# RANDOM EVEN GRAPHS

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Abstract. We study a random even subgraph of a finite graph $G$ with a general edge-weight $p \in (0,1)$. We demonstrate how it may be obtained from a certain random-cluster measure on $G$, and we propose a sampling algorithm based on coupling from the past. A random even subgraph of a planar lattice undergoes a phase transition at the parameter-value $\frac{1}{2}p_{\mathrm{c}}$, where $p_{\mathrm{c}}$ is the critical point of the $q = 2$ random-cluster model on the dual lattice. The properties of such a graph are discussed, and are related to Schramm–Löwner evolutions (SLE).

## 1. Introduction

Our purpose in this paper is to study a random even subgraph of a finite graph $G = (V, E)$, and to show how to sample such a subgraph. A subset $F$ of $E$ is called *even* if, for all $x \in V$, $x$ is incident to an even number of elements of $F$. We call the subgraph $(V, F)$ *even* if $F$ is even, and we write $\mathcal{E}$ for the set of all even subsets $F$ of $E$. It is standard that every even set $F$ may be decomposed as an edge-disjoint union of cycles. Let $p \in [0, 1)$. The *random even subgraph* of $G$ with parameter $p$ is that with law

$$\rho_p(F) = \frac{1}{Z^{\mathrm{E}}} p^{|F|}(1-p)^{|E\setminus F|}, \qquad F \in \mathcal{E}, \tag{1.1}$$

where $Z^{\mathrm{E}} = Z^{\mathrm{E}}_G(p)$ is the appropriate normalising constant.

We may express $\rho_p$ as follows in terms of product measure on $E$. Let $\phi_p$ be product measure with density $p$ on the configuration space $\Omega = \{0,1\}^E$. For $\omega \in \Omega$ and $e \in E$, we call $e$ *$\omega$-open* if $\omega(e) = 1$, and *$\omega$-closed* otherwise. Let $\partial\omega$ denote the set of vertices $x \in V$ that are incident to an odd number of $\omega$-open edges. Then

$$\rho_p(F) = \frac{\phi_p(\omega_F)}{\phi_p(\partial\omega = \varnothing)}, \qquad F \subseteq E, \tag{1.2}$$

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where $\omega_F$ is the edge-configuration whose open edge-set is $F$. In other words, $\phi_p$ describes the random subgraph of $G$ obtained by randomly and independently deleting each edge with probability $1-p$, and $\rho_p$ is the law of this random subgraph conditioned on being even.

Random even graphs are closely related to the Ising model and the random-cluster model on $G$, and we review these models briefly. Let $\beta \in (0,\infty)$ and

$$p = 1 - e^{-2\beta} = \frac{2\tanh\beta}{1+\tanh\beta}. \tag{1.3}$$

The Ising model on $G$ has configuration space $\Sigma = \{-1,+1\}^V$, and probability measure

$$\pi_\beta(\sigma) = \frac{1}{Z^{\mathrm{I}}} \exp\biggl\{\beta \sum_{e\in E} \sigma_x\sigma_y\biggr\}, \qquad \sigma \in \Sigma. \tag{1.4}$$

where $Z^{\mathrm{I}} = Z^{\mathrm{I}}_G(\beta)$ is the partition function that makes $\pi_\beta$ a probability measure. A *spin-cluster* of a configuration $\sigma \in \Sigma$ is a maximal connected subgraph of $G$ each of whose vertices $v$ has the same spin-value $\sigma_v$. A spin-cluster is termed a $k$ cluster if $\sigma_v = k$ for all $v$ belonging to the cluster. An important quantity associated with the Ising model is the 'two-point correlation function'

$$\tau_\beta(x,y) = \pi_\beta(\sigma_x = \sigma_y) - \tfrac12 = \tfrac12 \pi_\beta(\sigma_x\sigma_y), \qquad x,y \in V, \tag{1.5}$$

where $P(f)$ denotes the expectation of a random variable $f$ under the probability measure $P$.

The random-cluster measure on $G$ with parameters $p \in (0,1)$ and $q = 2$ is given as follows [it may be defined for general $q > 0$ but we are concerned here only with the case $q = 2$]. Let

$$\begin{aligned}\phi_{p,2}(\omega) &= \frac{1}{Z^{\mathrm{RC}}}\biggl\{\prod_{e\in E} p^{\omega(e)}(1-p)^{1-\omega(e)}\biggr\} 2^{k(\omega)} \\ &= \frac{1}{Z^{\mathrm{RC}}} p^{|\eta(\omega)|}(1-p)^{|E\setminus\eta(\omega)|} 2^{k(\omega)}, \qquad \omega \in \Omega,\end{aligned} \tag{1.6}$$

where $k(\omega)$ denotes the number of $\omega$-open components on the vertex-set $V$, $\eta(\omega) = \{e \in E : \omega(e) = 1\}$ is the set of open edges, and $Z^{\mathrm{RC}} = Z^{\mathrm{RC}}_G(p)$ is the appropriate normalizing factor.

The relationship between the Ising and random-cluster models on $G$ is well established, and hinges on the fact that, in the notation introduced above,

$$\tau_\beta(x,y) = \tfrac12 \phi_{p,2}(x \leftrightarrow y),$$

where $\{x \leftrightarrow y\}$ is the event that $x$ and $y$ are connected by an open path. See [11] for an account of the random-cluster model. There is a relationship between the Ising model and the random even graph also, known misleadingly as the 'high-temperature expansion'. This may be stated as follows (see [3]

for a proof). Consider the Ising model on $G$ with parameter $\beta$, and let $p = 1 - e^{-2\beta}$. Then (cf. (1.2))

$$2\tau_{\beta,2}(x,y) = \frac{\phi_{p/2}(\partial\omega = \{x,y\})}{\phi_{p/2}(\partial\omega = \varnothing)}, \qquad x,y \in V,\ x \neq y. \tag{1.7}$$

This note is laid out in the following way. In Section 2 we define a random even subgraph of a finite or infinite graph, and we explain how to sample a *uniform* even subgraph. In Section 3 we explain how to sample a non-uniform random even graph, starting with a sample from a random-cluster measure. An algorithm for exact sampling is presented in Section 4 based on the method of coupling from the past. The structure of random even subgraphs of the square and hexagonal lattices is summarized in Section 5.

In a second paper [13], we study the asymptotic properties of a random even subgraph of the complete graph $K_n$. Whereas the special relationship with the random-cluster and Ising models is the main feature of the current work, the analysis of [13] is more analytic, and extends to random graphs whose vertex degrees are constrained to lie in any given subsequence of the non-negative integers.

