THE INTEGRAL OF THE SUPREMUM PROCESS OF BROWNIAN MOTION

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Abstract. In this paper we study the integral of the supremum process of standard Brownian motion. We present an explicit formula for the moments of the integral (or area) $\mathcal{A}(T)$, covered by the process in the time interval $[0, T]$. The Laplace transform of $\mathcal{A}(T)$ follows as a consequence. The main proof involves a double Laplace transform of $\mathcal{A}(T)$ and is based on excursion theory and local time for Brownian motion.

1. Introduction

Let $B(t)$, $t \geq 0$, be a standard Brownian motion. Consider the following associated processes: the supremum process $S(t) := \max_{0 \leq s \leq t} B(t)$, and the local time $L(t)$, which can be regarded as a measure of the time $B(t)$ spends at 0 in the interval $[0, t]$, see Revuz and Yor [10, Chapter VI] for details. It is well-known that these two processes, although pathwise quite different, have the same distribution [10, Chapter VI.2],

$$\{S(t)\}_{t \geq 0} \overset{\mathrm{d}}{=} \{L(t)\}_{t \geq 0}.$$

The purpose of this paper is to study the distribution of the area under $S(t)$ or, equivalently, $L(t)$ over a given time interval $[0, T]$. That is, the integral

$$\mathcal{A}(T) := \int_0^T S(t)\,\mathrm{d}t \overset{\mathrm{d}}{=} \int_0^T L(t)\,\mathrm{d}t. \tag{1.1}$$

For ease of notation, let $\mathcal{A} := \mathcal{A}(1)$.

The area (1.1) appeared as a random parameter when analysing displacements for linear probing hashing. The Laplace transform of $\mathcal{A}$, which is presented in Corollary 2.4, provided the means to prove one of the main theorems in Petersson [9].

Note that the usual Brownian scaling

$$\{B(Tt)\}_{t \geq 0} \overset{\mathrm{d}}{=} \{T^{1/2} B(t)\}_{t \geq 0},$$

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for any $T > 0$, implies the corresponding scaling for the supremum process,

$$\big\{S(Tt)\big\}_{t\geq 0} \overset{\mathrm{d}}{=} \big\{T^{1/2}S(t)\big\}_{t\geq 0}.$$

Thus, for $T > 0$,

$$\mathcal{A}(T) = T\int_0^1 S(Tt)\,\mathrm{d}t \overset{\mathrm{d}}{=} T^{3/2}\mathcal{A}, \tag{1.2}$$

and it is enough to study $\mathcal{A}$.

2. Results

Let $\psi(s) := \mathbb{E}\, e^{-s\mathcal{A}}$ denote the Laplace transform of $\mathcal{A}$. An essential part of this paper is devoted to proving the following formula for the Laplace transform of a variation of ψ, or in other words, a *double* Laplace transform of $\mathcal{A}$. Such formulas have already been derived for the integral of $|B(t)|$ and other similar integrals of processes related to Brownian motion, see Perman and Wellner [8] and the survey by Janson [3].

Theorem 2.1. *Let ψ be the Laplace transform of $\mathcal{A}$. For all $\alpha, \lambda > 0$,*

$$\int_0^\infty \psi\big(\alpha s^{3/2}\big)e^{-\lambda s}\,\mathrm{d}s = \int_0^\infty \Big(1 + \frac{3\alpha s}{2\sqrt{2\lambda}}\Big)^{-2/3} e^{-\lambda s}\,\mathrm{d}s.$$

Remark 2.2. One of the parameters α and λ in Theorem 2.1 can be eliminated (by setting it equal to 1, for instance) without loss of generality. In fact, for any $\beta > 0$, the formula is preserved by the substitutions $\lambda \mapsto \beta\lambda$, $\alpha \mapsto \beta^{3/2}\alpha$ and $s \mapsto \beta^{-1}s$.

The proof is given in Section 5. It is based on excursion theory for Brownian motion and is inspired by similar arguments for other Brownian areas, see Perman and Wellner [8].

