag{4.6}$$

describing the horizontal profile, i.e. the distribution of the depths (after rescaling). It is known, as an immediate consequence of Aldous [2, 3], that as $n\to\infty$, $\nu_n \xrightarrow{\mathrm{d}} \nu_{\mathrm{EXC}}$, where the random probability measure ν_{EXC} is the occupation measure of the Brownian excursion, and thus has the local time of the Brownian excursion as density.

The similarity with the vertical profile and ISE is obvious, and we adopt below the same notation as before ($m_{i,n}, m_{\lambda,n}$, etc.) for the moments of ν_n and ν_{EXC}. In particular, now $m_i := \int x^i\, d\nu_{\mathrm{EXC}}(x) = \int_0^1 e(t)^i\, dt$, where $e(t)$ is a Brownian excursion. Then a result similar to Theorem 4.1 holds:

THEOREM 4.6. *As $n\to\infty$, the moments $m_{1,n}, m_{2,n},\dots$ of the horizontal profile (depth distribution) measure ν_n converge jointly in distribution to the moments $m_1, m_2,\dots$ of ν_{EXC}. The convergence of moments holds as well, and for all partitions λ, the joint λ-moment of the random variables $m_{i,n}$, defined by* (4.2), *satisfies*

$$\mathbb{E}(m_{\lambda,n}) \to \mathbb{E}(m_\lambda) = \frac{2^{-3|\lambda|/2}\, d_\lambda\, \Gamma(1/2)}{\Gamma(p+|\lambda|/2-1/2)}, \tag{4.7}$$

where the number d_λ is defined by

- $d_\emptyset = -2$,
- $d_\lambda = (p+|\lambda|/2-3/2)d_{\lambda'}$ *if* $\lambda_1 = 0$, *with* $\lambda' = (\lambda_2,\dots,\lambda_p)$,
- *if* $\lambda_1 > 0$,

$$d_\lambda = \frac{1}{4}\sum_{\emptyset\neq I\subsetneq[p]} d_{\lambda_I} d_{\lambda_J} + \sum_{\sigma\le\lambda,\, |\sigma|=|\lambda|-1} \binom{\lambda}{\sigma} d_{\bar\sigma}, \tag{4.8}$$

with the same notation as in (4.3).

The proof is very similar to the proof of Theorem 4.1, but simpler. Note that the binomial coefficient $\binom{\lambda}{\sigma}$ is simply equal to one of the λ_i. The proof is sketched at the end of Section 12.

The grand-moments in (4.7) have been computed by a different method by Richard [27]; a special case (moments of m_1 and m_2) is given by Nguyen The [25], and the moments of the Brownian excursion area m_1 were found already by Louchard [22], see also [11] and [15]. The grand-moments in

(4.7), as well as the grand-moments in (4.4) above, can also be derived by the method of [15, Section 5] and [16], which is related to the method used here but phrased in different terms. (Presumably, the method of [27] too applies to (4.4) as well.)

Again, the same result holds for random plane trees as well, provided we change the scale factor $2^{-3/2}$ in (4.6) to $2^{-1/2}$.

REMARK 4.7. A *Dyck path* of length $2n$ is a 1-dimensional walk starting and ending at 0, taking steps in $\{-1,+1\}$, and never reaching a negative position. There is a well-known correspondence between plane trees with $n+1$ vertices and Dyck paths of length $2n$, where the Dyck path gives the depths of the vertices along the depth-first walk on the tree. It follows easily that Theorem 4.6 holds for moments of a uniformly chosen random Dyck path w_n of length $2n$ too defined by

$$m_{k,n} := \frac{1}{2n}\sum_{i=1}^{2n}(2n)^{-k/2}w_n(i)^k;$$

this has previously been shown by Richard [27].

REMARK 4.8. It is possible to use our methods to obtain results on grand-moments of the vertical and horizontal occupation measures together, and thus on the joint distribution of the vertical and horizontal profile, and also on the asymptotic distribution of the pair of labels $\big(\ell(v), d(v)\big)$. We leave this to the reader.

5. The Brownian snake and CRT.

5.1. *The Brownian snake.* We begin by recalling the definition of the Brownian snake, see Le Gall [20, Chapter IV] or Le Gall and Weill [21] for further details; see also [16, Section 4.1]. Let ζ, the *lifetime*, be $\zeta := 2B^{\mathrm{ex}}$, where B^{ex} is a Brownian excursion on $[0,1]$. (In general, the lifetime ζ might be any (locally) Hölder continuous non-negative stochastic process on some interval I; in other contexts, ζ is often taken to be reflected Brownian motion on $[0,\infty)$ [20].) Let, for $s,t \in [0,1]$,

$$m(s,t;\zeta) := \min\{\zeta(u) : u \in [s,t]\} \qquad \text{when } s \le t,$$

and $m(s,t;\zeta) := m(t,s;\zeta)$ when $s > t$. The Brownian snake with lifetime ζ then can be defined as the continuous stochastic process $W(s,t)$ on $[0,1]\times[0,\infty)$ such that, conditioned on ζ, W is Gaussian with mean 0 and covariances

$$\operatorname{Cov}\big(W(s_1,t_1)W(s_2,t_2) \mid \zeta\big) = \min\big(t_1,t_2,m(s_1,s_2;\zeta)\big).$$

We have defined the Brownian snake as a random field with two parameters, but we are really only interested in the specialization $\overline{W}(s) := W(s, \zeta(s))$, $s \in [0,1]$; this stochastic process is called the *head* of the Brownian snake. (In fact, it is easily seen that the pair $(\zeta, \overline{W})$ determines W; see further [24].) Conditioned on ζ, $\overline{W}$ is a Gaussian process on $[0,1]$ with mean 0 and covariances $\mathbb{E}(\overline{W}(s)\overline{W}(t) \mid \zeta) = m(s,t;\zeta)$. Consequently, still conditioned on ζ, $\overline{W}(s) - \overline{W}(t)$ has a normal distribution with mean 0 and variance

$$\operatorname{Var}(\overline{W}(s) - \overline{W}(t) \mid \zeta) = \sigma^2(s,t;\zeta) := \zeta(s) + \zeta(t) - 2m(s,t;\zeta). \tag{5.1}$$

The random probability measure μ_{ISE} can be defined as the occupation measure of the process $\overline{W}$, see [20], [21] and the next subsection. Hence, f_{ISE} is the occupation density of $\overline{W}$, also called its *local time*.

5.2. *Brownian CRT.* The Brownian CRT (continuum random tree) was introduced by Aldous [1, 2, 3] as a natural limit of rescaled finite random trees. It is a random compact metric space that is a topological tree in the sense that every pair of points x, y are connected by a unique path (homeomorphic to $[0,1]$), and that path has length $d(x,y)$. We let here and later d denote the metric. The Brownian CRT is further equipped with a probability measure ν, which gives a meaning to "a random node" in the CRT.

One of Aldous's characterizations of the Brownian CRT [2, 3, 4] uses the distribution of the shape and edge lengths of the spanning subtree R_k spanned by the root o and k independent random nodes $X_1, \dots, X_k$ in the tree. (Here k is an arbitrary positive integer.) Then a.s., the subtree R_k admits the root and $X_1, \dots, X_k$ as leaves, and has exactly $k-1$ internal nodes, all of of degree 3; the leaves are labelled but not the internal nodes. If we ignore the edge lengths (which are positive real numbers), there are $(2k-3)!! = 1 \cdot 3 \cdot 5 \cdots (2k-3)$ possible "shapes" of R_k; for each shape we number the $2k-1$ edges in some order. Letting T^*_{2k} be the finite set of shapes, R_k can thus be described by a shape $\hat{t} \in T^*_{2k}$ and the edge lengths $x_1, \dots, x_{2k-1} > 0$, and for the Brownian CRT, R_k has density [3, Lemma 21]

$$f(\hat{t}; x_1, \dots, x_{2k-1}) = s e^{-s^2/2}, \qquad s = \sum_{i=1}^{2k-1} x_i. \tag{5.2}$$

Aldous [3, Corollary 22] also gives a construction of the Brownian CRT in terms of a (normalized) Brownian excursion B^{ex}. Let $\zeta = 2B^{\text{ex}}$. Then Aldous shows that there exists a function $\tilde{\zeta}$ mapping $[0,1]$ onto the Brownian CRT, with the Lebesgue measure mapped to ν and, cf. (5.1),

$$d(\tilde{\zeta}(s), \tilde{\zeta}(t)) = \zeta(s) + \zeta(t) - 2m(s,t;\zeta) = \sigma^2(s,t;\zeta). \tag{5.3}$$

Indeed, the Brownian CRT can be defined as the quotient space of $[0,1]$ with the semi-metric $\sigma^2(s,t;\zeta)$, identifying points of distance 0, see [21].

The function $\tilde{\zeta}$ is not injective, but if $\tilde{\zeta}(s) = \tilde{\zeta}(t)$, and thus $\sigma^2(s,t;\zeta) = 0$, then $\zeta(s) = \zeta(t) = m(s,t;\zeta)$, which implies $\overline{W}(s) = \overline{W}(t)$. Hence we can define a continuous random function $\widetilde{W}$ on the Brownian CRT by $\widetilde{W}(\tilde{\zeta}(s)) = \overline{W}(s)$; conditioned on the CRT, the $\widetilde{W}(x)$ are jointly Gaussian with mean 0 and, by (5.1) and (5.3), $\operatorname{Var}(\widetilde{W}(x) - \widetilde{W}(y)) = d(x,y)$. Thus $\widetilde{W}$ is the random mapping of the Brownian CRT into $\mathbb{R}$ considered by Aldous [4]; Aldous defines ISE as the measure on $\mathbb{R}$ that ν is mapped to by $\widetilde{W}$. This is clearly the same as the measure that $\overline{W}$ maps Lebesgue measure on $[0,1]$ to, i.e. the occupation measure of $\overline{W}$ as claimed above.

6. Existence of the density: Proof of Theorem 2.1. Although Theorem 2.1 follows easily from Theorem 3.1 and its proof, we find it interesting to give a different, self-contained proof. We use the standard Fourier method, see e.g. [13] and the references there, together with Aldous's theory of the Brownian CRT [2, 3]. We define the Fourier transform $\widehat{\mu}$ of a finite measure μ by $\widehat{\mu}(t) := \int e^{\mathrm{i}tx}\, d\mu(x)$.

