Line-of-sight percolation

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Abstract

Given $\omega \geq 1$, let $\mathbb{Z}^2_{(\omega)}$ be the graph with vertex set $\mathbb{Z}^2$ in which two vertices are joined if they agree in one coordinate and differ by at most ω in the other. (Thus $\mathbb{Z}^2_{(1)}$ is precisely $\mathbb{Z}^2$.) Let $p_c(\omega)$ be the critical probability for site percolation on $\mathbb{Z}^2_{(\omega)}$. Extending recent results of Frieze, Kleinberg, Ravi and Debany, we show that $\lim_{\omega\to\infty} \omega p_c(\omega) = \log(3/2)$. We also prove analogues of this result for the n-by-n grid and in higher dimensions, the latter involving interesting connections to Gilbert's continuum percolation model. To prove our results, we explore the component of the origin in a certain non-standard way, and show that this exploration is well approximated by a certain branching random walk.

1 Introduction and results

Frieze, Kleinberg, Ravi and Debany [9] proposed the following random graph as a model of an *ad hoc* network in an environment with (regular) obstructions. Given positive integers n and ω, let $G = [n]^2_{(\omega)}$ be the graph with vertex set $[n] \times [n]$ in which two vertices (x_1, y_1) and (x_2, y_2) are joined if $x_1 = x_2$ and $|y_1 - y_2| \leq \omega$ or $y_1 = y_2$ and $|x_1 - x_2| \leq \omega$. Let V be a random subset of $[n] \times [n]$ obtained by selecting each point (x, y) with probability p, independently of the other points, and let $G[V]$ denote the subgraph of G induced by the vertices in V. We write $G_{n,\omega,p}$ for $G[V]$, and $G_{\omega,p}$ for the infinite random subgraph of $\mathbb{Z}^2_{(\omega)}$ defined in the same way but starting from $\mathbb{Z}^2$ rather than $[n] \times [n]$.

The model $G'_{n,\omega,p}$ studied in [9] was defined on the n by n torus rather than the grid, but $G_{n,\omega,p}$ seems more natural in terms of the original motivation. For the results here, it will make no difference which variant we consider.

One interpretation of $G_{n,\omega,p}$ is as follows: sensors are dropped onto a random selection of crossroads in a regularly laid out city (or planted forest); two sensors

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can communicate if they are within distance ω and the line of sight between them is not blocked by a building (tree). The graph $G_{n,\omega,p}$ indicates which pairs of sensors can communicate directly, so we would like to know when $G_{n,\omega,p}$ has a giant component, and roughly how large it is. Alternatively, taking the point of view of percolation theory, we would like to study percolation in $G_{\omega,p}$, i.e., to know for which choices of the parameters $G_{\omega,p}$ has an infinite component.

Kolmogorov's 0/1–law implies that for each fixed ω there is a 'critical probability' $p_c(\omega)$ such that if $p > p_c(\omega)$ then $G_{\omega,p}$ contains an infinite component with probability 1, while if $p < p_c(\omega)$ then with probability 1 all components of $G_{\omega,p}$ are finite. Furthermore, by the 'uniqueness theorem' of Aizenman, Kesten and Newman [1] (see also [7, p. 121]), when an infinite cluster exists it is unique with probability 1. Among other results, Frieze, Kleinberg, Ravi and Debany [9] proved (essentially) that

$$1/(4e) \leq \liminf_{\omega\to\infty} \omega p_c(\omega) \leq \limsup_{\omega\to\infty} \omega p_c(\omega) \leq 1,$$

and posed the natural question of determining the value of $\lim \omega p_c(\omega)$, assuming it exists. (In fact, they worked with finite random graphs, as we shall in Theorem 2 below.) Here we shall answer this question, proving that

$$\lim_{\omega\to\infty} \omega p_c(\omega) = \log(3/2) = 0.4054\ldots, \tag{1}$$

as part of the more detailed Theorem 1 below.

In homogeneous cases such as this, where the expected degrees of all vertices are equal, the critical probability is perhaps more naturally described in terms of the 'critical expected degree'. Since the expected degree of a vertex of $G_{\omega,p}$ is $4p\omega$, (1) states that the critical expected degree of $G_{\omega,p}$ tends to $4\log(3/2) = 1.6218\ldots$ as $\omega \to \infty$.

Note that it is too much to hope to find $p_c(\omega)$ exactly for a given ω: with ω fixed, $p_c(\omega)$ is the critical probability for site percolation on a certain lattice $\mathbb{Z}^2_{(\omega)}$ (non-planar for $\omega \geq 2$); the exact values of such critical probabilities are known only in a few very special cases. Indeed, $p_c(1)$ is the critical probability for site percolation on $\mathbb{Z}^2$, and even this is not known exactly.

In spirit, Theorem 1 and its proof are very close to the arguments given in [6] for Penrose's spread-out percolation model [13]: when $\omega \to \infty$ with $p\omega$ bounded, we can approximate the neighbourhood exploration process in $G_{\omega,p}$ by a certain branching process, and describe the distribution of the small components in this graph in terms of the branching process. We then show by a rather *ad hoc* argument that almost all vertices not in 'small' components are in a single infinite component. An analogous assertion holds for $G_{n,\omega,p}$ and its unique giant component. In contrast to [6], where we could appeal to the general sparse inhomogeneous random graph model of [5], here even the local coupling to the branching process has to be done by hand. We shall discuss this later.

To state our results precisely, we first define the branching process alluded to above. Let $\mu > 0$ be a real parameter, and let $\Gamma = \Gamma_\mu$ denote (a random variable with) the geometric distribution with parameter $1 - e^{-\mu}$. Thus,

$$\mathbb{P}(\Gamma = k) = (1 - e^{-\mu})^k e^{-\mu}$$

for $k = 0, 1, \ldots$, and $\mathbb{E}(\Gamma) = e^{\mu} - 1$. Let $\Gamma_{\mu}^{(r)}$ denote the distribution of the sum of r independent copies of Γ_{μ}.

Let $\mathfrak{X}_{\mu}$ be the branching process $(X_0, X_1, \ldots)$ in which X_0 consists of a single particle x_0, each particle $x \in X_t$ has children independently of the other particles and of the history, the number of children of x_0 has the distribution $\Gamma_{\mu}^{(4)}$, and the number of children of a particle $x \in X_t$, $t \geq 1$, has the distribution $\Gamma_{\mu}^{(2)}$. Let $\phi(\mu)$ denote the *survival probability* of the branching process $\mathfrak{X}_{\mu}$, i.e.,

$$\phi(\mu) = \mathbb{P}\big(|X_t| > 0 \text{ for all } t\big).$$

Note that

$$\mathbb{E}\big(\Gamma_{\mu}^{(2)}\big) = 2\mathbb{E}(\Gamma_{\mu}) = 2(e^{\mu} - 1),$$

so the branching process $\mathfrak{X}_{\mu}$ is supercritical if and only if $\mu > \log(3/2)$, i.e., $\phi(\mu) > 0$ if and only if $\mu > \log(3/2)$.

In the result below, C_0 denotes the component of the origin in the random graph $G_{\omega,p}$; if the origin is not a vertex of this graph, then we set $C_0 = \emptyset$. We write $\mathbb{P}_{\omega,p}$ for the probability measure associated to $G_{\omega,p}$, and $\theta(\omega, p) = \mathbb{P}_{\omega,p}(|C_0| = \infty)$ for the probability that C_0 is infinite.

Theorem 1. *Let $0 < \lambda < \log(3/2)$ be constant. If ω is large enough then all components of $G_{\omega,\lambda/\omega}$ are finite with probability 1. Furthermore, there is a constant $a_{\lambda} > 0$ such that, for all large enough ω,*

$$\mathbb{P}_{\omega,\lambda/\omega}(|C_0| \geq k) \leq \frac{\lambda}{\omega} e^{-a_{\lambda}(k-1)}$$

holds for every $k \geq 1$.

Let $\lambda > \log(3/2)$ be constant. If ω is large enough, then $G_{\omega,\lambda/\omega}$ contains a unique infinite component with probability 1. Furthermore,

$$\theta(\omega, \lambda/\omega) \sim \phi(\lambda)\lambda/\omega \tag{2}$$

as $\omega \to \infty$.

Note that Theorem 1 certainly implies (1), which is equivalent (as noted earlier) to the statement that the critical expected degree tends to $4\log(3/2)$ as $\omega \to \infty$. The proof of Theorem 1 will form the bulk of Section 2. In fact, the upper bound on component sizes (giving the lower bound on the critical expected degree), is fairly easy. The lower bound on component sizes (giving the upper bound on the critical expected degree) will be derived in a way that will enable us to prove a finite analogue of Theorem 1, stated as Theorem 2 below.

For the upper bound on component sizes we shall prove a much more detailed statement than stated in Theorem 1, namely that

$$\mathbb{P}_{\omega,p}(|C_0| \geq k) \leq p\mathbb{P}(|\mathfrak{X}_{\mu}| \geq k) \tag{3}$$

for every $\omega \geq 1$, $0 < p < 1$ and $k \geq 1$, where

$$e^{-\mu} = (1 - p)^{\omega}, \tag{4}$$

and $|\mathfrak{X}_\mu|$ denotes the total number of particles in all generations of $\mathfrak{X}_\mu$. For $\mu < \log(3/2)$, the branching process $\mathfrak{X}_\mu$ is strictly subcritical, and standard results imply that there is some $b_\mu > 0$ such that $\mathbb{P}(|\mathfrak{X}_\mu| \geq k) \leq e^{-b_\mu(k-1)}$ holds for every k. If $\lambda < \log(3/2)$ is constant and we set $p = \lambda/\omega$, then as $\omega \to \infty$ we have $\mu \sim p\omega = \lambda$. In particular, μ is bounded above by some $\mu' < \log(3/2)$ when ω is large enough, so the first statement in Theorem 1 follows.

Letting k tend to infinity in (3) (or simply setting $k = \infty$, which makes perfect sense), we see that $\theta(\omega, p) \leq p\phi(\mu)$. Standard results on branching processes imply that $\phi(\cdot)$ is continuous. Thus (3) implies the upper bound implicit in (2), i.e., that

$$\theta(\omega, \lambda/\omega) \leq (1 + o(1))\phi(\lambda)\lambda/\omega$$

as $\omega \to \infty$ with $\lambda > \log(3/2)$ constant.

Note that for the exact comparison with a branching process above, it is natural to work with the parameter $\mu = -\log((1-p)^\omega)$. However, in the lower bounds on component sizes, we can obtain only approximate results, and it is more natural to use the asymptotically equivalent parameter $\lambda = p\omega$, which we have used throughout the statement of Theorem 1. We shall switch between these two parameters as and when convenient.

Although our main focus is the infinite random graph $G_{\omega,p}$, our other aim is to prove a result corresponding to Theorem 1 for the finite graphs $G_{n,\omega,p}$ and $G'_{n,\omega,p}$. Note that there is a natural coupling in which $G_{n,\omega,p}$, defined on the grid, is a subgraph of $G'_{n,\omega,p}$, defined on the torus. Let $C_1(G)$ denote the number of vertices in a largest component of a graph G.