Theorem 2.3. *The n:th moment of $\mathcal{A}$ is*

$$\mathbb{E}\,\mathcal{A}^n = \frac{n!\,\Gamma(n+2/3)}{\Gamma(2/3)\,\Gamma(3n/2+1)}\Big(\frac{3\sqrt{2}}{4}\Big)^n, \qquad n \in \mathbb{N}.$$

Proof. Set $\lambda = 1$ in Theorem 2.1 and denote the left and right hand side by

$$I(\alpha) := \int_0^\infty \psi\big(\alpha s^{3/2}\big)e^{-s}\,\mathrm{d}s$$

and

$$J(\alpha) := \int_0^\infty \Big(1 + \frac{3\alpha s}{2\sqrt{2}}\Big)^{-2/3} e^{-s}\,\mathrm{d}s.$$

The integrand of $I(\alpha)$ and all its derivatives with respect to α are dominated by functions of the form $s^K e^{-s}$, uniformly in $\alpha > 0$. Differentiation of $I(\alpha)$ is therefore allowed indefinitely due to dominated convergence. The same argument applies to $J(\alpha)$.

Also, the dominated convergence theorem shows that integration (with respect to s) can be interchanged with taking the limit $\alpha \to 0+$. Thus

$$\begin{aligned}
\lim_{\alpha\to 0+} \frac{\mathrm{d}^n I(\alpha)}{\mathrm{d}\alpha^n} &= \lim_{\alpha\to 0+} \int_0^\infty \frac{\mathrm{d}^n}{\mathrm{d}\alpha^n} \psi(\alpha s^{3/2}) e^{-s}\,\mathrm{d}s \\
&= \int_0^\infty \lim_{\alpha\to 0+} (-s^{3/2})^n\, \mathbb{E}\Big(\mathcal{A}^n \exp\{-\alpha s^{3/2}\mathcal{A}\}\Big) e^{-s}\,\mathrm{d}s \\
&= (-1)^n\, \mathbb{E}(\mathcal{A}^n) \int_0^\infty s^{3n/2} e^{-s}\,\mathrm{d}s \\
&= (-1)^n\, \Gamma(3n/2+1)\, \mathbb{E}\,\mathcal{A}^n
\end{aligned}$$

and

$$\begin{aligned}
\lim_{\alpha\to 0+} \frac{\mathrm{d}^n J(\alpha)}{\mathrm{d}\alpha^n} &= \lim_{\alpha\to 0+} \int_0^\infty \frac{\mathrm{d}^n}{\mathrm{d}\alpha^n} \Big(1+\frac{3\alpha s}{2\sqrt{2}}\Big)^{-2/3} e^{-s}\,\mathrm{d}s \\
&= \int_0^\infty \lim_{\alpha\to 0+} \frac{\Gamma(n+2/3)}{\Gamma(2/3)} \Big(\frac{-3s}{2\sqrt{2}}\Big)^n \Big(1+\frac{3\alpha s}{2\sqrt{2}}\Big)^{-n-2/3} e^{-s}\,\mathrm{d}s \\
&= \frac{\Gamma(n+2/3)}{\Gamma(2/3)} \Big(\frac{-3}{2\sqrt{2}}\Big)^n \int_0^\infty s^n e^{-s}\,\mathrm{d}s \\
&= \frac{\Gamma(n+2/3)}{\Gamma(2/3)} \Big(\frac{-3\sqrt{2}}{4}\Big)^n n!.
\end{aligned}$$

The fact that $I(\alpha) = J(\alpha)$ completes the proof. □

The first four moments of $\mathcal{A}$ are listed in Table 1. Further, Stirling's formula provides the asymptotic relation

$$\mathbb{E}\,\mathcal{A}^n \sim \frac{2\sqrt{3\pi}}{3\,\Gamma(2/3)} n^{1/6} \Big(\frac{n}{3e}\Big)^{n/2}, \qquad n \to \infty. \tag{2.1}$$

Corollary 2.4. *The Laplace transform of $\mathcal{A}$ is*

$$\psi(s) = \frac{1}{\Gamma(2/3)} \sum_{n=0}^\infty \frac{\Gamma(n+2/3)}{\Gamma(3n/2+1)} \Big(\frac{-3\sqrt{2}\,s}{4}\Big)^n. \tag{2.2}$$

Proof. The corollary follows from the identity

$$\psi(s) = \sum_{n=0}^\infty \frac{(-s)^n}{n!}\, \mathbb{E}\,\mathcal{A}^n.$$

Note that the sum converges absolutely for every complex s. □

The graph of $\psi(s)$ is shown in Figure 1.