LEMMA 6.1. *If* $0 \le \alpha < 3/2$, *then*

$$\mathbb{E} \int_{-\infty}^{\infty} \left(|t|^\alpha |\widehat{\mu}_{\mathrm{ISE}}(t)|\right)^2 dt < \infty.$$

PROOF. Since μ_{ISE} is the occupation measure of $\overline{W}$, the head of the Brownian snake, its Fourier transform can be expressed as

$$\widehat{\mu}_{\mathrm{ISE}}(t) := \int_{-\infty}^{\infty} e^{\mathrm{i}tx}\, d\mu_{\mathrm{ISE}}(x) = \int_0^1 e^{\mathrm{i}t\overline{W}(s)}\, ds.$$

Consequently, $|\widehat{\mu}_{\mathrm{ISE}}(t)|^2 = \int_0^1 \int_0^1 e^{\mathrm{i}t(\overline{W}(s) - \overline{W}(u))}\, ds\, du$. Conditioned on ζ, $\overline{W}(s) - \overline{W}(u)$ is by (5.1) a Gaussian random variable with mean 0 and variance $\sigma^2(s,u;\zeta)$. Hence,

$$\begin{aligned}
\mathbb{E}\left(|\widehat{\mu}_{\mathrm{ISE}}(t)|^2 \mid \zeta\right) &= \int_0^1 \int_0^1 \mathbb{E}\left(e^{\mathrm{i}t(\overline{W}(u) - \overline{W}(s))} \,\middle|\, \zeta\right) ds\, du \\
&= \int_0^1 \int_0^1 e^{-t^2 \sigma^2(s,u;\zeta)/2}\, ds\, du
\end{aligned}$$

and thus, letting U_1 and U_2 be independent uniform random variables on $[0,1]$,

$$\mathbb{E}\,|\widehat{\mu}_{\mathrm{ISE}}(t)|^2 = \mathbb{E}\int_0^1\int_0^1 e^{-t^2\sigma^2(s,u;\zeta)/2}\,ds\,du = \mathbb{E}\,e^{-t^2\sigma^2(U_1,U_2;\zeta)/2}. \tag{6.1}$$

Let $\tilde{\zeta}$ be as in Subsection 5.2. Then $X_i := \tilde{\zeta}(U_i)$, $i=1,2$, are two independent random nodes in the Brownian CRT, and (5.3) shows that (6.1) can be written

$$\mathbb{E}\,|\widehat{\mu}_{\mathrm{ISE}}(t)|^2 = \mathbb{E}\,e^{-t^2 d(X_1,X_2)/2}. \tag{6.2}$$

For $\alpha \ge 0$ we thus have, letting $y=t^2$,

$$\begin{aligned}\mathbb{E}\int_0^\infty \bigl(t^\alpha|\widehat{\mu}_{\mathrm{ISE}}(t)|\bigr)^2\,dt &= \mathbb{E}\int_0^\infty t^{2\alpha}e^{-t^2 d(X_1,X_2)/2}\,dt\\ &= \mathbb{E}\int_0^\infty \tfrac12 y^{\alpha-1/2}e^{-yd(X_1,X_2)/2}\,dy\\ &= \mathbb{E}\,\tfrac12\bigl(d(X_1,X_2)/2\bigr)^{-\alpha-1/2}\Gamma(\alpha+1/2)\\ &= C_1(\alpha)\,\mathbb{E}\,d(X_1,X_2)^{-\alpha-1/2}.\end{aligned} \tag{6.3}$$

From (5.2) (with $k=2$), or by the invariance of the CRT under random rerooting, follows the symmetry $d(X_1,X_2) \overset{\mathrm{d}}{=} d(X_1,o)$. Moreover, by the same formula (5.2) with $k=1$, $d(X_1,o)$ has a Rayleigh distribution with density $xe^{-x^2/2}$. Hence,

$$\mathbb{E}\,d(X_1,X_2)^{-\alpha-1/2} = \mathbb{E}\,d(X_1,o)^{-\alpha-1/2} = \int_0^\infty x^{-\alpha-1/2}xe^{-x^2/2}\,dx < \infty,$$

when $\alpha < 3/2$, and the result follows from (6.3) and the symmetry of $|\widehat{\mu}_{\mathrm{ISE}}|$. ∎

By Lemma 6.1, if $0 \le \alpha < 3/2$, then $\int_{-\infty}^{\infty}\bigl(|y|^\alpha|\widehat{\mu}_{\mathrm{ISE}}(y)|\bigr)^2\,dy < \infty$ a.s. Taking first $\alpha=0$, we see that $\widehat{\mu}_{\mathrm{ISE}} \in L^2(\mathbb{R})$; by Plancherel's theorem [28, Theorem 7.9] this shows that μ_{ISE} is absolutely continuous with a density $f_{\mathrm{ISE}} \in L^2$. Note that the Fourier transform $\widehat{f}_{\mathrm{ISE}}$ coincides with $\widehat{\mu}_{\mathrm{ISE}}$.

For $\alpha \ge 0$, we define the (generalized) Sobolev space $L^{2,\alpha}$ by

$$L^{2,\alpha} := \{f \in L^2(\mathbb{R}) : \|f\|_{2,\alpha}^2 := \int_{-\infty}^{\infty}\bigl((1+|t|)^\alpha|\hat{f}(t)|\bigr)^2\,dt < \infty\}, \tag{6.4}$$

where $\hat{f}$ is the Fourier transform of f. Lemma 6.1 thus shows that a.s. $f_{\mathrm{ISE}} \in L^{2,\alpha}$ for every $\alpha < 3/2$. (There is no problem with null sets, since it suffices to consider rational α, say.)

Further, for $0 < \alpha < 1$, we define the Hölder space H_α as the space of bounded continuous functions f on $\mathbb{R}$ such that $|f(x) - f(y)| \le C|x-y|^\alpha$ for some C and all x and y.

To show that f_{ISE} is (i.e. can be chosen) continuous with the regularity properties in Theorem 2.1, we use some general embedding properties of these spaces.

LEMMA 6.2. (i) *If $0 \le \alpha < 1/2$ and $1/2 \ge 1/p > 1/2 - \alpha$, then $L^{2,\alpha} \subset L^p$.*

(ii) *If $1/2 < \alpha < 3/2$, then $L^{2,\alpha} \subset H_{\alpha-1/2}$, and the inclusion map is bounded.*

(iii) *If $\alpha \ge 1$ and $f \in L^{2,\alpha}$, then f has a derivative f' in distribution sense and a.e., with $f' \in L^{2,\alpha-1}$.*

This lemma is well-known: (i) and (ii) are special cases of the Sobolev (or Besov) embedding theorem, see e.g. [5, Theorem 6.5.1] or [29, Chapter V]; indeed, we may also take $1/p = 1/2 - \alpha$ in (i). However, since the proof of the general embedding theorem is quite technical, we give a simple proof of this special case.

PROOF. (i): We may assume $p > 2$ since the case $p = 2$ is trivial. Define $p' \in (1,2)$ by $1/p' = 1 - 1/p$. If $f \in L^{2,\alpha}$, Hölder's inequality yields

$$\begin{aligned}\int_{-\infty}^{\infty} |\hat{f}|^{p'} &\le \Bigl(\int_{-\infty}^{\infty} \bigl((1+|t|)^{\alpha p'} |\hat{f}(t)|^{p'}\bigr)^{2/p'}\,dt\Bigr)^{p'/2} \\ &\qquad\qquad \times \Bigl(\int_{-\infty}^{\infty} \bigl((1+|t|)^{-\alpha p'}\bigr)^{2/(2-p')}\,dt\Bigr)^{1-p'/2} \\ &= \|f\|_{2,\alpha}^{p'} \Bigl(\int_{-\infty}^{\infty} (1+|t|)^{-2\alpha p'/(2-p')}\,dt\Bigr)^{1-p'/2} < \infty,\end{aligned}$$

since it is easy to check that $2\alpha p' > 2 - p'$ when $1/p > 1/2 - \alpha$. Consequently, $\hat{f} \in L^{p'}$, which by the Hausdorff–Young inequality yields $\hat{\hat{f}} \in L^p$. By the inversion theorem for the Fourier transform (defined for tempered distributions, say), this yields $f \in L^p$.

(ii): First, by Hölder's (Cauchy–Schwarz's) inequality,

$$\int_{-\infty}^{\infty} |\hat{f}| \le \Bigl(\int_{-\infty}^{\infty} \bigl((1+|t|)^{\alpha} |\hat{f}(t)|\bigr)^2\,dt\Bigr)^{1/2} \Bigl(\int_{-\infty}^{\infty} (1+|t|)^{-2\alpha}\,dt\Bigr)^{1/2} < \infty,$$

since $2\alpha > 1$. Hence f has an absolutely integrable Fourier transform, which shows that f is a continuous bounded function given by the inversion formula

$f(x) = (2\pi)^{-1} \int e^{-\mathrm{i}xt} \hat{f}(t)\, dt$. Hence, for any x and $h > 0$, using Hölder's inequality again,

$$
\begin{aligned}
|f(x+h) - f(x)| &= \frac{1}{2\pi} \Big| \int_{-\infty}^{\infty} (e^{-\mathrm{i}(x+h)t} - e^{-\mathrm{i}xt}) \hat{f}(t)\, dt \Big| \\
&\le \Big(\int_{-\infty}^{\infty} \big|e^{\mathrm{i}ht} - 1\big|^2 \, |t|^{-2\alpha}\, dt \Big)^{1/2} \Big(\int_{-\infty}^{\infty} |t|^{2\alpha} |\hat{f}(t)|^2 \, dt \Big)^{1/2} \\
&\le \Big(h^{2\alpha-1} \int_{-\infty}^{\infty} \big|e^{\mathrm{i}u} - 1\big|^2 \, |u|^{-2\alpha}\, du \Big)^{1/2} \|f\|_{2,\alpha} \\
&\le C_2(\alpha) \|f\|_{2,\alpha} h^{\alpha - 1/2}.
\end{aligned}
$$

(iii): We have $\widehat{f'}(t) = -\mathrm{i}t\hat{f}$, with f' taken as a distribution. Since $f \in L^{2,\alpha}$, this shows that $\widehat{f'} \in L^2$ and thus $f' \in L^2$ by Plancherel's theorem. Consequently, by elementary distribution theory, the derivative exists a.e., and equals the distributional derivative f'. Further, from the definition (6.4), $f' \in L^{2,\alpha-1}$. □

Since, as remarked above, Lemma 6.1 shows that a.s. $f_{\mathrm{ISE}} \in L^{2,\alpha}$ for every $\alpha < 3/2$, Theorem 2.1(ii) follows by Lemma 6.2(ii), while Theorem 2.1(iii) follows by Lemma 6.2(iii) and (i) (applied to f'_{ISE}).

Finally, Theorem 2.1(i) follows because μ_{ISE} has compact support, viz. the image of the compact set $[0,1]$ by the continuous function $\overline{W}$. □

7. Local limit law for the density: Proof of Theorem 3.1. The proof is based on the known convergence (1.1) of random measures. To obtain the stronger result in Theorem 3.1 on convergence of densities, we use a compactness argument as follows. We begin with a measure-theoretic lemma. Recall that a Polish space is a space with a topology that can be defined by a complete separable metric. For generalities on convergence of random elements of metric spaces (equipped with their Borel σ-fields), see e.g. Billingsley [6] or Kallenberg [18]. In particular, recall that a sequence (W_n) of random variables in a metric space $\mathcal{S}$ is *tight* if for every $\varepsilon > 0$, there exists a compact subset $K \subseteq \mathcal{S}$ such that $\mathbb{P}(W_n \in K) > 1 - \varepsilon$ for every n; in a Polish space, this is equivalent to relative compactness (of the corresponding distributions) by Prohorov's theorem [6, Theorems 6.1 and 6.2], [18, Theorem 16.3]. Recall further that "convergence in distribution" really means convergence of the corresponding distributions, but it is often convenient to talk about random variables instead of their distributions.

