COMPLEX INTERPOLATION OF COMPACT OPERATORS MAPPING INTO THE COUPLE (FL^∞, FL_1^∞).

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Abstract. If (A_0, A_1) and (B_0, B_1) are Banach couples and a linear operator $T : A_0 + A_1 \to B_0 + B_1$ maps A_0 compactly into B_0 and maps A_1 boundedly into B_1, does T necessarily also map $[A_0, A_1]_\theta$ compactly into $[B_0, B_1]_\theta$ for $\theta \in (0, 1)$?

After 42 years this question is still not answered, not even in the case where $T : A_1 \to B_1$ is also compact. But affirmative answers are known for many special choices of (A_0, A_1) and (B_0, B_1). Furthermore it is known that it would suffice to resolve this question in the special case where (B_0, B_1) is the special couple $(\ell^\infty(FL^\infty), \ell^\infty(FL_1^\infty))$. Here FL^∞ is the space of all sequences $\{\lambda_n\}_{n\in\mathbb{Z}}$ which are Fourier coefficients of essentially bounded functions, and FL_1^∞ is the weighted space of all sequences $\{\lambda_n\}_{n\in\mathbb{Z}}$ such that $\{e^n\lambda_n\}_{n\in\mathbb{Z}} \in FL^\infty$.

We provide an affirmative answer to this question in the related but simpler case where (B_0, B_1) is the special couple (FL^∞, FL_1^∞).

1. Introduction

This paper deals with the following question, which originated with Alberto Calderón's study [3] of complex interpolation spaces, and has been open for the 42 years which have elapsed since the writing of [3].

Question C: *Suppose that A_0 and A_1 are compatible Banach spaces, i.e., they form a Banach pair, and that so are B_0 and B_1. Suppose that $T : A_0 + A_1 \to B_0 + B_1$ is a linear operator such that $T : A_0 \to B_0$ compactly and $T : A_1 \to B_1$ boundedly. Does it follow that T maps the complex interpolation space $[A_0, A_1]_\theta$ into the compact interpolation space $[B_0, B_1]_\theta$ compactly for each $\theta \in (0, 1)$?*

Affirmative answers have been obtained over the years for a considerable number of particular cases of Question C. See, for example, [3], [16], [17], [5], [6], [9], and [11]. More recent studies of other aspects of this question can be found in [4], [12], [13] and [19]. We also refer to [10] and the website [7] for some general remarks and further details about this question and related questions, including a remarkable "almost counterexample" found by Fedor Nazarov.

Although we still cannot answer Question C, we can, apparently for the first time in nearly thirteen years, i.e., since the submission of [11], further enlarge the family of partial affirmative answers.

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Our main result in this paper, Theorem 14, immediately and obviously implies that the answer to Question C is yes in one more particular case, namely where (A_0, A_1) is an arbitrary Banach couple, and (B_0, B_1) is the special couple of sequence spaces (FL^∞, FL_1^∞) of Fourier coefficients which was introduced and studied in [14].

To the best of our knowledge, this particular case of Calderón's problem cannot be resolved via any other methods or cases treated in previous partial solutions of the problem.

Our main motivation for considering this particular case is that a corresponding affirmative (or negative!) result for the apparently related couple $(B_0, B_1) = (\ell^\infty(FL^\infty), \ell^\infty(FL_1^\infty))$ would resolve Calderón's problem in complete generality. (See [6], [9] and [12].) It would apparently also be sufficient to consider the couple of spaces of distributions $(B_0, B_1) = (FL^\infty(\mathbb{R}), FL_1^\infty(\mathbb{R}))$. As shown in [12], it would also be sufficient to consider the couple $(B_0, B_1) : (\ell^\infty(FC), \ell^\infty(FC_1))$.

We believe that there ought to be a simpler proof of Theorem 14 than the one given here. One of the main underlying ideas is the use of Lusin's theorem, which leads us to the intermediate result Theorem 4. Intuitively it seems rather clear that Theorem 4 implies Theorem 14. But, unfortunately, we have not found any way so far to bypass the lengthy explanation which we need for the technical details of this implication.

We thank Fedor Nazarov for many interesting discussions and enlightening comments about Question C in general.

2. Some preliminaries

As usual we let $\mathbb{T}$ denote the torus $\{z \in \mathbb{C} : |z| = 1\}$. Define the σ-algebra $\mathcal{T}$ of measurable subsets of $\mathbb{T}$ to consist of all images of Lebesgue measurable subsets of $[0, 2\pi)$ under the bijection $\phi(t) = e^{it}$ and define μ to be the measure on these sets defined by taking $\mu(E)$ to be the Lebesgue measure of $\phi^{-1}(E)$. In other words, μ is simply arc length measure on $\mathbb{T}$.

The *distance* $d(z_1, z_2)$ between two points z_1 and z_2 of $\mathbb{T}$ will be taken to be the length of the shortest arc in $\mathbb{T}$ joining them, i.e., $d(z_1, z_2) = d(1, z_2/z_1) = |t|$ where t is the unique number in $(-\pi, \pi]$ which satisfies $z_2/z_1 = e^{it}$.

For each $z \in \mathbb{T}$ and each $r \in (0, \pi]$, we let $\Gamma(z, r)$ be the closed arc in $\mathbb{T}$ of length $2r$ centred at z, i.e., $\Gamma(z, r) = \left\{ze^{it} : |t| \le r\right\} = \{w \in \mathbb{T} : d(w, z) \le r\}$.

Let $L^1(\mathbb{T}) = L^1(\mathbb{T}, \mathcal{T}, \mu)$ and $L^\infty(\mathbb{T}) = L^\infty(\mathbb{T}, \mathcal{T}, \mu)$ denote the usual Lebesgue spaces of (equivalence classes of) measurable functions $f : \mathbb{T} \to \mathbb{C}$ normed by $\|f\|_{L^1} = \int_{\mathbb{T}} |f| \, d\mu = \int_0^{2\pi} \left|f(e^{it})\right| dt$ and $\|f\|_{L^\infty} = \operatorname{ess\,sup}_{t \in [0,2\pi)} \left|f(e^{it})\right|$ respectively. For each $r \in (0, \pi]$ and $z \in \mathbb{T}$ the formula

$$\int_{\Gamma(z,r)} f d\mu = \int_{-r}^{r} f(ze^{it}) dt \tag{1}$$

obviously holds whenever $f = \chi_E$ and $E \in \mathcal{T}$, and therefore it also holds for all $f \in L^1(\mathbb{T})$.

We conclude this section by recalling some standard facts related to the Lebesgue differentiation theorem, and expressing them in ways that will be convenient for our purposes here.

Definition 1. *For each μ-integrable function $f : \mathbb{T} \to \mathbb{C}$, let Ω_f denote the set of all points $z \in \mathbb{T}$ for which the limit $\lim_{r\to 0} \frac{1}{2r}\int_{-r}^{r} f(ze^{it})dt$ exists. Let $f^\circ : \mathbb{T} \to \mathbb{C}$ denote the function defined by $f^\circ(z) = \lim_{r\to 0} \frac{1}{2r}\int_{-r}^{r} f(ze^{it})dt \cdot \chi_{\Omega_f}(z)$ for all $z \in \mathbb{T}$. It will be convenient to refer to the points of Ω_f as the weak canonical Lebesgue points of f.*

If two μ-integrable functions f and g coincide almost everywhere on $\mathbb{T}$, then of course $\Omega_f = \Omega_g$ and so $f^\circ(z) = g^\circ(z)$ for *all* $z \in \mathbb{T}$. Thus our definitions of Ω_f and f° extend unambiguously to the case where f is an element of $L^1(\mathbb{T})$ rather than a single function. It will be convenient to refer to the function f° as the *canonical representative* of the element f. It is an obvious consequence of the Lebesgue differentiation theorem that $\Omega_f \in \mathcal{T}$ with $\mu(\mathbb{T}\backslash\Omega_f) = 0$ and that $f(z) = f^\circ(z)$ for almost every $z \in \mathbb{T}$. This in turn implies that $\operatorname{ess\,sup}_{t\in[0,2\pi)} \left|f(e^{it})\right| = \sup_{t\in[0,2\pi)} \left|f^\circ(e^{it})\right|$.

Definition 2. *For each set $E \in \mathcal{T}$, let $E^{(d)}$ denote the set of all points of density of E, i.e., the points z in $\mathbb{T}$ for which the limit $\lim_{r\to 0} \frac{1}{2r}\int_{-r}^{r} \chi_E(ze^{it})dt$ exists and is equal to 1, i.e.,*

$$\lim_{r\to 0} \frac{\mu(E \cap \Gamma(z,r))}{2r} = 1.$$

It is of course a classical result, deduced by applying the Lebesgue differentiation theorem to the function χ_E, that $E^{(d)} \in \mathcal{T}$ and $\mu(E^{(d)}\backslash E) = \mu(E\backslash E^{(d)}) = 0$. Since $E = (E \cap E^{(d)}) \cup E\backslash E^{(d)}$, this implies in turn that $\mu(E \cap E^{(d)}) = \mu(E)$ and so, of course,

$$\mu(\mathbb{T}\backslash(E \cap E^{(d)})) = \mu(\mathbb{T}\backslash E). \tag{2}$$

3. The main ingredient

Our goal in this section is to prove Theorem 4 which will subsequently be the principal tool in the proof of our main result. But first we need the following lemma:

Lemma 3. *Let K be a relatively compact subset of $L^\infty(\mathbb{T})$. Then, for each $\epsilon > 0$, there exists a number $\delta = \delta(\epsilon)$ and a set $U = U_\epsilon \subset \mathbb{T}$ such that*

$$\mu(\mathbb{T}\backslash U) < \epsilon \tag{3}$$

and, for each $f \in K$,

$$\operatorname{ess\,sup}\{|f(z_1) - f(z_2)| : z_1, z_2 \in U, d(z_1, z_2) < \delta\} \le \epsilon.$$

More precisely,

$$\sup\{|f^\circ(z_1) - f^\circ(z_2)| : z_1, z_2 \in U \cap \Omega_f, d(z_1, z_2) < \delta\} \le \epsilon. \tag{4}$$

Proof. Since K is relatively compact, there exists some integer N and a collection of N elements $\{f_1, f_2, ..., f_N\} \subset K$, such that $K \subset \bigcup_{m=1}^{N} B(f_j, \epsilon/3)$. (Here of course $B(f, r)$ denotes the set $\{g \in L^\infty(\mathbb{T}) : \|f - g\|_{L^\infty(\mathbb{T})} < r\}$.)