Theorem 2. *Let $0 < \lambda < \log(3/2)$ be constant. There are constants A_λ and ω_λ such that*

$$C_1(G_{n,\omega,\lambda/\omega}) \leq C_1(G'_{n,\omega,\lambda/\omega}) \leq A_\lambda \log n$$

holds with probability $1 - o(1)$ as $n \to \infty$, whenever $\omega = \omega(n)$ satisfies $\omega_\lambda \leq \omega \leq n$.

Let $\lambda > \log(3/2)$ be constant. If $\omega = \omega(n)$ is such that ω and $n/\omega \to \infty$, then

$$C_1(G_{n,\omega,\lambda/\omega}),\ C_1(G'_{n,\omega,\lambda/\omega}) = (\phi(\lambda) + o_p(1))n\lambda/\omega$$

as $n \to \infty$.

We shall also state results for the case $\omega \to \infty$ with $n = \Theta(\omega)$; for details see Subsection 2.3.

2 Proofs

2.1 Upper bounds on component sizes

We start with some simple upper bounds on the component sizes. Let p and ω be arbitrary. To prove (3), it suffices to couple $G_{\omega,p}$ and $\mathfrak{X}_\mu$ so that if we explore the component C_0 of the origin in $G_{\omega,p}$ in a suitable manner, then the

number of vertices reached at each stage is at most the number of particles of $\mathfrak{X}_\mu$ in the corresponding generation, where $\mu = -\omega \log(1-p)$ is given by (4).

In each step of our overall exploration process (except for the first) we either 'explore vertically', first up and then down, or 'explore horizontally', first to the left and then to the right. To 'explore upwards' from a vertex $v = (x, y)$ of $G_{\omega,p}$, we test the points $(x, y+1)$, $(x, y+2)$, ..., one by one. Each test 'succeeds' if the relevant point is a vertex of $G_{\omega,p}$ that was not previously reached (in an earlier stage of the overall exploration). We stop the upwards exploration as soon as ω consecutive tests fail.

If $v_1, \dots, v_r$ denote the vertices of $G_{\omega,p}$ reached during this upwards exploration, corresponding to the successful tests, then vv_1, $v_1v_2, \dots, v_{r-1}v_r$ are all edges of $G_{\omega,p}$. Also, there is typically no edge of $G_{\omega,p}$ from $v_r = (x, y')$ to a point (x, y'') with $y'' > y'$; this is not always true as a test may fail due to the presence of a vertex we reached earlier. If we have not previously tested the relevant points, so each is present in $G_{\omega,p}$ with probability p, independently of the others, then the probability that the next ω tests fail is exactly $(1-p)^\omega = e^{-\mu}$, so r has the geometric distribution Γ_μ. In general, we may have previously tested some of the points; a repeated test of a point always fails by definition, so, conditional on previous steps of the exploration, the distribution of r is stochastically dominated by Γ_μ.

To 'explore vertically' from a vertex v, we explore upwards from v, and then explore downwards from v. Similarly, to explore horizontally, we first explore to the left, and then to the right.

Let us condition on $v_0 = (0, 0)$ being a vertex of $G_{\omega,p}$, an event of probability p. Set $Y_0 = \{v_0\}$. Explore vertically and horizontally from v_0, and let Y_1 be the set of vertices of $G_{\omega,p}$ reached in these explorations. From the remarks above, $|Y_1|$ has exactly the distribution $\Gamma_\mu^{(4)}$. Suppose that we have defined Y_t, $t \geq 0$. For every vertex $v \in Y_t$ that was reached during a horizontal exploration, we explore vertically from v. Similarly, we explore horizontally from each $v \in Y_t$ reached during a vertical exploration. Let Y_{t+1} be the set of new vertices of $G_{\omega,p}$ reached.

It is not hard to check that the exploration just defined does indeed uncover the whole component C_0. Indeed, let $C_0' = \bigcup_{t \geq 0} Y_t$. Then we certainly have $v_0 \in C_0' \subset C_0$. Since C_0 is a connected subgraph of $G_{\omega,p}$, if $C_0' \neq C_0$ then there are vertices $v \in C_0'$ and $w \in C_0 \setminus C_0'$ that are adjacent in $G_{\omega,p}$. Choosing v and w at minimal (Euclidean) distance, there are no vertices of $G_{\omega,p}$ between v and w. We may suppose without loss of generality that $v = (x, y)$ and $w = (x', y)$ with $0 < x' - x \leq \omega$. When exploring, we reach v at some stage; if we do so during a vertical exploration, then exploring from v to the right we reach w at the first step. On the other hand, if we reach v while exploring to the right or to the left, the same exploration would also reach w, either just after or just before reaching v. This shows that $w \in C_0'$, contradicting our assumptions.

From the remarks above, conditional on previous explorations, the number of vertices reached in a horizontal or vertical exploration is stochastically dominated by the $\Gamma_\mu^{(2)}$ distribution. Thus the whole sequence $|Y_0|, |Y_1|, |Y_2|, \dots$ is

stochastically dominated by $|X_0|, |X_1|, |X_2|, \ldots$, where $X_0, X_1, \ldots$ are the generations of the branching process $\mathfrak{X}_\mu$. Since $C_0 = \bigcup_{t \geq 0} Y_t$, we thus have

$$\mathbb{P}\big(|C_0| \geq k \mid v_0 \in V(G_{\omega,p})\big) \leq \mathbb{P}(|\mathfrak{X}_\mu| \geq k)$$

for every $k \geq 1$, which gives (3).

As noted earlier, the first part of Theorem 1 follows, along with the upper bound implicit in (2).

2.2 The lower bound on component sizes

Throughout this section we shall set $p = \lambda/\omega$, where $\lambda > \log(3/2)$ is constant and we let $\omega \to \infty$. All asymptotic notation refers to this limit.

Usually, in showing that the neighbourhood exploration process in some random graph is well approximated by a branching process, one first shows that a given vertex is unlikely to be in a short cycle. Here the latter statement is false: given that $v = (x, y)$ is a vertex of $G_{\omega,p}$, it has probability $\Theta(1)$ of being in a triangle of the form (x, y), $(x, y + a)$, $(x, y + b)$, $0 < a < b \leq \omega$. However, these triangles (and similar cycles) have already been accounted for in the construction of the branching process. If we delete all edges of $G_{\omega,p}$ corresponding to line segments whose interiors contain vertices of $G_{\omega,p}$, then it is easy to check that the density of short cycles in the resulting graph is very low. We shall show something more or less equivalent to this statement.

Let k be fixed. As before, let us condition on the event that $v_0 = (0, 0)$ is a vertex of $G_{\omega,p}$. We claim that we can couple the first k steps $Y_0, Y_1, \ldots, Y_k$ of the exploration defined above with the first k generations of $\mathfrak{X}_\lambda$ so that they agree (in the sense that $|Y_t| = |X_t|$ for $0 \leq t \leq k$) with probability $1 - o(1)$. (Here the asymptotic notation refers to the limit $\omega \to \infty$ with k and $\lambda = p\omega$ fixed.) We have already shown that the exploration is stochastically dominated by $\mathfrak{X}_\mu$, where $\mu = -\omega \log(1 - p)$. Since $\mu \sim \lambda$, the processes $\mathfrak{X}_\mu$ and $\mathfrak{X}_\lambda$ may be coupled so that their first k generations agree whp, so it suffices to 'bound (Y_t) from below', i.e., to couple a subset of this exploration process with $\mathfrak{X}_\lambda$ so as to agree whp for k generations.

In doing so, we must be careful: suppose that as part of our exploration we find vertices of C_0 located as in Figure 1. Then we are in trouble – since w and w' lie on the same horizontal line, the horizontal explorations from these points will necessarily interfere. In particular, the expected number of new vertices reached from w and w' together will be significantly less than $2\mathbb{E}(\Gamma_\lambda^{(2)})$. Thus, if we reach w first, we do not allow ourselves to test the point w'.

In general, we consider the *lines* of $\mathbb{Z}^2$, i.e., the sets of the form $\{x\} \times \mathbb{Z}$ or $\mathbb{Z} \times \{y\}$. Initially, we mark all lines as 'unclaimed', except the two lines through v_0, which we mark as 'claimed by v_0'. Whenever we find a vertex $v \in C_0$ during a horizontal exploration, this vertex immediately claims the vertical line it lies on. Similarly, when we find a vertex v during a vertical exploration, v claims the horizontal line it lies on. We modify our exploration as follows: when exploring horizontally or vertically from a vertex v, we omit testing any point w on a

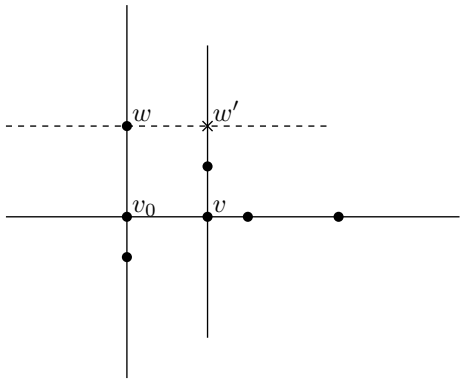

Figure 1: Part of our exploration, including a possible clash (between w and w') that we must avoid. The solid lines indicate regions that have already been tested for vertices of $G_{\omega,p}$; these lines extend a distance ω from the last vertex found.

line already claimed by a vertex other than v. This ensures that the same line cannot be claimed twice: when we find a vertex $w \in C_0$ by exploring from v, exactly one line through w has been claimed, namely that joining v to w; the vertex w then claims the other unclaimed line.

In this modified exploration, a point (x, y) can only be tested once, so each tested point is present in $G_{\omega,p}$ with (conditional) probability p. Since the expected number of vertices reached within k steps of the exploration is $O(1)$, whp there are at most $\log\omega$ such vertices, say, and hence at most $2\log\omega$ claimed lines. In each horizontal or vertical exploration, we are exploring along a claimed line, and we omit testing at most one point on each other claimed line. Thus we omit testing $O(\log\omega) = o(1/p)$ points. Since we would have expected only $o(1)$ successes among these tests, omitting them makes essentially no difference: the number r of new vertices found may be coupled with a $\Gamma_\lambda^{(2)}$ distribution so as to agree with probability $1 - o(1)$. As we expect to reach $O(1)$ vertices during the first k steps of our exploration, the sum of these $o(1)$ error probabilities is still $o(1)$, so the claim follows.

The truth of the statement '$|C_0| \geq k$' is certainly determined by the numbers $|Y_0|, \ldots, |Y_{k-1}|$. Thus, for every $1 \leq k < \infty$ we have

$$\mathbb{P}_{\omega,\lambda/\omega}\big(|C_0| \geq k \mid v_0 \in V(G_{\omega,\lambda/\omega})\big) \to \mathbb{P}(|\mathfrak{X}_\lambda| \geq k)$$

as $\omega \to \infty$, i.e.,

$$\mathbb{P}_{\omega,\lambda/\omega}(|C_0| \geq k) \sim \mathbb{P}(|\mathfrak{X}_\lambda| \geq k)\lambda/\omega.$$

It follows that there is some $K = K(\omega) \to \infty$ such that

$$\mathbb{P}_{\omega,\lambda/\omega}(|C_0| \geq K) \sim \mathbb{P}(|\mathfrak{X}_\lambda| \geq K)\lambda/\omega.$$

Since the branching process $\mathfrak{X}_\lambda$ does not depend on ω, the right-hand side above

is asymptotically $\phi(\lambda)\lambda/\omega$. To complete the proof of (2) it remains to show that

$$\mathbb{P}_{\omega,\lambda/\omega}(K \le |C_0| < \infty) = o(\lambda/\omega).$$

In other words, roughly speaking, we must show that almost all vertices of $G_{\omega,\lambda/\omega}$ in 'large' components are in infinite components. As noted in the introduction, the probability that there is an infinite component is either 0 or 1, and when there is an infinite component it is unique with probability 1, so the remaining assertion of Theorem 1 follows.