LEMMA 7.1. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two Polish spaces, and let $\phi : \mathcal{S}_1 \to \mathcal{S}_2$ be an injective continuous map. If (W_n) is a tight sequence of random elements* *of $\mathcal{S}_1$ such that $\phi(W_n) \xrightarrow{\mathrm{d}} Z$ in $\mathcal{S}_2$ for some random $Z \in \mathcal{S}_2$, then $W_n \xrightarrow{\mathrm{d}} W$ in $\mathcal{S}_1$ for some W with $\phi(W) \overset{\mathrm{d}}{=} Z$.*

PROOF. By Prohorov's theorem, each subsequence of (W_n) has a subsequence that converges in distribution to some limit. Let W' and W'' be limits in distribution of two such subsequences $W_{n'_i}$ and $W_{n''_i}$. Since ϕ is continuous, $\phi(W_{n'_i}) \xrightarrow{\mathrm{d}} \phi(W')$ and $\phi(W_{n''_i}) \xrightarrow{\mathrm{d}} \phi(W'')$. Hence, $\phi(W') \overset{\mathrm{d}}{=} Z \overset{\mathrm{d}}{=} \phi(W'')$.

Let K be a compact subset of $\mathcal{S}_1$. Then $\phi(K) \subseteq \mathcal{S}_2$ is compact, and thus measurable, and, using the injectivity of ϕ,

$$\mathbb{P}(W' \in K) = \mathbb{P}\big(\phi(W') \in \phi(K)\big) = \mathbb{P}\big(\phi(W'') \in \phi(K)\big) = \mathbb{P}(W'' \in K).$$

Since a probability measure on a Polish space is determined by its values on compact sets, see e.g. [8, Proposition 8.1.10], it follows that $W' \overset{\mathrm{d}}{=} W''$.

In other words, there is a unique distribution of the subsequence limits. Thus, if W is one such limit, then every subsequence of (W_n) has a subsequence that converges in distribution to W; this is equivalent to $W_n \xrightarrow{\mathrm{d}} W$. $\square$

Let Y_n denote the random probability measure on the left-hand side of (1.1), let ν_h be the probability measure with the triangular density function $h^{-1}(1-|x|/h)_+$, and let $\bar{Y}_n$ be the convolution $Y_n * \nu_{\gamma n^{-1/4}}$. Note that $\bar{Y}_n$ has the density $g_n(x) := n^{-1}\gamma^{-1}n^{1/4}\bar{X}_n(\gamma^{-1}n^{1/4}x) \in C_0(\mathbb{R})$. Since $Y_n \xrightarrow{\mathrm{d}} \mu_{\mathrm{ISE}}$ by (1.1), and $\nu_{\gamma n^{-1/4}} \xrightarrow{\mathrm{p}} \delta_0$, it follows easily that $\bar{Y}_n \xrightarrow{\mathrm{d}} \mu_{\mathrm{ISE}}$ too.

Let $\mathcal{S}_1 := \{f \in C_0(\mathbb{R}) : f \geq 0\}$, with the uniform topology inherited from $C_0(\mathbb{R})$, let $\mathcal{S}_2$ be the space of locally finite measures on $\mathbb{R}$ with the vague topology, see e.g. Kallenberg [18, Appendix A2], and let ϕ map a function f to the corresponding measure $f\,dx$, i.e., $\phi(f)$ is the measure with density f. Then $\mathcal{S}_1$ is a closed subset of the separable Banach space $C_0(\mathbb{R})$, and is thus Polish, and so is $\mathcal{S}_2$ by [18, Theorem A2.3]. Further, ϕ is continuous and injective.

Take $W_n := g_n$ in Lemma 7.1. We have just shown that $\phi(g_n) = \bar{Y}_n \xrightarrow{\mathrm{d}} \mu_{\mathrm{ISE}}$ in the space of probability measures on $\mathbb{R}$ and thus also in the larger space $\mathcal{S}_2$. If we can show that the sequence g_n is tight in $\mathcal{S}_1$, or, equivalently, in $C_0(\mathbb{R})$, then Lemma 7.1 shows that $g_n \xrightarrow{\mathrm{d}} g$ for some random function $g \in C_0(\mathbb{R})$, which further equals (in distribution) the density f_{ISE} of μ_{ISE}; hence the conclusion of Theorem 3.1 follows. It thus remains only to prove the following lemma.

LEMMA 7.2. *The sequence* $g_n(x) := n^{-1}\gamma^{-1}n^{1/4}\bar{X}_n(\gamma^{-1}n^{1/4}x)$, $n = 1, 2, \dots$, *is tight in* $C_0(\mathbb{R})$.

The central estimate in the proof of Lemma 7.2, and thus of Theorem 3.1, is the following, which will be proved in Section 8. For a sequence $x(j)$, we define its Fourier transform by $\widehat{x}(u) := \sum_j x(j)e^{\mathrm{i}ju}$; this equals the Fourier transform of the measure $\sum_j x(j)\delta_j$ on $\mathbb{R}$.

LEMMA 7.3. *There exists a constant* C_1 *such that for all* $n \geq 1$ *and* $u \in [-\pi, \pi]$,

$$\mathbb{E}\,|n^{-1}\widehat{X_n}(u)|^2 \leq \frac{C_1}{1+nu^4}. \tag{7.1}$$

We can now prove Lemma 7.2 as follows. We have $\widehat{Y_n}(y) = n^{-1}\widehat{X_n}(\gamma n^{-1/4}y)$. Consequently, $\widehat{Y_n}$ is a periodic function with period $2\pi\gamma^{-1}n^{1/4}$, and Lemma 7.3 translates to

$$\mathbb{E}\,|\widehat{Y_n}(y)|^2 \leq \frac{C_1}{1+\gamma^4 y^4}, \qquad |y| \leq \gamma^{-1}n^{1/4}\pi. \tag{7.2}$$

Further,

$$\widehat{g_n}(y) = \widehat{\bar{Y}_n}(y) = \widehat{Y_n}(y)\widehat{\nu}_{\gamma n^{-1/4}}(y). \tag{7.3}$$

LEMMA 7.4. *Suppose that* $0 \leq a < 3$. *Then there exists a constant* $C(a)$ *such that if* $h > 0$ *and* f *is a function with period* $2\pi/h$, *then*

$$\int_{-\infty}^{\infty} |y|^a |f(y)|^2 |\widehat{\nu}_h(y)|^2\,dy \leq C(a)\int_{-\pi/h}^{\pi/h} |y|^a|f(y)|^2\,dy.$$

PROOF. By the change of variables $y \mapsto h^{-1}y$, we may assume $h = 1$. Then $\widehat{\nu}_1(y) = (\sin(y/2)/(y/2))^2$. Hence, for $k \neq 0$ and $|y| \leq \pi$,

$$|\widehat{\nu}_1(y+2k\pi)| = \frac{\sin^2(y/2)}{(k\pi+y/2)^2} \leq \frac{y^2}{k^2}$$

and

$$\begin{aligned}\int_{(2k-1)\pi}^{(2k+1)\pi} |y|^a|f(y)|^2|\widehat{\nu}_1(y)|^2\,dy &\leq (3\pi|k|)^a \int_{-\pi}^{\pi} |f(y)|^2 \frac{y^4}{k^4}\,dy \\ &\leq C_1(a)|k|^{a-4}\int_{-\pi}^{\pi}|y|^a|f(y)|^2\,dy.\end{aligned}$$

For the case $k = 0$, we use instead the estimate $|\widehat{\nu}_1(y)| \leq 1$. The result follows by summing over all k. ∎

Let $h := \gamma n^{-1/4}$. Then, by Lemma 7.4, (7.3) and (7.2), for any fixed a with $0 \le a < 3$,

$$\begin{aligned}\mathbb{E}\int_{-\infty}^{\infty} |y|^a |\widehat{g_n}(y)|^2\,dy &\le C(a)\,\mathbb{E}\int_{-\pi/h}^{\pi/h} |y|^a |\widehat{Y_n}(y)|^2\,dy \\ &= C(a)\int_{-\pi/h}^{\pi/h} |y|^a\,\mathbb{E}\,|\widehat{Y_n}(y)|^2\,dy \\ &\le C_2(a)\int_{-\infty}^{\infty} \frac{|y|^a}{1+\gamma^4 y^4}\,dy \le C_3(a).\end{aligned}$$

We have proved the following, taking $a = 2\alpha$.

LEMMA 7.5. *If $0 \le \alpha < 3/2$, then $\mathbb{E}\,\|g_n\|_{2,\alpha}^2 \le C(\alpha)$, for some $C(\alpha)$ not depending on n.* □

Next, fix $\beta \in (0,1)$, and let $\alpha = \beta + 1/2 < 3/2$. For $A, M > 0$, let $K_{M,A}$ be the set of all functions f in $C_0(\mathbb{R})$ such that $f(x) = 0$ for $|x| \ge M$ and $\|f\|_{2,\alpha} \le A$. By Lemma 6.2(ii), the functions in $K_{M,A}$ are all Hölder(β)-continuous with uniformly bounded norm; they thus form an equicontinuous family. We may regard $K_{M,A}$ as a subset of $C[-M,M]$, the space of continuous functions on the compact interval $[-M,M]$, and it follows by the Arzela–Ascoli theorem [28, A5] that $K_{M,A}$ is a relatively compact subset of $C[-M,M]$, and thus of $C_0(\mathbb{R})$ too. (Note that the functions in $K_{M,A}$ all vanish at $\pm M$.)

Let $\varepsilon > 0$. It follows from Lemma 7.5 that there exists A such that $\mathbb{P}(\|g_n\|_{2,\alpha} > A) \le \varepsilon/2$ for every n. Moreover, Marckert [23, Theorem 5] also showed that

$$n^{-1/4}\sup\{|j| : X_n(j) \ne 0\} = n^{-1/4}\sup\{|\ell(v)| : v \in T_n\} \xrightarrow{\mathrm{d}} W \tag{7.4}$$

for some random variable W. It follows from (7.4) that there exists M such that

$$\begin{aligned}&\mathbb{P}(g_n(x) \ne 0 \text{ for some } x \text{ with } |x| > M) \\ &\qquad = \mathbb{P}\big(X_n(j) \ne 0 \text{ for some } j \text{ with } |j| > \gamma^{-1} n^{1/4} M - 1\big) < \varepsilon/2.\end{aligned}$$

Consequently, $\mathbb{P}(g_n \in K_{M,A}) > 1 - \varepsilon$ for every n, which shows that the sequence (g_n) is tight. This completes the proof of Lemma 7.2, and thus of Theorem 3.1, except for the proof of Lemma 7.3.

REMARK 7.6. A more concrete alternative to the compactness argument (Lemma 7.1) used above is to define regularizations of functions f on $\mathbb{R}$ by

$f^{(h)}(x) := h^{-1}\int_x^{x+h} f$ for $h > 0$. Note that $f^{(h)}_{\mathrm{ISE}}(x) = h^{-1}\mu_{\mathrm{ISE}}[x, x+h]$. For each fixed $h > 0$, the version of (3.1) with both left- and right-hand side regularized holds since the corresponding distribution functions converge. We may then let $h \to 0$, using the Hölder estimate obtained by Lemmas 7.5 and 6.2 together with [6, Theorem 4.2].