**Remark 1.8.** The definition (1.1) may be generalized by replacing the single parameter $p$ by a family $\mathbf{p} = (p_e : e \in E)$, just as sometimes is done for the random-cluster measure (1.6), see for example [24]; we let

$$\rho_{\mathbf{p}}(F) = \frac{1}{Z} \prod_{e \in F} p_e \prod_{e \notin F} (1 - p_e). \tag{1.9}$$

For simplicity we will mostly consider the case of a single $p$.

## 2. Uniform random even subgraphs

In the case $p = \frac{1}{2}$ in (1.1), every even subgraph has the same probability, so $\rho_{\frac{1}{2}}$ describes a uniform random even subgraph of $G$. Such a random subgraph can be obtained as follows.

We identify the family of all spanning subgraphs of $G = (V, E)$ with the family of all subsets of $E$. This family can further be identified with $\{0,1\}^E = \mathbb{Z}_2^E$, and is thus a vector space over $\mathbb{Z}_2$; the addition is componentwise addition modulo 2 in $\{0,1\}^E$, which translates into taking the symmetric difference of edge-sets: $F_1 + F_2 = F_1 \triangle F_2$ for $F_1, F_2 \subseteq E$.

The family of even subgraphs of $G$ forms a subspace $\mathcal{E}$ of this vector space $\{0,1\}^E$, since $F_1 + F_2 = F_1 \triangle F_2$ is even if $F_1$ and $F_2$ are even. (In fact, $\mathcal{E}$ is the cycle space $Z_1$ in the $\mathbb{Z}_2$-homology of $G$ as a simplical complex.) In particular, the number of even subgraphs of $G$ equals $2^{c(G)}$ where $c(G) = \dim(\mathcal{E})$; $c(G)$ is thus the number of independent cycles in $G$, and, as is well known,

$$c(G) = |E| - |V| + k(G). \tag{2.1}$$

**Proposition 2.2.** *Let $C_1, \dots, C_c$ be a maximal set of independent cycles in $G$. Let $\xi_1, \dots, \xi_c$ be independent* $\mathrm{Be}(\frac{1}{2})$ *random variables (i.e., the results of fair coin tosses). Then $\sum_i \xi_i C_i$ is a uniform random even subgraph of $G$.*

*Proof.* $C_1, \dots, C_c$ is a basis of the vector space $\mathcal{E}$ over $\mathbb{Z}_2$. □

One standard way of choosing $C_1, \dots, C_c$ is exploited in the next proposition. Another, for planar graphs, is given by the boundaries of the finite faces; this will be used in Section 5. In the following proposition, we use the term *spanning subforest* of $G$ to mean a maximal forest of $G$, that is, the union of a spanning tree from each component of $G$.

**Proposition 2.3.** *Let $(V, F)$ be a spanning subforest of $G$. Each subset $X$ of $E \setminus F$ can be completed by a unique $Y \subseteq F$ to an even edge-set $E_X = X \cup Y \in \mathcal{E}$. Choosing a uniform random subset $X \subseteq E \setminus F$ thus gives a uniform random even subgraph $E_X$ of $G$.*

*Proof.* It is easy to see, and well known, that each edge $e_i \in E \setminus F$ can be completed by edges in $F$ to a unique cycle $C_i$; these cycles form a basis of $\mathcal{E}$ and the result follows by Proposition 2.2. (It is also easy to give a direct proof.) □

There is a natural definition of a uniform random even subgraph of an *infinite*, locally finite graph $G = (V, E)$. In fact, we can define the subspace $\mathcal{E}$ of $\{0,1\}^E$ as before. (Note that we need $G$ to be locally finite in order to do so.) Further, let $E_1$ be a finite subset of $E$. The natural projection $\pi_{E_1} : \{0,1\}^E \to \{0,1\}^{E_1}$ given by $\pi_{E_1}(\omega) = (\omega_e)_{e \in E_1}$ maps $\mathcal{E}$ onto a subspace $\mathcal{E}_{E_1} = \pi_{E_1}(\mathcal{E})$ of $\{0,1\}^{E_1}$.

**Theorem 2.4.** *Let $G$ be a locally finite, countably infinite graph. There exists a unique probability measure $\rho$ on $\Omega = \{0,1\}^E$ such that, for every finite set $E_1 \subset E$, $(\omega_e)_{e \in E_1}$ is uniformly distributed on $\mathcal{E}_{E_1}$, i.e.,*

$$\rho(\pi_{E_1}^{-1}(A)) = |A \cap \mathcal{E}_{E_1}| / |\mathcal{E}_{E_1}|, \qquad A \subseteq \{0,1\}^{E_1}. \tag{2.5}$$

*Proof.* We have defined $\rho$ on all cylinder events by (2.5). By the Kolmogorov extension theorem, it suffices to show that this definition is consistent as $E_1$ varies, which amounts to showing that if $E_1 \subseteq E_2 \subset E$ with $E_1, E_2$ finite, then the projection $\pi_{E_2 E_1} : \{0,1\}^{E_2} \to \{0,1\}^{E_1}$ maps the uniform distribution on $\mathcal{E}_{E_2}$ to the uniform distribution on $\mathcal{E}_{E_1}$. This is an immediate consequence of the fact that $\pi_{E_2 E_1}$ is a linear map of $\mathcal{E}_{E_2}$ onto $\mathcal{E}_{E_1}$. □

## 3. Random even subgraphs via coupling

We return to the random even subgraph with parameter $p \in [0, 1)$ defined by (1.1) for a finite graph $G = (V, E)$. We show next how to couple the $q = 2$ random-cluster model and the random even subgraph of $G$. Let $p \in [0, \frac{1}{2}]$, and

let $\omega$ be a realization of the random-cluster model on $G$ with parameters $2p$ and $q = 2$. Let $R = (V, \gamma)$ be a uniform random even subgraph of $(V, \eta(\omega))$.

**Theorem 3.1.** *Let $p \in [0, \frac{1}{2}]$. The graph $R = (V, \gamma)$ is a random even subgraph of $G$ with parameter $p$.*

This recipe for random even subgraphs provides a neat method for their simulation, provided $p \le \frac{1}{2}$. One may sample from the random-cluster measure by the method of coupling from the past (see [19]), and then sample a uniform random even subgraph by either Proposition 2.2 or Proposition 2.3.