So far we have coupled the neighbourhood exploration process in $G_{\omega,p}$ with a branching process $\mathfrak{X}_\lambda$. If we keep track of the locations of the points in $Y_0, Y_1, \ldots$ as well as their number, then the appropriate limit object is a branching random walk.

Indeed, let us turn $\mathfrak{X}_\lambda$ into a branching random walk $\widetilde{\mathfrak{X}}_\lambda$ on $\mathbb{R}^2$ (or, more formally, on $\mathbb{R}^2 \times \{\mathrm{v}, \mathrm{h}\}$) as follows. Each generation $\widetilde{X}_t$, $t \ge 1$, will consist of a finite set of points of $\mathbb{R}^2$, each labelled with either 'v' (for reached by a vertical step) or 'h' (reached by a horizontal step). A particle of type 'v' at a point (x, y) has children in the next generation according to the following rule: generate a Poisson process $\mathcal{P}$ of intensity λ on $\mathbb{R} \times \{y\}$. Starting from (x, y) and working to the right, include as children of (x, y) all points of $\mathcal{P}$ until the first time that we come to a gap of length greater than 1. Do the same to the left. All children have type 'h'. The rule for children of particles of type 'h' is similar, using the line $\{x\} \times \mathbb{R}$ instead. Sometimes we start with two particles at the origin, one of type 'h' and one of type 'v'; we write $\widetilde{\mathfrak{X}}^2_\lambda$ for this branching random walk. At other times, we consider the same branching rule but start with a single particle, writing $\widetilde{\mathfrak{X}}^1_\lambda$. When the starting rule is clear or unimportant, we write simply $\widetilde{\mathfrak{X}}_\lambda$. Since the number of offspring of a particle in $\widetilde{\mathfrak{X}}_\lambda$ is independent of the position of this particle, we may view $\widetilde{\mathfrak{X}}^2_\lambda$ as our original branching process $\mathfrak{X}_\lambda$ by simply ignoring the positions of the particles.

It is easy to check that the rules above correspond to a certain limit of the horizontal and vertical explorations we considered earlier, where we take $\omega \to \infty$ and rescale by dividing the coordinates of our lattice points by ω. In particular, the coupling argument above shows that for any fixed k, we may couple $(Y_t)_{t \le k}$ with the first k generations $(\widetilde{X}_t)_{t \le k}$ of the branching random walk $\widetilde{\mathfrak{X}}^2_\lambda$ so that with probability $1 - o(1)$ there is exactly one particle $(x', y') \in \widetilde{X}_t$ for each $(x, y) \in Y_t$, $1 \le t \le k$, and $|x/\omega - x'|, |y/\omega - y'| \le k/\omega$.

We shall only study the branching random walk $\widetilde{\mathfrak{X}}_\lambda$ in a trivial way, calculating certain expectations; moreover, we shall avoid all detailed calculations, needing only 'soft' arguments.

We start with a simple lemma showing that the supercritical branching random walk $\widetilde{\mathfrak{X}}_\mu$, $\mu > \log(3/2)$, remains supercritical even when restricted to a large enough square. By the *restriction* of $\widetilde{\mathfrak{X}}_\mu$ to a region R we mean the branching random walk obtained from $\widetilde{\mathfrak{X}}_\mu$ by deleting all particles that lie outside R, along with all their descendants. We write $\widetilde{\mathfrak{X}}_\mu(C)$ for the restriction of $\widetilde{\mathfrak{X}}_\mu$ to $[-C, C]^2$.

Lemma 3. *Let $\mu > \log(3/2)$ and $\varepsilon > 0$ be fixed. There are constants A, B and T_1 (depending on μ) with the following properties.*

(i) For any point (x_0, y_0) of $[-A, A]^2$, if we start the branching random walk $\widetilde{\mathfrak{X}}_\mu$ with a single point (of type 'h', say) at (x_0, y_0) and restrict to $[-B, B]^2$, then the expected number of points of generation T_1 inside $[-A, A]^2$ is at least 2.

(ii) We have

$$\mathbb{P}\big(\widetilde{\mathfrak{X}}_\mu^2(B) \text{ survives forever}\big) \geq \phi(\mu) - 3\varepsilon, \tag{5}$$

where $\phi(\mu)$ is the survival probability of the branching process $\mathfrak{X}_\mu$.

Proof. Starting the unrestricted walk $\widetilde{\mathfrak{X}}_\mu^1$ with a single particle at the origin, the expected size of generation t is $\mathbb{E}\big(\Gamma_\mu^{(2)}\big)^t = (2e^\mu - 2)^t$. Since $\mu > \log(3/2)$, there is some T_1 such that this expectation is at least 10. Fix such a T_1.

Let $\widetilde{\mathfrak{X}}_\mu^1(A)$ denote the restriction of $\widetilde{\mathfrak{X}}_\mu^1$ to $[-A, A]^2$. As $A \to \infty$, the first T_1 generations of $\widetilde{\mathfrak{X}}_\mu^1(A)$ converge in distribution to the first T_1 generations of $\widetilde{\mathfrak{X}}_\mu^1$. Hence there is some A_0 such that the expected size of generation T_1 of $\widetilde{\mathfrak{X}}_\mu^1(A)$ is at least 8 whenever $A \geq A_0$.

By symmetry, the distributions of the number of points of generation T_1 of $\widetilde{\mathfrak{X}}_\mu^1(A_0)$ in each of the four squares $[0, \pm A_0] \times [0, \pm A_0]$ are identical. Let $\mathcal{D}$ denote this distribution, noting that $\mathbb{E}(\mathcal{D}) \geq 2$.

Let $A \geq A_0$ be a constant depending on μ and ε, to be chosen later. Set $B = 2A$, and let (x_0, y_0) be any point of $[-A, A]^2$. Without loss of generality, we may assume that $x_0, y_0 \leqq 0$, so $[x_0, x_0 + A] \times [y_0, y_0 + A]$ is contained in $[-A, A]^2$; see Figure 2. Let $\widetilde{\mathfrak{X}}'_\mu$ denote the branching random walk $\widetilde{\mathfrak{X}}_\mu^1$ started at (x_0, y_0) and restricted to $[-B, B]^2$. Since the square of side $2A$ centered at (x_0, y_0) is contained in $[-B, B]^2$, the process $\widetilde{\mathfrak{X}}'_\mu$ stochastically dominates the translated process $\widetilde{\mathfrak{X}}_\mu^1(A) + (x_0, y_0)$. In particular, the number of points of $\widetilde{\mathfrak{X}}'_\mu$ in $[-A, A]^2$, which is at least the number of points in $[x_0, x_0 + A] \times [y_0, y_0 + A]$, stochastically dominates $\mathcal{D}$. Since $\mathbb{E}(\mathcal{D}) \geq 2$, this proves (i).

We have shown that, for any $A \geq A_0$, if we start our branching walk restricted to $[-2A, 2A]^2$ at any point of $[-A, A]^2$, and look at the number of points of $[-A, A]^2$ reached at times 0, T_1, $2T_1$, $3T_1$,..., then the numbers we see stochastically dominate a certain supercritical Galton–Watson branching process (with offspring distribution $\mathcal{D}$). This process survives forever with some probability $p_0 > 0$ depending on μ only.

Recall that ignoring the positions of the particles turns the (unrestricted) branching walk $\widetilde{\mathfrak{X}}_\mu^2$ into our original branching process $\mathfrak{X}_\mu$. Except that the offspring distribution is different for the first generation, $\mathfrak{X}_\mu$ is a standard Galton–Watson process, so standard results (for example, Theorem 2 in Athreya and Ney [2]) imply that with probability 1, either $\mathfrak{X}_\mu$ dies out (an event of probability $1 - \phi(\mu)$), or $|X_t| \to \infty$ as $t \to \infty$. It follows that there is a time T such that with probability at least $\phi(\mu) - \varepsilon$ generation T contains at least $\log(1/\varepsilon)/p_0$ particles. Hence, choosing $A \geq A_0$ large enough, the walk $\widetilde{\mathfrak{X}}_\mu^2(A)$ has probability

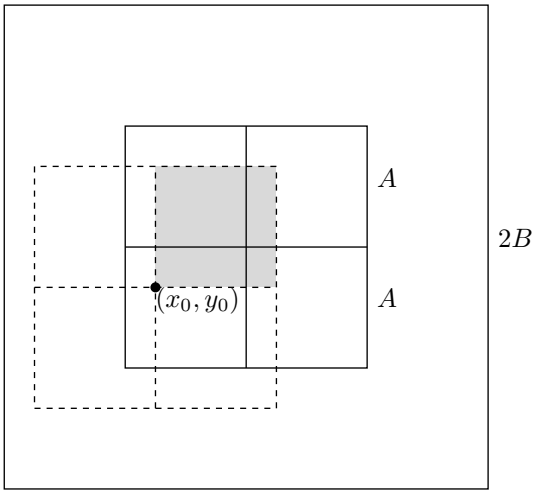

Figure 2: The dotted lines show the square S of side $2A$ centered at (x_0, y_0), which is contained within the outer square of side $2B = 4A$ centered at the origin. The shaded quadrant of S is contained in $[-A, A]^2$.

at least $\phi(\mu) - 2\varepsilon$ of generating at least $\log(1/\varepsilon)/p_0$ particles in generation T; these particles lie in $[-A, A]^2$ by definition.

Let N denote the number of particles of $\widetilde{\mathfrak{X}}^2_\mu(B)$ in generation T in $[-A, A]^2$. Since $\widetilde{\mathfrak{X}}^2_\mu(B)$ may be regarded as a superset of $\widetilde{\mathfrak{X}}^2_\mu(A)$, we have $N \geq \log(1/\varepsilon)/p_0$ with probability at least $\phi(\mu) - 2\varepsilon$. Each particle of $\widetilde{\mathfrak{X}}^2_\mu(B)$ in $[-A, A]^2$ survives forever with probability at least p_0, independently of the others, so

$$\begin{aligned} \mathbb{P}\big(\widetilde{\mathfrak{X}}^2_\mu(B) \text{ dies out}\big) &\leq \mathbb{P}\big(N < \log(1/\varepsilon)/p_0\big) + (1-p_0)^{\log(1/\varepsilon)/p_0} \\ &< 1 - (\phi(\mu) - 2\varepsilon) + \varepsilon = 1 - (\phi(\mu) - 3\varepsilon). \end{aligned}$$

In other words, $\widetilde{\mathfrak{X}}^2_\mu(B)$ survives with probability at least $\phi(\mu) - 3\varepsilon$, proving (5). □

In the next lemma we consider the restricted walk $\widetilde{\mathfrak{X}}_\mu(B)$ started at a particle v of type 'h' just below the bottom edge of $[-B, B]^2$; making sense of this requires a slight modification of our definition of restriction. Recalling that we label particles by the kind of exploration that reached them, the first step from v is vertical, so the first generation $\widetilde{X}_1$ of the unrestricted walk started at v consists of points on the same vertical line as v. To define the first generation $\widetilde{X}'_1$ of our restriction $\widetilde{\mathfrak{X}}_\mu(B)$, first delete all points of $\widetilde{X}_1$ outside $[-B, B]^2$. Then, if some point in the remaining set $\widetilde{X}_1 \cap [-B, B]^2$ is within distance 1 of v, let $\widetilde{X}'_1 = \widetilde{X}_1 \cap [-B, B]^2$; otherwise, delete all remaining points too and let $\widetilde{X}'_1 = \emptyset$. All points in $\widetilde{X}'_1$ are inside $[-B, B]^2$, and from here we continue the restricted walk as usual. The reason for the somewhat fussy definition of the

first generation is that when we return to the graph $G_{\omega,p}$, we will wish to find certain paths starting at a vertex v just outside $[-\omega B, \omega B]^2$, with all remaining vertices inside $[-\omega B, \omega B]^2$; the first edge of such a path must join v to a point in $[-\omega B, \omega B]^2$ at distance at most ω from v.