8. Proof of Lemma 7.3. It remains to prove Lemma 7.3. We consider first the case of random binary trees with the natural labelling. We introduce the sequence of generating functions

$$F_k(t, x_1, \dots, x_k) := \sum_{T \in \mathcal{T}} t^{|T|} \prod_{i=1}^{k} \Big(\sum_{v \in T} x_i^{\ell(v)} \Big), \tag{8.1}$$

where $\mathcal{T}$ is the family of all (possibly empty) binary trees and $|T|$ is the number of nodes in T. Thus F_k is a power series in t, with coefficients in $\mathbb{Z}[x_1, \dots, x_k, 1/x_1, \dots, 1/x_k]$, the ring of Laurent polynomials in the x_i with integer coefficients. For $k = 0$, the product in the definition of F_0 reduces to 1, so that F_0 is simply the generating function of binary trees. In what follows, we often denote $\boldsymbol{x} = (x_1, \dots, x_k)$ and $F_k(\boldsymbol{x}) = F_k(t, x_1, \dots, x_k)$. Moreover, for any subset I of $[k] := \{1, 2, \dots, k\}$, we denote $\boldsymbol{x}_I = (x_{i_1}, x_{i_2}, \dots, x_{i_p})$ if $I = \{i_1, \dots, i_p\}$ with $i_1 < \cdots < i_p$.

PROPOSITION 8.1. *The series F_k can be determined by induction on $k \geq 0$ using*

$$F_k(\boldsymbol{x}) = \mathbf{1}_{[k=0]} + t \sum_{(I,J)} \Big(\prod_{i \in I} \bar{x}_i \Big) \Big(\prod_{j \in J} x_j \Big) F_{|I|}(\boldsymbol{x}_I) F_{|J|}(\boldsymbol{x}_J) \tag{8.2}$$

where the sum runs over all ordered pairs (I, J) of subsets of $[k]$ such that $I \cap J = \emptyset$, and $\bar{x}_i = 1/x_i$. In particular,

$$F_0 = \frac{1 - \sqrt{1 - 4t}}{2t} \tag{8.3}$$

and each $F_k(\boldsymbol{x})$ admits a rational expression in F_0 and the x_i.

PROOF. The equation satisfied by F_0 reads $F_0 = 1 + tF_0^2$ and is of course very classical: it is obtained by splitting a binary tree into its left and right subtrees. Note that the empty binary tree does not contribute to F_k when $k > 0$. Then, every non-empty binary tree is formed of a root with a left subtree T_1 and a right subtree T_2. Hence, for $k \geq 1$,

$$\begin{aligned} F_k(\boldsymbol{x}) &= \sum_{T_1,T_2} t^{1+|T_1|+|T_2|} \prod_{i=1}^{k} \left(1 + \sum_{v\in T_1} x_i^{\ell(v)-1} + \sum_{v\in T_2} x_i^{\ell(v)+1}\right) \\ &= \sum_{T_1,T_2} t^{1+|T_1|+|T_2|} \sum_{(I,J)} \prod_{i\in I} \left(\sum_{v\in T_1} x_i^{\ell(v)-1}\right) \prod_{j\in J} \left(\sum_{v\in T_2} x_j^{\ell(v)+1}\right) \end{aligned}$$

where the sets I and J are as in the statement of the proposition. The result follows upon exchanging the two sums. □

Actually, for the proof of Lemma 7.3 we need only a special case of F_2. The above proposition gives a simple explicit expression of $F_2(t, x, y)$ in terms of F_0 (and x and y), or, equivalently, in terms of the generating function $B = F_0 - 1$ of non-empty binary trees:

$$B = B(t) = \frac{1 - 2t - \sqrt{1-4t}}{2t}. \tag{8.4}$$

Corollary 8.2. *For any real u,*

$$F_2(t, e^{\mathrm{i}u}, e^{-\mathrm{i}u}) = \frac{B(1+B)(1+2B-B^2)}{(1-B)(1+B-2B\cos u)^2}. \tag{8.5}$$

Proof. The cases $k = 1$ and $k = 2$ of the previous proposition give

$$F_1(x) = tF_0^2 + t(x+\bar{x})F_0F_1(x)$$

and

$$\begin{aligned} F_2(x,y) = tF_0^2 &+ t(x+\bar{x})F_0F_1(x) + t(y+\bar{y})F_0F_1(y) \\ &+ t(\bar{x}y + x\bar{y})F_1(x)F_1(y) + t(xy+\bar{x}\bar{y})F_0F_2(x,y). \end{aligned}$$

Using $F_0 = 1 + B$ and $t = B/(1+B)^2$, this gives

$$\begin{aligned} F_1(x) &= \frac{B(1+B)}{1+B(1-x-\bar{x})}, \\ F_2(x,y) &= \frac{B(1+B)(1+2B+B^2(1-xy-\bar{x}\bar{y}))}{(1+B(1-x-\bar{x}))(1+B(1-y-\bar{y}))(1+B(1-xy-\bar{x}\bar{y}))}. \end{aligned}$$

Specializing to $x = 1/y = e^{\mathrm{i}u}$ provides the result. □

By definition,

$$\widehat{X_n}(u) = \sum_j X(j;T_n)e^{\mathrm{i}ju} = \sum_{v\in T_n} e^{\mathrm{i}\ell(v)u}. \tag{8.6}$$

Hence, if $\mathcal{T}_n := \{T \in \mathcal{T} : |T| = n\}$ is the family of binary trees of size n,

$$\mathbb{E}\,|\widehat{X_n}(u)|^2 = |\mathcal{T}_n|^{-1} \sum_{T\in\mathcal{T}_n} \Big|\sum_{v\in T} e^{\mathrm{i}\ell(v)u}\Big|^2$$

and

$$F_2(t, e^{\mathrm{i}u}, e^{-\mathrm{i}u}) = \sum_{n=1}^{\infty} t^n |\mathcal{T}_n|\,\mathbb{E}\,|\widehat{X_n}(u)|^2. \tag{8.7}$$

Let $[t^n]F(t)$ denote the coefficient of t^n in a power series $F(t)$. Since $|\mathcal{T}_n| = [t^n]B(t) = \frac{1}{n+1}\binom{2n}{n} \sim \pi^{-1/2}n^{-3/2}4^n$, (7.1) is equivalent to

$$[t^n]F_2(t, e^{\mathrm{i}u}, e^{-\mathrm{i}u}) \le C_2 4^n \frac{n^{1/2}}{1+nu^4}, \qquad |u| \le \pi. \tag{8.8}$$

We will prove this using complex analysis. We begin by studying B.

Lemma 8.3. *$B = B(t)$ is a bounded analytic function of t in the domain $\mathcal{D} := \mathbb{C} \setminus [1/4, +\infty)$. Moreover, for $t \in \partial\mathcal{D} = [1/4, +\infty)$, B has continuous boundary values $B_+(t)$ and $B_-(t)$ from the upper and lower side. $B(t)$ (extended by B_+ or B_-) is real if and only if $t \in (-\infty, 1/4]$; on this interval $B(t)$ is strictly increasing from -1 to $B(1/4) = 1$.*

Proof. The first assertions are immediate from (8.4). Next, if $B(t)$ is real, then so is $t = B/(1+B)^2$. It follows further from (8.4) that $B(t)$ is real for $t \le 1/4$, but $B_\pm(t)$ is not real for $t > 1/4$. The formula $t = B/(1+B)^2$ shows further that $B = -1$ is impossible, and that $B = 1$ if and only if $t = 1/4$. Since $B(t) \to -1$ as $t \to -\infty$, it follows by continuity that $-1 < B(t) < 1$ for $t < 1/4$. For such t we have $dB/dt = (dt/dB)^{-1} = (1+B)^3/(1-B) > 0$, which completes the proof. □

Let us for simplicity write $F_u(t) := F_2(t, e^{\mathrm{i}u}, e^{-\mathrm{i}u})$.

We first observe that, for any real u, $F_u(t)$ is an analytic function of t in the domain $\mathcal{D}' := \mathcal{D} \setminus (-\infty, -3/4]$. Indeed, by Corollary 8.2 and Lemma 8.3, $F_u(t)$ is meromorphic in $\mathcal{D}$ with poles when $1-B = 0$ or $1+B-2B\cos u = 0$. In the first case, $B = 1$ and thus $t = B(1+B)^{-2} = 1/4$, which is outside $\mathcal{D}$. In the second case, $B = 1/(2\cos u - 1)$. Since $2\cos u - 1 \in [-3, 1]$, this means that B is real and either $B \geq 1$ or $B \leq -1/3$. By Lemma 8.3, $B \geq 1$ is impossible in $\mathcal{D}$, while $B \leq -1/3$ implies $t = B(1+B)^{-2} \leq -3/4$.

At this stage, we can apply, for any *fixed* value of u, the standard results of *singularity analysis* [12]. For $u = 0$, we find

$$[t^n]F_u(t) = \frac{4^n n^{1/2}}{\sqrt{\pi}} \left(1 + O(1/n)\right),$$

while for $u \neq 0$,

$$[t^n]F_u(t) = \frac{1}{2(1-\cos u)^2} \frac{4^n n^{-1/2}}{\sqrt{\pi}} \left(1 + O(1/n)\right).$$

These results are certainly compatible with the desired bound (8.8), but, as we need a *uniform* bound, valid for all u, we have to resort to the basic principles of singularity analysis.

By the Cauchy integral formula,

$$[t^n]F_u(t) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma F_u(t) \frac{dt}{t^{n+1}} \tag{8.9}$$

for any contour Γ in $\mathcal{D}'$ that loops once around 0. We assume $n > 4$ and choose a contour $\Gamma = \Gamma^{(n)}$ that depends on n: $\Gamma^{(n)} := \Gamma_1 \cup \Gamma_+ \cup \Gamma_2 \cup \Gamma_-$, where Γ_1 is the circle $|t - 1/4| = 1/n$ in negative direction, Γ_+ and Γ_- are both the interval $[1/4 + 1/n, 1/2]$, taken in opposite directions and using the boundary values B_+ and B_-, respectively, and Γ_2 is the circle $|t| = 1/2$ in positive direction. (For convenience, we have pushed the contour to include part of the boundary of $\mathcal{D}'$; the reader who prefers staying strictly inside $\mathcal{D}'$ may replace $\Gamma_\pm$ by line segments close to the real axis.)

Next we estimate $|F_u(t)|$ on $\Gamma^{(n)}$.

LEMMA 8.4. *For all $t \in \Gamma^{(n)}$ and $u \in [-\pi, \pi]$,*

$$|F_u(t)| \leq C_3 \frac{n^{3/2}}{1 + nu^4}. \tag{8.10}$$

PROOF. We claim that for $t \in \Gamma^{(n)}$

$$|1 - B(t)| \geq c_1 n^{-1/2}, \tag{8.11}$$

$$\left|1 + B(t) - 2\cos u B(t)\right| \geq c_2 \max(n^{-1/2}, 1 - \cos u). \tag{8.12}$$

The result then follows from (8.5) and $1 - \cos u \geq c_3 u^2$.