The σ-algebra $\mathcal{T}$ contains all the Borel sets in $\mathbb{T}$, and the measure μ is finite, regular and complete on $\mathcal{T}$, i.e., it satisfies conditions (b), (c) and (d) of Theorem 2.14 of [18] pp. 40–41. Thus we can apply Lusin's Theorem ([18] Theorem 2.24 p. 55) to obtain, for each $j \in \{1, 2, ..., N\}$, a continuous function $g_j : \mathbb{T} \to \mathbb{C}$ such that the set $G_j = \{z \in \mathbb{T} : f_j^\circ(z) \neq g_j(z)\}$ has measure $\mu(G_j) < \epsilon/N$. Since each g_j is uniformly continuous, we can choose $\delta = \delta(\epsilon)$ to be a number with the property that

$$z_1, z_2 \in \mathbb{T}, d(z_1, z_2) < \delta \Longrightarrow \max\{|g_j(z_1) - g_j(z_2)| : j = 1, 2, ..., N\} < \epsilon/3.$$

Let E be the set $E = \mathbb{T} \backslash \bigcup_{j=1}^N G_j$. Then we will choose the set U to be $U = E \cap E^{(d)}$. Clearly $\mathbb{T} \backslash E = \bigcup_{j=1}^N G_j$ has measure $\mu\left(\bigcup_{j=1}^N G_j\right)$ not exceeding $\sum_{j=1}^N \mu(G_j) < \epsilon$, and so (3) follows immediately from (2).

For each point $z \in \mathbb{T}$ and each $r \in (0, \pi]$ and each $j \in \{1, 2, ..., N\}$, we have

$$\begin{aligned}
& \int_{\Gamma(z,r)} \left|f_j^\circ(w) - g_j(w)\right| d\mu(w) \\
= & \int_{\Gamma(z,r)\cap E} \left|f_j^\circ(w) - g_j(w)\right| d\mu(w) + \int_{\Gamma(z,r)\backslash E} \left|f_j^\circ(w) - g_j(w)\right| d\mu(w) \\
= & \ 0 + \int_{\Gamma(z,r)\backslash E} \left|f_j^\circ(w) - g_j(w)\right| d\mu(w) \\
\leq & \ \mu(\Gamma(z,r)\backslash E) \left(\|f_j\|_{L^\infty} + \|g_j\|_{L^\infty}\right)
\end{aligned}$$

If $z \in U$, and is therefore a point of density of E, then the preceding estimates show that

$$\begin{aligned}
& \limsup_{r\to 0} \frac{1}{\mu(\Gamma(z,r))} \int_{\Gamma(z,r)} \left|f_j^\circ(w) - g_j(w)\right| d\mu(w) \\
& \qquad \leq \left(\|f_j\|_{L^\infty} + \|g_j\|_{L^\infty}\right) \lim_{r\to 0} \frac{\mu(\Gamma(z,r)) - \mu((\Gamma(z,r)\cap E))}{\mu(\Gamma(z,r))} = 0.
\end{aligned}$$

In other words (cf. (1), we have

$$\lim_{r\to 0} \frac{1}{2r} \int_{-r}^{r} \left|f_j^\circ(ze^{it}) - g_j(ze^{it})\right| dt = 0 \quad \text{for all } z \in U \text{ and all } j \in \{1, 2, ..., N\}. \tag{5}$$

Now suppose that f is some arbitrary element of K. Then there exists some integer j depending on f such that

$$f \in B(f_j, \epsilon/3). \tag{6}$$

Suppose that z_1 and z_2 are arbitrary points in $U \cap \Omega_f$ satisfying $d(z_1, z_2) < \delta$. For each $w \in \mathbb{T}$ we have

$$\begin{aligned}
|f^\circ(wz_1) - f^\circ(wz_2)| \leq & \left|f^\circ(wz_1) - f_j^\circ(wz_1)\right| + \left|f_j^\circ(wz_1) - g_j(wz_1)\right| \\
& + |g_j(wz_1) - g_j(wz_2)| + \left|g_j(wz_2) - f_j^\circ(wz_2)\right| + \left|f_j^\circ(wz_2) - f^\circ(wz_2)\right|.
\end{aligned} \tag{7}$$

Since $d(wz_1, wz_2) < \delta$, we have $|g_j(wz_1) - g_j(wz_2)| < \epsilon/3$ for each $w \in \mathbb{T}$. In view of (6) we also have $\left|f^\circ(wz_1) - f_j^\circ(wz_1)\right| < \epsilon/3$ and $\left|f^\circ(wz_2) - f_j^\circ(wz_2)\right| < \epsilon/3$ for every $w \in \mathbb{T}$. So, for each $r \in (0, \pi]$, we can set $w = e^{it}$ and integrate the inequality (7) with respect to t on the interval $[-r, r]$ to obtain that

$$\begin{aligned}
& \frac{1}{2r} \int_{-r}^{r} \left|f^\circ(e^{it}z_1) - f^\circ(e^{it}z_2)\right| dt \\
& \leq \epsilon/3 + \frac{1}{2r} \int_{-r}^{r} \left|f_j^\circ(e^{it}z_1) - g_j(e^{it}z_1)\right| dt + \epsilon/3 + \frac{1}{2r} \int_{-r}^{r} \left|g_j(e^{it}z_2) - f_j^\circ(e^{it}z_2)\right| dt + \epsilon/3.
\end{aligned}$$

In view of (5) both of the integrals in the preceding line tend to 0 as r tends to 0, and so we have shown that

$$\limsup_{r\to 0}\frac{1}{2r}\int_{-r}^{r}\left|f(e^{it}z_1)-f(e^{it}z_2)\right|dt\leq\epsilon. \tag{8}$$

Finally, since z_1 and z_2 are both in Ω_f, we see that

$$\begin{aligned}|f^{\circ}(z_1)-f^{\circ}(z_2)| &= \left|\lim_{r\to 0}\frac{1}{2r}\int_{-r}^{r}f(e^{it}z_1)dt-\lim_{r\to 0}\frac{1}{2r}\int_{-r}^{r}f(e^{it}z_2)dt\right| \\ &= \lim_{r\to 0}\left|\frac{1}{2r}\int_{-r}^{r}\left(f(e^{it}z_1)-f(e^{it}z_2)\right)dt\right| \\ &\leq \limsup_{r\to 0}\frac{1}{2r}\int_{-r}^{r}\left|f(e^{it}z_1)-f(e^{it}z_2)\right|dt.\end{aligned}$$

This, combined with (8) establishes (4) and completes the proof of Lemma 3. □

In order to be able to conveniently apply the next theorem later in the proof of our main result, we will have to distinguish, more pedantically than is usually necessary, between elements of $L^\infty(\mathbb{T})$ and the functions that represent them. So here we will use bold lower case letters for elements (equivalent classes of complex valued functions) in $L^\infty(\mathbb{T})$ and usual italic lower case letters for the complex valued functions in those equivalence classes.

Theorem 4. *Let K be a subset of $L^\infty(\mathbb{T})$. Let $\mathcal{E}_K$ be the set of all functions $u:\mathbb{T}\times\mathbb{T}\to\mathbb{C}$ which are finite sums of the form*

$$u(\zeta,w)=\sum_{n=-M}^{M}\zeta^n v_n(w), \tag{9}$$

where each function $v_n:\mathbb{T}\to\mathbb{C}$ is essentially bounded and where the elements $\mathbf{v}_n\in L^\infty(\mathbb{T})$, which are represented by v_n respectively for each integer $n\in[-M,M]$, satisfy

$$\sum_{n=-M}^{M}\zeta^n\mathbf{v}_n\in K \quad \textit{for each } \zeta\in\mathbb{T}. \tag{10}$$

For each $\delta>0$, define

$$\rho_K(\delta):=\sup\left\{\int_{\mathbb{T}}|u(w,z_1w)-u(w,z_2w)|\,d\mu(w):u\in\mathcal{E}_K,z_1,z_2\in\mathbb{T},d(z_1,z_2)\leq\delta\right\}. \tag{11}$$

If K is a relatively compact subset of $L^\infty(\mathbb{T})$ then

$$\lim_{\delta\to 0}\rho_K(\delta)=0. \tag{12}$$

Proof. Let us begin by checking the almost obvious fact that the integral $\int_{\mathbb{T}}|u(w,z_1w)-u(w,z_2w)|\,d\mu(w)$ does not depend on our particular choices of the representatives v_n of $\mathbf{v}_n$ in the formula (9).

More explicitly, we claim that if $\phi(\zeta,w)=\sum_{n=-M}^{M}\zeta^n f_n(w)$ where $f_n(w)=v_n(w)$ for a.e. $w\in\mathbb{T}$ and for each n, then

$$\int_{\mathbb{T}}|u(w,z_1w)-u(w,z_2w)|\,d\mu(w)=\int_{\mathbb{T}}|\phi(w,z_1w)-\phi(w,z_2w)|\,d\mu(w). \tag{13}$$

To establish (13) we first observe that, for each fixed $z \in \mathbb{T}$,

$$\begin{aligned}
\int_{\mathbb{T}} |\phi(w, zw) - u(w, zw)|\, d\mu(w) &= \int_{\mathbb{T}} \left| \sum_{n=-M}^{M} w^n (f_n(zw) - v_n(zw)) \right| d\mu(w) \\
&\le \sum_{n=-M}^{M} \int_{\mathbb{T}} |w^n (f_n(zw) - v_n(zw))|\, d\mu(w) \\
&= \sum_{n=-M}^{M} \int_{\mathbb{T}} |f_n(zw) - v_n(zw)|\, d\mu(w) \\
&= \sum_{n=-M}^{M} \int_{\mathbb{T}} |f_n(w) - v_n(w)|\, d\mu(w) = 0. \qquad (14)
\end{aligned}$$

Next we note that, for each fixed z_1 and z_2,

$$\begin{aligned}
|\phi(w, z_1 w) - \phi(w, z_2 w)| \le{}& |\phi(w, z_1 w) - u(w, z_1 w)| + \\
& |u(w, z_1 w) - u(w, z_2 w)| + |u(w, z_2 w) - \phi(w, z_2 w)| .
\end{aligned}$$

Integrating, and using (14), we obtain that

$$\int_{\mathbb{T}} |\phi(w, z_1 w) - \phi(w, z_2 w)|\, d\mu(w) \le \int_{\mathbb{T}} |u(w, z_1 w) - u(w, z_2 w)|\, d\mu(w). \qquad (15)$$

The reverse inequality to (15) is obtained exactly analogously and so indeed we obtain (13).

We will use the usual notation $\eta G = \{\eta w : w \in G\}$ for each $\eta \in \mathbb{C}$ and for each subset G of some complex vector space W (which will often also be simply $\mathbb{C}$).

Obviously the sets $\mathcal{E}_K$ and the quantities $\rho_K(\delta)$ satisfy $\mathcal{E}_{\eta K} = \eta \mathcal{E}_K$ and so $\rho_{\eta K}(\delta) = \eta \rho_K(\delta)$ for each $\eta > 0$. So we can suppose, without loss of generality, that the given relatively compact set K is contained in the unit ball of $L^\infty(\mathbb{T})$.