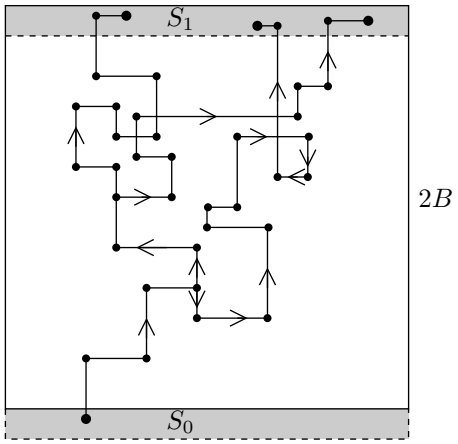

Figure 3: The shaded regions are the sets S_0 and S_1 considered in Lemma 4 and Corollary 5; the solid square is the boundary of $[-B, B]^2$. The paths are for $T_2 = 16$. Note that there may be additional points of the branching process in the interiors of the line segments of these paths; indeed, there must be in any line segment of length more than 1.

Lemma 4. *Let $\mu > \log(3/2)$ be given. There are constants B and T_2 (with B as in Lemma 3) with the following property. Whenever we start $\widetilde{\mathfrak{X}}_\mu(B)$ at a point of type 'h' in the set $S_0 = [-B, B] \times [-B - 1/2, -B]$ shown in Figure 3, the expected number of points of generation T_2 of type 'h' in the set $S_1 = [-B, B] \times [B - 1/2, B]$ is at least 2.*

Proof. Let A, B and T_1 be as in Lemma 3, and let $(\widetilde{X}'_t)$ denote the generations of the branching random walk under consideration. From the definition of $\widetilde{\mathfrak{X}}_\mu$, the (non-normalized) probability density function describing the x- or y- displacement in a single step is positive on the whole real line. Hence the probability that $\widetilde{X}'_2$ contains a point of $[-A, A]^2$, which depends on (x_0, y_0), is bounded below by some constant $c = c(B) > 0$, independent of (x_0, y_0). Using part (i) of Lemma 3 and induction on k, it follows that

$$\mathbb{E}\big(|\widetilde{X}'_{kT_1+2} \cap [-A, A]^2|\big) \geq 2^k c$$

for every k.

Arguing as at the start of the proof, there is some $c' > 0$ such that for any point $(x, y) \in [-A, A]^2$, if we start a copy of $\widetilde{\mathfrak{X}}^1_\mu$ at a point (x, y) of type 'h' and restrict to $[-B, B]^2$, the expected number of points of S_1 that we obtain two

generations later is at least c'. It follows that

$$\mathbb{E}\big(|\widetilde{X}'_{kT_1+4} \cap S_1|\big) \geq 2^k cc'.$$

Choosing k large enough that $2^k cc' > 2$, and taking k even so that all points of generation $T_2 = kT_1 + 4$ have type 'h', the result follows. □

The same proof shows that Lemma 4 still holds if, instead of looking for points of type 'h' in S_1, we look for points of type 'v' in the vertical strip $S'_1 = [B - 1/2, B] \times [-B, B]$ just inside the right hand edge of $[-B, B]^2$. The same is true in the following two corollaries.

Corollary 5. *Let $\mu > \log(3/2)$ and $\varepsilon > 0$ be given, and let B and T_2 be as in Lemma 4. Then every large enough constant N has the following property. Whenever we start $\widetilde{\mathfrak{X}}_\mu$ at N points of S_0 of type 'h' and restrict to $[-B, B]^2$, the probability that generation T_2 contains at least N points of S_1 with distinct x-coordinates is at least $1 - \varepsilon$.*

Proof. Let B and T_2 be as in Lemma 4, and let Z_i be the number of T_2th generation descendants of the ith starting point that lie in S_1. Since we start N independent (restricted) branching walks, the random variables Z_i are independent; Lemma 4 shows that each has expectation at least 2. Although the Z_i are not identically distributed, as T_2 is constant we have a common upper bound (of order $(2e^\mu - 2)^{2T_2}$) on $\mathbb{E}(Z_i^2)$ for any i. Hence $Z = \sum_{i=1}^N Z_i$ has expectation at least $2N$ and variance $O(N)$. Consequently, $\mathbb{P}(Z \geq N) \geq 1 - \varepsilon$ if N is large enough.

Recalling that T_2 is even, in going from generation $T_2 - 1$ to T_2 we take horizontal steps, so the x-coordinates of the points we reach are distinct with probability 1. □

The supercriticality of the restricted walk and the survival probability bound (5) have the following consequence.

Corollary 6. *Let $\mu > \log(3/2)$ and $\varepsilon > 0$ be fixed, let B be as above, and let N be a constant. There is a constant T_3 such that, if $\widetilde{\mathfrak{X}}^2_\mu(B)$ is started (as usual) with two particles at $(0, 0)$, one of type 'h' and one of type 'v', then with probability at least $\phi(\mu) - 4\varepsilon$ either generation T_3 or generation $T_3 + 1$ contains at least N points of S_1 with type 'h' with distinct x-coordinates.*

Proof. This follows easily from Lemma 3(ii), using arguments similar to those in the proofs of Lemma 4 and Corollary 5. We omit the details. Note that we need to consider two generations, T_3 and $T_3 + 1$, as it may be that only one of the initial particles survives, in which case, after a while, either only odd generations or only even generations contain particles of type 'h', and we cannot say in advance which it will be. □

We are now ready to complete the proof of Theorem 1.

Proof of Theorem 1. We have already proved the upper bounds on component sizes; it remains to prove the lower bound implicit in (2), i.e., to show that

$$\liminf_{\omega\to\infty} \theta(\omega,\lambda/\omega)(\lambda/\omega)^{-1} \geq \phi(\lambda) \tag{6}$$

whenever $\lambda > \log(3/2)$. As remarked above, it follows by standard results that if ω is large enough (so $\theta(\omega,\lambda/\omega) > 0$), then with probability 1 there is a unique infinite component.

Let $\lambda > \log(3/2)$ be fixed, and set $\mu = \lambda$ for compatibility with our branching process notation above. (Often, at this point in the proof one would fix an arbitrary $\mu < \lambda$, but here the usual 'elbow room' turns out not to be needed.)

Fix $\varepsilon > 0$ with $10\varepsilon < \phi(\mu)$, and let B, N, T_2 and T_3 be defined as in the corollaries above. As before, we condition on $(0,0)$ being a vertex of $G_{\omega,p}$, and let (Y_t) denote the neighbourhood exploration process in $G_{\omega,p}$, $p = \mu/\omega$. We have already seen that, for any fixed k, the first k generations of the rescaled process (Y_t/ω) (defined by $Y_t/\omega = \{(x/\omega, y/\omega) : (x,y) \in Y_t\}$) may be coupled with the branching random walk $\widetilde{\mathfrak{X}}^2_\mu = (\widetilde{X}_t)$ so that with probability $1 - o(1)$ as $\omega \to \infty$ the first k generations agree up to displacements of individual points by up to k/ω.

This coupling result holds also if we explore only the subgraph of $G_{\omega,p}$ induced by vertices in $[-\omega B, \omega B]^2$, and replace $\widetilde{\mathfrak{X}}^2_\mu$ by its restriction $\widetilde{\mathfrak{X}}^2_\mu(B)$ to $[-B,B]^2$. Combining this observation with Corollary 6 above, we see that if ω is large enough then, with probability at least $\phi(\mu) - 5\varepsilon$, exploring $G_{\omega,p}$ from $(0,0)$ within $[-\omega B, \omega B]^2$, in either T_3 or T_3+1 steps we reach at least N vertices $v_1, \ldots, v_N$ of $\omega S_1 = [-\omega B, \omega B] \times [\omega(B-1/2), \omega B]$, with the last step to each vertex being horizontal. Note that we may assume that $v_1, \ldots, v_N$ have distinct x-coordinates (since the corresponding particles of $\widetilde{\mathfrak{X}}_\mu$ do).

For $(a,b) \in \mathbb{Z}^2$, let $Q_{a,b}$ be the square

$$Q_{a,b} = \big((2a-1)\omega B, (2a+1)\omega B\big) \times \big((2b-1)\omega B, (2b+1)\omega B\big).$$

Since we exclude the boundaries, the squares $Q_{a,b}$ are disjoint. Since N is constant (i.e., independent of ω), the coupling above extends to explorations started at any N points of $\mathbb{Z}^2$, provided that there are no clashes at the first step, i.e., that we do not attempt to explore vertically from two points on the same vertical line, or horizontally from two points on the same horizontal line. In what follows, we can always assume that this proviso is satisfied (from the 'distinct x-coordinates' conclusions of Corollaries 5 and 6, and the corresponding 'distinct y-coordinates' conclusions of the variants where we end with points near the right of $[-B,B]^2$). We shall not comment on this annoying technicality further.

Suppose that, without testing points in $Q_{a,b}$, we have found N vertices $v_1, \ldots, v_N$ of $G_{\omega,p}$ lying just below $Q_{a,b}$, i.e., in the strip of width $\omega/2$ bordering $Q_{a,b}$ from below. Then, from Corollary 5, provided ω is large enough, with probability at least $1 - 2\varepsilon$ we may find paths in $G_{\omega,p}$ from $\{v_i\}$ to N points w_i of $Q_{a,b}$ just below the top of $Q_{a,b}$, using only vertices in $Q_{a,b}$. Similarly (using the variant of Corollary 5 with S_1' in place of S_1), with probability at least $1-2\varepsilon$

we may find such paths to N points just inside the right-hand side of $Q_{a,b}$. By symmetry, the same conclusions hold if we start with N points v_i just to the left of $Q_{a,b}$.