In fact, (8.11) is the special case $u = 0$ of (8.12), so it suffices to prove the latter. Since $B(t)$ vanishes only for $t = 0$, compactness shows that $|B(t)| \geq c_4 > 0$ for $t \in \Gamma \subset \{t : \frac{1}{20} \leq |t| \leq \frac{1}{2}\}$. Hence, it is enough to prove

$$|B(t)^{-1} + 1 - 2\cos u| \geq c_5 \max(n^{-1/2}, 1 - \cos u). \tag{8.13}$$

Indeed,

$$\begin{aligned} B(t)^{-1} + 1 - 2\cos u &= \frac{1 - 2t + \sqrt{1-4t}}{2t} + 1 - 2\cos u \\ &= \frac{1 - 4t + \sqrt{1-4t}}{2t} + 2(1 - \cos u). \end{aligned} \tag{8.14}$$

For $t \in \Gamma_1$, this is $2\sqrt{1-4t} + O(1/n) + 2(1 - \cos u)$. Since $\operatorname{Re}\sqrt{1-4t} \geq 0$ and $1 - \cos u \geq 0$, then

$$\begin{aligned} |B(t)^{-1} + 1 - 2\cos u| &\geq |2\sqrt{1-4t} + 2(1 - \cos u)| - O(1/n) \\ &\geq \max\{2|\sqrt{1-4t}|, 2(1 - \cos u)\} - O(1/n) \\ &= \max\{4n^{-1/2}, 2(1 - \cos u)\} - O(1/n), \end{aligned}$$

which yields (8.13) for $t \in \Gamma_1$.

On $\Gamma_\pm$, $\sqrt{1-4t}$ is imaginary, and

$$\bigl|\operatorname{Im}\bigl(B(t)^{-1} + 1 - 2\cos u\bigr)\bigr| = \frac{|\sqrt{1-4t}|}{2t} \geq \sqrt{4t-1} \geq 2n^{-1/2}. \tag{8.15}$$

Moreover, if further $1 - \cos u \leq 2\sqrt{4t-1}$, (8.15) also yields $\bigl|\operatorname{Im}\bigl(B(t)^{-1} + 1 - 2\cos u\bigr)\bigr| \geq \sqrt{4t-1} \geq \frac{1}{2}(1 - \cos u)$. If, on the contrary, $1 - \cos u > 2\sqrt{4t-1}$, then, because $0 \leq 4t - 1 \leq 1$ and thus $4t - 1 \leq \sqrt{4t-1}$,

$$\begin{aligned} \bigl|\operatorname{Re}\bigl(B(t)^{-1} + 1 - 2\cos u\bigr)\bigr| &= 2(1 - \cos u) - \frac{4t-1}{2t} \\ &\geq 2(1 - \cos u) - 2(4t - 1) \geq 1 - \cos u. \end{aligned}$$

In both cases, $\bigl|B(t)^{-1} + 1 - 2\cos u\bigr| \geq \frac{1}{2}(1 - \cos u)$, which together with (8.15) completes the verification of (8.13) for $t \in \Gamma_\pm$.

Finally, for $t \in \Gamma_2$, we use compactness. We observed above that $1 + B(t)(1 - 2\cos u) = 0$ is possible only for $t \in (-\infty, -3/4] \cup \{1/4\}$, and in particular not for $t \in \Gamma_2$; hence $\inf_{t \in \Gamma_2,\, u \in [0,2\pi]} |B(t)^{-1} + 1 - 2\cos u| = c_6 > 0$, which implies (8.13) for $t \in \Gamma_2$. □

By (8.9) and (8.10),

$$[t^n]F_u(t) \leq \int_\Gamma |F_u(t)|\,|t|^{-n-1}\,|dt| \leq C_3 \frac{n^{3/2}}{1 + nu^4} \int_\Gamma |t|^{-n-1}\,|dt|. \tag{8.16}$$

For $t \in \Gamma_1$, $|t|^{-n-1} = O(4^n)$, and thus $\int_{\Gamma_1} |t|^{-n-1}\,|dt| = O(n^{-1}4^n)$. Secondly, $\int_{\Gamma_\pm} |t|^{-n-1}\,|dt| \le \int_{1/4}^{1/2} t^{-n-1}\,dt \le n^{-1}4^n$. Finally, $\int_{\Gamma_2} |t|^{-n-1}\,|dt| = O(2^n)$. Summing these estimates we find $\int_{\Gamma} |t|^{-n-1}\,|dt| = O(n^{-1}4^n)$, which together with (8.16) completes the proof of (8.8) and thus Lemma 7.3 in the case of binary trees.

For complete binary trees, we use the well-known equivalence between binary and complete binary trees, where a binary tree T of order n is identified with the internal nodes in a complete binary tree T^c of order $2n+1$. With this identification, one has

$$X(j;T^c) = \begin{cases} X(j-1;T) + X(j+1;T) & \text{if } j \neq 0, \\ 1 + X(-1;T) + X(1;T) & \text{otherwise.} \end{cases}$$

Hence, temporarily using X_n^c instead of X_n for the complete binary trees, it follows from (8.6) that $\widehat{X_{2n+1}^c}(u) = 1 + 2\cos u \widehat{X_n}(u)$. Hence the estimate in Lemma 7.3 holds for complete binary trees too (possibly with a different constant).

9. Proof of Corollaries 3.2–3.4.

Proof of Corollary 3.2. This is immediate from (3.2) and the fact that $f_n \to f$ in $C_0(\mathbb{R})$ implies $f_n(j_n/n^{1/4}) \to f(x)$, see e.g. [6, Theorem 5.5]. $\square$

Proof of Corollary 3.3. By symmetry, $f_{\mathrm{ISE}}(x) \overset{\mathrm{d}}{=} f_{\mathrm{ISE}}(-x)$, so we may suppose $x \ge 0$. Then, as shown by Bousquet-Mélou [7, §6.2.2, Conjecture 15 and Theorem 14], for naturally embedded random binary trees, $X_n(\lfloor xn^{1/4}\rfloor)/n^{3/4} \xrightarrow{\mathrm{d}} 2^{-1/2}Y(2^{-1/2}x)$ for a family of random variables $Y(u)$, $u \ge 0$, with moment generating functions $\mathbb{E}\, e^{aY(u)} = L(u,a)$. Combining this with Corollary 3.2, we find $2^{-1/4} f_{\mathrm{ISE}}(2^{-1/4}x) \overset{\mathrm{d}}{=} 2^{-1/2}Y(2^{-1/2}x)$, $x \ge 0$, and thus $f_{\mathrm{ISE}}(x) \overset{\mathrm{d}}{=} 2^{-1/4}Y(2^{-1/4}|x|)$. (The normalization of f_{ISE} in [7] is different.) $\square$

Proof of Corollary 3.4. By the proof of Corollary 3.3, $f_{\mathrm{ISE}}(0) \overset{\mathrm{d}}{=} 2^{-1/4}Y(0)$, where, by [7, Proposition 12 and Theorem 14] $Y(0) \overset{\mathrm{d}}{=} 2^{1/2}3^{-1}T^{-1/2}$.

The (negative) moments of T are given by the standard formula $\mathbb{E}\, T^{-s} = \Gamma(3s/2+1)/\Gamma(s+1)$, $s > -2/3$. $\square$

10. Other tree models. Consider a randomly labelled conditioned Galton–Watson tree as in Conjecture 3.5. We know that the global limit result (1.1) holds, and the proof in Section 7 holds *verbatim* in this case too and shows that to prove Conjecture 3.5, it is sufficient to verify that the estimate of Lemma 7.3 holds. We have not been able to do so in general, but we can show the required estimate in the two special cases in Theorem 3.6.

We consider thus in this section the two families of *labelled plane trees* that were studied in [7]. In the first family $\mathcal{T}^1$, the root is labelled 0, and the labels of two adjacent nodes differ by ± 1. In the second family $\mathcal{T}^2$, the latter condition is generalized by allowing the increments along edges to be $0, \pm 1$.

Again, we introduce a sequence of generating functions:

$$F_k(t, x_1, \dots, x_k) \equiv F(\boldsymbol{x}) := \sum_{T \in \mathcal{T}} t^{|T|} \prod_{i=1}^{k} \Big(\sum_{v \in T} x_i^{\ell(v)} \Big), \tag{10.1}$$

where $\mathcal{T}$ is either $\mathcal{T}^1$ or $\mathcal{T}^2$ and $|T|$ is the number of *edges* in T. The following proposition is the counterpart, for each of the two new families, of Proposition 8.1.

Proposition 10.1. *For plane trees with increments ± 1, the series F_k can be determined by induction on $k \geq 0$ using*

$$F_k(\boldsymbol{x}) = 1 + t \sum_{I \subseteq [k]} \left(\prod_{i \in I} \bar{x}_i + \prod_{i \in I} x_i \right) F_{|I|}(x_I) F_{|J|}(x_J)$$

where $J = [k] \setminus I$ and $\bar{x}_i = 1/x_i$. For trees with increments $0, \pm 1$, the above equation becomes

$$F_k(\boldsymbol{x}) = 1 + t \sum_{I \subseteq [k]} \left(1 + \prod_{i \in I} \bar{x}_i + \prod_{i \in I} x_i \right) F_{|I|}(x_I) F_{|J|}(x_J),$$

with the same notation as above. In both cases, each $F_k(\boldsymbol{x})$ admits a rational expression in F_0 and the x_i.

Proof. The proof is very similar to that of Proposition 8.1. We now use the standard recursive description of plane trees based on the deletion of the leftmost subtree T_1 of a tree T (not reduced to a single node). This leaves another plane tree T_2. Also, one has to take into account the fact that the label of the root of T_1 may now take two (or three) different values (depending on the family of trees under consideration). Finally, the tree reduced to a single node contributes 1 in each F_k. When $\mathcal{T} = \mathcal{T}^1$ and $k \geq 0$, this gives

$$F_k(\boldsymbol{x}) = 1 + \sum_{T_1,T_2} t^{1+|T_1|+|T_2|} \prod_{i=1}^{k} \left(\sum_{v\in T_1} x_i^{\ell(v)-1} + \sum_{v\in T_2} x_i^{\ell(v)} \right) + \sum_{T_1,T_2} t^{1+|T_1|+|T_2|} \prod_{i=1}^{k} \left(\sum_{v\in T_1} x_i^{\ell(v)+1} + \sum_{v\in T_2} x_i^{\ell(v)} \right),$$

and the result follows after expanding the products, and then exchanging the sums. ☐

We easily find explicit formulas for F_1 and F_2 from Proposition 10.1, cf. the proof of Corollary 8.2. We leave the details to the reader and state only the result that we need, in terms of the series $T = T(t)$ that counts labelled trees not reduced to a single node. Depending on which tree family is studied, one has

$$T = T^{(1)} := B(2t) = \frac{1 - 4t - \sqrt{1-8t}}{4t} \qquad \text{for } \mathcal{T}^1,$$
$$T = T^{(2)} := B(3t) = \frac{1 - 6t - \sqrt{1-12t}}{6t} \qquad \text{for } \mathcal{T}^2,$$

where the series $B(t)$ is defined by (8.4).