Choose a positive number ϵ and let $U = U_\epsilon$ and $\delta = \delta(\epsilon)$ be a set and a number with the properties listed in the statement of Lemma 3, corresponding to our current choices of ϵ and of the relatively compact set K. We will show that, for this choice of δ,

$$\sup \left\{ \int_{\mathbb{T}} |u(w, z_1 w) - u(w, z_2 w)|\, d\mu(w) : u \in \mathcal{E}_K, z_1, z_2 \in \mathbb{T}, d(z_1, z_2) \le \delta \right\} \le (2\pi + 4)\epsilon. \qquad (16)$$

Since ϵ can be chosen arbitrarily small, this will of course suffice to establish (12).

Let $u(\zeta, w) = \sum_{n=-M}^{M} \zeta^n v_n(w)$ be an arbitrary element of $\mathcal{E}_K$. In view of (13) we may assume without loss of generality that it is in fact given by the formula

$$u(\zeta, w) = \sum_{n=-M}^{M} \zeta^n v_n^\circ(w) \qquad (17)$$

where, for each n, v_n° is the canonical representative of v_n, and of $\mathbf{v}_n$ the element of $L^\infty(\mathbb{T})$ represented by v_n).

Let $\Omega = \bigcap_{n=-M}^{M} \Omega_{v_n}$. Clearly

$$\mu(\mathbb{T}\backslash\Omega) = \mu\left(\bigcup_{n=-M}^{M} \mathbb{T}\backslash\Omega_{v_n}\right) = 0, \tag{18}$$

and, for each fixed $\zeta \in \mathbb{T}$, every point in Ω is a weak canonical Lebesgue point for the function $w \mapsto u(\zeta, w)$. (Of course the set of all weak canonical Lebesgue points for this function may be strictly larger than Ω.)

In view of (10) and the fact that K is a subset of the unit ball of $L^\infty(\mathbb{T})$, it follows that for each fixed $\zeta \in \mathbb{T}$, we have $|u(\zeta, w)| \leq 1$ for almost every $w \in \mathbb{T}$. The exceptional null set may depend on ζ, but the next simple lemma shows that the exceptional null sets for different ζ are contained in a common null set.

Lemma 5. *Suppose that $u(\zeta, w) = \sum_{n=-M}^{M} \zeta^n v_n(w)$ for some fixed integer M and all $\zeta, w \in \mathbb{T}$, where v_n are some measurable functions on $\mathbb{T}$, and suppose that, for each fixed $\zeta \in \mathbb{T}$, $|u(\zeta, w)| \leq 1$ for almost all $w \in \mathbb{T}$. Then, for almost all $w \in \mathbb{T}$, $|u(\zeta, w)| \leq 1$ holds for all $\zeta \in \mathbb{T}$.*

Proof. Let, for $\zeta \in \mathbb{T}$, N_ζ be the null set $\{w : |u(\zeta, w)| > 1\}$. Let Q be a countable dense subset of $\mathbb{T}$ (for example, the set of points with rational argument), and let $N = \bigcup_{\zeta \in Q} N_\zeta$. The $\mu(N) = 0$, and if $w \notin N$, then $|u(\zeta, w)| \leq 1$ for all $\zeta \in Q$. However, for fixed w, $u(\zeta, w)$ is a continuous function of ζ, so the inequality holds for all $\zeta \in \mathbb{T}$. $\square$

In particular, for every $z \in \mathbb{T}$, $|u(w, zw)| \leq 1$ for a.e. w, and thus

$$\int_F |u(w, zw)|\, d\mu(w) \leq \mu(F) \quad \text{for every set } F \in \mathcal{T} \text{ and } z \in \mathbb{T}. \tag{19}$$

Obviously $\mu(\eta G) = \mu(G)$ for every measurable set $G \subset \mathbb{T}$ and every $\eta \in \mathbb{T}$. Let z_1 and z_2 be arbitrary points in $\mathbb{T}$ satisfying

$$d(z_1, z_2) < \delta \tag{20}$$

and consider the set $F = z_1^{-1}(U \cap \Omega) \cap z_2^{-1}(U \cap \Omega)$. Whenever $w \in F$, both the points $z_1 w$ and $z_2 w$ are in $U \cap \Omega$. In view of (20) we also have $d(z_1 w, z_2 w) < \delta$. Furthermore, (by (10) and (17)), for any fixed value of w in F, the function $f(z) := u(w, z)$ is a representative of an element of K, the points $z_1 w$ and $z_2 w$ are both in Ω_f, and $f^\circ(z_1 w) = f(z_1 w)$, $f^\circ(z_2 w) = f(z_2 w)$. Consequently, applying (4), we have

$$|u(w, z_1 w) - u(w, z_2 w)| = |f(z_1 w) - f(z_2 w)| \leq \epsilon \quad \text{for all } w \in F. \tag{21}$$

Now we need to estimate the measure of the complement of the set F. We write $F = F_1 \cap F_2$, where $F_j = z_j^{-1}(U \cap \Omega)$ for $j = 1, 2$ First, since $U = (U \cap \Omega) \cup (U \backslash \Omega)$, we deduce from (18) that $\mu(U \cap \Omega) = \mu(U)$. Therefore, $\mu(F_1) = \mu(F_2) = \mu(U)$ and $\mu(\mathbb{T}\backslash F_1) = \mu(\mathbb{T}\backslash F_2) = \mu(\mathbb{T}\backslash U)$. We also recall that the properties listed in the statement of Lemma 3, which U must satisfy, include the condition $\mu(\mathbb{T}\backslash U) < \epsilon$. From these remarks, and the fact that $\mathbb{T}\backslash(F_1 \cap F_1) = (\mathbb{T}\backslash F_1) \cup (\mathbb{T}\backslash F_2)$, we can now see that

$$\mu\left(\mathbb{T}\backslash F\right) < 2\epsilon. \tag{22}$$

Finally, by (21), (19) and (22), we obtain that

$$\begin{aligned}
& \int_{\mathbb{T}} |u(w, z_1 w) - u(w, z_2 w)|\, d\mu(w) \\
= \quad & \int_F |u(w, z_1 w) - u(w, z_2 w)|\, d\mu(w) + \int_{\mathbb{T}\setminus F} |u(w, z_1 w) - u(w, z_2 w)|\, d\mu(w) \\
\leq \quad & \int_F \epsilon d\mu(w) + \int_{\mathbb{T}\setminus F} |u(w, z_1 w)|\, d\mu(w) + \int_{\mathbb{T}\setminus F} |u(w, z_2 w)|\, d\mu(w) \\
\leq \quad & \mu(F)\epsilon + 2\mu(\mathbb{T}\setminus F) \leq (2\pi + 4)\epsilon.
\end{aligned}$$

Since the element u of $\mathcal{E}_K$ and the points z_1 and z_2 in $\mathbb{T}$ satisfying $d(z_1, z_2) < \delta$ were chosen arbitrarily, this establishes (16) and so completes the proof of Theorem 4. □

4. Some more preliminaries

Next we recall some standard things about Alberto Calderón's complex interpolation spaces. We choose versions of the notation and definitions to suit our particular purposes. Some of the facts that we prove are implicit or even sometimes explicit in Calderón's fundamental paper [3] about these spaces, or in other papers. But we prefer to give rather full explanations here. We will assume some familiarity with the basic notions and terminology of interpolation space theory as presented, for example, in the early chapters of [1] or of [2].

Let $\mathbb{S}$ be the "unit strip", i.e., $\mathbb{S} = \{z \in \mathbb{C} : 0 \leq \operatorname{Re} z \leq 1\}$ and let $\mathbb{A}$ denote the "unit annulus", $\mathbb{A} = \{z \in \mathbb{C} : 1 \leq |z| \leq e\}$. Calderón constructed his complex interpolation spaces in [3] via certain analytic vector valued functions defined on $\mathbb{S}$. Most of the time here we prefer to use an alternative construction (cf. [8]) where $\mathbb{S}$ is replaced by $\mathbb{A}$.

Let $\vec{B} = (B_0, B_1)$ be an arbitrarily Banach couple of complex Banach spaces. Let $\mathcal{F}_{\mathbb{A}}(\vec{B}) = \mathcal{F}_{\mathbb{A}}(B_0, B_1)$ be the space of all continuous functions $f : \mathbb{A} \to B_0 + B_1$ such that

$$f \text{ is analytic in the interior } \mathbb{A}^\circ \text{ of } \mathbb{A} \tag{23}$$

and

(24)
for $j = 0, 1$, the restriction of f to the circle $e^j\mathbb{T}$ is a continuous map of $e^j\mathbb{T}$ into B_j .

We norm $\mathcal{F}_{\mathbb{A}}(\vec{B})$ by $\|f\|_{\mathcal{F}_{\mathbb{A}}(\vec{B})} = \sup\left\{\left\|f(e^{j+it})\right\|_{B_j} : t \in [0, 2\pi), j = 0, 1\right\}$.

For each $\theta \in (0, 1)$, let $[\vec{B}]_{\theta,\mathbb{A}}$ denote the space of all elements in $B_0 + B_1$ of the form $b = f(e^\theta)$, for some $f \in \mathcal{F}_{\mathbb{A}}(\vec{B})$. It is normed by

$$\|b\|_{[\vec{B}]_{\theta,\mathbb{A}}} := \inf\left\{\|f\|_{\mathcal{F}_{\mathbb{A}}(\vec{B})} : f \in \mathcal{F}_{\mathbb{A}}(\vec{B}), f(e^\theta) = b\right\}.$$

As shown in [8], the space $[\vec{B}]_{\theta,\mathbb{A}}$, coincides, to within equivalence of norms, with Calderón's complex interpolation space $[\vec{B}]_\theta$.

Each function $f \in \mathcal{F}_{\mathbb{A}}(\vec{B})$, can be expressed as a Laurent series $f(z) = \sum_{n=-\infty}^{\infty} z^n \widehat{f}(n)$, which converges absolutely and uniformly in $B_0 + B_1$ norm on every compact subset of $\mathbb{A}^\circ$. The coefficients $\widehat{f}(n)$ are given by the formulæ

$$\widehat{f}(n)=\frac{1}{2\pi i}\oint_{\beta\mathbb{T}}\frac{1}{z^{n+1}}f(z)dz, \tag{25}$$

where here, and in what follows, (cf. [15] p. 10 and pp. 257–8) we are using Riemann integrals of continuous Banach space valued functions. Here β is a number in the interval $(1,e)$ and the value of $\widehat{f}(n)$ is of course independent of its particular value in $(1,e)$. We can of course rewrite (25) as

$$\widehat{f}(n)=\frac{1}{2\pi}\int_0^{2\pi}e^{-n(\alpha+it)}f(e^{\alpha+it})dt \text{ where } \alpha=\ln\beta\in(0,1). \tag{26}$$

For each fixed n the function $\alpha\mapsto\frac{1}{2\pi}\int_0^{2\pi}e^{-n(\alpha+it)}f(e^{\alpha+it})dt$ is also defined for $\alpha=0$ and $\alpha=1$, and is a continuous map of the closed interval $[0,1]$ into B_0+B_1. Consequently, the formula (26) also holds for $\alpha=0$ and $\alpha=1$. For these two values, the integral $\frac{1}{2\pi}\int_0^{2\pi}e^{-n(\alpha+it)}f(e^{\alpha+it})dt$ defines an element of B_0 or of B_1 respectively. Thus we see that $\widehat{f}(n)\in B_0\cap B_1$ for each $n\in\mathbb{Z}$.