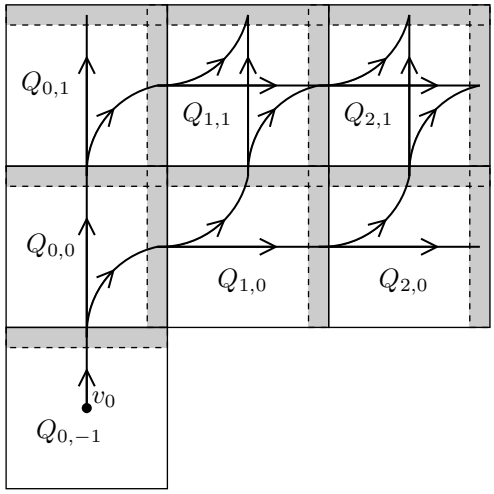

Figure 4: A schematic depiction of the proof that percolation occurs. Roughly speaking, whenever there is an arrow joining two shaded regions, from N vertices in the first shaded region we are very likely to find N in the second. We start by showing that from v_0 we find N vertices in the shaded region of $Q_{0,-1}$ with probability close to $\phi(\mu)$.

To complete the proof, it is convenient to explore the component C_0 of $G_{\omega,p}$ containing the point $v_0 = (0, -2B\omega) \in Q_{0,-1}$, in a manner shown schematically in Figure 4. Recalling that v_0 is present in $G_{\omega,p}$ with probability p, by Corollary 6, with unconditional probability at least $p(\phi(\mu) - 5\varepsilon)$ we find paths within $G_{\omega,p} \cap Q_{0,-1}$ from v_0 to N points just below the top of $Q_{0,-1}$. Let us denote this event by E.

Let us now explore within the squares $Q_{a,b}$, $a, b \geq 0$, working upwards in layers, i.e., in increasing order of $a + b$. When we come to a particular square $Q_{a,b}$, if we have found N vertices of $G_{\omega,p}$ near the top of the square $Q_{a,b-1}$ below, or N vertices near the right of the square $Q_{a-1,b}$, then with probability at least $1 - 4\varepsilon$, testing only points inside $Q_{a,b}$, we find paths from these vertices to N points near the right of $Q_{a,b}$, and to N points near the top of $Q_{a,b}$. If this happens, we declare the point $(a, b) \in \mathbb{N}^2$ to be *open*. If we did not reach N vertices of $G_{\omega,p}$ near the top of the square below or the right of the square to the left, we declare (a, b) to be open regardless of what happens inside $Q_{a,b}$.

Because each exploration is confined to its own square, each (a, b) is declared open with conditional probability at least $1 - 4\varepsilon$. Thus the distribution of open points $(a, b) \in \mathbb{N}^2$ stochastically dominates a distribution in which each point is open independently with probability $1 - 4\varepsilon$.

Suppose that E holds and that there is an infinite oriented path P in $\mathbb{N}^2$

starting at $(0,0)$, in which every vertex is open and every edge goes up or to the right. Then v_0 lies in an infinite component of $G_{\omega,p}$. It is easy to check that the probability of the existence of such a path P tends to 1 as $\varepsilon \to 0$. (For example, one can bound the expected number of 'blocking cycles' as in the proof of Lemma 6 in [7], and show that this expectation tends to 0 as $\varepsilon \to 0$.) Thus there is a function $f(\varepsilon)$ with $f(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that

$$\theta(\omega, \mu/\omega) \geq (1 - f(\varepsilon))\mathbb{P}(E) \geq (1 - f(\varepsilon)(\mu/\omega)(\phi(\mu) - 5\varepsilon)$$

whenever ω is large enough. Since $\varepsilon > 0$ was arbitrary, this proves (6). □

We now turn to the finite graphs $G_{n,\omega,p}$ and $G'_{n,\omega,p}$.

Proof of Theorem 2. Let $v = (x, y)$, $1 \leq x, y \leq n$, be any potential vertex of $G_{n,\omega,p}$, and let C_v and C'_v denote the components of v in the graphs $G_{n,\omega,p}$ and $G_{\omega,p}$. We may regard $G_{n,\omega,p}$ as a subgraph of $G_{\omega,p}$, so

$$\mathbb{P}(|C_v| \geq k) \leq \mathbb{P}(|C'_v| \geq k).$$

If $\lambda < \log(3/2)$ is constant and ω is large enough, then by Theorem 1 the second probability above is at most $(\lambda/\omega)e^{-a_\lambda(k-1)}$ for every $k \geq 1$, where $a_\lambda > 0$ depends on λ only. Setting $A = 3/a_\lambda$, say, taking $k = A \log n$, and recalling that there are n^2 choices for v, it follows that with probability $1 - o(1)$ the graph $G_{n,\omega,\lambda/\omega}$ contains no components of size larger than $A \log n$, whenever ω is large enough.

Turning to $G'_{n,\omega,p}$, although this cannot be seen as a subgraph of $G_{\omega,p}$, for any given vertex v of $G'_{n,\omega,p}$ we may couple the neighbourhood exploration processes in $G'_{n,\omega,p}$ and in $G_{\omega,p}$ started at v so that the latter dominates. (Alternatively, note that the branching process domination argument we gave in $G_{\omega,p}$ applies just as well in $G'_{n,\omega,p}$.) Thus the bound above holds for $G'_{n,\omega,p}$ also. Since $G_{n,\omega,p}$ is a subgraph of $G'_{n,\omega,p}$, the first statement in Theorem 2 follows.

It remains to show that in the supercritical case, if $\lambda > \log(3/2)$ is constant and $\omega = \omega(n)$ with $\omega \to \infty$ and $n/\omega \to \infty$, then $G_{n,\omega,\lambda/\omega}$ and $G'_{n,\omega,\lambda/\omega}$ contain a unique giant component of the 'expected' size, namely $(\phi(\lambda) + o(1))n\lambda/\omega$. As the arguments are mostly rather standard, we shall only sketch them.

The local coupling arguments above show that we have the expected number of vertices in 'small' components: it remains only to show that whp there is a component of size at least $(\phi(\lambda) + o(1))n\lambda/\omega$. Fix $\varepsilon > 0$ and set $\lambda' = \lambda - \varepsilon$; we assume that ε is such that $\lambda - 2\varepsilon > \log(3/2)$. It is also convenient to assume that $\lambda < 1$. We start by analyzing components in $G_{n,\omega,\lambda'/\omega}$, using the extra vertices of $G_{n,\omega,\lambda/\omega}$ for later sprinkling.

We know from Lemma 3 that if C is a large enough constant (depending on λ' and ε), then the branching random walk $\widetilde{\mathfrak{X}}_{\lambda'-\varepsilon}(C)$ restricted to $[-C, C]^2$ is supercritical, and indeed survives with probability at least $\phi(\lambda'-\varepsilon)-\varepsilon$ whenever we start it with two particles of type 'h' and 'v' at a point not too close to the boundary of $[-C, C]^2$. Let G be the subgraph of $G_{n,\omega,\lambda'/\omega}$ induced by vertices in $[0, 2\omega C]^2$. When we explore the component of a vertex of G not too close

to the boundary, there are two sources of error in coupling this exploration to the corresponding (restricted) branching random walk: the first comes from approximating the discrete distribution of points on a horizontal or vertical line by a Poisson process, which gives rise to an error probability of $O(\lambda'/\omega) = O(1/\omega)$ at each step. The second comes from omitting tests for points on lines already claimed by other points; we may account for this by reducing the branching process parameter slightly. If we have found m points so far, each claims only one line (apart from the first). It follows that for some $L = \Theta(\varepsilon\omega)$, we can couple our exploration to dominate $\widetilde{\mathfrak{X}}_{\lambda'-\varepsilon}(C)$ as long as $m \le L$, with an error probability of at most λ'/ω in each step (coming from the approximation of a binomial distribution by a Poisson), and thus a total error probability that is $O(\varepsilon)$.

Let N be the number of vertices of G that are in components of size at least L. From the remarks above it follows that $\mathbb{E}N \ge xN_0$, where $x = \phi(\lambda' - \varepsilon) - O(\varepsilon) = \phi(\lambda) - O(\varepsilon)$, and $N_0 = (2C\omega)^2\lambda'/\omega$ is the expected number of vertices of G. Furthermore, the number N is concentrated in the sense that $\mathbb{E}|N - \mathbb{E}N| = O(\varepsilon N_0)$ (at least for large ω); this can be seen, for example, by comparing with N_k, the number of vertices of G in components of order at least k, for some fixed but large k, and noting that starting our exploration from two different vertices and using a corresponding upper bound, we find that the variance of N_k is $o(N_0^2)$. Hence, $N > (x - \varepsilon^{1/2})N_0$ with probability at least $1 - O(\varepsilon^{1/2})$.

Sprinkling extra vertices with density $(\lambda - \lambda')/\omega$, it is easy to check that, whp, all components in G of order at least L join up to form a single component, which thus has size at least N. One crude argument is as follows: suppose that before sprinkling we have components C_1 and C_2 each with at least $L = \Theta(\omega)$ vertices. Let A denote the set of x-coordinates of vertices of C_1, and B the set of y-coordinates of vertices of C_2. Since the expected number of vertices of G an any horizontal or vertical line in $[0, 2\omega C]^2$ is $O(1)$, whp no line contains more than $O(\log\omega)$ vertices, so we may assume that $|A|, |B| = \Omega(\omega/\log\omega)$, say. Let S be the set of sprinkled vertices in $A \times B$. Then $|S|$ is concentrated about its mean, which is of order at least $\omega/(\log\omega)^2$, so we may assume that $|S| \ge \omega/(\log\omega)^3$, say. Using once more the fact that there aren't too many vertices on any line, we may find a subset S' in which all coordinates are distinct, with $|S'| = \Omega(\omega/(\log\omega)^5)$, say. Finally, for each $(x, y) \in S'$, we look for a vertical path of sprinkled vertices (outside $A \times B$) joining (x, y) to the (nearest) vertex of C_1 with the same x-coordinate, and a horizontal path to C_2. We find such paths if each of at most $8C$ intervals of length $\omega/2$ contains a sprinkled vertex, an event whose probability is bounded away from zero. These events are independent for different $(x, y) \in S'$, so with very high probability at least one such pair of paths is present.

Returning to $G_{n,\omega,\lambda/\omega}$, or $G'_{n,\omega,\lambda/\omega}$, we may cover the vertex set of this graph by squares of side-length $2C\omega$, overlapping in regions of width $C\omega$, say. The argument above shows that within each square, we almost certainly find a component of the right size, i.e., containing at least a proportion $x - \varepsilon^{1/2}$

of the vertices. The sprinkling above also shows that the giant components of overlapping squares are very likely to meet. Considering either mixed or 1-independent percolation on a K by K grid in which each vertex/bond is open with probability at least p, by estimating the expected number of blocking cycles it is easy to check that the probability that any two vertices are joined is at least $f(p)$, for some function $f(p)$ tending to 1 as $p \to 1$, independent of K. Comparison with such a model shows that as $n, n/\omega \to \infty$, whp the large components in almost all squares are linked up, giving a giant component in $G_{n,\omega,\lambda/\omega}$ or $G'_{n,\omega,\lambda/\omega}$ containing at least a proportion $\phi(\lambda) - \delta(\varepsilon)$ of all vertices, for some function $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. □

Note that the condition $n/\omega \to \infty$ in the second part of Theorem 2 is essential. Indeed, if n/ω is bounded, say $n \leq C\omega$, then the approximation of the component exploration by the branching random walk $\widetilde{\mathfrak{X}}_\lambda$ breaks down even during the first step: this branching random walk has probability $\Theta(1)$ of starting by moving to a point at distance at least C (corresponding to distance at least $C\omega \geq n$ in the graph) from the initial point. This condition was not needed in the much weaker result of Frieze, Kleinberg, Ravi and Debany [9], but this is not surprising. The very simple analysis in [9] ignored these long-range steps of the branching random walk (as well as other complications), which is why it does not give the correct limiting size of the giant component, or indeed the asymptotic critical probability.