*Proof.* Let $g \subseteq E$ be even. By the observations in Section 2, with $c(\omega) = c(V, \eta(\omega))$ denoting the number of independent cycles in the open subgraph,

$$\mathbb{P}(\gamma = g \mid \omega) = \begin{cases} 2^{-c(\omega)} & \text{if } g \subseteq \eta(\omega), \\ 0 & \text{otherwise,} \end{cases}$$

so that

$$\mathbb{P}(\gamma = g) = \sum_{\omega : g \subseteq \eta(\omega)} 2^{-c(\omega)} \phi_{2p,2}(\omega).$$

Now $c(\omega) = |\eta(\omega)| - |V| + k(\omega)$, so that, by (1.6),

$$\begin{aligned} \mathbb{P}(\gamma = g) &\propto \sum_{\omega : g \subseteq \eta(\omega)} (2p)^{|\eta(\omega)|} (1-2p)^{|E \setminus \eta(\omega)|} 2^{k(\omega)} \frac{1}{2^{|\eta(\omega)| - |V| + k(\omega)}} \\ &\propto \sum_{\omega : g \subseteq \eta(\omega)} p^{|\eta(\omega)|} (1-2p)^{|E \setminus \eta(\omega)|} \\ &= [p + (1-2p)]^{|E \setminus g|} p^{|g|} \\ &= p^{|g|} (1-p)^{|E \setminus g|}, \qquad g \subseteq E. \end{aligned}$$

The claim follows. □

Let $p \in (\frac{1}{2}, 1)$. If $G$ is even, we can sample from $\rho_p$ by first sampling a subgraph $(V, \widetilde{F})$ from $\rho_{1-p}$ and then taking the complement $(V, E \setminus \widetilde{F})$, which has the distribution $\rho_p$. If $G$ is not even, we adapt this recipe as follows. For $W \subseteq V$ and $H \subseteq E$, we say that $H$ is *$W$-even* if each component of $(V, H)$ contains an even number of members of $W$. Let $W \neq \varnothing$ be the set of vertices of $G$ with odd degree, so that, in particular, $E$ is $W$-even. Let $\Omega^W = \{\omega \in \Omega : \eta(\omega) \text{ is } W\text{-even}\}$. For $\omega \in \Omega^W$, we pick disjoint subsets $P^i = P^i_\omega$, $i = 1, 2, \ldots, \frac{1}{2}|W|$, of $\eta(\omega)$, each of which constitutes an open non-self-intersecting path with distinct endpoints lying in $W$, and such that every member of $W$ is the endpoint of exactly one such path. Write $P_\omega = \bigcup_i P^i_\omega$.

Let $r = 2(1-p)$, and let $\phi^W_{r,2}$ be the random-cluster measure on $\Omega$ with parameters $r$ and $q = 2$ conditional on the event $\Omega^W$. We sample from $\phi^W_{r,2}$

to obtain a subgraph $(V, \eta(\omega))$, from which we select a uniform random even subgraph $(V, \gamma)$ by the procedure of the previous section.

**Theorem 3.2.** *Let $p \in (\frac{1}{2}, 1)$. The graph $S = (V, E \setminus (\gamma \triangle P_\omega))$ is a random even subgraph of $G$ with parameter $p$.*

The recipes in Theorems 3.1 and 3.2 can be combined as follows, if we consider the generalized model mentioned in Remark 1.8 with one parameter $p_e \in (0,1)$ for each edge $e \in E$. Let $A = \{e \in E : p_e > \frac{1}{2}\}$. Define $r_e = 2p_e$ when $e \notin A$ and $r_e = 2(1-p_e)$ when $e \in A$. (Thus $0 < r_e \le 1$.) Let $W = W_A$ be the set of vertices that are $A$-*odd*, i.e., endpoints of an odd number of edges in $A$. Sample $\omega$ from the random-cluster measure with parameters $\mathbf{r} = (r_e : e \in E)$ and $q = 2$, conditioned on $\eta(\omega)$ being $W$-even, let $P_\omega$ be as above (for $W = W_A$), and sample a uniform random even subgraph $(V, \gamma)$ of $(V, \eta(\omega))$.

**Theorem 3.3.** *The graph $S = (V, \gamma \triangle P_\omega \triangle A)$ is a random even subgraph of $G$ with the distribution $\rho_{\mathbf{p}}$ given in* (1.9).

Note that Theorems 3.1 and 3.2 are special cases of Theorem 3.3, with $A = \varnothing$ and $A = E$ respectively. We find it more illuminating to present the proof of Theorem 3.2 in this more general setup.

*Proof of Theorem 3.3, and thus of Theorem 3.2.* Let $F = \gamma \triangle P_\omega \triangle A$ be the resulting edge-set, and note that

$$\eta(\omega) \supseteq \gamma \triangle P_\omega = F \triangle A. \tag{3.4}$$

Furthermore, if $F$ is even, then $F \triangle A$ has odd degree exactly at vertices in $W = W_A$; hence (3.4) implies that necessarily $\omega \in \Omega^W$.

Given an even edge set $f \subseteq E$, we thus obtain $F = f$ if we first choose $\omega \in \Omega^W$ with $\eta(\omega) \supseteq f \triangle A$ and then (having chosen $P_\omega$) select $\gamma$ as the even subgraph $f \triangle A \triangle P_\omega$. Hence, for every $\omega \in \Omega^W$ with $\eta(\omega) \supseteq f \triangle A$, we have $\mathbb{P}(F = f \mid \omega) = 2^{-c(\omega)}$, and summing over such $\omega$ we find

$$\begin{aligned}
\mathbb{P}(F = f) &\propto \sum_{\omega : \eta(\omega) \supseteq f \Delta A} 2^{-c(\omega)} \phi_{\mathbf{r},2}(\omega) \\
&\propto \sum_{\omega : \eta(\omega) \supseteq f \triangle A} 2^{-c(\omega)} 2^{k(\omega)} \prod_{e \in E} r_e^{\omega(e)} (1 - r_e)^{1-\omega(e)} \\
&\propto \sum_{\omega : \eta(\omega) \supseteq f \triangle A} 2^{-|\eta(\omega)|} \prod_{e \in E} r_e^{\omega(e)} (1 - r_e)^{1-\omega(e)} \\
&= \sum_{\omega : \eta(\omega) \supseteq f \triangle A} \prod_{e \in E} \Big(\frac{r_e}{2}\Big)^{\omega(e)} (1 - r_e)^{1-\omega(e)} \\
&= \prod_{e \in f \triangle A} \Big(\frac{r_e}{2}\Big) \prod_{e \notin f \triangle A} \Big(1 - \frac{r_e}{2}\Big).
\end{aligned}$$