For each $f\in\mathcal{F}_{\mathbb{A}}(\vec{B})$ and each $N\in\mathbb{N}$, we define the Nth Fejér mean of f to be the function $\sigma_N(f):\mathbb{A}\to B_0+B_1$ given by the formula

$$\sigma_N(f)(e^{\alpha+it})=\frac{1}{2\pi}\int_0^{2\pi}\kappa_N(t-s)f(e^{\alpha+is})ds \quad \text{for each } \alpha\in[0,1] \text{ and } t\in[0,2\pi),$$

where κ_N is the usual Fejér kernel, $\kappa_N(t)=\frac{1}{N+1}\left(\frac{\sin\frac{N+1}{2}t}{\sin\frac{t}{2}}\right)^2$. Since (cf. [15] p. 12) $\kappa_N(t)=\sum_{n=-N}^{N}\left(1-\frac{|n|}{N+1}\right)e^{int}$, it follows immediately that $\sigma_N(f)(e^{\alpha+it})=\sum_{n=-N}^{N}\left(1-\frac{|n|}{N+1}\right)e^{n(\alpha+it)}\widehat{f}(n)$, i.e., $\sigma_N(f)(z)=\sum_{n=-N}^{N}\left(1-\frac{|n|}{N+1}\right)z^n\widehat{f}(n)$ for all $z\in\mathbb{A}$. It follows obviously from (24) that $f|_{e^j\mathbb{T}}:e^j\mathbb{T}\to B_j$ is in fact uniformly continuous for $j=0,1$. This combined with the fact that $\{\kappa_N\}_{N\in\mathbb{N}}$ is a summability kernel ([15] p. 9) readily implies, using standard estimates like those in the proof of [15] Lemma 2.2 p. 10, that $\lim_{N\to\infty}\sup\left\{\left\|\sigma_N(f)(e^{j+it})-f(e^{j+it})\right\|_{B_j}:t\in[0,2\pi)\right\}=0$ for $j=0,1$. Thus we have that $\lim_{N\to\infty}\|\sigma_N(f)-f\|_{\mathcal{F}_{\mathbb{A}}(\vec{B})}=0$ and consequently also

$$\lim_{N\to\infty}\left\|\sigma_N(f)(e^\theta)-f(e^\theta)\right\|_{[\vec{B}]_{\theta,\mathbb{A}}}=0 \quad \text{for each } \theta\in(0,1). \tag{27}$$

Let $\mathcal{G}_{\mathbb{A}}(\vec{B})$ denote the subspace of $\mathcal{F}_{\mathbb{A}}(\vec{B})$ consisting of all functions $g:\mathbb{A}\to B_0\cap B_1$ which are finite sums of the form $g(z)=\sum_{n=-N}^{N}z^nb_n$, for elements $b_n\in B_0\cap B_1$. The remarks in the preceding paragraph show that $\sigma_N(f)\in\mathcal{G}_{\mathbb{A}}(\vec{B})$ for every $f\in\mathcal{F}_{\mathbb{A}}(\vec{B})$, and that hence $\mathcal{G}_{\mathbb{A}}(\vec{B})$ is dense in $\mathcal{F}_{\mathbb{A}}(\vec{B})$.

Definition 6. *For any given subsets K_0 of B_0 and $K_1\subset B_1$, we let $\mathcal{F}_{\mathbb{A}}(K_0,K_1)$ denote the subset of functions $f\in\mathcal{F}_{\mathbb{A}}(\vec{B})$ with the additional property that $f(e^{j+it})\in K_j$ for all $t\in[0,2\pi)$ and $j=0,1$.*

Fact 7. *Let K_j be an arbitrary subset of B_j for $j=0,1$. Let $\widetilde{K_j}$ denote the closure in B_j of the convex hull of K_j and let K_j^* denote the closure in B_j of the absolutely convex hull of K_j. Let f be an arbitrary element of $\mathcal{F}_{\mathbb{A}}(K_0,K_1)$. Then*

(i) *$\widehat{f}(n)\in K_0^*\cap K_1^*$ for each $n\in\mathbb{Z}$, and*
(ii) *the function $\sigma_N(f)$ is in $\mathcal{F}_{\mathbb{A}}(\widetilde{K_0},\widetilde{K_1})$ for each $N\in\mathbb{N}$.*

Proof of Fact 7. Perhaps the quickest and easiest way to see (ii) is to consider the function $f_M : \mathbb{A} \to B_0$ defined by, for each $t \in [0, 2\pi)$ and each $\alpha \in [0, 1]$.

$$f_M(e^{\alpha+it}) = \frac{1}{2\pi} \sum_{m=1}^{M} \int_{2\pi(m-1)/M}^{2\pi m/M} \kappa_N(t-s) f(e^{\alpha+i2\pi m/M})\, ds.$$

Since $\kappa_N \geq 0$ and $\frac{1}{2\pi} \sum_{m=1}^{M} \int_{2\pi(m-1)/M}^{2\pi m/M} \kappa_N(t-s) ds = 1$, it is clear that, for $j = 0, 1$, the element $f_M(e^{j+it})$ is a convex combination of elements of K_j. Furthermore, the uniform continuity of $f|_{e^j\mathbb{T}} : e^j\mathbb{T} \to B_j$ ensures that $\lim_{M\to\infty} \left\| f_M(e^{j+it}) - \sigma_N(f)(e^{j+it}) \right\|_{B_0} = 0$. So indeed $\sigma_N(f)(e^{j+it}) \in \widetilde{K_j}$.

Analogously, for (i), for each fixed $n \in \mathbb{Z}$ and each $M \in \mathbb{N}$, we consider the element

$$f_M(n, j) = \frac{1}{2\pi} \sum_{m=1}^{M} \int_{2\pi(m-1)/M}^{2\pi m/M} e^{-n(j+i2\pi m/M)} f(e^{j+i2\pi m/M}) dt \quad \text{for each } j \in \{0, 1\}.$$

Since $f_M(n, j) = \frac{1}{M} \sum_{m=1}^{M} e^{-n(j+i2\pi m/M)} f(e^{j+i2\pi m/M})$, it is clear that $f_M(n, j) \in K_j^*$ for $j = 0, 1$. The uniform continuity of the function $g : \mathbb{T} \to B_j$ defined by $g(e^{it}) = e^{-n(j+it)} f(e^{j+it})$ ensures that $\lim_{M\to\infty} \left\| f_M(n, j) - \frac{1}{2\pi} \int_0^{2\pi} e^{-n(j+it)} f(e^{j+it}) dt \right\|_{B_j} = 0$. Since $\frac{1}{2\pi} \int_0^{2\pi} e^{-n(j+it)} f(e^{j+it}) dt = \widehat{f}(n)$ for $j = 0$ and also for $j = 1$ we deduce that $\widehat{f}(n) \in K_0^* \cap K_1^*$. □

Let FL^1 be the space of complex sequences $\lambda = \{\lambda_n\}_{n\in\mathbb{Z}}$ which are Fourier coefficients of functions in $L^1(\mathbb{T})$, i.e., such that, for some function $u = u_\lambda \in L^1(\mathbb{T})$,

$$\lambda_n = \frac{1}{2\pi} \int_0^{2\pi} e^{-int} u_\lambda(e^{it}) dt \quad \text{for all } n \in \mathbb{Z}.$$

We norm FL^1 by setting $\|\lambda\|_{FL^\infty} = \|u_\lambda\|_{L^1(\mathbb{T})}$. Analogously, we let FL^∞ be the space of complex sequences $\lambda = \{\lambda_n\}_{n\in\mathbb{Z}}$ which are Fourier coefficients of functions in $L^\infty(\mathbb{T})$, i.e., such that the above function u_λ is also in $L^\infty(\mathbb{T})$. Here we use the norm $\|\lambda\|_{FL^\infty} = \|u_\lambda\|_{L^\infty(\mathbb{T})}$. Let FC be the closed subspace of FL^∞ consisting of those sequences $\{\lambda_n\}_{n\in\mathbb{Z}}$ which are the Fourier coefficients of a continuous function on $\mathbb{T}$, i.e., a function in $C(\mathbb{T})$.

We will use the following basic properties of Fourier series ([15], I.2 and I.3):

Fact 8. (i) *for each sequence $\lambda = \{\lambda_n\}_{n\in\mathbb{Z}} \in FL^1$ the (Laurent or trigonometric) polynomials $P_N(e^{it}) = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N+1}\right) e^{int}\lambda_n$ converge to the associated function $u_\lambda(e^{it})$ for a.e. $t \in [0, 2\pi]$, and also in the norm of $L^1(\mathbb{T})$.*

(ii) *If, furthermore, $\lambda \in FL^\infty$, then* $\sup_{t\in[0,2\pi)} \left|P_N(e^{it})\right| \leq \|u_\lambda\|_{L^\infty(\mathbb{T})}$.

(iii) *If $\lambda \in FC$, then the convergence of the above sequence $\left\{P_N(e^{it})\right\}_{N\in\mathbb{N}}$ occurs for every $t \in [0, 2\pi]$ and furthermore it is uniform on $\mathbb{T}$, i.e., the sequence converges in $L^\infty(\mathbb{T})$ norm.*

For $p = 1, \infty$ and each fixed $\alpha \in \mathbb{R}$ let FL_α^p be the space of complex sequences $\{\lambda_n\}_{n\in\mathbb{Z}}$ such that $\{e^{\alpha n}\lambda_n\}_{n\in\mathbb{Z}} \in FL^p$. It is normed by $\left\|\{\lambda_n\}_{n\in\mathbb{Z}}\right\|_{FL_\alpha^p} = \left\|\{e^{\alpha n}\lambda_n\}_{n\in\mathbb{Z}}\right\|_{FL^p}$. We also define FC_α to be the closed subspace of FL_α^∞ of sequences $\{\lambda_n\}_{n\in\mathbb{Z}}$ such that $\{e^{\alpha n}\lambda_n\}_{n\in\mathbb{Z}} \in FC$.

These spaces are used extensively in [14], in particular to define the complex interpolation method as both a minimal and as a maximal method.

Obviously $FL^\infty \subset FL^1 \subset \ell^\infty$ and both these inclusions are continuous. So, for $j = 0, 1$ we have the continuous inclusions $FL^\infty_j \subset FL^1_j \subset \ell^\infty_{\min\{1,e^n\}} = \left\{\{\lambda_n\}_{n\in\mathbb{Z}} : \sup_{n\in\mathbb{Z}} \min\{1, e^n\} |\lambda_n| < \infty\right\}$. Consequently, $\vec{FL^p} = (FL^p_0, FL^p_1)$ is a Banach couple for $p = 1$, and for $p = \infty$.