2.3 The case $n = \Theta(\omega)$

In this subsection we briefly consider the emergence of the giant component in $G_{n,\omega,p}$ or $G'_{n,\omega,p}$ when $\omega \to \infty$ and $n = \Theta(\omega)$. We assume throughout that $n \sim C\omega$ for some constant $C > 0$. As before, the relevant normalization is to take $p = p(n) = \lambda/\omega$, with λ constant.

As noted above, Theorem 2 does not apply in this setting. However, our methods still allow us to obtain the critical point, and indeed the size of the giant component, in terms of the solutions to certain non-linear equations. As this problem is of rather specialized interest, we shall only outline the results and arguments.

Let $\mathcal{P}_\lambda$ be a Poisson process on $\mathbb{R}$ with intensity λ. For $d > 1$, let $g_\lambda(d)$ be the probability that within the interval $[0, d]$ there is an interval of length 1 containing no points of $\mathcal{P}$, and let $r_\lambda(d) = 1 - g_\lambda(d)$. For $0 \leq d \leq 1$ we set $g_\lambda(d) = 0$ and $r_\lambda(d) = 1$. Given that both 0 and d are points of $\mathcal{P}_\lambda$, then $r_\lambda(d)$ is the probability that they are joined in the graph with vertex set $\mathcal{P}_\lambda$ and edges between all points at distance at most 1. The function $r_\lambda(d)$ will be key to our analysis. (This function $r_\lambda(d)$ has been studied in other contexts. It is the probability that the largest gap between $\mathrm{Po}(\lambda d)$ points uniformly and independently distributed in $[0, 1]$ is at most $1/d$; this largest gap is also a version of a scan statistic; see Glaz, Naus and Wallenstein [11], for example. An explicit, but rather complicated, expression for $r_\lambda(d)$ is easily derived from

the formula given by Stevens [14] for the case of the largest gap between a fixed number of random points on a circle.)

Let us start with the simpler toroidal case. Let $\mathcal{P}_{\lambda,C}$ be a Poisson process of intensity λ on the circle $\mathbb{T}_C = \mathbb{R}/(C\mathbb{Z})$ with circumference C. Adding in the extra point 0, let $Z = Z(\lambda, C)$ denote (the distribution of) the number of other points of $\mathcal{P}_{\lambda,C}$ that may be reached from 0 in steps of size at most 1. Let $\mathfrak{X}_{\lambda,C}$ be the branching process in which the number of children of initial particle has the distribution of the sum of two independent copies of Z, and the number of children of each later particle has the distribution of Z. Approximating the component exploration in $G'_{n,\omega,p}$ by the corresponding branching random walk on the torus, the proof of Theorem 2 adapts easily to show that if $p = \lambda/\omega$ and $n \sim C\omega$, then the asymptotic fraction of vertices of $G'_{n,\omega,p}$ that are in the giant component is simply the survival probability $\rho(\lambda, C)$ of $\mathfrak{X}_{\lambda,C}$. In particular, the critical value $\lambda_c = \lambda_c(C)$ of λ may be found by solving $\mathbb{E}(Z(\lambda, C)) = 1$.

If the point $x \in \mathbb{T}_C$ is present, then it may be reached from 0 with probability $1 - g_\lambda(x)g_\lambda(C - x)$, so λ_c is given by the solution to

$$\int_0^C \big(1 - g_\lambda(x)g_\lambda(C - x)\big)\, dx = 1.$$

The case of the grid is more interesting, but also more complicated. In this case we must consider the restriction $\widetilde{\mathfrak{X}}_{\lambda,C}$ of our original branching random walk $\widetilde{\mathfrak{X}}_\lambda$ to a square $[0, C]^2$. Previously, we used a similar restriction with C large as a tool in our analysis. Here we are forced to analyze the restriction itself, and have no control over C.

Fixing λ and C, for $0 \le x, y \le C$, let $\rho_h(x, y)$, $\rho_v(x, y)$ and $\rho(x, y)$ denote respectively the survival probabilities in the cases where we start with one particle of type 'h' at (x, y), one particle of type 'v' at (x, y), and one particle of each type at (x, y). By symmetry we have $\rho_h(x, y) = \rho_v(y, x)$. Also,

$$\rho(x, y) = 1 - (1 - \rho_h(x, y))(1 - \rho_v(x, y)) = \rho_h(x, y) + \rho_v(x, y) - \rho_h(x, y)\rho_v(x, y).$$

Starting with two particles, one of each type, at a point (x, y) chosen uniformly from $[0, C]^2$, the survival probability ρ satisfies

$$\rho = \int_0^C \int_0^C \rho(x, y) = 2 \int_0^C \int_0^C \rho_h(x, y) - \int_0^C \int_0^C \rho_h(x, y)\rho_h(y, x),$$

so to understand ρ it suffices to understand $\rho_h(x, y)$.

It is not hard to convince oneself that the proof of Theorem 2 can be modified to show that if $p = \lambda/\omega$ and $n \sim C\omega$, then the asymptotic fraction of vertices of $G_{n,\omega,p}$ in the largest component is simply $\rho = \rho(\lambda, C)$. Filling in the details may well require considerable work, however: for example, to make the sprinkling argument work one needs to show that $\rho(\lambda, C)$ is continuous in λ. Let us omit the details and turn instead to the study of $\rho(\lambda, C)$.

It is easy to see that one can express $\rho_h(x, y)$ as the maximum solution to a certain non-linear integral equation; as the details are not very illuminating, we

ignore the size of the giant component and consider only the question of when it emerges, i.e., when $\rho(\lambda, C) > 0$. At this point, it is tempting to claim that arguments of the type given in [5], for example, show that $\rho(\lambda, C) > 0$ if and only if the corresponding linearized operator has norm strictly greater than 1. However, there are some complications: in [5], we assumed that the linear operator was symmetric and compact; here it is neither! These complications seem to be connected to the interchange of x and y coordinates, or, putting it another way, to the one-dimensional action of the operator on the two-dimensional space $[0, C]^2$. It turns out that one can simplify things greatly by considering two steps of the branching process at once.

Starting with a point at (x, y) (of type 'h', say, although it doesn't matter), and taking two steps in our restricted branching random walk, the expected number of points in $[x' + dx'] \times [y' + dy']$ that we reach is easily seen to be $\lambda^2 r_\lambda(|x - x'|) r_\lambda(|y - y'|)\, dx'\, dy'$. Let T_2 be the corresponding integral operator on $L^2([0, C]^2)$, so

$$(T_2(f))(x', y') = \int_0^C \int_0^C \lambda^2 r_\lambda(|x - x'|) r_\lambda(|y - y'|) f(x, y)\, dx\, dy.$$

The operator T_2 is very well behaved: the kernel is clearly symmetric (with respect to swapping (x, y) and (x', y')) and, since $0 \le r \le 1$, the operator is compact. Thus arguments of the type given [5] do show that $\rho(\lambda, C) > 0$ if and only if $\|T_2\| > 1$.

At this point, we can return to one dimension: the operator T_2 has a unique (up to normalization) eigenfunction ϕ with eigenvalue the norm of T_2. Let T be the operator on $L^2([0, C])$ defined by

$$(T(f))(x') = \int_0^C \lambda r_\lambda(|x - x'|) f(x)\, dx.$$

From the form of T_2, it is not hard to check that $\phi(x, y) = \psi(x)\psi(y)$, where ψ is the eigenfunction of T with maximum eigenvalue. In particular, $\|T_2\| = \|T\|^2$. Although we only sketched the details, one has $\rho(\lambda, C) > 0$ if and only if $\|T_2\| > 1$, so in this setting the critical value of λ is given by the solution to $\|T\| = 1$, noting that $\|T\|$ depends both on C and on λ.

Unfortunately it seems unlikely that one can find $\|T\|$ explicitly; this is to be expected. The conclusion is that we find the same sort of connection between the giant component in a certain random graph and the norm of an integral operator on $L^2([0, C])$ as seen in [5]. This is especially interesting as one cannot directly apply the results of [5]. Indeed, here, the approximating branching process/random walk is *not* of the multi-type Poisson form considered there.

3 Discussion and variants

The line-of-sight model $G_{\omega,p}$ is unusual in various ways. There are several random graph models in which one can determine the exact threshold for the

emergence of a giant component in terms of the survival probability of a suitable a branching process – the simplest example is $G(n,p)$; a very general inhomogeneous model with this property is that of [5]. In lattice percolation models, on the other hand, as a rule one cannot determine the exact critical probability. Penrose [13] defined a 'spread-out' percolation model as follows: let S be a fixed bounded symmetric set in $\mathbb{R}^2$. Given parameters ω and p, for every pair of points v, w of $\mathbb{Z}^2$, join them by an edge with probability p if $v - w \in \omega S$, independently of all other such pairs. (Penrose's model is somewhat more general, but this is the essence.) For fixed ω, this is a lattice percolation model, and the critical probability cannot be found exactly. However, Penrose showed that as $\omega \to \infty$, the asymptotic form of the critical probability can be found – the 'critical expected degree' tends to 1. In [6], it was shown that this result follows easily from those of [5]. Given the homogeneous nature of the model, the critical expected degree tending to 1 shows that asymptotically, cycles do not matter; this is far from the case in the line-of-sight model.

In terms of the critical probability, the line-of-sight model $G_{\omega,p}$ behaves differently from Penrose's model: even when ω is large, $G_{\omega,p}$ contains many short cycles, so the critical expected degree, $4\log(3/2)$, is larger than 1. There are two important differences between the models. Firstly, $G_{\omega,p}$ is a site percolation model (vertices are selected at random), and Penrose's model is a bond percolation one (edges are selected at random). Secondly, in $G_{\omega,p}$ the geometric condition for joining vertices involves scaling a set $S \subset \mathbb{R}^2$ of measure zero. We shall examine the effect of these differences separately, by considering two variants of $G_{\omega,p}$.

There is a natural bond percolation variant of $G_{\omega,p}$. Given ω and p, let $\widetilde{G}_{\omega,p}$ be the graph on $\mathbb{Z}^2$ defined as follows. For every pair of vertices v, w such that the line segment vw has length at most ω and is horizontal or vertical, join v and w with probability p, independently of all other such pairs. We consider the limit $\omega \to \infty$ with $p = \lambda/\omega$, where λ is constant. In this model the density of short cycles is very low. Thus, as we explore the component of a vertex in the usual way, the number of vertices reached at each step may be approximated by a Galton–Watson branching process in which each particle has a Poisson number of children with mean 4λ. Moreover, the location of these vertices may be approximated by a corresponding branching random walk $\widetilde{\mathfrak{X}}_{4\lambda}$. The arguments above for $G_{\omega,p}$ carry over to this setting, showing that the critical probability $p_c(\omega)$ is asymptotically $1/(4\omega)$, and that if $p = \lambda/\omega$ and $\omega \to \infty$ then the infinite component contains a fraction $\widetilde{\phi}(4\lambda) + o(1)$ of the vertices, where $\widetilde{\phi}(\mu)$ is the survival probability of the branching process underlying $\widetilde{\mathfrak{X}}_\mu$, i.e., the maximal solution to $\widetilde{\phi}(\mu) = 1 - e^{-\mu\widetilde{\phi}(\mu)}$. One can also obtain corresponding results for finite graphs, showing that when $\omega \to \infty$, they behave much like the Erdős-Rényi (or Gilbert) model $G(n, 4\lambda/n)$.