COROLLARY 10.2. *For plane trees with increments* ± 1*,*

$$F_2(t, e^{\mathrm{i}u}, e^{-\mathrm{i}u}) = \frac{(1+T)\left(1 + T^2\cos^2 u\right)}{(1-T)(1-T\cos u)^2}. \tag{10.2}$$

For plane trees with increments $0, \pm 1$*,*

$$F_2(t, e^{\mathrm{i}u}, e^{-\mathrm{i}u}) = \frac{(1+T)(9 + T^2\,(1+2\cos u)^2)}{(1-T)(3 - T(1+2\cos u))^2}. \tag{10.3}$$

We may now complete the proof of Theorem 3.6 by the argument in Section 8; we give a sketch only and leave again the details to the reader. First, the functions $F_2(t/2, e^{\mathrm{i}u}, e^{-\mathrm{i}u})$ (for $\mathcal{T}^1$) and $F_2(t/3, e^{\mathrm{i}u}, e^{-\mathrm{i}u})$ (for $\mathcal{T}^2$) are analytic functions of $t \in \mathcal{D}$ for every real u. Next, in analogy with Lemma 8.4, with the same contour $\Gamma^{(n)}$ as there, for $t \in \Gamma$ and $|u| \leq \pi$,

$$\left|F_2(t/2, e^{\mathrm{i}u}, e^{-\mathrm{i}u})\right| \leq C_4 \frac{n^{3/2}}{1+nu^4} \qquad \text{for } \mathcal{T}^1,$$
$$\left|F_2(t/3, e^{\mathrm{i}u}, e^{-\mathrm{i}u})\right| \leq C_5 \frac{n^{3/2}}{1+nu^4} \qquad \text{for } \mathcal{T}^2.$$

Indeed, the proof is almost exactly the same; we replace the left-hand side of (8.12) by $|1-\cos uB(t)|$ and $|3-B(t)-2\cos uB(t)|$ and similarly the left-hand side (8.13) by $|B(t)^{-1}-\cos u|$ and $|3B(t)^{-1}-1-2\cos u|$, note the corresponding changes in (8.15) and argue as before.

The Cauchy integral formula (8.9) then leads to, cf. (8.8), for $|u|\le\pi$,

$$[t^n]F_2(t,e^{\mathrm{i}u},e^{-\mathrm{i}u})\le C_6 8^n\frac{n^{1/2}}{1+nu^4}, \qquad \text{for } \mathcal{T}^1,$$
$$[t^n]F_2(t,e^{\mathrm{i}u},e^{-\mathrm{i}u})\le C_7 12^n\frac{n^{1/2}}{1+nu^4}, \qquad \text{for } \mathcal{T}^2.$$

By (8.7) and $|\mathcal{T}_n^1| = 2^n[t^n]B(t)\sim\pi^{-1/2}n^{-3/2}8^n$, $|\mathcal{T}_n^2| = 3^n[t^n]B(t)\sim\pi^{-1/2}n^{-3/2}12^n$, this yields (7.1) for these two families. (Note that we have let $|T|$ be the number of edges for $\mathcal{T}^1$ and $\mathcal{T}^2$; thus we now should replace X_n by X_{n+1} in (8.7), but this makes no difference for (7.1).)

This completes the proof of Theorem 3.6.

11. Moments of the density of ISE. We know by Corollary 3.3 that $f_{\mathrm{ISE}}(\lambda)$ has a moment generating function (defined in an interval containing 0), and thus finite moments of all orders. We next present a formula for these moments, and more generally for mixed moments involving several values of λ. We use a general method for occupation densities of Gaussian processes. To state the formula, we introduce more notation.

Given ζ (which as always is $2B^{\mathrm{ex}}$), and k points $s_1,\dots,s_k\in[0,1]$, the random vector $(\overline{W}(s_1),\dots,\overline{W}(s_k))$ has a Gaussian distribution with mean 0 and covariance matrix

$$\Sigma_{\zeta;s_1,\dots,s_k} := \big(m(s_i,s_j;\zeta)\big)_{i,j=1}^k. \tag{11.1}$$

We let $\varphi_{\zeta;s_1,\dots,s_k}$ denote the density function of this distribution. (We may ignore the cases when the distribution is degenerate; if $s_1,\dots,s_p$ are distinct, the distribution is non-degenerate a.s. with respect to ζ.)

Using the construction in Subsection 5.2 of the Brownian CRT from ζ, we can transfer these notations to the CRT. Given ζ and k points $x_1,\dots,x_k$ in the corresponding CRT, the random vector $(\widetilde{W}(x_1),\dots,\widetilde{W}(x_k))$ has a Gaussian distribution with mean 0 and covariance matrix

$$\Sigma_{\zeta;x_1,\dots,x_k} := \big(m(x_i,x_j;\zeta)\big)_{i,j=1}^k, \tag{11.2}$$

where $m(x,y;\zeta)$ is the length of the common part of the paths from the root to x and y in the CRT. We let $\varphi_{\zeta;x_1,\dots,x_k}$ denote the density function of this distribution, and note that if $x_i = \tilde{\zeta}(s_i)$, $i = 1, \dots, k$, then $m(x_i, x_j; \zeta) = m(s_i, s_j; \zeta)$ and $\varphi_{\zeta;x_1,\dots,x_k} = \varphi_{\zeta;s_1,\dots,s_k}$.

We further let $X_1, \dots, X_k$ denote k independent random nodes in the Brownian CRT (with the uniform distribution ν).

THEOREM 11.1. *For any real numbers $\lambda_1, \dots, \lambda_k$,*

(11.3)

$$\begin{aligned}\mathbb{E}\big(f_{\text{ISE}}(\lambda_1)\cdots f_{\text{ISE}}(\lambda_k)\big) &= \mathbb{E}\int_0^1\cdots\int_0^1 \varphi_{\zeta;s_1,\dots,s_k}(\lambda_1,\dots,\lambda_k)\,ds_1\cdots ds_k\\ &= \mathbb{E}\,\varphi_{\zeta;X_1,\dots,X_k}(\lambda_1,\dots,\lambda_k).\end{aligned}$$

PROOF. The equality of the last two expressions follows by the construction of the Brownian CRT and the definitions above.

We define, for $\lambda \in \mathbb{R}$ and $h > 0$,

$$\begin{aligned}Z_h(\lambda) &= h^{-1}\int_0^1 \mathbf{1}_{[\overline{W}(s)\in[\lambda,\lambda+h]]}\,ds = h^{-1}\mu_{\text{ISE}}[\lambda,\lambda+h]\\ &= h^{-1}\int_\lambda^{\lambda+h} f_{\text{ISE}}(y)\,dy.\end{aligned} \tag{11.4}$$

Since f_{ISE} is continuous by Theorem 2.1, $Z_h(\lambda) \to f_{\text{ISE}}(\lambda)$ a.s. as $h \to 0$. From this definition follows

$$\begin{aligned}&\mathbb{E}\big(Z_h(\lambda_1)\cdots Z_h(\lambda_k)\mid\zeta\big)\\ &= \int_0^1\cdots\int_0^1 h^{-k}\,\mathbb{P}\big(\overline{W}(s_i)\in[\lambda_i,\lambda_i+h],\ i=1,\dots,k\mid\zeta\big)\,ds_1\cdots ds_k\\ &= \int_0^1\cdots\int_0^1 h^{-k}\int_{\lambda_1}^{\lambda_1+h}\cdots\int_{\lambda_k}^{\lambda_k+h}\varphi_{\zeta;s_1,\dots,s_k}(y_1,\dots,y_k)\,dy_1\cdots dy_k\,ds_1\cdots ds_k\\ &= \mathbb{E}\bigg(h^{-k}\int_{\lambda_1}^{\lambda_1+h}\cdots\int_{\lambda_k}^{\lambda_k+h}\varphi_{\zeta;X_1,\dots,X_k}(y_1,\dots,y_k)\,dy_1\cdots dy_k\,\bigg|\,\zeta\bigg)\end{aligned}$$

and thus

$$\begin{aligned}&\mathbb{E}\big(Z_h(\lambda_1)\cdots Z_h(\lambda_k)\big)\\ &\qquad= \mathbb{E}\,h^{-k}\int_{\lambda_1}^{\lambda_1+h}\cdots\int_{\lambda_k}^{\lambda_k+h}\varphi_{\zeta;X_1,\dots,X_k}(y_1,\dots,y_k)\,dy_1\cdots dy_k.\end{aligned} \tag{11.5}$$

To obtain the conclusion, we now let $h \to 0$; however, we have to justify taking the limit inside the expectations on both sides. For the right-hand side, we use the fact that a non-degenerate Gaussian distribution in $\mathbb{R}^k$ with mean 0 has a density function that has its maximum at 0; hence we can, by Lemma 11.2 below, use dominated convergence with $\varphi_{\zeta;X_1,\dots,X_k}(0,\dots,0)$ as dominating function. Since $\varphi_{\zeta;X_1,\dots,X_k}$ is continuous, the right-hand side of (11.5) thus converges to the right-hand side of (11.3).

For the left-hand side we begin by applying Fatou's lemma, which now shows that the left-hand side of (11.3) is *at most* equal to the right-hand side. By Lemma 11.2 below, this yields a uniform bound, C_k say, of the left-hand side for all $\lambda_1,\dots,\lambda_k$. It follows from (11.4) that $\mathbb{E}(Z_h(\lambda_1)\cdots Z_h(\lambda_k)) \le C_k$ too, for every $h>0$. If we here replace k by $2k$, repeating every λ_i twice, we see that the random variables $V_h := Z_h(\lambda_1)\cdots Z_h(\lambda_k)$ satisfy $\mathbb{E}\,V_h^2 \le C_{2k}$. The variables V_h are thus uniformly integrable, and from $V_h \to f_{\mathrm{ISE}}(\lambda_1)\cdots f_{\mathrm{ISE}}(\lambda_k)$ as $h\to 0$ follows $\mathbb{E}\,V_h \to \mathbb{E}(f_{\mathrm{ISE}}(\lambda_1)\cdots f_{\mathrm{ISE}}(\lambda_k))$, see e.g. [14, Theorems 5.4.2 and 5.5.2]. □

LEMMA 11.2. *For every $k\ge 1$, $\mathbb{E}\,\varphi_{\zeta;X_1,\dots,X_k}(0,\dots,0)<\infty$.*

PROOF. The subtree R_k of the Brownian CRT spanned by $X_1,\dots,X_k$ and the root o has $k-1$ internal nodes. Let R_k' be the subtree spanned by o and the internal nodes of R_k, and let $\ell_1,\dots,\ell_k$ be the lengths of the k edges that attach $X_1,\dots,X_k$ to R_k'. The values of $\widetilde{W}$ along R_k form a branching Brownian motion, i.e., $\widetilde{W}$ is a Brownian motion along each edge of R_k and all increments are independent. In particular, conditioned on R_k and the values of $\widetilde{W}$ on R_k', the values $\widetilde{W}(X_1),\dots,\widetilde{W}(X_k)$ at the leaves are independent Gaussian variables with some means $b_1,\dots,b_k$ and variances $\ell_1,\dots,\ell_k$. The conditional density function is thus at most $\prod_1^k(2\pi\ell_i)^{-1/2}$, and thus, taking the expectation and using (5.2),

$$
\begin{aligned}
&\mathbb{E}\,\varphi_{\zeta;X_1,\dots,X_k}(0,\dots,0) \le \mathbb{E}\prod_1^k(2\pi\ell_i)^{-1/2}\\
&= (2k-3)!!\,(2\pi)^{-k/2}\int\cdots\int\prod_1^k \ell_i^{-1/2}\Big(\sum_{i=1}^{2k-1}\ell_i\Big)e^{-\frac12\left(\sum_1^{2k-1}\ell_i\right)^2}\,d\ell_1\cdots d\ell_{2k-1}\\
&<\infty.
\end{aligned}
$$

□

Since the distribution of the covariance matrix $\Sigma_{\zeta;X_1,\dots,X_k}$ is given by (11.2) and (5.2), it is in principle possible to write the right-hand side of (11.3) as a multiple integral. However, the expression becomes rather complicated for higher moments. In the simplest case $\lambda_1=\cdots=\lambda_k=0$, (11.3) reduces to $\mathbb{E}\,f_{\mathrm{ISE}}(0)^k=(2\pi)^{-k/2}\,\mathbb{E}\big(\det\big(\Sigma_{\zeta;X_1,\dots,X_k}\big)^{-1/2}\big)$, but even this seem difficult to compute in general. (These moments were found by another method in Corollary 3.4.)