With $1_e$ denoting the indicator function of the event $\{e \in F\}$, this can be rewritten as

$$\begin{aligned}\mathbb{P}(F = f) &\propto \prod_{e \notin A} (r_e/2)^{1_e}(1 - r_e/2)^{1-1_e} \prod_{e \in A} (r_e/2)^{1-1_e}(1 - r_e/2)^{1_e} \\ &= \prod_{e \notin A} p_e^{1_e}(1 - p_e)^{1-1_e} \prod_{e \in A} (1 - p_e)^{1-1_e} p_e^{1_e} \\ &= \prod_{e \in E} p_e^{1_e}(1 - p_e)^{1-1_e} \\ &\propto \rho_{\mathbf{p}}(f).\end{aligned}$$

The claim follows. □

There is a converse to Theorem 3.1. Take a random even subgraph $(V, F)$ of $G = (V, E)$ with parameter $p \leq \frac{1}{2}$. To each $e \notin F$, we assign an independent random colour, *blue* with probability $p/(1 - p)$ and *red* otherwise. Let $H$ be obtained from $F$ by adding in all blue edges.

**Theorem 3.5.** *The graph $(V, H)$ has law $\phi_{2p,2}$.*

*Proof.* For $h \subseteq E$,

$$\begin{aligned}\mathbb{P}(H = h) &\propto \sum_{J \subseteq h,\ J \text{ even}} \left(\frac{p}{1-p}\right)^{|J|} \left(\frac{p}{1-p}\right)^{|h \setminus J|} \left(\frac{1-2p}{1-p}\right)^{|E \setminus h|} \\ &\propto p^{|h|}(1 - 2p)^{|E \setminus h|} N(h),\end{aligned}$$

where $N(h)$ is the number of even subgraphs of $(V, h)$. As in the above proof, $N(h) = 2^{|h|-|V|+k(h)}$ where $k(h)$ is the number of components of $(V, h)$, and the proof is complete. □

An edge $e$ of a graph is called *cyclic* if it belongs to some cycle of the graph.

**Corollary 3.6.** *For $p \in [0, \frac{1}{2}]$ and $e \in E$,*

$$\rho_p(e \text{ is open}) = \tfrac{1}{2}\phi_{2p,2}(e \text{ is a cyclic edge of the open graph}).$$

By summing over $e \in E$, we deduce that the mean number of open edges under $\rho_p$ is one half of the mean number of cyclic edges under $\phi_{2p,2}$.

*Proof.* Let $\omega \in \Omega$ and let $\mathcal{C}$ be a maximal family of independent cycles of $\omega$. Let $R = (V, \gamma)$ be a uniform random even subgraph of $(V, \eta(\omega))$, constructed using Proposition 2.2 and $\mathcal{C}$. For $e \in E$, let $M_e$ be the number of elements of $\mathcal{C}$ that include $e$. If $M_e \geq 1$, the number of these $M_e$ cycles of $\gamma$ that are selected in the construction of $\gamma$ is equally likely to be even as odd. Therefore,

$$\mathbb{P}(e \in \gamma \mid \omega) = \begin{cases} \frac{1}{2} & \text{if } M_e \geq 1, \\ 0 & \text{if } M_e = 0. \end{cases}$$

The claim follows by Theorem 3.1. □

## 4. Sampling an even subgraph

It was remarked earlier that Theorem 3.1 gives a neat way of sampling an even subgraph of $G$ according to the probability measure $\eta_p$ with $p \le \frac{1}{2}$. Simply use coupling-from-the-past (cftp) to sample from the random-cluster measure $\phi_{2p,2}$, and then flip a fair coin once for each member of some maximal independent set of cycles of $G$. The situation is slightly more complicated when $p > \frac{1}{2}$ and $G$ is not itself even, since the conditioned random-cluster measure used in Theorems 3.2 and 3.3 is neither monotone nor anti-monotone. We indicate briefly in this section how to adapt the technique of cftp to such a situation.

Let $E$ be a non-empty finite set, and let $\mu$ be a probability measure on the product space $\Omega = \{0,1\}^E$. We call $\mu$ *monotone* (respectively, *anti-monotone*) if $\mu(1_e \mid \xi_e)$ is non-decreasing (respectively, non-increasing) in $\xi \in \Omega$. Here, $1_e$ is the indicator function that $e$ is open, and $\xi_e$ is the configuration obtained from $\xi$ on $E \setminus \{e\}$. For $e \in E$, $\psi \in \Omega$, and $b = 0, 1$, we write $\psi_e^b$ for the configuration that agrees with $\psi$ off $e$ and takes the value $b$ on $e$.

It is standard that cftp may be used to sample from $\mu$ if $\mu$ is monotone (see [26, 19]), and it is explained in [14] how to adapt this when $\mu$ is anti-monotone. We propose below a mechanism that results in an exact sample from $\mu$ without any assumption of (anti-)monotonicity. This mechanism may be cast in the more general framework of the 'bounding chain' of [17], but, unlike in that work, it makes use of the fact that $\Omega$ is partially ordered. Write $S_\mu = \{\omega \in \Omega : \mu(\omega) > 0\}$, the subset of $\Omega$ on which $\mu$ is strictly positive, and *assume for simplicity that $S_\mu$ is increasing, and that* $1 \in S_\mu$, where 1 (respectively, 0) denotes the configuration of 'all 1s' (respectively, 'all 0s'). This assumption is valid in the current setting, but is not necessary for all that follows.

We start with the usual Gibbs sampler for $\mu$. This is a discrete-time Markov chain $G = (G_n : n \ge 0)$ on the state space $\Omega$ that updates as follows. Suppose $G_n = \xi$. A uniformly distributed member of $E$ is chosen, $e$ say, and also a random variable $U$ with the uniform distribution on $[0,1]$. Then $G_{n+1} = \xi'$ where $\xi'(f) = \xi(f)$ for $f \ne e$, and

$$\xi'(e) = \begin{cases} 0 & \text{if } U > \mu(1_e \mid \xi_e), \\ 1 & \text{if } U \le \mu(1_e \mid \xi_e). \end{cases}$$

The transition rule is well defined whenever $\xi_e^1 \in S_\mu$. It is convenient to use the device of [14] to extend this definition to configurations not in $S_\mu$, and to this end we set

$$\mu(1_e \mid \xi_e) = \max\bigl\{\mu(1_e \mid \psi_e) : \psi_e \ge \xi_e,\ \psi_e^1 \in S_\mu\bigr\} \tag{4.1}$$

when $\xi_e^1 \notin S_\mu$. There is a degree of arbitrariness about this definition, which we follow for consistency with [14].