The continuous inclusion $FL^1_\theta \subset [\vec{FL^1}]_\theta$ was shown in [14], and it is not difficult to also obtain the reverse continuous inclusion. As explained in [12], the formula

$$[\vec{FL^\infty}]_\theta = FC_\theta \text{ to within equivalence of norms} \tag{28}$$

can be deduced from $[\vec{FL^1}]_\theta = FL^1_\theta$ with the help of a duality argument. We need (28) crucially in this paper, and since existing proofs of it in the literature are not completely explicit, we provide a new proof below, in Appendix A. In fact there we prove the following slightly stronger result:

Proposition 9. *For each $\theta \in (0,1)$ the three spaces $[\vec{FL^\infty}]_\theta$, $[\vec{FL^\infty}]_{\theta,\mathbb{A}}$ and FC_θ coincide isometrically.*

Recall that the equality $[\vec{A}]_{\theta,\mathbb{A}} = [\vec{A}]_\theta$ holds for all Banach couples $\vec{A}$ [8], but in general the norms are equivalent only and not identical. (At least, this is proven for thick annuli in [8]; it is undoubtedly true for all annuli, but it seems that no-one has published a proof of this.)

We next consider some special and natural maps on the spaces FL^p_α and FC_α.

Definition 10. *For each fixed $s \in \mathbb{R}$, let $\mathcal{M}_s$ be the "multiplier" map on sequences defined by $\mathcal{M}_s \{\lambda_n\}_{n\in\mathbb{Z}} = \left\{e^{ins}\lambda_n\right\}_{n\in\mathbb{Z}}$ and for each fixed $m \in \mathbb{Z}$ let $\mathcal{S}_m$ be the "shift" map on sequences defined by $\mathcal{S}_m \{\lambda_n\}_{n\in\mathbb{Z}} = \{\lambda_{n-m}\}_{n\in\mathbb{Z}}$.*

Lemma 11. *For each $s \in \mathbb{R}$ and each $m \in \mathbb{Z}$ both $\mathcal{M}_s$ and $\mathcal{S}_m$ are bounded maps of FL^p_α onto itself and of FC_α onto itself, for each fixed α and for $p = 1, \infty$. In fact $\mathcal{M}_s$ is an isometry, and $\mathcal{S}_m$ has norm $e^{\alpha m}$ on each of these spaces.*

Proof. This is obvious from the following simple observation: Whenever $\{e^{\alpha n}\lambda_n\}_{n\in\mathbb{Z}}$ is the sequence of Fourier coefficients of the function $v(e^{it}) \in L^1(\mathbb{T})$, then

(i) $\{e^{ins}e^{\alpha n}\lambda_n\}_{n\in\mathbb{Z}} = \{e^{\alpha n}\left(\mathcal{M}_s\{\lambda_n\}\right)_n\}$ is the sequence of Fourier coefficients of the "rotated" function $e^{it} \mapsto v(e^{i(t+s)})$, and

(ii) $\{e^{\alpha(n-m)}\lambda_{n-m}\}_{n\in\mathbb{Z}} = e^{-\alpha m}\{e^{\alpha n}\left(\mathcal{S}_m\{\lambda_n\}\right)_n\}$ is the sequence of Fourier coefficients of the function $e^{it} \mapsto e^{imt}v(e^{it})$. □

The results in the following theorem and proposition are slight reformulations of straightforward and classical properties of analytic functions. But it seems preferable to state them explicitly in the notation needed for our purposes, and to provide explicit proofs.

Theorem 12. *Suppose that the sequence $\lambda = \{\lambda_n\}_{n\in\mathbb{Z}} \in FL^1$ is also an element of FL^1_1. Then $\lambda \in FC_\theta$ for all $\theta \in (0,1)$. Furthermore, the limit*

$$v(z) = \lim_{N\to\infty} \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N+1}\right) z^n \lambda_n \tag{29}$$

exists for every $z \in \mathbb{A}^\circ$ and almost every $z \in \mathbb{T}$ and almost every $z \in e\mathbb{T}$, and the complex valued function v which it defines has the following properties:

(i) *$v(e^{it}) = u_\lambda(e^{it})$ for a.e. $t \in [0, 2\pi)$,*
(ii) *the function $e^{it} \mapsto v(e^{1+it})$ is in $L^1(\mathbb{T})$ and $\frac{1}{2\pi}\int_0^{2\pi} e^{-int} v(e^{1+it})dt = e^n \lambda_n$ for all $n \in \mathbb{Z}$,*
(iii) *for each $\theta \in (0,1)$, the function $e^{it} \mapsto v(e^{\theta+it})$ is in $C(\mathbb{T})$ and $\frac{1}{2\pi}\int_0^{2\pi} e^{-int} v(e^{\theta+it})dt = e^{\theta n}\lambda_n$ for all $n \in \mathbb{Z}$, and*
(iv) *for each $\theta \in (0,1)$,*

$$\left|v(e^\theta)\right| \le C_\theta \left(\int_0^{2\pi} \left|v(e^{it})\right| dt\right)^{1-\theta} \left(\int_0^{2\pi} \left|v(e^{1+it})\right| dt\right)^{\theta}, \tag{30}$$

where the constant C_θ depends only on θ.

Proof. Since $\lambda \in FL^1 \cap FL^1_1$, it follows that
(31)
$$|\lambda_m| \le \frac{1}{2\pi}\min\{1, e^{-m}\}\max\left\{\|\lambda\|_{FL^1(\mathbb{T})}, \left\|\{e^n\lambda_n\}_{n\in\mathbb{Z}}\right\|_{FL^1(\mathbb{T})}\right\} \quad \text{for each } m \in \mathbb{Z}.$$

So, for each fixed $\theta \in (0,1)$, the sequence $\left\{e^{(\theta+it)n}\lambda_n\right\}_{n\in\mathbb{Z}}$ satisfies

$$\sup_{t\in\mathbb{R}} \left|e^{(\theta+it)n}\lambda_n\right| = \left|e^{\theta n}\lambda_n\right| \le const. \min\{e^{\theta n}, e^{(\theta-1)n}\} \quad \text{for each } n \in \mathbb{Z}. \tag{32}$$

For each $N \in \mathbb{N}$ we define the function (Laurent polynomial) $P_N : \mathbb{C}\backslash\{0\} \to \mathbb{C}$ by

$$P_N(z) = \sum_{n=-N}^{N}\left(1 - \frac{|n|}{N+1}\right) z^n \lambda_n. \tag{33}$$

In view of Fact 8 (i), the sequence $\left\{P_N(e^{it})\right\}_{N\in\mathbb{N}}$ converges a.e. to $u_\lambda(e^{it})$ for a.e. $t \in [0, 2\pi)$ and also in the norm of $L^1(\mathbb{T})$. For exactly the same reason, the sequence $\left\{e^{it} \mapsto P_N(e^{1+it})\right\}_{N\in\mathbb{N}} = \left\{e^{it} \mapsto \sum_{n=-N}^{N}\left(1 - \frac{|n|}{N+1}\right)e^{(1+it)n}\lambda_n\right\}_{N\in\mathbb{N}}$ converges a.e. on $\mathbb{T}$, and also in $L^1(\mathbb{T})$ norm, to a function, (which we shall temporarily call $u_{\lambda,1}(e^{it})$) whose Fourier coefficients are $\{e^n\lambda_n\}$.

For each fixed $\theta \in (0,1)$ we see from (32) that the sequence of functions $e^{it} \mapsto P_N(e^{\theta+it})$ converges absolutely and uniformly on $\mathbb{T}$ to a continuous function on $\mathbb{T}$ (which we shall temporarily call $u_{\lambda,\theta}(e^{it})$) whose Fourier coefficients are $\left\{e^{\theta n}\lambda_n\right\}$.

In view of the preceding remarks, it is clear that $\lambda \in FC_\theta$ for each $\theta \in (0,1)$. It is also clear that the function v defined as in (29) i.e., by $v(z) := \lim_{N\to\infty} P_N(z)$, exists for all $z \in \mathbb{A}^\circ$ and for almost every z on each of the circles $\mathbb{T}$ and $e\mathbb{T}$, and it satisfies $v(e^{it}) = u_\lambda(e^{it})$ and $v(e^{1+it}) = u_{\lambda,1}(e^{it})$ for a.e. $t \in [0, 2\pi)$ and $v(e^{\theta+it}) = u_{\lambda,\theta}(e^{it})$ for all $t \in [0, 2\pi)$ and all $\theta \in (0,1)$. So parts (i), (ii) and (iii) of Theorem 12 have been established.

It still remains to prove part (iv). For this let us first remark that all continuous functions $f : \mathbb{A} \to \mathbb{C}$ which are analytic in the interior of $\mathbb{A}$ satisfy

$$\left|f(e^\theta)\right| \le C_\theta \left(\int_0^{2\pi} \left|f(e^{it})\right| dt\right)^{1-\theta} \left(\int_0^{2\pi} \left|f(e^{1+it})\right| dt\right)^{\theta} \tag{34}$$

for each $\theta \in (0,1)$, where the constant C_θ depends only on θ. We can show this via classical results. Or we can deduce it as a very very special case (where $X_0 = X_1 = \mathbb{C}$) of the estimate (1) of part (i) of Lemma 2 of [11] p. 265. (Note that in the proof of part (i) of Lemma 2 of [11, p. 265], there is a small misprint: the function

$F(z)=f(e^z)e^{z^2}$ should of course be $F(z)=f(e^z)e^{-z^2}$. Cf. also the proof of the related estimate $(**)$ on p. 363 of [12].)

Applying (34) to the analytic polynomials P_N, we have

$$\left|P_N(e^\theta)\right| \le C_\theta \left(\int_0^{2\pi}\left|P_N(e^{it})\right| dt\right)^{1-\theta}\left(\int_0^{2\pi}\left|P_N(e^{1+it})\right| dt\right)^{\theta}. \tag{35}$$

Finally we use the fact that the above-mentioned convergence of $\left\{P_N(e^{j+it})\right\}_{N\in\mathbb{N}}$ to $v(e^{j+it})$ in $L^1(\mathbb{T})$ implies that $\lim_{N\to\infty}\int_0^{2\pi}\left|P_N(e^{j+it})\right| dt=\int_0^{2\pi}\left|v(e^{j+it})\right| dt$ for $j=0,1$. Thus we obtain (30) from (35) by taking the limit as N tends to ∞. This completes the proof of Theorem 12. $\square$

We conclude this section with a proposition which paraphrases and expands upon some of the things in the statements and the proofs of Theorem 12 and Lemma 11. Since in our application of these things in the next section we will only need to deal with sequences λ in the smaller space $FL^\infty\cap FL_1^\infty$ instead of in $FL^1\cap FL_1^1$, we only consider such sequences here. Also, it will be convenient for us to use a slightly exotic variant (see (36)) of the formula (29) so that all the functions that we have to deal with will be defined unambiguously at *every* point of $\mathbb{A}$

Proposition 13. *For each element $\lambda=\{\lambda_n\}_{n\in\mathbb{Z}}$ of the sequence space $FL^\infty\cap FL_1^\infty$, let $T\lambda$ be the function defined by*

(36)

$$T\lambda(z)=\begin{cases}\lim_{N\to\infty}\sum_{n=-N}^{N}\left(1-\frac{|n|}{N+1}\right)z^n\lambda_n & \text{if this limit exists}\\ 0 & \text{if the above limit does not exist}\end{cases}$$

for each $z\in\mathbb{A}$.