Turning to our second variant, let C_ε be the cross with (four) arms of length 1 and width 2ε, so

$$C_\varepsilon = [-1,1] \times [-\varepsilon,\varepsilon] \ \cup \ [-\varepsilon,\varepsilon] \times [-1,1].$$

Let $G_{\omega,p,\varepsilon}$ be the random graph defined as follows: define a graph G on $\mathbb{Z}^2$ by joining two vertices v and w if $v-w \in \omega C_\varepsilon$. Then select vertices independently with probability p to form $G_{\omega,p,\varepsilon}$. When $\varepsilon = 0$, this graph is exactly $G_{\omega,p}$. For $\varepsilon > 0$, the area of ωC_ε is proportional to ω^2, so we consider the limit with $p = \lambda/\omega^2$ and $\omega \to \infty$. In this context it is natural to rescale the vertex set, selecting vertices of $(1/\omega)\mathbb{Z}^2$ with probability λ/ω^2 and joining them if their difference lies in C_ε. In the limit, the selected vertices form a Poisson process on $\mathbb{R}^2$ with intensity λ. This gives us another natural random graph model, $G_{\lambda,\varepsilon}$. The vertex set of $G_{\lambda,\varepsilon}$ is a Poisson process $\mathcal{P}_\lambda$ on $\mathbb{R}^2$ with intensity λ, and two vertices v and w are joined if and only if $v-w \in C_\varepsilon$. It is not hard to see that as $\omega \to \infty$ with ε fixed, ω^2 times the critical probability for percolation in $G_{\omega,p,\varepsilon}$ approaches the critical density λ_c for percolation in $G_{\lambda,\varepsilon}$.

The random graph $G_{\lambda,\varepsilon}$ we have just defined is a special case of a percolation model introduced by Gilbert [10] in 1961, Gilbert's *disc model*. To define this, let S be a symmetric (in the sense $S = -S$) set in $\mathbb{R}^2$ with Lebesgue measure $0 < \text{area}(S) < \infty$, and let $G(\lambda, S)$ be the random graph whose vertex set is the Poisson process $\mathcal{P}_\lambda$, in which vertices are joined if their difference lies in S, so $G_{\lambda,\varepsilon} = G(\lambda, C_\varepsilon)$. Gilbert asked the following question: given S, what is the critical value $\lambda_c = \lambda_c(S)$ above which $G(\lambda_c, S)$ contains an infinite component? Since $G(\lambda, S)$ and $G(\lambda/c^2, cS)$ have the same distribution as graphs, one often works with the 'critical expected degree', or *critical area* $a_c(S) = \lambda_c(S)\text{area}(S)$, instead. (The term critical area is used because of the standard normalization $\lambda = 1$.)

As with many such percolation questions, it seems impossible to determine $a_c(S)$ exactly. The most studied cases are when S is a disc D, or when S is a square $[-1,1]^2$. The best rigorous bounds for $a_c(D)$, the bounds $2.184 \leq a_c(D) \leq 10.588$ proved by Hall [12], are not much better than Gilbert's original bounds; although these bounds are far apart, it seems hard to improve them significantly. In practice, however, $a_c(D)$ is known quite precisely; Balister, Bollobás and Walters [4] *proved* that $[4.508, 4.515]$ is a 99.99% confidence interval for $a_c(D)$, and $[4.392, 4.398]$ for $a_c([-1,1]^2)$. See [7, Section 8.1] for more details, as well as many references to heuristic bounds.

Recall that we introduced $G_{\lambda,\varepsilon} = G(\lambda, C_\varepsilon)$ as a variant of $G_{\omega,p}$. In this context it is natural to take the limit $\varepsilon \to 0$, and ask whether the critical average degree converges to the corresponding value for $G_{\omega,p}$. Since the area of the cross C_ε is $8\varepsilon - 4\varepsilon^2$, one might expect that

$$8\varepsilon\lambda_c(C_\varepsilon) \sim 4\log(3/2) \tag{7}$$

as $\varepsilon \to 0$. We shall see that this is not the case. In fact, rather surprisingly, we can describe the limit of $\varepsilon\lambda_c(C_\varepsilon)$ in terms of the Gilbert model with S a square!

Given $\lambda > 0$, let G be the graph $G(\lambda, [-1,1]^2)$ conditioned on the origin being a vertex. In other words, G is the graph on $\mathcal{P}_\lambda \cup \{(0,0)\}$ in which we join two vertices if their ℓ_∞ distance is at most 1. Let

$$f_2(\lambda) = \mathbb{E}(|C_0| - 1),$$

where C_0 is the component of G containing $(0,0)$, and $|C_0|$ is the number of vertices in C_0. Note that we subtract 1 to avoid counting the 'extra' vertex at the origin. Later we shall consider an analogous quantity defined in terms of a d-dimensional form of Gilbert's model; this is the reason for the notation f_2.

Theorem 7. *The critical densities $\lambda_c(C_\varepsilon)$ satisfy $\varepsilon\lambda_c(C_\varepsilon) \to \mu$ as $\varepsilon \to 0$, where μ is the unique solution to $f_2(\mu) = 1$.*

Proof. Note first that the definition of μ makes sense: we have $f_2(0) = 0$, while $f_2(\lambda)$ diverges at $\lambda_c([-1,1]^2) < \infty$. Furthermore, up to this point $f_2(\lambda)$ is increasing and continuous; hence there is a unique solution to $f_2(\mu) = 1$.

Fix $\lambda > 0$ and $\varepsilon > 0$; we shall take $\varepsilon^{-1}\lambda$ as our density parameter.

Let us condition on $v_0 = (0,0)$ being a vertex of $G_{\lambda/\varepsilon,\varepsilon} = G(\lambda/\varepsilon, C_\varepsilon)$, and explore the component of v_0 in this graph. As in our study of $G_{\omega,p}$, we break the exploration down into steps; in each step, we explore either horizontally or vertically from a vertex v. As before, we set $Y_0 = \{v_0\}$, explore horizontally and vertically from v_0, writing Y_1 for the set of vertices reached, and then explore horizontally from those vertices of Y_t reached during vertical explorations, and *vice versa*. We write Y_{t+1} for the new vertices found during such explorations, so the vertex set of the component containing v_0 is the disjoint union of the sets Y_t, $t \geq 0$. This time, however, the horizontal and vertical explorations are a little more complicated.

Let us say that an edge vw of $G_{\lambda/\varepsilon,\varepsilon}$ is *horizontal* if $v - w \in [-1,1] \times [-\varepsilon,\varepsilon]$. The definition of *vertical* edges is analogous. Note that some edges are both horizontal and vertical, but the proportion of such edges tends to 0 as $\varepsilon \to 0$.

To explore horizontally from v, let H_v consist of all new (not reached in previous explorations) vertices v' of $G_{\lambda/\varepsilon,\varepsilon}$ joined to v by horizontal edges, together with all new vertices v'' joined to such vertices v' by horizontal edges, and so on. In other words, writing $G^{\mathrm{h}}_{\lambda/\varepsilon,\varepsilon}$ for the subgraph of $G_{\lambda/\varepsilon,\varepsilon}$ formed by all horizontal edges, $H_v \cup \{v\}$ is simply the component containing v in the subgraph of $G^{\mathrm{h}}_{\lambda/\varepsilon,\varepsilon}$ induced by the v and the new vertices. In particular, if this is our first exploration, then $v = v_0 = (0,0)$, and $H_v \cup \{v\}$ is simply the component C_0 of the origin in $G^{\mathrm{h}}_{\lambda/\varepsilon,\varepsilon}$.

Now $G^{\mathrm{h}}_{\lambda/\varepsilon,\varepsilon}$ is exactly the Gilbert graph $G(\lambda/\varepsilon, [-1,1] \times [-\varepsilon,\varepsilon])$, conditioned on $(0,0)$ being a vertex. Scaling vertically by a factor ε, the distribution of this random graph (as an abstract graph) is identical to that of $G(\lambda, [-1,1]^2)$. Let $\mathcal{D}_2(\lambda)$ denote the distribution of $|C_0| - 1$, where C_0 is the component of the origin in this graph, so $f_2(\lambda) = \mathbb{E}(\mathcal{D}_2(\lambda))$ by definition. Then $|H_{v_0}|$ has the distribution $\mathcal{D}_2(\lambda)$.

In later steps of the exploration, the restriction to new vertices ensures that the distribution of H_v, given the exploration so far, is stochastically dominated by that of H_{v_0}, i.e., by $\mathcal{D}_2(\lambda)$. The same holds for vertical explorations. It follows that the sequence $|Y_0|, |Y_1|, \ldots$ is stochastically dominated by $|X_0|, |X_1|, \ldots$, where (X_t) is a branching process in which the number of children of each particle except the initial one has the distribution $\mathcal{D}_2(\lambda)$, and different particles have children independently. The number of children of the initial particle is

distributed as the sum of two independent copies of $\mathcal{D}_2(\lambda)$, since we explore horizontally and vertically from v_0.

Since $\mathbb{E}(\mathcal{D}_2(\lambda)) = f_2(\lambda)$, this shows that $G_{\lambda/\varepsilon,\varepsilon}$ has no giant component when $\lambda < \mu$, where μ satisfies $f_2(\mu) = 1$. Hence, $\varepsilon\lambda_c(C_\varepsilon) \leq \mu$.

It remains to show that if $\lambda > \mu$ is constant, then for ε small enough we have percolation in the graph $G_{\lambda/\varepsilon,\varepsilon}$. The argument is very similar to that for $G_{\omega,p}$, so we give only the briefest outline, emphasizing the differences. In doing so we may assume that $\lambda < \lambda_c([-1,1]^2)$: if $\lambda > \lambda_c([-1,1]^2)$ then the horizontal subgraph of $G_{\lambda/\varepsilon,\varepsilon}$ has the same distribution as a graph as the supercritical Gilbert graph $G(\lambda,[-1,1]^2)$, so this subgraph already contains an infinite component.

When exploring horizontally in $G_{\lambda/\varepsilon,\varepsilon}$, there is *a priori* no bound on how far we may drift vertically. To deal with this, let us choose a large constant A (depending only on λ), and limit each horizontal or vertical exploration to A steps. Let $\mathcal{D}_2^{(A)}(\lambda)$ be the distribution of the number of points reached from v_0 in this restricted horizontal exploration. Since the horizontal subgraph of $G_{\lambda/\varepsilon,\varepsilon}$ is subcritical, the distribution $\mathcal{D}_2^{(A)}(\lambda)$ converges to $\mathcal{D}_2(\lambda)$ as $A \to \infty$. In particular, since $\lambda > \mu$, we may choose A so that $\mathbb{E}\big(\mathcal{D}_2^{(A)}(\lambda)\big) > 1$.