Let $(e_n, U_n)$ be an independent sequence as above. Let $(A_n, B_n : n \geq 0)$ be a Markov chain with state space $\Omega^2$, and $(A_0, B_0) = (0, 1)$. Suppose $(A_n, B_n) = (\xi, \eta)$ where $\xi \leq \eta$. We set $(A_{n+1}, B_{n+1}) = (\xi', \eta')$ where $\xi'(f) = \xi(f)$, $\eta'(f) = \eta(f)$ for $f \neq e_{n+1}$. At $e = e_{n+1}$ we set

$$\begin{aligned} \xi'(e) = 1 \quad &\text{if and only if} \quad U_{n+1} \leq \alpha, \\ \eta'(e) = 1 \quad &\text{if and only if} \quad U_{n+1} \leq \beta, \end{aligned}$$

where

$$\begin{aligned} \alpha = \alpha(\xi, \eta) &= \min\big\{\mu(1_e \mid \psi_e) : \xi_e \leq \psi_e \leq \eta_e\big\}, \\ \beta = \beta(\xi, \eta) &= \max\big\{\mu(1_e \mid \psi_e) : \xi_e \leq \psi_e \leq \eta_e\big\}. \end{aligned} \tag{4.2}$$

Since $\alpha \leq \beta$, we have that $\xi' \leq \eta'$.

We run the chain $(A, B)$ starting at negative times, in the manner prescribed by cftp, and let $T$ be the coalescence time. More precisely, for $m \geq 0$, let $(A_k(m), B_k(m) : -m \leq k \leq 0)$ denote the chain beginning with $A_{-m}(m) = 0$, $B_{-m}(m) = 1$, using a fixed random sequence $(e_n, U_n)_{-\infty}^0$ for all $m$, and set

$$T = \min\{m \geq 0 : A_0(m) = B_0(m)\},$$

so that $A_0(T) = B_0(T)$.

**Theorem 4.3.** *If $S_\mu$ is increasing and $1 \in S_\mu$, then $P(T < \infty) = 1$, and $A_0(T)$ has law $\mu$.*

*Proof.* We prove only that $P(T < \infty) = 1$. The second part is a standard exercise in cftp, and is easily derived as in [14], Theorem 2.2. By the definition of $S_\mu$ and (4.1), there exists $\eta = \eta(E, \mu) > 0$ such that $\mu(1_e \mid \xi_e) \geq \eta$ for all $e \in E$ and $\xi \in \Omega$. In any given time-interval of length $|E|$, there is a strictly positive probability that the corresponding sequence $(e_i, U_i)$ satisfies $E = \{e_i\}$ and $U_i < \eta$ for all $i$. On this event, the lower process $A$ takes the value 1 after the interval is past, so that coalescence has taken place. The corresponding events for distinct time-intervals are independent, whence the tail of $T$ is no greater than geometric. □

The above recipe is exactly that of [19] when $\mu$ is monotone, and that of [14] when $\mu$ is anti-monotone.

Let $G = (V, E)$ be a finite graph, and $W \subseteq V$ a non-empty set of vertices with $|W|$ even. Let $\mathbf{r} = (r_e : e \in E)$ be a vector of numbers from $(0, 1]$, and let $\phi_{\mathbf{r},q}$ be the random-cluster measure on $G$ with edge-parameters $\mathbf{r}$ and $q \geq 1$. We write $\phi_{\mathbf{r},q}^W$ for $\phi_{\mathbf{r},q}$ conditioned on the event that the open graph is $W$-even, and note that $\phi_{\mathbf{r},q}^W$ is neither monotone nor anti-monotone. The event $S_\mu$ is easily seen to be increasing, and $1 \in S_\mu$. We may therefore apply Theorem 4.3 to the measure $\mu = \phi_{\mathbf{r},q}^W$.

When implementing the above, it is convenient to have a quick way to calculate $\alpha$ and $\beta$ in (4.2). We shall not go into this here, but note that there is no serious difficulty in the case in question.

## 5. Random even subgraphs of planar lattices

In this section, we consider random even subgraphs of the square and hexagonal lattices. We show that properties of the Ising models on these lattices imply properties of the random even graphs. In so doing, we shall review certain known properties of the Ising model, and we include a 'modern' proof of the established fact that the Ising model on the square lattice has a unique Gibbs state at the critical point.

Let $G = (V, E)$ be a planar graph embedded in $\mathbb{R}^2$, with dual graph $G_{\mathrm{d}} = (V_{\mathrm{d}}, E_{\mathrm{d}})$, and write $e_{\mathrm{d}}$ for the dual edge corresponding to the primal edge $e \in E$. [See [11] for an account of planar duality in the context of the random-cluster model.] Let $p \in (0, \frac{1}{2}]$ and let $\omega \in \Omega = \{0,1\}^E$ have law $\phi_{2p,2}$. There is a one–one correspondence between $\Omega$ and $\Omega_{\mathrm{d}} = \{0,1\}^{E_{\mathrm{d}}}$ given by $\omega(e)+\omega_{\mathrm{d}}(e_{\mathrm{d}}) = 1$. It is well known that $\omega_{\mathrm{d}}$ has the law of the random-cluster model on $G_{\mathrm{d}}$ with parameters $(1-2p)/(1-p)$ and 2, see [11] for example. For $A \subseteq V$, the *boundary* of $A$ is given by $\partial A = \{v \in A : v \sim w \text{ for some } w \notin A\}$. For $\omega \in \Omega$, let $f_0, f_1, \dots, f_c$ be the faces of $(V, \eta(\omega))$, with $f_0$ the infinite face. These faces are in one–one correspondence with the clusters of $(V_{\mathrm{d}}, \eta(\omega_{\mathrm{d}}))$, which we thus denote by $K_0, K_1, \dots, K_c$, and the boundaries of the finite faces form a basis of $\mathcal{E} = \mathcal{E}(V, \eta(\omega))$. More precisely, the boundary of each finite face $f_i$ consists of an 'outer boundary' and zero, one or several 'inner boundaries'; each of these parts is a cycle (and two parts may have up to one vertex in common). If we orient the outer boundary cycle counter-clockwise (positive) and the inner boundary cycles clockwise (negative), the face will always be on the left side along the boundaries, and the winding numbers of the boundary cycles sum up to 1 at every point inside the face and to 0 outside the face. It is easy to see that the outer boundary cycles form a maximal family of independent cycles of $(V, \eta(\omega))$, and thus a basis of $\mathcal{E}$; another basis is obtained by the complete boundaries $C_i$ of the finite faces. We use the latter basis, and select a random subset of the basis by randomly assigning (by fair coin tosses) $+$ and $-$ to each cluster in the dual graph $(V_{\mathrm{d}}, \eta(\omega_{\mathrm{d}}))$, or equivalently to each face $f_i$ of $(V, \eta(\omega))$. We then select the boundaries $C_i$ of the finite faces $f_i$ that have been given a sign different from the sign of the infinite face $f_0$. The union (modulo 2) of the selected boundaries is by Proposition 2.2 and Theorem 3.1 a random even subgraph of $G$ with parameter $p$. On the other hand, this union is exactly the dual boundary of the $+$ clusters of $G_{\mathrm{d}}$, that is, the set of open edges $e \in E$ with the property that one endpoint of the corresponding dual edge $e_{\mathrm{d}}$ is labelled $+$ and the other is labelled $-$. [Such an edge $e_{\mathrm{d}}$ is called a $+/-$ edge.]