Then:

(i) *The formula (36) defines a map T from $FL^\infty\cap FL_1^\infty$ into a certain class of functions on $\mathbb{A}$, which is "linear almost everywhere". I.e., for each λ and ξ in $FL^\infty\cap FL_1^\infty$ and each α and β in $\mathbb{C}$,*

(37)

$$T(\alpha\lambda+\beta\xi)(z)=\alpha T\lambda(z)+\beta T\xi(z) \text{ for all } z\in\mathbb{A}^\circ \text{ and for almost all } z\in\mathbb{T}\cup e\mathbb{T}.$$

Furthermore, for each $\lambda\in FL^\infty\cap FL_1^\infty$, the function $v=T\lambda$ has the following properties:

(ii) *For each $\alpha\in[0,1]$, v satisfies*

$$\|\lambda\|_{FL_\alpha^\infty}=\operatorname*{ess\,sup}_{t\in[0,2\pi)}\left|v(e^{\alpha+it})\right|. \tag{38}$$

(iii) *When $\alpha=\theta\in(0,1)$ the function $v(e^{\theta+it})$ depends continuously on t and* (38) *can be rewritten as*

$$\|\lambda\|_{FL_\theta^\infty}=\|\lambda\|_{FC_\theta}=\sup_{t\in[0,2\pi)}\left|v(e^{\theta+it})\right|. \tag{39}$$

(iv) *v is analytic in $\mathbb{A}^\circ$.*

(v) *The function $T(\mathcal{M}_s\lambda)$ coincides with the function $z\mapsto v(e^{is}z)$ for each fixed $s\in\mathbb{R}$.*

(vi) *Except possibly on some null subset of $\mathbb{T}\cup e\mathbb{T}$, the function $T(\mathcal{S}_m\lambda)$ coincides with the function $z\mapsto z^m v(z)$ for each fixed $m\in\mathbb{Z}$.*

(vii) *The function v satisfies the estimate* (30).

Proof. From Theorem 12 and its proof it is immediately clear that the right sides of (29) and of (36) coincide for all $z \in \mathbb{A}^\circ$ and for almost all $z \in \mathbb{T} \cup e\mathbb{T}$ and that (keeping in mind the definitions of the spaces FL^∞_α and FC_α) properties (i), (ii), (iii) and (vii) all hold.

To show property (iv) we note that, for any α and β satisfying $0 < \alpha < \beta < 1$, and for all z in the closed annulus $\left\{z : e^\alpha \le |z| \le e^\beta\right\}$, the sequence $\{z^n\lambda_n\}_{n\in\mathbb{Z}}$ satisfies (cf. (31)) $|z^n\lambda_n| \le const. \min\{1, e^{-n}\} \max\{e^{\alpha n}, e^{\beta n}\} = const. \min\left\{e^{\alpha n}, e^{(\beta-1)n}\right\}$. Consequently the sequence of Laurent polynomials $\{P_N(z)\}_{N\in\mathbb{N}}$ (cf. (33)) converges uniformly on each such annulus and so the function $v = T\lambda = \lim_{N\to\infty} P_N$ is indeed analytic in $\mathbb{A}^\circ$.

Property (v) is an immediate consequence of the definitions of T and of $\mathcal{M}_s$ (cf. Definition 10).

To show (vi), we choose an arbitrary $m \in \mathbb{Z}$ and let $w = T(\mathcal{S}_m\lambda)$. We have to show that

$$w(z) = z^m v(z) \quad \text{for all } z \in \mathbb{A}^\circ \text{ and almost all } z \in \mathbb{T} \cup e\mathbb{T}. \tag{40}$$

This is obvious for each $z \in \mathbb{A}^\circ$, since, when $1 < |z| < e$, we have $\lim_{N\to\infty} \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N+1}\right) z^n\lambda_n = \sum_{n=-\infty}^{\infty} z^n\lambda_n$ and this series is absolutely convergent. It remains to deal with $z \in \mathbb{T} \cup e\mathbb{T}$. When $z \in \mathbb{T}$, we apply part (i) of Theorem 12 to the sequence $\mathcal{S}_m\lambda$, and see that the function $w = T(\mathcal{S}_m\lambda)$ must satisfy $\frac{1}{2\pi}\int_0^{2\pi} e^{-int} w(e^{it})dt = \lambda_{n-m}$ for each $n \in \mathbb{Z}$. Since we also have $\frac{1}{2\pi}\int_0^{2\pi} e^{-int}[e^{imt}v(e^{it})]dt = \lambda_{n-m}$ for all $n \in \mathbb{Z}$, it follows from the uniqueness theorem for Fourier series that

$$w(e^{it}) = e^{imt}v(e^{it}) \quad \text{for a.e. } t \in [0, 2\pi). \tag{41}$$

Similarly, from part (ii) of Theorem 12 applied to both $\mathcal{S}_m\lambda$ and λ, we have

$$\frac{1}{2\pi}\int_0^{2\pi} e^{-int} w(e^{1+it})dt = e^n\lambda_{n-m} = \frac{1}{2\pi}\int_0^{2\pi} e^{-int}[e^{(1+it)m}v(e^{1+it})]dt$$

for all $n \in \mathbb{Z}$. So another application of the uniqueness theorem gives us that

$$w(e^{1+it}) = e^{(1+it)m}v(e^{1+it}) \quad \text{for a.e. } t \in [0, 2\pi). \tag{42}$$

In view of (41) and (42) we have now established (40). This completes the proof of Proposition 13. □

5. The main result

We are finally ready to state and prove our main result.

Theorem 14. *Let $\vec{B} = (B_0, B_1)$ be the couple $F\vec{L}^\infty$. Let K_0 be a relatively compact subset of the unit ball of FL^∞ and let K_1 be the unit ball of $F\vec{L}^\infty_1$. Let θ be a constant satisfying $0 < \theta < 1$ and let K_θ be the set of all sequences $\lambda \in FL^\infty + FL^\infty_1$ of the form $\lambda = f(e^\theta)$ for $f \in \mathcal{F}_{\mathbb{A}}(K_0, K_1)$. Then K_θ is a relatively compact subset of FC_θ.*

Remark 15. As stated in the introduction, Theorem 14 immediately applies an affirmative answer to Question C in the case where (B_0, B_1) is the couple of sequence spaces (FL^∞, FL^∞_1). This is because of (28) (cf. also Proposition 9). We will also use (28) in the proof of Theorem 14.

Proof. K_1 is of course closed and convex, and, since the closed convex hull of a relatively compact subset of a normed linear space is compact, we may also suppose without loss of generality that K_0 is closed and convex. It then follows from Fact 7 and (27) that the set of sequences

$$K_{\theta,\mathcal{G}} := \{\lambda = f(e^\theta) : f \in \mathcal{F}_{\mathbb{A}}(K_0, K_1) \cap \mathcal{G}_{\mathbb{A}}(F\vec{L^\infty})\}$$

is a dense subset of K_θ in the norm of $[F\vec{L}^\infty]_{\theta,\mathbb{A}} = FC_\theta$. Thus we have reduced the proof of the theorem to showing that the set $K_{\theta,\mathcal{G}}$ is relatively compact in FC_θ.

Obviously $K_{\theta,\mathcal{G}} \subset FL^\infty \cap FL_1^\infty$. Let T be the map defined in (36) and let $T(K_{\theta,\mathcal{G}})$ be the collection of all functions $v = T\lambda$ obtained as λ ranges over $K_{\theta,\mathcal{G}}$. Let $T(K_{\theta,\mathcal{G}})|_{e^\theta\mathbb{T}}$ denote the collection of all functions on the circle $e^\theta\mathbb{T}$ which are restrictions to that circle of functions in $T(K_{\theta,\mathcal{G}})$. It follows from (39) and the fact that $T\xi(z) - T\lambda(z) = T(\xi - \lambda)(z)$ for all $z \in e^\theta\mathbb{T}$ and all $\xi, \lambda \in FL^\infty \cap FL_1^\infty$ (cf. (37)) that, in order to show that $K_{\theta,\mathcal{G}}$ is relatively compact in FC_θ, it suffices to show that $T(K_{\theta,\mathcal{G}})|_{e^\theta\mathbb{T}}$ is a relatively compact subset of $C(e^\theta\mathbb{T})$. By definition, $K_{\theta,\mathcal{G}}$ is contained in the unit ball of $[F\vec{L}^\infty]_{\theta,\mathbb{A}}$ and so $T(K_{\theta,\mathcal{G}})|_{e^\theta\mathbb{T}}$ is a bounded set of functions in $C(e^\theta\mathbb{T})$. So, by the Arzelà–Ascoli theorem, it remains only to show that $T(K_{\theta,\mathcal{G}})|_{e^\theta\mathbb{T}}$ is an equicontinuous set of functions. Intuitively at least, this is a rather obvious consequence of Theorem 4 and Theorem 12, in particular (30). But let us carefully go through all the steps to show it:

Consider some fixed but arbitrary element ψ of $T(K_{\theta,\mathcal{G}})|_{e^\theta\mathbb{T}}$. It must be a continuous function $\psi : e^\theta\mathbb{T} \to \mathbb{C}$ which is the restriction to $e^\theta\mathbb{T}$ of some function $T\lambda$ where the sequence λ is a fixed element of $K_{\theta,\mathcal{G}}$. It must, by the definition of $\mathcal{G}_{\mathbb{A}}$, be of the form

$$\lambda = \sum_{n=-M}^{M} e^{\theta n}\lambda^n \tag{43}$$

for some $M \in \mathbb{N}$, where $\lambda^n \in FL^\infty \cap FL_1^\infty$ for each n. (Here we are using a superscript index in λ^n to avoid confusion with the previous notation λ_n for individual terms of the sequence λ.) Furthermore, we have $\sum_{n=-M}^{M} e^{(j+it)n}\lambda^n \in K_j$ for $j = 0, 1$ and for each fixed $t \in [0, 2\pi)$. In fact, for these values of j and t, we have

$$\sum_{n=-M}^{M} e^{(j+it)n}\lambda^n \in K_j \cap FL^\infty \cap FL_1^\infty. \tag{44}$$

Let $K_* = T(K_0 \cap FL^\infty \cap FL_1^\infty)|_{\mathbb{T}}$, i.e., K_* is the set of all functions defined on $\mathbb{T}$ which are the restrictions to $\mathbb{T}$ of functions of the form $T\xi$ for $\xi \in K_0 \cap FL^\infty \cap FL_1^\infty$. Let K be the set of all elements (i.e., equivalence classes of functions) in $L^\infty(\mathbb{T})$ which have a representative in K_*. In view of (38) for $\alpha = 0$ and the fact that $T\xi(z) - T\lambda(z) = T(\xi - \lambda)(z)$ for a.e. $z \in \mathbb{T}$ and all $\xi, \lambda \in FL^\infty \cap FL_1^\infty$ (cf. (37)), we can assert that K is a relatively compact subset of the unit ball of $L^\infty(\mathbb{T})$. We now set $v_n = T(\lambda^n)$ for each $n \in [-M, M]$, (where λ^n are the elements appearing in (43)) and, for each $\zeta \in \mathbb{A}$, we define $u(\zeta) = \sum_{n=-M}^{M} \zeta^n v_n$. Correspondingly, we define $u(\zeta, w) = \sum_{n=-M}^{M} \zeta^n v_n(w)$ for each ζ and w in $\mathbb{A}$.