Remark 2.5. The Laplace transform of $\mathcal{A}$ can also be expressed in terms of generalized hypergeometric functions,

$$\psi(s) = {}_1F_1\Big(\frac{5}{6}; \frac{4}{6}; \frac{s^2}{6}\Big) - \frac{4s}{3\sqrt{2\pi}}\, {}_2F_2\Big(\frac{6}{6}, \frac{8}{6}; \frac{7}{6}, \frac{9}{6}; \frac{s^2}{6}\Big).$$

$\mathbb{E}\,\mathcal{A} = \dfrac{4}{3\sqrt{2\pi}}$	$\mathbb{E}\,\mathcal{A}^2 = \dfrac{5}{12}$	$\mathbb{E}\,\mathcal{A}^3 = \dfrac{64}{63\sqrt{2\pi}}$	$\mathbb{E}\,\mathcal{A}^4 = \dfrac{11}{24}$

TABLE 1. The first four moments of $\mathcal{A}$.

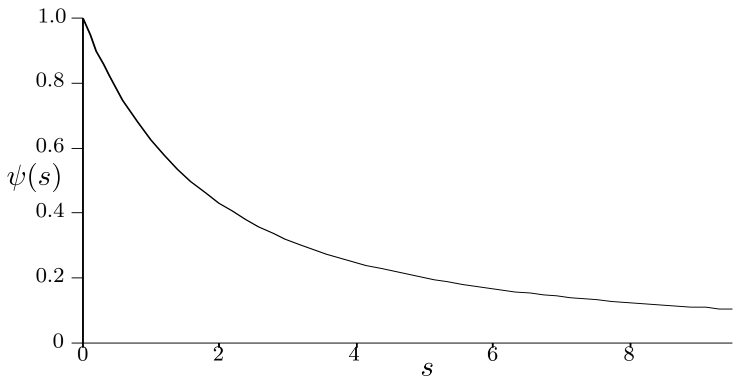

FIGURE 1. The Laplace transform of $\mathcal{A}$.

3. Tail asymptotics

Tauberian theorems by Davies [1] and Kasahara [7] (see Janson [4, Theorem 4.5] for a convenient version) show that the moment asymptotics (2.1) implies the estimate $\ln \mathbb{P}(\mathcal{A} > x) \sim -3x^2/2$ for the tail of the distribution function. Thus, the following corollary is obtained.

Corollary 3.1. *$\mathcal{A}$ has the tail estimate*

$$\mathbb{P}(\mathcal{A} > x) = \exp\{-3x^2/2 + o(x^2)\}, \qquad x \to \infty.$$

(This result can also be proved by large deviation theory; cf. similar results in Fill and Janson [2].)

It seems difficult to obtain more precise tail asymptotics from the moment asymptotics, but it is natural to make a conjecture.

Conjecture 3.2. *$\mathcal{A}$ has a density function $f_{\mathcal{A}}(x)$ satisfying*

$$f_{\mathcal{A}}(x) \sim \frac{2 \cdot 3^{1/6}}{\Gamma(2/3)}\, x^{1/3} e^{-3x^2/2}, \qquad x \to \infty.$$

In fact, if $\mathcal{A}$ has a density with $f_{\mathcal{A}}(x) \sim a x^b e^{-cx^d}$ for some constants a, b, c, d, then it is the only possible choice that yields the moment asymptotics (2.1), cf. Janson and Louchard [5].