It is standard that the $+/-$ configuration on $G_{\mathrm{d}}$ is distributed as the Ising model on $G_{\mathrm{d}}$ with parameter $\beta$ satisfying

$$\frac{1-2p}{1-p} = 1 - e^{-2\beta} = \frac{2\tanh\beta}{1+\tanh\beta}. \tag{5.1}$$

In summary, we have the following.

**Theorem 5.2.** *Let $G$ be a finite planar graph with dual $G_{\mathrm{d}}$. A random even subgraph of $G$ with parameter $p \in (0, \frac{1}{2}]$ is dual to the $+/-$ edges of the Ising model on $G_{\mathrm{d}}$ with $\beta$ satisfying* (5.1).

Much is known about the Ising model on subsets of two-dimensional lattices, and the above fact permits an analysis of random even subgraphs of their dual lattices.

Consider the case when $G$ is a box in the square lattice $\mathbb{Z}^2$. That is, $G = G_{m,n}$ is the subgraph of $\mathbb{Z}^2$ induced by the vertex-set $[-m, m] \times [-n, n]$, where $m, n \in \mathbb{Z}_+$ and $[a, b]$ is to be interpreted as $[a, b] \cap \mathbb{Z}$. It is a mild inconvenience that $G_{m,n}$ is not an even graph, and we adjust the 'boundary' to rectify this. For definiteness, we consider the so-called 'wired boundary condition' on $G_{m,n}$, which is to say that we consider the random-cluster measure on the graph $G^{\mathrm{w}}_{m,n}$ obtained from $G_{m,n}$ by identifying as one the set of vertices lying in its boundary $\partial G_{m,n}$.

The Ising model on $\mathbb{Z}^2$ with parameter $\beta$ is *critical* when $e^{2\beta} = 1 + \sqrt{2}$, or equivalently when the above random-cluster model on the dual lattice has parameter satisfying

$$\frac{1-2p}{1-p} = \frac{\sqrt{2}}{1+\sqrt{2}} = 2 - \sqrt{2},$$

i.e., $p = p_{\mathrm{c}}$ where

$$p_{\mathrm{c}} = \frac{1}{2+\sqrt{2}} = 1 - \frac{1}{\sqrt{2}}. \tag{5.3}$$

The Ising model has been studied extensively in the physics literature, and physicists have a detailed knowledge of the two-dimensional case particularly. There is a host of 'exact calculations', rigorous proofs of which can present challenges to mathematicians, see [3, 18].

We shall use the established facts stated in the following theorem. The continuity of the magnetization at the critical point contributes to the proof that the boundary of a large spin-cluster of the critical Ising model converges weakly to the Schramm–Löwner curve $\mathrm{SLE}_3$, see [22, 23].

**Theorem 5.4.** *The critical value of the Ising model on the square lattice is $\beta_{\mathrm{c}} = \beta_{\mathrm{sd}}$ where $\beta_{\mathrm{sd}} = \frac{1}{2}\log(1+\sqrt{2})$ is the 'self-dual point'. The magnetization (and therefore the corresponding random-cluster percolation-probability also) is a continuous function of $\beta$ on $[0, \infty)$.*

We note the corollary that the wired and free random-cluster measures on $\mathbb{Z}^2$ are identical for $p \in [0,1]$; see [11], Thms 5.33 and 6.17.

*Proof.* These facts are 'classical' and have received much attention, see [18] for example; they may be proved as follows using 'modern' arguments. Recall first that the magnetization equals the percolation probability of the corresponding wired random-cluster model, and the two-point correlation function of the Ising model equals the two-point connectivity function of the random-cluster model (see [11]). We have that $\beta_{\mathrm{sd}} \le \beta_{\mathrm{c}}$, by Theorem 6.17(a) of [11] or otherwise, and similarly the random-cluster model with free boundary conditions has percolation-probability 0 whenever either $\beta \le \beta_{\mathrm{sd}}$ or $\beta < \beta_{\mathrm{c}}$. One may deduce by a coupling argument that the Ising model with $\beta < \beta_{\mathrm{c}}$ has a unique Gibbs state which we denote by $\pi_\beta$.

By the results of [2, 20], the two-point correlation function $\pi_\beta(\sigma_x\sigma_y)$ of the spins at $x$ and $y$ decays exponentially as $|x-y| \to \infty$ when $\beta < \beta_{\mathrm{c}}$, and it follows by the final statement of [12] or otherwise that $\beta_{\mathrm{c}} = \beta_{\mathrm{sd}}$. The continuity of the magnetization at $\beta \ne \beta_{\mathrm{c}}$ is standard, see for example [11], Theorems 5.16 and 6.17(b). When $\beta = \beta_{\mathrm{c}}$, it suffices to show that the $\pm$ boundary-condition Gibbs states $\pi^{\pm}_{\beta_{\mathrm{c}}}$ and the free boundary-condition Gibbs state $\pi^0_{\beta_{\mathrm{c}}}$ satisfy $\pi^+_{\beta_{\mathrm{c}}} = \pi^-_{\beta_{\mathrm{c}}} = \pi^0_{\beta_{\mathrm{c}}}$. Suppose this does not hold, so that $\pi^+_{\beta_{\mathrm{c}}} \ne \pi^-_{\beta_{\mathrm{c}}} \ne \pi^0_{\beta_{\mathrm{c}}}$. By the random-cluster representation or otherwise, the two-point correlation functions $\pi^{\pm}_{\beta_{\mathrm{c}}}(\sigma_x\sigma_y)$ are bounded away from 0 for all pairs $x$, $y$ of vertices. By the main result of [1, 15] (see also [9]) and the symmetry of $\pi^0_{\beta_{\mathrm{c}}}$, we have that $\pi^0_{\beta_{\mathrm{c}}} = \frac{1}{2}\pi^+_{\beta_{\mathrm{c}}} + \frac{1}{2}\pi^-_{\beta_{\mathrm{c}}}$, whence $\pi^0_{\beta_{\mathrm{c}}}(\sigma_x\sigma_y)$ is bounded away from 0. By [11], Theorem 5.17, this contradicts the above remark that the percolation-probability of the free-boundary condition random-cluster measure is 0 at $\beta = \beta_{\mathrm{sd}} = \beta_{\mathrm{c}}$. □