With this modified exploration, the coupling argument used in $G_{\omega,p}$ carries over to show that we may couple the first $O(1)$ steps of the exploration with the appropriate branching process with an error probability of $O(\varepsilon)$. The main difference is that rather than claiming lines, new vertices we find claim horizontal or vertical strips of width $2A\varepsilon$. As before, we can in fact couple with a supercritical branching random walk, and use oriented percolation to find an infinite component. □

Let A_ε be the annulus centered on the origin with outer radius 1 and inner radius $1-\varepsilon$. Independently, Franceschetti, Booth, Cook, Meester and Bruck [8], and Balister, Bollobás and Walters [3] showed that $\lim_{\varepsilon\to 0} a_c(A_\varepsilon) = 1$. This result is similar in spirit to the results here: one shows local approximation by a branching process (in this case, a very simple process, since the graph contains few short cycles), and then must work to deduce percolation in the supercritical case.

Let S_ε be the 'square annulus' $[-1,1]^2 \setminus [-(1-\varepsilon),1-\varepsilon]^2$. One might expect $\lim_{\varepsilon\to 0} a_c(S_\varepsilon)$ to also equal 1, but this is not true: in [3] the bound $a_c(S_\varepsilon) \geq 1.014$ is proved for every $\varepsilon > 0$. Our methods here show that $a_c(S_\varepsilon)$ does converge as $\varepsilon \to 0$, and give a description of the limiting value.

Let $F = [-1,1]^2 \times \{-1,1\}$, so $F \subset \mathbb{R}^2 \times \mathbb{Z}$ consists of two opposite faces of the cube $[-1,1]^3$. We define a variant $G(\lambda,F)$ of Gilbert's model as follows: the vertex set is a Poisson process of intensity λ on $\mathbb{R}^2 \times \mathbb{Z}$, with the origin added, and two vertices v and w are joined if and only if $v - w \in F$. Let C_0 be the component of the origin in $G(\lambda,F)$, and let $f(\lambda) = \mathbb{E}(|C_0| - 1)$.

Theorem 8. *The critical area $a_c(S_\varepsilon)$ tends to 2μ as $\varepsilon \to 0$, where μ is the unique solution to $f(\mu/8) = 1$.*

Note that the expected degree in $G(\lambda, F)$ is 8λ; thus the limiting expected area or degree in Theorem 8 is twice the expected degree at which $\mathbb{E}(|C_0|-1) = 1$ holds in $G(\lambda, F)$. In this sense, Theorem 8 is analogous to Theorem 7.

Proof. The proof is very similar to that of Theorem 7, so we omit the details. The key observation is as follows: call an edge vw of $G(\lambda, S_\varepsilon)$ 'horizontal' if $v-w$ lies in one of the vertical sides of the annulus, i.e., if $\pm(v-w) \in [1-\varepsilon, 1]\times[-1, 1]$, and define vertical edges similarly. As $\varepsilon \to 0$, it is easy to check that almost all short cycles in $G(\lambda/\varepsilon, S_\varepsilon)$ consist entirely of horizontal edges, or entirely of vertical edges. As before, we break down the neighbourhood exploration process in $G(\lambda/\varepsilon, S_\varepsilon)$ into horizontal and vertical explorations. In a horizontal exploration, say, we look for all new vertices that can be reached by horizontal edges. As before, the asymptotic condition for criticality is that the expected number of new vertices found in a single horizontal (or vertical) exploration is 1. Rescaling horizontally by $2/\varepsilon$, a horizontal exploration may be coupled with an exploration of the component of the origin in the Gilbert model $G(\lambda/2, T_\varepsilon)$, where T_ε consists of two squares of side 2 centered at $(\pm(2\varepsilon^{-1} - 1), 0)$. If we take $k \leq 1/\varepsilon$ steps from the origin, with the displacement of each step in T_ε, then we can tell from our final position how many steps were to the right and how many to the left. It follows that for the first $1/\varepsilon$ steps, the explorations in $G(\lambda/2, T_\varepsilon)$ and in $G(\lambda/2, F)$ may be regarded as identical. It follows that $\varepsilon\lambda_c(S_\varepsilon) \sim 2(\mu/8)$, where $f(\mu/8) = 1$. Since the area of S_ε is asymptotically 8ε, the result follows. □

Since it is much easier to estimate (by simulation) the point at which an expectation crosses 1 than the point at which it diverges, Theorem 8 makes it much easier to estimate $\lim_{\varepsilon\to 0} a_c(S_\varepsilon)$; a simple simulation suggests that the limit is 1.11406 ± 0.00001.

The models above make just as good sense in any dimension. For $d \geq 1$ and $\lambda \geq 0$, let $G = G(\lambda, [-1, 1]^d)$ be the graph whose vertex set consists of a Poisson process in $\mathbb{R}^d$ of intensity λ together with the origin, in which two vertices are joined if they are within ℓ_∞ distance 1. Let

$$f_d(\lambda) = \mathbb{E}(|C_0|) - 1,$$

where C_0 is the component of G containing the origin, so $f_d(\lambda) = \infty$ if $\lambda \geq \lambda_c$, where λ_c is the critical density for $G(\lambda, [-1, 1]^d)$. In general, one cannot hope to evaluate $f_d(\lambda)$ exactly. However, when $d = 1$, $|C_0| - 1$ has exactly the distribution $\Gamma^{(2)}_\lambda$ defined earlier, so $f_1(\lambda) = 2(e^\lambda - 1)$, and $f_1(\log(3/2)) = 1$. It is easy to check that $f_2(\lambda/2) > f_1(\lambda)$ for any $\lambda > 0$. (This is the natural comparison, as the expected degree of a vertex of the relevant graph is $2^d\lambda$.) It follows that the quantity μ defined in Theorem 7 is strictly less than $\log(3/2)/2$, so Theorem 7 shows that our 'guess' (7) does not hold. (A quick simulation suggests that μ is around 0.177635.) In other words, the critical expected degree in models defined using a cross with very thin arms does not approach that of a

model defined using a cross whose arms have width 0. (Compare $8 \times 0.177635 = 1.421\ldots$ with $4\log(3/2) = 1.621\ldots$.)

We finish by turning to one final class of variants of $G_{\omega,p}$, namely the natural generalizations to higher dimensions. Of course, there is more than one natural equivalent of a 2-dimensional cross in higher dimensions: a union of line segments, or a union of $(d-1)$-dimensional hypercubes. Fortunately one can cover both generalizations in a single definition.

Given $d \geq 2$, $1 \leq r \leq d-1$, $\omega \geq 1$ and $0 < p < 1$, let $G_{d,r,\omega,p}$ be the random graph defined as follows: for the vertex set, select points of $\mathbb{Z}^d$ independently with probability p. Join two vertices v and w if and only if v and w differ in at most r coordinates, and differ in each of these coordinates by at most ω. If $d = 2$ and $r = 1$ then we obtain $G_{\omega,p}$.

Theorem 9. *Let $p_c(d,r,\omega)$ denote the critical probability p for percolation in $G_{d,r,\omega,p}$. Then*

$$\omega^r p_c(d,r,\omega) \to \lambda_{d,r}$$

when $\omega \to \infty$ with $d \geq 2$ and $1 \leq r \leq d-1$ fixed, where $\lambda_{d,r}$ is the unique solution to

$$\left(\binom{d}{r} - 1\right) f_r(\lambda_{d,r}) = 1.$$

Recalling that $f_1(\lambda) = 2(e^\lambda - 1)$, we see that

$$\lambda_{d,1} = \log\left(\frac{2d-1}{2d-2}\right),$$

so Theorem 9 generalizes the asymptotic critical probability result for $G_{\omega,p}$ given by Theorem 1. Note that the function $f_2(\mu)$, which appeared in Theorem 7 in the analysis of a certain 2-dimensional graph, appears here in the analysis of the very different graphs $G_{d,2,\omega,p}$, $d \geq 3$.

The proof of Theorem 9 is very similar to that of Theorem 1, so we omit it. The main difference is that instead of exploring horizontally or vertically, i.e., always exploring within a line, we always explore within an affine subspace of dimension r generated by r of the coordinate axes. The factor $\binom{d}{r} - 1$ above appears because there are $\binom{d}{r}$ such subspaces through any given point, and we reach any vertex other than the initial one by exploring within one of them. This time, when we find a vertex v of $G_{d,r,\omega,p}$ by exploring within a certain subspace, this vertex immediately claims the other $\binom{d}{r} - 1$ subspaces in which it lies. As before, we omit testing points w in two (or more) claimed subspaces. When exploring from v in a subspace S, any such point w lies in a subspace S' through a vertex v' of C_0 that we have already found, with S' and S unequal and therefore not parallel. Since S and S' intersect in a space of lower dimension, in the early stages of our exploration almost all points of S are claimed only by v, and the coupling goes through as before.

Just as for $G_{\omega,p}$, we could 'thicken' the generalized crosses defining $G_{d,r,\omega,p}$ by ε and let $\varepsilon \to 0$, obtaining a generalization of Theorem 7. In this case the limit μ in Theorem 7 is replaced by the solution to $\left(\binom{d}{r} - 1\right) f_d(\mu) = 1$.

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References

[1] M. Aizenman, H. Kesten and C.M. Newman, Uniqueness of the infinite cluster and related results in percolation, in *Percolation Theory and Ergodic Theory of Infinite Particle Systems (Minneapolis, Minn., 1984–1985)*, Springer (1987), pp. 13–20.

[2] K.B. Athreya and P.E. Ney, *Branching processes*, Springer, Berlin, 1972, xi+287 pp.

[3] P. Balister, B. Bollobás and M. Walters, Continuum percolation with steps in an annulus, *Ann. Appl. Probab.* **14** (2004), 1869–1879.

[4] P. Balister, B. Bollobás and M. Walters, Continuum percolation with steps in the square or the disc, *Random Structures and Algorithms* **26** (2005), 392–403.

[5] B. Bollobás, S. Janson and O. Riordan, The phase transition in inhomogeneous random graphs, *Random Structures and Algorithms*, **31** (2007), 3–122.

[6] B. Bollobás, S. Janson and O. Riordan, Spread-out percolation in $\mathbb{R}^d$, *Random Struct. Algorithms* **31** (2007), 239–246.

[7] B. Bollobás and O. Riordan, *Percolation*, Cambridge University Press, 2006, x + 323 pp.

[8] M. Franceschetti, L. Booth, M. Cook, R. Meester and J. Bruck, Continuum percolation with unreliable and spread-out connections, *J. Stat. Phys.*, **118** (2005), 721–734.

[9] A. Frieze, J. Kleinberg, R. Ravi and W. Debany, Line-of-sight networks, *Proc. 18th ACM-SIAM Symposium on Discrete Algorithms* (2007), 968–977.

[10] E.N. Gilbert, Random plane networks, *J. Soc. Indust. Appl. Math.* **9** (1961), 533–543.

[11] J. Glaz, J. Naus and S. Wallenstein, *Scan statistics*, Springer, New York, 2001, xvi+370 pp.

[12] P. Hall, On continuum percolation, *Ann. Probab.* **13** (1985), 1250–1266.

[13] M.D. Penrose, On the spread-out limit for bond and continuum percolation, *Ann, Appl. Probab.* **3** (1993), 253–276.

[14] W. L. Stevens, Solution to a geometrical problem in probability, *Ann. Eugenics* **9** (1939), 315–320.