In the case $k = 1$, Theorem 11.1 yields a simple formula for the average $\mathbb{E} f_{\mathrm{ISE}}$ of the density, which equals the density of the average $\mathbb{E}\mu_{\mathrm{ISE}}$, i.e. the density of a random point chosen according to the random ISE. In the latter formulation, it was found by Aldous [4].

COROLLARY 11.3. *For any real* λ,

$$\mathbb{E} f_{\mathrm{ISE}}(\lambda) = (2\pi)^{-1/2} \int_0^\infty y^{1/2} \exp\Bigl(-\frac{\lambda^2}{2y} - \frac{y^2}{2}\Bigr)\, dy.$$

PROOF. $\varphi_{\zeta;X_1}(\lambda) = (2\pi y)^{-1/2} e^{-\lambda^2/(2y)}$, where $y = d(X_1, o)$, and y has the density function $ye^{-y^2/2}$ by (5.2). $\square$

Alternatively, expanding the Laplace transform of Corollary 3.3 in a gives (see [7, Proposition 13]):

$$\mathbb{E} f_{\mathrm{ISE}}(\lambda) = \frac{2^{-1/4}}{\sqrt{\pi}} \sum_{m\ge 0} \frac{(-2^{3/4}|\lambda|)^m}{m!} \cos\frac{(m+1)\pi}{4} \Gamma\left(\frac{m+3}{4}\right).$$

Both expressions yield $\mathbb{E} f_{\mathrm{ISE}}(0) = 2^{-3/4}\pi^{-1/2}\Gamma(3/4)$, as given by Corollary 3.4.

From Corollary 11.3 follows easily by integration another formula by Aldous [4]; we leave the proof to the reader.

COROLLARY 11.4. *For every real* $a > -1$,

$$\mathbb{E}\int_{-\infty}^{\infty} |x|^a \, d\mu_{\mathrm{ISE}}(x) = \mathbb{E}\int_{-\infty}^{\infty} |x|^a f_{\mathrm{ISE}}(x)\, dx = \frac{2^{3a/4}}{\sqrt{\pi}} \Gamma\Bigl(\frac{a}{2} + \frac{1}{2}\Bigr)\Gamma\Bigl(\frac{a}{4} + 1\Bigr).$$

$\square$

This extends (4.5). (To see the equivalence when $a = 2k$, use the duplication formula for the Gamma function twice.)

12. The grand-moments of the ISE: Proofs.

PROOF OF THEOREM 4.1. Let f be a continuous function on $\mathbb{R}$. First, if f is bounded, then $\mu \mapsto \int f\, d\mu$ is a continuous functional on the space of probability measures on $\mathbb{R}$, and since $\mu_n \xrightarrow{\mathrm{d}} \mu_{\mathrm{ISE}}$ in this space, see (1.1), it follows that

$$\int f\, d\mu_n \xrightarrow{\mathrm{d}} \int f\, d\mu_{\mathrm{ISE}}. \tag{12.1}$$

We need to extend this to unbounded f. Thus, let f_u, for $u > 0$, be the function that is equal to f on $[-u, u]$, and is constant on $(-\infty, -u]$ and on $[u, \infty)$. Since f_u is bounded, (12.1) applies to f_u, i.e. $\int f_u \, d\mu_n \xrightarrow{\mathrm{d}} \int f_u \, d\mu_{\mathrm{ISE}}$ for every $u > 0$. Moreover, let $V_n := \sup\{|x| : x \in \operatorname{supp} \mu_n\} = (2n)^{-1/4} \sup\{|\ell(v)| : v \in T_n\}$. By Marckert [23, Theorem 5], $V_n \xrightarrow{\mathrm{d}} V$ for some random variable V. Consequently,

$$\limsup_{n\to\infty} \mathbb{P}\Big(\int f_u \, d\mu_n \neq \int f \, d\mu_n\Big) \le \limsup_{n\to\infty} \mathbb{P}(V_n > u) \le \mathbb{P}(V \ge u),$$

which tends to 0 as $u \to \infty$. Finally, $\int f_u \, d\mu_{\mathrm{ISE}} \to \int f \, d\mu_{\mathrm{ISE}}$ as $u \to \infty$, since μ_{ISE} has compact support. Consequently, [6, Theorem 4.2] shows that (12.1) holds for any continuous f.

Taking $f(x) = x^k$ in (12.1), we obtain the convergence $m_{k,n} \xrightarrow{\mathrm{d}} m_k$ of the moments asserted in Theorem 4.1. Moreover, taking f to be a linear combination of such monomials, we see that joint convergence holds by the Cramér–Wold device [6, Theorem 7.7].

In particular, for any partition λ, $m_{\lambda,n} \xrightarrow{\mathrm{d}} m_\lambda$. We will show that the expectation $\mathbb{E}(m_{\lambda,n})$ converges as $n \to \infty$. Applying this to the partition λ' where each part in λ is repeated twice, we see that also $\mathbb{E}(m_{\lambda,n}^2) = \mathbb{E}(m_{\lambda',n})$ converges. The variables $m_{\lambda,n}$ are thus uniformly integrable, and the limit of their expectations $\mathbb{E}(m_{\lambda,n})$ equals the expectation $\mathbb{E}(m_\lambda)$ of their limit, see e.g. [14, Theorems 5.4.2 and 5.5.2].

To complete the proof of Theorem 4.1, we thus have to show that the grand-moments $\mathbb{E}(m_{\lambda,n})$ of μ_n converge to the limits stated in the theorem. We introduce the non-normalized moments of μ_n:

$$\bar{M}_{i,n} = \sum_{v \in T_n} \ell(v)^i = 2^{i/4} n^{1+i/4} \, m_{i,n},$$

as well as their factorial version, which is simpler to handle via generating functions:

$$M_{i,n} = \sum_{v \in T_n} \ell(v) \, (\ell(v) - 1) \cdots (\ell(v) - i + 1) \, .$$

We also use the notation $M_{\lambda,n}$, analogous to (4.2), for a partition $\lambda = (\lambda_1, \dots, \lambda_p)$. Then the remaining part of Theorem 4.1 easily follows from:

Proposition 12.1. *As $n \to \infty$, the non-normalized factorial moments of μ_n satisfy*

$$\mathbb{E}(M_{\lambda,n}) = \frac{\Gamma(1/2) n^{p+|\lambda|/4}}{\Gamma(p + |\lambda|/4 - 1/2)} \, (c_\lambda + o(1))$$

PROOF. Let us first relate $M_{\lambda,n}$ to the generating functions of Proposition 8.1. It is simple to see that

$$\begin{aligned}\partial_\lambda F_p &:= \frac{\partial^{|\lambda|} F_p}{\partial x_1^{\lambda_1} \cdots \partial x_p^{\lambda_p}}(t, 1, \ \ldots, 1) \\ &= \sum_{T \in \mathcal{T}} t^{|T|} \prod_{i=1}^{p} \left(\sum_{v \in T} \ell(v)(\ell(v)-1) \cdots (\ell(v)-\lambda_i+1) \right) \qquad (12.2) \\ &= \sum_{n \geq 0} t^n C_n \, \mathbb{E}(M_{\lambda,n}),\end{aligned}$$

where $C_n = \binom{2n}{n}/(n+1)$ is the number of binary trees with n nodes, known as the nth Catalan number. By Proposition 8.1, the series $\partial_\lambda F_p$ is a rational function of t and $\sqrt{1-4t}$. We want to study the singularities of these series. We will prove that, for $p > 0$,

$$\partial_\lambda F_p = \frac{P_\lambda(t) + Q_\lambda(t)\sqrt{1-4t}}{(1-4t)^{e_\lambda}}, \qquad (12.3)$$

where $P_\lambda(t)$ and $Q_\lambda(t)$ are two Laurent polynomials in t, and

$$e_\lambda = p + \frac{1}{2}\left\lfloor \frac{|\lambda|}{2} \right\rfloor - \frac{1}{2} = \begin{cases} p + |\lambda|/4 - 1/2 & \text{if } |\lambda| \text{ is even,} \\ p + |\lambda|/4 - 3/4 & \text{if } |\lambda| \text{ is odd.} \end{cases}$$

(Note that P_λ and Q_λ may be singular at $t = 0$, although $\partial_\lambda F_p$ is analytic there.) From (12.3), it follows that the only possible singularity of $\partial_\lambda F_p$ is at $t = 1/4$, and that, as $t \to 1/4$,

$$\partial_\lambda F_p = \frac{c_\lambda + o(1)}{(1-4t)^{p+|\lambda|/4-1/2}},$$

where $c_\lambda = P_\lambda(1/4)$ when $|\lambda|$ is even and $c_\lambda = 0$ when $|\lambda|$ is odd. We will further show that the numbers c_λ satisfy the recurrence relation (4.3). The form (12.3) and the above singular behaviour do not hold when $p = 0$, and should be replaced in this case by the expression (8.3) of F_0 and the singular behaviour

$$F_0 = 2 - 2\sqrt{1-4t} + O(1-4t).$$

Assume for the moment that we have proved (12.3). Then the standard results of singularity analysis [12] provide

$$[t^n]\partial_\lambda F_p = C_n \, \mathbb{E}(M_{\lambda,n}) = \frac{4^n n^{p+|\lambda|/4-3/2}}{\Gamma(p+|\lambda|/4-1/2)} \left(c_\lambda + o(1)\right).$$

Given that $C_n \sim 4^n n^{-3/2}/\Gamma(1/2)$, this gives the result stated in the proposition. Note that this asymptotic behaviour also holds for $p = 0$, with $c_\emptyset = -2$.

Let us thus focus on (12.3). Our proof works by induction on $p + |\lambda|$.

• If $p = 0$, then λ is the empty partition, and we have worked out above the value of F_0 and its asymptotic behaviour when $t \to 1/4$.