Conjecture 3.2 may be compared with similar results for several Brownian areas in Janson and Louchard [5], see also Janson [3]. Note that in these result for Brownian areas, the exponent of x is always an integer (0, 1 or 2). It is therefore a small surprise that here, the exponent seems to be 1/3, corresponding to the power $n^{1/6}$ in (2.1).

4. Preliminaries on point processes

Let $\mathfrak{S}$ be a measurable space. (In this paper, $\mathfrak{S}$ is either an interval of the real line or the product of two such intervals.) Although a point process Ξ will be regarded as a random set $\{\xi_i\} \subset \mathfrak{S}$, it is technically convenient to formally define it as an integer-valued random measure $\sum_i \delta_{\xi_i}$. Hence, $\Xi(A)$ denotes the number of points ξ_i that belong to a (measurable) subset $A \subseteq \mathfrak{S}$. Also, $x \in \Xi$ is equivalent to $\Xi(\{x\}) > 0$. See further e.g. Kallenberg [6].

A *Poisson process with intensity* $\mathrm{d}\mu$, where $\mathrm{d}\mu$ is a measure on $\mathfrak{S}$, is a point process Ξ such that $\Xi(A)$ has a Poisson distribution with mean $\mu(A)$ for every measurable $A \subseteq \mathfrak{S}$, and $\Xi(A_1), \dots, \Xi(A_k)$ are independent for every family $A_1, \dots, A_k$ of disjoint measurable sets. Lemma 4.1 is a standard formula for Laplace functionals, see for instance [6, Lemma 12.2(i)].

Lemma 4.1. *If Ξ is a Poisson process with intensity $\mathrm{d}\mu$ on a set $\mathfrak{S}$, and $f : \mathfrak{S} \to [0, \infty)$ is a measurable function, then*

$$\mathbb{E} \exp\Big\{-\sum_{\xi\in\Xi} f(\xi)\Big\} = \exp\Big\{-\int_{\mathfrak{S}} \big(1 - e^{-f(x)}\big)\,\mathrm{d}\mu(x)\Big\}.$$

Lemma 4.2, on the other hand, is more of a digression. The result follows from a standard Gamma integral by integration by parts. (The result can also be written as $2\Gamma\big(1/2\big)\lambda^{1/2}$.)

Lemma 4.2. *If $\lambda > 0$, then*

$$\int_0^\infty \big(1 - e^{-\lambda x}\big) x^{-3/2}\,\mathrm{d}x = 2\sqrt{\pi\lambda}.$$

5. Proof of Theorem 2.1

The set $\{t : B(t) = 0\}$ is a.s. closed and unbounded, so its complement $\{t : B(t) \neq 0\}$ is an infinite union of finite open intervals, denoted by $I_\nu = (g_\nu, d_\nu)$, $\nu = 1, 2, \dots$, in some order. (The intervals cannot be ordered by appearance, since there is a.s. an infinite number of them in, say, [0,1]. Fortunately, the order does not matter.) The restrictions of $B(t)$ to these intervals are called the *excursions* of $B(t)$. Let $\mathbf{e}_\nu$ be the excursion during I_ν.

The local time $L(t)$ is constant during each excursion. Let τ_ν be the local time during $\mathbf{e}_\nu$ and let $\ell_\nu := d_\nu - g_\nu$ be the length of $\mathbf{e}_\nu$. It is well-known, see Revuz and Yor [10, Chapter XII], that the collection of pairs $\{(\tau_\nu, \ell_\nu)\}_{\nu=1}^\infty$ forms a Poisson process in $[0, \infty) \times (0, \infty)$ with intensity

$$\mathrm{d}\Lambda = (2\pi\ell^3)^{-1/2}\,\mathrm{d}\tau\,\mathrm{d}\ell.$$

Note also that, a.s., if the excursion $\mathbf{e}_{\nu_1}$ comes before $\mathbf{e}_{\nu_2}$, then $\tau_{\nu_1} < \tau_{\nu_2}$.