The following property of $u(\zeta,w)$ is an obvious consequence of (37):

(45)
$$\text{For each fixed } \zeta\in\mathbb{A},\ u(\zeta,w)=T\left(\sum_{n=-M}^{M}\zeta^n\lambda^n\right)(w)\ \begin{cases}\text{(i)} & \text{for a.e. } w\in\mathbb{T}.\\ \text{(ii)} & \text{for a.e. } w\in e\mathbb{T}.\\ \text{(iii)} & \text{for all } w\in\mathbb{A}^\circ.\end{cases}$$

Using part (i) of (45) we see that, for each fixed $\zeta\in\mathbb{T}$, we can write $\zeta=e^{it}$ for some fixed real t and apply (44) for the case where $j=0$ to deduce that the restriction of the function $w\mapsto u(e^{it},w)$ to the circle $\mathbb{T}$ coincides for a.e. $w\in\mathbb{T}$ with a function in K_* (namely $T\left(\sum_{n=-M}^{M}e^{itn}\lambda^n\right)(w)$ restricted to $\mathbb{T}$). In other words, if we let $\mathbf{v}_n$ denote the equivalence class in $L^\infty(\mathbb{T})$ containing $v_n|_{\mathbb{T}}$ for each n, then, for each fixed $t\in[0,2\pi)$, the equivalence class $\sum_{n=-M}^{M}e^{int}\mathbf{v}_n$ in $L^\infty(\mathbb{T})$, which contains the restriction of $u(e^{it},w)$ to $\mathbb{T}$, is an element of K.

This last condition means exactly that the restriction of $u(\zeta,w)$ to $\mathbb{T}\times\mathbb{T}$ is an element of the class $\mathcal{E}_K$ defined in Theorem 4. Thus, if ρ_K is the function defined by (11), it follows from that definition that

$$\begin{aligned}&\int_0^{2\pi}\left|u(e^{it},z_1e^{it})-u(e^{it},z_2e^{it})\right|dt\\ &= \int_{\mathbb{T}}|u(w,z_1w)-u(w,z_2w)|d\mu(w)\le\rho_K\left(d(z_1,z_2)\right)\quad\text{for all } z_1,z_2\in\mathbb{T}.\end{aligned}\tag{46}$$

Next, we consider the properties of the function $w\mapsto u(\zeta,w)$, when w is restricted to $e\mathbb{T}$ and for each constant $\zeta\in e\mathbb{T}$. This time we use part (ii) of (45) to obtain that this function coincides, for a.e. $w\in e\mathbb{T}$, with $T\left(\sum_{n=-M}^{M}\zeta^n\lambda^n\right)(w)$. By (44) for $j=1$, we know that $\sum_{n=-M}^{M}\zeta^n\lambda^n$ is in the unit ball of FL_1^∞. In turn this implies, using (38) for $\alpha=1$, that $|u(\zeta,w)|\le 1$ for a.e. $w\in e\mathbb{T}$.

Consider the function $U(\zeta,w)$ defined for all $\zeta,w\in\mathbb{T}$ by $U(\zeta,w)=u(e\zeta,ew)$. Clearly $U(\zeta,w)=\sum_{n=-M}^{M}\zeta^nV_n(w)$ where $V_n(w):=e^nv_n(ew)$. We also have, for each fixed $\zeta\in\mathbb{T}$, that $|U(\zeta,w)|\le 1$ for a.e. $w\in\mathbb{T}$. Thus Lemma 5 implies that for a.e. $w\in e\mathbb{T}$, $|u(\zeta,w)|\le 1$ for all $\zeta\in e\mathbb{T}$. It follows that for every $z\in\mathbb{T}$, $|u(w,zw)|\le 1$ for a.e. $w\in e\mathbb{T}$, and thus

$$\int_0^{2\pi}\left|u(e^{1+it},ze^{1+it})\right|dt\le 2\pi.\tag{47}$$

The reason that we need (47) is that it immediately implies that

$$\int_0^{2\pi}\left|u(e^{1+it},z_1e^{1+it})-u(e^{1+it},z_2e^{1+it})\right|dt\le 4\pi,\quad\text{for all } z_1,z_2\in\mathbb{T}.\tag{48}$$

Now we will use part (iii) of (45) with $\zeta=e^\theta$. It gives us that

$$\psi(w)=u(e^\theta,w)=\sum_{n=-M}^{M}e^{\theta n}v_n(w)\quad\text{for all } w\in e^\theta\mathbb{T}.\tag{49}$$

Let s and σ be two arbitrary fixed real numbers and consider the sequence $\xi=\sum_{n=-M}^{M}\mathcal{S}_n\left(\mathcal{M}_s-\mathcal{M}_\sigma\right)\lambda^n$ where the λ^n's are the same fixed sequences in $FL^\infty\cap FL_1^\infty$ which were introduced in (43). It follows from Lemma 11 that $\xi\in FL^\infty\cap FL_1^\infty$.

In view of parts (i), (v) and (vi) of Proposition 13 and the definitions of v_n and $u(\zeta, w)$ given above, we can assert that the following sequence of equalities hold for all $w \in \mathbb{A}^\circ$ and for almost all $w \in \mathbb{T} \cup e\mathbb{T}$.

$$\begin{aligned} T\xi(w) &= \sum_{n=-M}^{M} T\left(\mathcal{S}_n\left(\mathcal{M}_s - \mathcal{M}_\sigma\right)\lambda^n\right)(w) \\ &= \sum_{n=-M}^{M} T\left(\mathcal{S}_n\mathcal{M}_s\lambda^n\right)(w) - \sum_{n=-M}^{M} T\left(\mathcal{S}_n\mathcal{M}_\sigma\lambda^n\right)(w) \\ &= \sum_{n=-M}^{M} w^n T\left(\mathcal{M}_s\lambda^n\right)(w) - \sum_{n=-M}^{M} w^n T\left(\mathcal{M}_\sigma\lambda^n\right)(w) \\ &= \sum_{n=-M}^{M} w^n T\lambda^n(e^{is}w) - \sum_{n=-M}^{M} w^n T\lambda^n(e^{i\sigma}w) \\ &= \sum_{n=-M}^{M} w^n v_n(e^{is}w) - \sum_{n=-M}^{M} w^n v_n(e^{i\sigma}w) \\ &= u(w, e^{is}w) - u(w, e^{i\sigma}w). \end{aligned} \tag{50}$$

We now apply part (vii) of Proposition 13 to the function $T\xi(w)$. In view of (50), this gives us that

$$\begin{aligned} &\left|u(e^\theta, e^{is}e^\theta) - u(e^\theta, e^{i\sigma}e^\theta)\right| \\ &\quad\le C_\theta\left(\int_0^{2\pi}\left|u(e^{it}, e^{is}e^{it}) - u(e^{it}, e^{i\sigma}e^{it})\right|dt\right)^{1-\theta}\cdot \\ &\qquad\left(\int_0^{2\pi}\left|u(e^{1+it}, e^{is}e^{1+it}) - u(e^{1+it}, e^{i\sigma}e^{1+it})\right|dt\right)^{\theta}. \end{aligned}$$

In view of (49), the first term in the preceding inequality is $\left|\psi(e^{\theta+is}) - \psi(e^{\theta+i\sigma})\right|$. So, using (46) and (48), we deduce that

$$\left|\psi(e^{\theta+is}) - \psi(e^{\theta+i\sigma})\right| \le C_\theta\left(\rho_K(d(e^{is}, e^{i\sigma}))\right)^{1-\theta}(4\pi)^\theta \quad \text{for all } s, \sigma \in \mathbb{R}. \tag{51}$$

The relative compactness of K implies, via Theorem 4, that $\lim_{\delta\to 0}\rho_K(\delta) = 0$. Thus the inequality (51) establishes the equicontinuity of the set of functions $T(K_{\theta,\mathcal{G}})\big|_{e^\theta\mathbb{T}}$. As explained earlier, this suffices to complete the proof of Theorem 14. □

Appendix A. A proof of Proposition 9.

Until now we have not related very explicitly to Calderón's original definition of his spaces $[A_0, A_1]_\theta$. We shall assume that the reader is familiar with their construction via a certain space $\mathcal{F}(A_0, A_1)$ of functions $f : \mathbb{S} \to A_0 + A_1$, where $\mathbb{S} = \{z \in \mathbb{C} : 0 \le \operatorname{Re} z \le 1\}$, and with the convenient dense subspace $\mathcal{G}(A_0, A_1)$. We refer to [3] for the details.

We will begin our proof of Proposition 9, by establishing the inclusion

$$FC_\theta \subset [FL^\infty, FL_1^\infty]_\theta \tag{52}$$

and the two norm inequalities

$$\|\lambda\|_{[FL^\infty,FL_1^\infty]_\theta} \le \|\lambda\|_{FC_\theta} \quad \text{for all } \lambda \in FC_\theta, \tag{53}$$

and

$$\|\lambda\|_{[FL^\infty,FL_1^\infty]_{\theta,\mathbb{A}}} \le \|\lambda\|_{FC_\theta} \quad \text{for all } \lambda \in FC_\theta. \tag{54}$$

In fact it suffices to show that (53) and (54) each hold for all finitely supported sequences λ. Since these form a dense subset of FC_θ, this immediately implies that they both hold for all $\lambda \in FC_\theta$ and also establishes (52).