We consider now the so-called thermodynamic limit of the random even graph on $G^{\mathrm{w}}_{m,n}$ as $m, n \to \infty$. It is long established that the (free boundary-condition) Ising measure on $G_{m,n}$ converges weakly (in the product topology) to an infinite-volume limit measure denoted $\pi_\beta$. This may be seen as follows using the theory of the corresponding random-cluster model on $\mathbb{Z}^2$ (see [11]). When $\beta \le \beta_{\mathrm{c}}$, the existence of the limit follows more or less as discussed above, using the coupling with the random-cluster measure, and the fact that the percolation probability of the latter measure is 0 whenever $\beta \le \beta_{\mathrm{c}}$. We write $\pi_\beta$ for the limit Ising measure as $m, n \to \infty$.

The thermodynamic limit is slightly more subtle when $\beta > \beta_{\mathrm{c}}$, since the infinite-volume Ising model has a multiplicity of Gibbs states in this case. The (wired) random-cluster measure on $G^{\mathrm{w}}_{m,n}$ converges to the wired limit measure. By the uniqueness of infinite-volume random-cluster measures, the limit Ising measure is obtained by allocating random spins to the clusters of the infinite-volume random-cluster model (see Section 4.6 of [11]). Once

again, we write $\pi_\beta$ for the ensuing measure on $\{-1,+1\}^{\mathbb{Z}^2}$, and we note that $\pi_\beta = \frac{1}{2}\pi_\beta^+ + \frac{1}{2}\pi_\beta^-$ where $\pi_\beta^\pm$ denotes the infinite-volume Ising measure with $\pm$ boundary conditions.

It has been shown in [6] (see also [8], Corollary 8.4) that there exists (with strictly positive $\pi_\beta$-probability) an infinite spin-cluster in the Ising model if and only if $\beta > \beta_c$. More precisely:

(a) if $\beta \le \beta_c$, there is $\pi_\beta$-probability 1 that all spin-clusters are finite,
(b) if $\beta > \beta_c$, there is $\pi_\beta$-probability 1 that there exists a unique infinite spin-cluster, which is equally likely to be a $+$ cluster as a $-$ cluster. Furthermore, by the main theorem of [7] or otherwise, for any given finite set $S$ of vertices, the infinite spin-cluster contains, $\pi_\beta$-a.s., a cycle containing $S$ in its interior.

On passing to the dual graph, one finds that the random even subgraph of $G^{\mathrm{w}}_{m,n}$ with parameter $p \in (0, \frac{1}{2}]$ converges weakly as $m, n \to \infty$ to a probability measure $\rho_p$ that is concentrated on even subgraphs of $\mathbb{Z}^2$ and satisfies:

(a′) if $p \ge p_c$, there is $\rho_p$-probability 1 that all faces of the graph are bounded,
(b′) if $p < p_c$, there is $\rho_p$-probability 1 that the graph is the vertex-disjoint union of finite clusters.

(Note that (5.1) defines $\beta$ as a decreasing function of $p$, so the order relations are reversed.)

We have thus obtained a description of the weak-limit measure $\rho_p$ when $p \le \frac{1}{2}$, and we note the phase transition at the parameter-value $p = p_c$. When $p > \frac{1}{2}$, a random even subgraph of $G^{\mathrm{w}}_{m,n}$ is the complement of a random even subgraph with parameter $1-p$. [It is a convenience at this point that $G^{\mathrm{w}}_{m,n}$ is an even graph.] Hence the weak-limit measure $\rho_p$ exists for all $p \in [0,1]$ and gives meaning to the expression "a random even subgraph on $\mathbb{Z}^2$ with parameter $p$". [It is easily verified that $\rho_{\frac{1}{2}}$ equals the measure defined in Theorem 2.4 for $\mathbb{Z}^2$, and thus describes a uniform random even subgraph of $\mathbb{Z}^2$.] There is a sense in which the random even subgraph on $\mathbb{Z}^2$ has two points of phase transition, corresponding to the values $p_c$ and $1-p_c$.

We consider finally the question of the size of a typical face of the random even graph on $\mathbb{Z}^2$ when $p_c \le p \le \frac{1}{2}$. When $p > p_c$, this amounts to asking about the size of a (sub)critical Ising spin-cluster. Higuchi [16] has proved an exponential upper bound for the radius of such a cluster, and this has been extended by van den Berg [4] to the cluster-volume. Thus, the law of the area of a typical face has an exponential tail.

The picture is quite different when the square lattice is replaced by the hexagonal lattice $\mathbb{H}$. Any even subgraph of $\mathbb{H}$ has vertex degrees 0 and/or 2, and thus comprises a vertex-disjoint union of cycles, doubly infinite paths, and isolated vertices. The (dual) Ising model inhabits the (Whitney) dual lattice

of $\mathbb{H}$, namely the triangular lattice $\mathbb{T}$. Once again there exists a critical point $p_c = p_c(\mathbb{T}) < \frac{1}{2}$ such that the random even subgraph of $\mathbb{H}$ satisfies (a′) and (b′) above. In particular, the random even subgraph has a.s. only cycles and isolated vertices but no infinite paths. Recall that site percolation on $\mathbb{T}$ has critical value $\frac{1}{2}$. Therefore, for $p = \frac{1}{2}$, the face $F_x$ of the random even subgraph containing the dual vertex $x$ corresponds to a *critical* percolation cluster. It follows that its volume and radius have polynomially decaying tails, and that the boundary of $F_x$, when conditioned to be increasingly large, approaches $\mathrm{SLE}_6$. See [21, 23] and [5, 25]. The spin-clusters of the Ising model on $\mathbb{T}$ are 'critical' (in a certain sense described below) for all $p \in (p_c(\mathbb{T}), \frac{1}{2}]$, and this suggests the possibility that the boundary of $F_x$, when conditioned to be increasingly large, approaches $\mathrm{SLE}_6$ for any such $p$. This is supported by the belief in the physics community that the so-called universality class of the spin-clusters of the subcritical Ising model on $\mathbb{T}$ is the same as that of critical percolation.