Next, consider a Poisson process $\{T_i\}_{i=1}^\infty$ on $[0, \infty)$ with intensity $\lambda\,\mathrm{d}t$, independent of $\{B(t)\}$. Assume that the points are ordered with $0 < T_1 < T_2 < \cdots$. Then T_1, $T_2 - T_1$, … are i.i.d. $\mathrm{Exp}(\lambda)$ random variables with density function $\lambda e^{-\lambda t}$. Furthermore, T_1 is independent of $\{B(t)\}$ and thus of $\{\mathcal{A}(T)\}$. It follows from (1.2) that $\mathcal{A}(T_1) \overset{\mathrm{d}}{=} T_1^{3/2}\mathcal{A}$ and consequently

$$\mathbb{E}\, e^{-\alpha\mathcal{A}(T_1)} = \mathbb{E}\, e^{-\alpha T_1^{3/2}\mathcal{A}} = \mathbb{E}\,\psi\bigl(\alpha T_1^{3/2}\bigr) = \lambda\int_0^\infty e^{-\lambda s}\psi\bigl(\alpha s^{3/2}\bigr)\,\mathrm{d}s. \tag{5.1}$$

The times T_i are called *marks*, and an excursion is called *marked* if it contains at least one of the marks T_i. The marks $\{T_i\}$ are placed by first constructing $\{B(t)\}$ and then adding marks according to independent Poisson processes with intensities $\lambda\,\mathrm{d}t$ in each excursion. Thus, given the excursions $\{\mathbf{e}_\nu\}$, each excursion $\mathbf{e}_\nu$ is marked with probability $1-e^{-\lambda\ell_\nu}$, independently of the other excursions. The Poisson process $\Xi := \{(\tau_\nu,\ell_\nu)\}$ defined by the excursions can be written as the union $\Xi'\cup\Xi''$, where

$$\begin{aligned}\Xi' &:= \bigl\{(\tau_\nu,\ell_\nu) : \mathbf{e}_\nu \text{ is unmarked}\bigr\},\\ \Xi'' &:= \bigl\{(\tau_\nu,\ell_\nu) : \mathbf{e}_\nu \text{ is marked}\bigr\}.\end{aligned}$$

By the general independence properties of Poisson processes, Ξ' and Ξ'' are *independent* Poisson processes with intensities

$$\mathrm{d}\Lambda' := e^{-\lambda\ell}\,\mathrm{d}\Lambda = (2\pi)^{-1/2}\ell^{-3/2}e^{-\lambda\ell}\,\mathrm{d}\tau\,\mathrm{d}\ell \tag{5.2}$$

and

$$\mathrm{d}\Lambda'' := \bigl(1-e^{-\lambda\ell}\bigr)\,\mathrm{d}\Lambda = (2\pi)^{-1/2}\ell^{-3/2}\bigl(1-e^{-\lambda\ell}\bigr)\,\mathrm{d}\tau\,\mathrm{d}\ell, \tag{5.3}$$

respectively. In particular, if the lengths are ignored, the local times of the marked excursions form a Poisson process $\widetilde{\Xi}$ on $(0,\infty)$ with intensity

$$\int_{\ell=0}^\infty \bigl(1-e^{-\lambda\ell}\bigr)\,\mathrm{d}\Lambda = \widetilde{\lambda}\,\mathrm{d}\tau,$$

where, using Lemma 4.2,

$$\widetilde{\lambda} = \int_0^\infty (2\pi)^{-1/2}\ell^{-3/2}\bigl(1-e^{-\lambda\ell}\bigr)\,\mathrm{d}\ell = \sqrt{2\lambda}. \tag{5.4}$$