• If $p > 0$ and $\lambda_1 = 0$, then

$$\partial_\lambda F_p = \sum_{n\geq 0} t^n C_n \,\mathbb{E}(M_{\lambda,n}) = \sum_{n\geq 0} t^n C_n n \,\mathbb{E}(M_{\lambda',n}) = t\frac{\partial}{\partial t}\partial_{\lambda'} F_{p-1},$$

where $\lambda' = (\lambda_2, \dots, \lambda_p)$. Then the form (12.3) follows by a simple calculation from the induction hypothesis, and the fact that $e_\lambda = e_{\lambda'} + 1$. (We do not give the details.) This calculation also provides the value of c_λ in terms of $c_{\lambda'}$. The case $p = 1$ and $\lambda = (0)$ has to be treated separately, since in that case $\lambda' = \emptyset$ and the form (12.3) is not valid.

• If $p > 0$ and $\lambda_1 > 0$, then all the parts of λ are positive. Let us differentiate (8.2) λ_1 times with respect to x_1, then λ_2 times with respect to x_2, and so on, and then set $x_i = 1$ in the result. Since $\lambda_i > 0$ for all i, the terms for which $I \cup J \neq [p]$ do not contribute, and we are left with

$$\partial_\lambda F_p = t \sum_{I\subseteq[p]} \partial_{\lambda_I}\left(F_{|I|}(x_I)\prod_{i\in I}\bar{x}_i\right)\partial_{\lambda_J}\left(F_{|J|}(x_J)\prod_{j\in J} x_j\right) \tag{12.4}$$

where $J = [p]\setminus I$ and, for any function $G(t, x_I)$, we denote

$$\partial_{\lambda_I} G = \frac{\partial^{|\lambda_I|} G}{\partial x_{i_1}^{\lambda_{i_1}} \cdots \partial x_{i_r}^{\lambda_{i_r}}}(t, 1, \dots, 1)$$

if $I = \{i_1, \dots, i_r\}$ with $i_1 < \dots < i_r$. Now

$$\partial_{\lambda_I}\left(F_{|I|}(x_I)\prod_{i\in I}\bar{x}_i\right) = \sum_{\sigma\leq\lambda_I}(-1)^{|\lambda_I|-|\sigma|}\partial_\sigma F_{|I|}\prod_{i\in I}\frac{\lambda_i!}{\sigma_i!},$$

where the sum runs over all non-negative $|I|$-tuples $\sigma = (\sigma_i)_{i\in I}$ that are less than or equal to λ_I. The second derivative contains fewer terms:

$$\partial_{\lambda_J}\left(F_{|J|}(x_J)\prod_{j\in J} x_j\right) = \sum_{\varepsilon}\partial_{\lambda_J-\varepsilon}F_{|J|}\prod_{j\in J}\lambda_j^{\varepsilon_j},$$

where the sum runs over all $|J|$-tuples $\varepsilon = (\varepsilon_j)_{j\in J}$ such $\varepsilon_j \in \{0, 1\}$ for all j.

Let us now bravely replace the two derivatives occurring in (12.4) by their sum-expressions given above, and (mentally) expand the product of these sums. This gives $\partial_\lambda F_p$ as a sum over I, σ and ε. In this sum, the series $\partial_\lambda F_p$ appears twice, namely

(i) for $I = \emptyset$, $\sigma = \emptyset$ and $\varepsilon = (0, \dots, 0)$,
(ii) for $I = [p]$, $\sigma = \lambda$ and $\varepsilon = \emptyset$.

The corresponding summands are the same in both cases, namely $tF_0 \partial_\lambda F_p$. Hence (12.4) can be rewritten as

$$(1 - 2tF_0)\partial_\lambda F_p = t \sum_{I,\sigma,\varepsilon} \text{SUMMAND},$$

where the sum now excludes Cases (i) and (ii). In this sum, all terms of the form $\partial_\tau F_k$ now satisfy $k + |\tau| < p + |\lambda|$, so that the induction hypothesis applies to them. Note also that $(1 - 2tF_0) = \sqrt{1-4t}$, so that the previous equation really reads

$$\sqrt{1-4t}\ \partial_\lambda F_p = t \sum_{I,\sigma,\varepsilon} \text{SUMMAND} = \text{RHS}. \tag{12.5}$$

The latter observation is the key in our proof of (12.3).

In the right-hand side of the equation, let us study separately the cases where I or J are empty.

First case: I or J is empty. The contribution of the terms for which $I = \emptyset$ is

$$tF_0 \sum_{\varepsilon \neq 0} \partial_{\lambda - \varepsilon} F_p \prod_{j=1}^{p} \lambda_j^{\varepsilon_j}. \tag{12.6}$$

The contribution of the terms for which $J = \emptyset$ (that is, $I = [p]$) is

$$tF_0 \sum_{\sigma < \lambda} (-1)^{|\lambda| - |\sigma|} \partial_\sigma F_p \prod_{i=1}^{p} \frac{\lambda_i!}{\sigma_i!}. \tag{12.7}$$

We observe that the terms for which $|\varepsilon| = 1$ in (12.6) cancel out with the terms for which $|\sigma| = |\lambda| - 1$ in (12.7). (More generally, the term associated with ε, when $|\varepsilon|$ is odd, cancels out with the term associated with $\sigma = \lambda - \varepsilon$, but we do not need this property). After these cancellations, all the terms $\partial_\tau F_k$ that appear in this part of RHS satisfy $k = p$ and $|\tau| \leq |\lambda| - 2$. In particular, $e_\tau \leq e_\lambda - 1/2$ for each of them. The induction hypothesis then guarantees that this part of RHS is of the form

$$\text{RHS}_1 = \frac{P_1(t) + Q_1(t)\sqrt{1-4t}}{(1-4t)^{e_\lambda - 1/2}},$$

for two Laurent polynomials $P_1(t)$ and $Q_1(t)$. Given that we still have to divide RHS by $\sqrt{1-4t}$ to obtain the expression of $\partial_\lambda F_p$ (see (12.5)), this part of RHS is compatible with the expected form (12.3).

Before turning our attention to the case $\emptyset \neq I \neq [p]$, let us work out the value of $P_1(1/4)$, at least when $|\lambda|$ is even. In RHS_1, the only terms $\partial_\tau F_p$ for which $e_\tau = e_\lambda - 1/2$ are those for which $|\tau| = |\lambda| - 2$. That is, the terms for which $|\varepsilon| = 2$ in (12.6), and the terms for which $|\sigma| = |\lambda| - 2$ in (12.7). As $F_0 \to 2$ when $t \to 1/4$, this means that

$$\mathrm{RHS}_1 = \frac{1}{2(1-4t)^{e_\lambda - 1/2}} \left(\sum_{1 \le i < j \le p} c_{\lambda - \varepsilon_{i,j}} \lambda_i \lambda_j + \sum_{\sigma \le \lambda, |\sigma| = |\lambda| - 2} c_\sigma \prod_{i=1}^{p} \frac{\lambda_i!}{\sigma_i!} + o(1) \right),$$

where $\varepsilon_{i,j}$ is the p-tuple that has a one at positions i and j, and zeros elsewhere. In the second sum, the partitions σ such that $\sigma_i = \lambda_i - 2$ for some i contribute $\lambda_i(\lambda_i - 1)$, while those for which $\sigma_i = \lambda_i - 1$ and $\sigma_j = \lambda_j - 1$, so that $\sigma = \lambda - \varepsilon_{i,j}$, contribute $\lambda_i \lambda_j$, as in the first sum. A concise way of merging both sums consists in using the notation of (4.3) and writing

$$\mathrm{RHS}_1 = \frac{1}{(1-4t)^{e_\lambda - 1/2}} \left(\sum_{\sigma \le \lambda, |\sigma| = |\lambda| - 2} c_\sigma \binom{\lambda}{\sigma} + o(1) \right),$$

so that the polynomial $P_1(t)$ satisfies

$$P_1(1/4) = \sum_{\sigma \le \lambda, |\sigma| = |\lambda| - 2} c_\sigma \binom{\lambda}{\sigma},$$

where we recognize the second part of (4.3).

Second case: $I \neq \emptyset$ and $J \neq \emptyset$. In that case, the induction hypothesis (12.3) applies both to $\partial_\sigma F_{|I|}$ and $\partial_{\lambda_J - \varepsilon} F_{|J|}$. Moreover,

$$e_\sigma + e_{\lambda_J - \varepsilon} \le e_{\lambda_I} + e_{\lambda_J} = p + \frac{1}{2} \left\lfloor \frac{|\lambda_I|}{2} \right\rfloor + \frac{1}{2} \left\lfloor \frac{|\lambda_J|}{2} \right\rfloor - 1 \le e_\lambda - \frac{1}{2}. \tag{12.8}$$

This implies that the part of RHS for which $\emptyset \neq I \neq [p]$ can be written as

$$\mathrm{RHS}_2 = \frac{P_2(t) + Q_2(t)\sqrt{1-4t}}{(1-4t)^{e_\lambda - 1/2}},$$

for two Laurent polynomials $P_2(t)$ and $Q_2(t)$. Given that RHS_1 has also this form, we can conclude at last that (12.3) holds.

Let us finally work out the value of $P_2(1/4)$, at least when $|\lambda|$ is even. The only way for the inequalities (12.8) to be equalities is to take $\sigma = \lambda_I$,

$\varepsilon = 0$, with $|\lambda_I|$ and $|\lambda_J|$ even. Going back to (12.4), this means that the dominant contribution in RHS_2 is given by

$$\frac{1}{4(1-4t)^{e_\lambda - 1/2}} \sum_{\emptyset \neq I \subsetneq [p]} c_{\lambda_I} c_{\lambda_J}.$$

In other words,

$$P_2(1/4) = \frac{1}{4} \sum_{\emptyset \neq I \subsetneq [p]} c_{\lambda_I} c_{\lambda_J},$$

which gives the first part in (4.3). This completes the proof of Proposition 12.1 and thus of Theorem 4.1. $\square$

The proof of Theorem 4.5 is almost the same, using the generating functions and recursion relations in Proposition 10.1 and replacing $1 - 4t$ by $1 - 8t$ and $1 - 12t$, respectively. We omit the details.

To conclude this section, let us sketch the proof of Theorem 4.6. We define the generating functions F_k as in (8.1), but replacing $\ell(v)$ by the depth $d(v)$. Then (8.2) holds with $\bar{x}_i$ replaced by x_i. In particular, F_0 is still given by (8.3) and each $F_k(\boldsymbol{x})$ admits a rational expression in F_0 and the x_i. We claim that then (12.3) holds, with $e_\lambda = p + |\lambda|/2 - 1/2$, and

$$\partial_\lambda F_p = \frac{d_\lambda + o(1)}{(1-4t)^{p+|\lambda|/2-1/2}}.$$

This is proved by induction as above. Note that, after (12.4), the ∂_{λ_I} term expands exactly as the ∂_{λ_J} term, without $(-1)^{|\lambda_I|-|\sigma|}$ and thus without cancellation; this ultimately explains why the exponents e_λ increase faster for the horizontal profile than for the vertical. The rest is as above.

The same applies to plane trees, with F_k defined as in (10.1) with $\ell(v)$ replaced by $d(v)$, and the recursion relation

$$F_k(\boldsymbol{x}) = 1 + t \sum_{I \subseteq [k]} \left(\prod_{i \in I} x_i \right) F_{|I|}(x_I) F_{|J|}(x_J),$$

where $J = [k] \setminus I$. We omit the details.

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