The 'criticality' of such Ising spin-clusters (mentioned above) may be obtained as follows. Note first that, since $\beta < \beta_c$, there is a unique Gibbs state $\pi_\beta$ for the Ising model. Therefore, $\pi_\beta$ is invariant under the interchange of spin-values $-1 \leftrightarrow +1$. Let $R_n$ be a rhombus of the lattice with side-lengths $n$ and axes parallel to the horizontal and one of the diagonal lattice directions, and consider the event $A_n$ that $R_n$ is traversed from left to right by a $+$ path (i.e., a path $\nu$ satisfying $\sigma_y = +1$ for all $y \in \nu$). It is easily seen that the complement of $A_n$ is the event that $R_n$ is crossed from top to bottom by a $-$ path (see Lemma 11.21 of [10] for the analogous case of bond percolation on the square lattice). Therefore,

$$\pi_\beta(A_n) = \tfrac{1}{2}, \qquad 0 \le \beta < \beta_c. \tag{5.5}$$

For $x \in \mathbb{Z}^2$, let $S_x$ denote the spin-cluster containing $x$, and define

$$\mathrm{rad}(S_x) = \max\{|z - x| : z \in S\},$$

where $|y|$ is the graph-theoretic distance from 0 to $y$. By (5.5), there exists a vertex $x$ such that $\pi_\beta(\mathrm{rad}(S_x) \ge n) \ge (2n)^{-1}$. By the translation-invariance of $\pi_\beta$,

$$\pi_\beta\big(\mathrm{rad}(S_0) \ge n\big) \ge \frac{1}{2n}, \qquad 0 \le \beta < \beta_c, \tag{5.6}$$

where 0 denotes the origin of the lattice. The left side of (5.6) tends to 0 as $n \to \infty$, and the polynomial lower bound is an indicator of the criticality of the model.

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## References

[1] M. Aizenman. Translation invariance and instability of phase coexistence in the two dimensional Ising system. *Communications in Mathematical Physics*, 73:83–94, 1980.

[2] M. Aizenman, D. J. Barsky, and R. Fernández. The phase transition in a general class of Ising-type models is sharp. *Journal of Statistical Physics*, 47:343–374, 1987.

[3] R. J. Baxter. *Exactly Solved Models in Statistical Mechanics*. Academic Press, London, 1982.

[4] J. van den Berg. Approximate zero–one laws and sharpness of the percolation transition in a class of models including two-dimensional Ising percolation. *Annals of Probability*, 36:1880–1903, 2008.

[5] F. Camia and C. M. Newman. Two-dimensional critical percolation: the full scaling limit. *Communications in Mathematical Physics*, 238:1–38, 2006.

[6] A. Coniglio, C. R. Nappi, F. Peruggi, and L. Russo. Percolation and phase transitions in the Ising model. *Communications in Mathematical Physics*, 51:315–323, 1976.

[7] A. Gandolfi, M. Keane, and L. Russo. On the uniqueness of the occupied cluster in dependent two-dimensional site percolation. *Annals of Probability*, 16:1147–1157, 1988.

[8] H.-O. Georgii, O. Häggström, and C. Maes. The random geometry of equilibrium phases. In C. Domb, J. L. Lebowitz, editor, *Phase Transitions and Critical Phenomena*, volume 18, pages 1–142. Academic Press, London, 2000.

[9] H.-O. Georgii and Y. Higuchi. Percolation and number of phases in the two-dimensional Ising model. *Journal of Mathematical Physics*, 41:1153–1169, 2000.

[10] G. R. Grimmett. *Percolation*. Springer, Berlin, 2nd edition, 1999.

[11] G. R. Grimmett. *The Random-Cluster Model*. Springer, Berlin, 2006.

[12] G. R. Grimmett and B. T. Graham. Influence and sharp threshold theorems for monotonic measures. *Annals of Probability*, 34:1726–1745, 2006.

[13] G. R. Grimmett and S. Janson. Random graphs with forbidden vertex degrees. 2007. arxiv:0712.1270.

[14] O. Häggström and K. Nelander. Exact sampling from anti-monotone systems. *Statistica Neerlandica*, 52:360–380, 1998.

[15] Y. Higuchi. On the absence of non-translation invariant Gibbs states for the two-dimensional Ising model. In D. Szász J. Fritz, J. L. Lebowitz, editor, *Random Fields, Esztergom (Hungary)*, volume I, pages 517–534. North-Holland, Amsterdam, 1979.

[16] Y. Higuchi. A sharp transition for the two-dimensional Ising percolation. *Probability Theory and Related Fields*, 97:489–514, 1993.

[17] M. Huber. Exact sampling and approximate counting techniques. In *Proceedings of 30th ACM Symposium on the Theory of Computing*, volume 52, pages 31–40. 1998.

[18] B. M. McCoy and T. T. Wu. *The Two-Dimensional Ising Model*. Harvard University Press, Cambridge, Mass., 1973.

[19] J. G. Propp and D. B. Wilson. Exact sampling with coupled Markov chains and applications to statistical mechanics. *Random Structures & Algorithms*, 9:223–252, 1996.

[20] B. Simon. Correlation inequalities and the decay of correlations in ferromagnets. *Communications in Mathematical Physics*, 77:111–126, 1980.

[21] S. Smirnov. Critical percolation in the plane: conformal invariance, Cardy's formula, scaling limits. *Comptes Rendus des Séances de l'Académie des Sciences, Série I, Mathématiques*, 333:239–244, 2001.

[22] S. Smirnov. Conformal invariance in random-cluster models. I. Holomorphic fermions in the Ising model. 2007. arxiv:0708.0039.

[23] S. Smirnov. Towards conformal invariance of 2D lattice models. In *Proceedings of the International Congress of Mathematicians, Madrid, 2006*, volume II, pages 1421–1452. European Mathematical Society, Zurich, 2007.

[24] A. Sokal. The multivariate Tutte polynomial (alias Potts model) for graphs and matroids. In B. S. Webb, editor, *Surveys in Combinatorics, 2005*, volume 327 of *London Mathematical Society Lecture Notes Series*, pages 173–226. Cambridge University Press, 2005.

[25] W. Werner. Lectures on two-dimensional critical percolation. 2007. arxiv:0710.0856.

[26] D. B. Wilson. Layered multishift coupling for use in perfect sampling algorithms (with a primer on CFTP). In N. Madras, editor, *Monte Carlo Methods*, volume 26 of *Fields Institute Communications*, pages 141–176. American Mathematical Society, 2000.

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