Due to the fact that $B(T_1)\neq 0$ a.s., there exists a unique excursion $\mathbf{e}_{\nu^*}$ that contains the first mark T_1, i.e., $T_1\in I_{\nu^*}$. Let $\zeta := L(T_1) = \tau_{\nu^*}$ be the local time at T_1 (and thus during $\mathbf{e}_{\nu^*}$). Since $\mathbf{e}_{\nu^*}$ is the first marked excursion, its local time ζ is the first of the points in the Poisson process $\widetilde{\Xi}$ and hence

$$\zeta \sim \operatorname{Exp}\bigl(\sqrt{2\lambda}\bigr). \tag{5.5}$$

The restriction of $B(t)$ to the interval $[0,T_1]$ consists of all excursions $\mathbf{e}_\nu$ with local time $\tau_\nu < \tau_{\nu^*} = \zeta$ and the part of $\mathbf{e}_{\nu^*}$ on (g_{ν^*},T_1), plus the set

$$[0,T_1]\setminus\bigcup_\nu I_\nu = \{t\le T_1 : B(t)=0\}$$

which a.s. has measure 0 and thus may be ignored. Consequently, since $L(t) = \tau_\nu$ on I_ν,

$$\begin{aligned}\mathcal{A}(T_1) := \int_0^{T_1} L(t)\,\mathrm{d}t &= \sum_{\nu:\tau_\nu<\tau_{\nu^*}} \int_{I_\nu} L(t)\,\mathrm{d}t + \int_{g_{\nu^*}}^{T_1} L(t)\,\mathrm{d}t \\ &= \sum_{\nu:\tau_\nu<\zeta} \tau_\nu \ell_\nu + \zeta(T_1 - g_{\nu^*}) := \mathcal{A}' + \mathcal{A}''.\end{aligned}$$

The sum defined as $\mathcal{A}' = \sum_{\nu:\tau_\nu<\zeta} \tau_\nu\ell_\nu$ only contains terms for unmarked excursions $\mathbf{e}_\nu$. Thus

$$\mathcal{A}' = \sum_{(\tau_\nu,\ell_\nu)\in\Xi':\ \tau_\nu<\zeta} \tau_\nu\ell_\nu.$$

Recall that ζ is determined by Ξ'' (as the smallest τ with $(\tau,\ell)\in\Xi''$ for some ℓ) and that Ξ' and Ξ'' are independent. Hence, Ξ' and ζ are independent. It follows from Lemma 4.1, with $\mathfrak{S} = (0,\zeta)\times(0,\infty)$ and $f((\tau,\ell)) = \alpha\tau\ell$, that

$$\mathbb{E}\big(e^{-\alpha\mathcal{A}'} \mid \zeta\big) = \exp\Big\{-\int_{\tau=0}^{\zeta}\int_{\ell=0}^{\infty}\big(1-e^{-\alpha\tau\ell}\big)\,\mathrm{d}\Lambda'(\tau,\ell)\Big\}.$$

By (5.2) and Lemma 4.2,

$$\begin{aligned}&\int_{\tau=0}^{\zeta}\int_{\ell=0}^{\infty}\big(1-e^{-\alpha\tau\ell}\big)\,\mathrm{d}\Lambda'(\tau,\ell) \\ &\qquad= \int_{\tau=0}^{\zeta}\int_{\ell=0}^{\infty}\big(1-e^{-\alpha\tau\ell}\big)(2\pi)^{-1/2}\ell^{-3/2}e^{-\lambda\ell}\,\mathrm{d}\ell\,\mathrm{d}\tau \\ &\qquad= (2\pi)^{-1/2}\int_{\tau=0}^{\zeta}\int_{\ell=0}^{\infty}\big(e^{-\lambda\ell}-e^{-(\lambda+\alpha\tau)\ell}\big)\ell^{-3/2}\,\mathrm{d}\ell\,\mathrm{d}\tau \\ &\qquad= \int_{\tau=0}^{\zeta}\sqrt{2}\left(\sqrt{\lambda+\alpha\tau}-\sqrt{\lambda}\right)\mathrm{d}\tau \\ &\qquad= \frac{2\sqrt{2}}{3\alpha}\left((\lambda+\alpha\zeta)^{3/2}-\lambda^{3/2}\right)-\sqrt{2\lambda}\,\zeta,\end{aligned}$$

and it follows that

$$\mathbb{E}\big(e^{-\alpha\mathcal{A}'} \mid \zeta\big) = \exp\Big\{\sqrt{2\lambda}\,\zeta - \frac{2\sqrt{2}}{3\alpha}\left((\lambda+\alpha\zeta)^{3/2}-\lambda^{3/2}\right)\Big\}. \tag{5.6}$$