We use essentially the same simple reasoning as was used in [14] to show that $FL_\theta^1 \subset [FL^1, FL_1^1]_\theta$. Given an arbitrary sequence $\lambda = \{\lambda_n\}_{n\in\mathbb{Z}}$ with finite support, we introduce the sequence valued function $f(z) = \left\{e^{n(\theta-z)}\lambda_n\right\}_{n\in\mathbb{Z}}$. Clearly $f(\theta) = \lambda$, and for all real t and $j = 0,1$ we have

$$\begin{aligned}\|f(j+it)\|_{FL_j^\infty} &= \|f(j+it)\|_{FC_j^\infty} = \left\|\left\{e^{nj+n(\theta-j-it)}\lambda_n\right\}_{n\in\mathbb{Z}}\right\|_{FC} \\ &= \left\|\left\{e^{n\theta}e^{-int}\lambda_n\right\}_{n\in\mathbb{Z}}\right\|_{FC} = \left\|\left\{e^{n\theta}\lambda_n\right\}_{n\in\mathbb{Z}}\right\|_{FC} \\ &= \left\|\{\lambda_n\}_{n\in\mathbb{Z}}\right\|_{FC_\theta}.\end{aligned} \tag{55}$$

Since $f(z)$ is an entire function taking values in a finite dimensional subspace of $FL^\infty \cap FL_1^\infty$ and is bounded on $\mathbb{S}$, it is clear that, for each $\delta > 0$, the function $f_\delta(z) = e^{\delta(z-\theta)^2}f(z)$ is an element of Calderón's space $\mathcal{F}(FL^\infty, FL_1^\infty)$. So $\lambda = f_\delta(\theta) \in [FL^\infty, FL_1^\infty]_\theta$. By (55) we have

$$\begin{aligned}\|\lambda\|_{[FL^\infty,FL_1^\infty]_\theta} &\le \|f_\delta\|_{\mathcal{F}(FL^\infty,FL_1^\infty)} \\ &= \sup\left\{e^{\delta(j+it-\theta)^2}\|f(j+it)\|_{FL_j^\infty} : j = 0,1;\, t \in \mathbb{R}\right\} \\ &\le \max\left\{e^{\delta\theta^2}, e^{\delta(1-\theta)^2}\right\}\left\|\{\lambda_n\}_{n\in\mathbb{Z}}\right\|_{FC_\theta}.\end{aligned}$$

Since δ can be chosen arbitrarily small, this gives (53).

Let us also consider the function $g : \mathbb{A} \to FL^\infty \cap FL_1^\infty$ defined by $g(\zeta) = \left\{e^{n\theta}\zeta^{-n}\lambda_n\right\}_{n\in\mathbb{Z}}$. Since $g(e^z) = f(z)$ we of course have $g(e^\theta) = \lambda$ and also $g \in \mathcal{F}_{\mathbb{A}}(FL^\infty, FL_1^\infty)$ with norm

$$\|g\|_{\mathcal{F}_{\mathbb{A}}(FL^\infty,FL_1^\infty)} = \sup\left\{\left\|g(e^{j+it})\right\|_{FL_j^\infty} : j = 0,1;\, t \in [0,2\pi)\right\} = \left\|\{\lambda_n\}_{n\in\mathbb{Z}}\right\|_{FC_\theta}.$$

This immediately gives us (54).

We now know that (53) and (54) hold for all finitely supported sequences and so, as already explained, this establishes all of (52), (53) and (54).

Now we turn to proving the inclusion and inequalities which are the reverse of (52), (53) and (54) respectively. Again we will use density properties, more explicitly, the facts that $\mathcal{G}(FL^\infty, FL_1^\infty)$ is dense in $\mathcal{F}(FL^\infty, FL_1^\infty)$ and $\mathcal{G}_{\mathbb{A}}(FL^\infty, FL_1^\infty)$ is dense in $\mathcal{F}_{\mathbb{A}}(FL^\infty, FL_1^\infty)$. Our main step will be to show that, for each $g \in \mathcal{G}(FL^\infty, FL_1^\infty)$, we have $g(\theta) \in FC_\theta$ and

$$\|g(\theta)\|_{FC_\theta} \le \|g\|_{\mathcal{F}(FL^\infty,FL_1^\infty)}. \tag{56}$$

Analogously, for every $g \in \mathcal{G}_{\mathbb{A}}(FL^\infty, FL_1^\infty)$ we will show that $g(e^\theta) \in FC_\theta$ and

$$\left\|g(e^\theta)\right\|_{FC_\theta} \le \|g\|_{\mathcal{F}_{\mathbb{A}}(FL^\infty,FL_1^\infty)}. \tag{57}$$

If $g \in \mathcal{G}(FL^\infty, FL_1^\infty)$ then $g(z)$ is a finite sum $g(z) = \sum_{m=1}^{M} \phi_m(z) a^m$ where each $a^m = \{a_n^m\}_{n\in\mathbb{Z}}$ is a sequence in the space $FL^\infty \cap FL_1^\infty$ and each ϕ_m is a scalar valued entire function which is bounded on $\mathbb{S}$. Applying Theorem 12 to each sequence a^m, we see that $a^m \in FC_\theta$ for each m. (Note that $FL_j^\infty \subset FL_j^1$ and that the proof of Theorem 12 does not use Proposition 9. In fact, we only need the simple argument at the beginning of the proof.) So obviously also $g(\theta) \in FC_\theta$.

Given any sequence $\lambda = \{\lambda_n\}_{n\in\mathbb{Z}} \in FL^\infty$ and any other sequence $\sigma = \{\sigma_n\}_{n\in\mathbb{Z}}$ whose support $\mathbb{Z}_\sigma = \{n \in \mathbb{Z} : \sigma_n \neq 0\}$ is a finite set, we of course have that the finite sum $\sum_{n\in\mathbb{Z}} \lambda_n\sigma_n = \sum_{n\in\mathbb{Z}_\sigma} \lambda_n\sigma_n = \frac{1}{2\pi}\int_0^{2\pi} f_\lambda(e^{it}) f_\sigma(e^{-it})dt$, where f_λ and f_σ are the functions whose sequences of Fourier coefficients are respectively λ and σ. In particular this means that

$$\left|\sum_{n\in\mathbb{Z}} \lambda_n\sigma_n\right| \leq \frac{1}{2\pi} \|\lambda\|_{FL^\infty} \|\sigma\|_{FL^1} \tag{58}$$

and that

$$\|\lambda\|_{FL^\infty} = \sup_{\sigma\in Q}\left|\sum_{n\in\mathbb{Z}} \lambda_n\sigma_n\right| \tag{59}$$

where Q is the set of all finitely supported sequences σ such that $\|\sigma\|_{FL^1} = \int_0^{2\pi} \left|\sum_{n\in\mathbb{Z}_\sigma} \sigma_n e^{int}\right| dt = 2\pi$.

For each $\sigma \in Q$, we define the function $\psi_\sigma : \mathbb{C} \to \mathbb{C}$ by

$$\psi_\sigma(z) = \sum_{m=1}^{M} \phi_m(z) \sum_{n\in\mathbb{Z}_\sigma} a_n^m \sigma_n e^{nz}. \tag{60}$$

This is an entire function which is bounded on $\mathbb{S}$. So, by the Phragmèn–Lindelöf theorem, we have that

$$|\psi_\sigma(\theta)| \leq \sup\{|\psi_\sigma(j+it)| : j = 0, 1, t \in \mathbb{R}\}. \tag{61}$$

Applying (58), we see that, for $j = 0, 1$ and for all $t \in \mathbb{R}$,

$$\begin{aligned}
|\psi_\sigma(j+it)| &\leq \frac{1}{2\pi}\left\|\left\{\sum_{m=1}^{M} \phi_m(j+it) a_n^m e^{n(j+it)}\right\}_{n\in\mathbb{Z}}\right\|_{FL^\infty} \|\sigma\|_{FL^1} \\
&= \left\|\left\{\sum_{m=1}^{M} \phi_m(j+it) a_n^m e^{nj}\right\}_{n\in\mathbb{Z}}\right\|_{FL^\infty} = \left\|\left\{\sum_{m=1}^{M} \phi_m(j+it) a_n^m\right\}_{n\in\mathbb{Z}}\right\|_{FL_j^\infty} \\
&= \|g(j+it)\|_{FL_j^\infty} \leq \|g\|_{\mathcal{F}(FL^\infty, FL_1^\infty)}.
\end{aligned} \tag{62}$$

By (59) we have that

$$\begin{aligned}
\sup_{\sigma\in Q} |\psi_\sigma(\theta)| &= \left\|\left\{\sum_{m=1}^{M} \phi_m(\theta) a_n^m e^{n\theta}\right\}_{n\in\mathbb{Z}}\right\|_{FL^\infty} = \left\|\left\{\sum_{m=1}^{M} \phi_m(\theta) a_n^m\right\}_{n\in\mathbb{Z}}\right\|_{FL_\theta^\infty} \\
&= \|g(\theta)\|_{FL_\theta^\infty}.
\end{aligned} \tag{63}$$

Since $g(\theta) \in FC_\theta$ we have $\|g(\theta)\|_{FL_\theta^\infty} = \|g(\theta)\|_{FC_\theta}$ and so (56) follows from (63), (61) and (62). The proof of (57) is almost the same. The only differences are that this time the functions ϕ_m are each of the form $\phi_m(z) = z^{k_m}$ for some $k_m \in \mathbb{Z}$, and we have to use the function $\psi_\sigma(z) = \sum_{m=1}^{M} \phi_m(z) \sum_{n\in\mathbb{Z}_\sigma} a_n^m \sigma_n z^n$. By the maximum modulus principle for functions which are analytic on $\mathbb{A}$, we have $\left|\psi_\sigma(e^\theta)\right| \leq \max\left\{\left|\psi_\sigma(e^{j+it})\right| : j = 0, 1, t \in [0, 2\pi)\right\}$. Analogously to (62) this last expression is bounded from above by $\|g\|_{\mathcal{F}_{\mathbb{A}}(FL^\infty, FL_1^\infty)}$, and we obtain (57).

Finally, if f is an arbitrary element of $\mathcal{F}(FL^\infty, FL_1^\infty)$ then we approximate it in $\mathcal{F}(FL^\infty, FL_1^\infty)$ norm by a sequence $\{g_n\}_{n\in\mathbb{N}}$ in $\mathcal{G}(FL^\infty, FL_1^\infty)$. Thus, by (56), $\{g_n(\theta)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in FC_θ. Since it converges to $f(\theta)$ in the norm of $FL^\infty + FL_1^\infty$ it follows that $f(\theta) \in FC_\theta$ and $\|f(\theta)\|_{FC_\theta} \leq \|f\|_{\mathcal{F}(FL^\infty, FL_1^\infty)}$. Consequently $[FL^\infty, FL_1^\infty]_\theta \subset FC_\theta$ and $\|\lambda\|_{FC_\theta} \leq \|\lambda\|_{[FL^\infty, FL_1^\infty]_\theta}$ for all $\lambda \in [FL^\infty, FL_1^\infty]_\theta$. Since $\mathcal{G}_{\mathbb{A}}(FL^\infty, FL_1^\infty)$ is dense in $\mathcal{F}_{\mathbb{A}}(FL^\infty, FL_1^\infty)$, an exactly analogous argument using (57) shows that $\|\lambda\|_{FC_\theta} \leq \|\lambda\|_{[FL^\infty, FL_1^\infty]_{\theta,\mathbb{A}}}$. (Alternatively, one could use the first case and the general fact that the inclusion $[\vec{A}]_{\theta,\mathbb{A}} \to [\vec{A}]_\theta$ has norm 1, see [8].)

This completes the proof of Proposition 9.

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