Now consider $\mathcal{A}'' = \zeta(T_1 - g_{\nu^*})$. Note that $T_1 - g_{\nu^*}$ is the location (relative to the left endpoint of the excursion) of the first mark in the first marked excursion. Since Ξ is a Poisson process with intensity independent of τ, the location $T_1 - g_{\nu^*}$ is independent of the local time ζ of the first marked excursion. Further, the joint distribution of $(\ell_{\nu^*}, T_1 - g_{\nu^*})$ has density

$$(\widetilde{\lambda})^{-1}\lambda e^{-\lambda y}(2\pi)^{-1/2}\ell^{-3/2}\,\mathrm{d}\ell\,\mathrm{d}y, \qquad 0<y<\ell<\infty,$$

where the normalization constant $\widetilde{\lambda}$ is given by (5.4). Consequently,

$$\begin{aligned}\mathbb{E}\big(e^{-\alpha\mathcal{A}''} \mid \zeta\big) &= \mathbb{E}\big(e^{-\alpha\zeta(T_1-g_{\nu^*})} \mid \zeta\big) \\ &= \int_{y=0}^{\infty}\int_{\ell=y}^{\infty} e^{-\alpha\zeta y}(\widetilde{\lambda})^{-1}\lambda e^{-\lambda y}(2\pi)^{-1/2}\ell^{-3/2}\,\mathrm{d}\ell\,\mathrm{d}y \\ &= \pi^{-1/2}\lambda^{1/2}\int_{y=0}^{\infty} e^{-(\lambda+\alpha\zeta)y}y^{-1/2}\,\mathrm{d}y \\ &= \lambda^{1/2}(\lambda+\alpha\zeta)^{-1/2}. \end{aligned} \tag{5.7}$$

Again, since Ξ' and Ξ'' are independent, $\mathcal{A}'$ and $\mathcal{A}''$ are conditionally independent given ζ. Thus, equation (5.6) and (5.7) yield

$$\begin{aligned}\mathbb{E}\big(e^{-\alpha\mathcal{A}(T_1)} \mid \zeta\big) &= \mathbb{E}\big(e^{-\alpha\mathcal{A}'} \mid \zeta\big)\,\mathbb{E}\big(e^{-\alpha\mathcal{A}''} \mid \zeta\big) \\ &= \Big(\frac{\lambda}{\lambda+\alpha\zeta}\Big)^{1/2}\exp\Big\{\sqrt{2\lambda}\,\zeta - \frac{2\sqrt{2}}{3\alpha}\Big((\lambda+\alpha\zeta)^{3/2}-\lambda^{3/2}\Big)\Big\}.\end{aligned}$$

By (5.5), ζ has the density $\sqrt{2\lambda}e^{-\sqrt{2\lambda}x}$, $x>0$, and it follows that

$$\mathbb{E}\,e^{-\alpha\mathcal{A}(T_1)} = \lambda\sqrt{2}\int_0^{\infty}(\lambda+\alpha x)^{-1/2}\exp\Big\{-\frac{2\sqrt{2}}{3\alpha}\Big((\lambda+\alpha x)^{3/2}-\lambda^{3/2}\Big)\Big\}\,\mathrm{d}x.$$

Finally, the substitution

$$\frac{2\sqrt{2}}{3\alpha\lambda}\Big((\lambda+\alpha x)^{3/2}-\lambda^{3/2}\Big)\mapsto s$$

provides the slightly simpler formula

$$\mathbb{E}\,e^{-\alpha\mathcal{A}(T_1)} = \lambda\int_0^{\infty}\Big(1+\frac{3\alpha s}{2\sqrt{2\lambda}}\Big)^{-2/3}e^{-\lambda s}\,\mathrm{d}s.$$

The result now follows by a comparison with (5.1). □

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