STANDARD REPRESENTATION OF MULTIVARIATE FUNCTIONS ON A GENERAL PROBABILITY SPACE

SVANTE JANSON

Abstract. It is well-known that a random variable, i.e. a function defined on a probability space, with values in a Borel space, can be represented on the special probability space consisting of the unit interval with Lebesgue measure. We show an extension of this to multivariate functions. This is motivated by some recent constructions of random graphs.

A fundamental feature of probability theory is that random variables are defined as functions on a probability space $(\Omega, \mathcal{F}, P)$, but the actual probability space has very little importance, and often no importance at all. For example, we have the following well-known basic result, showing that as long as we consider a single random variable, we may assume that the probability space is $([0,1], \mathcal{B}, dx)$, the unit interval with the Borel σ-field and Lebesgue measure.

Theorem 1. *If $f : (\Omega, \mathcal{F}, P) \to \mathbb{R}$ is any random variable, then there exists a random variable $\tilde{f} : ([0,1], \mathcal{B}, dx) \to \mathbb{R}$ such that f and $\tilde{f}$ have the same distribution.*

A standard construction of $\tilde{f}$ is to take the right-continuous inverse of the distribution function $\mathbb{P}(f \le x)$, $\tilde{f}(x) := \inf\{y : \mathbb{P}(f \le y) > x\}$.

Moreover, it is also well-known that Theorem 1 extends to vector-valued random variables and random variables with values in more general spaces. To state a precise result, we recall that a *Borel space* [9, Appendix A1], also called Lusin space [6, III.16, III.20(b)], is a measurable space that is isomorphic to a Borel subset of [0,1]. Every Polish space (a complete separable metric space) with its Borel σ-field is a Borel space [9, Appendix A1], [6, III.17]. (For example, this includes $\mathbb{R}^n$, which gives the extension to vector valued random variables.)

Theorem 2. *If $f : (\Omega, \mathcal{F}, P) \to S$ is any random variable with values in a Borel space S, then there exists a random variable $\tilde{f} : ([0,1], \mathcal{B}, dx) \to S$ such that f and $\tilde{f}$ have the same distribution.*

Proof. This is an almost trivial extension of Theorem 1: By assumption, there exists an isomorphism $\varphi : S \to E$, where $E \subseteq [0,1]$ is a Borel set. This means that φ is a bijection, and that φ and $\varphi^{-1} : E \to S$ are measurable.

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Extend φ^{-1} to $\mathbb{R}$ by defining $\varphi^{-1}(x) = s_0$ when $x \notin E$ for some arbitrary $s_o \in S$. Then, $g := \varphi \circ f : (\Omega, \mathcal{F}, P) \to E \subset \mathbb{R}$ is a random variable, and Theorem 1 gives $\tilde{g} : ([0,1], \mathcal{B}, dx) \to \mathbb{R}$ with $\tilde{g} \overset{\mathrm{d}}{=} g$. Let $\tilde{f} := \varphi^{-1} \circ \tilde{g} : ([0,1], \mathcal{B}, dx) \to S$. $\square$

Remark 3. In fact, every uncountable Borel space T is isomorphic to [0,1] [6, Appendix III.80]. If μ is any continuous probability measure (i.e., a probability measure such that every point has measure 0) on a Borel space T, then T necessarily is uncountable, and thus there is an isomorphism $\alpha : T \to [0,1]$ which maps μ to some continuous Borel measure ν on [0,1], and hence α is an isomorphism $(T, \mu) \to ([0,1], \nu)$. Further, if we let $F_\nu(x) := \nu[0,x]$, then $F_\nu : [0,1] \to [0,1]$ is a continuous function that maps ν to the Lebesgue measure dx. Hence, in Theorem 1, $\tilde{f} \circ F_\nu \circ \alpha : (T, \mu) \to \mathbb{R}$ has $\tilde{f} \circ F_\nu \circ \alpha \overset{\mathrm{d}}{=} \tilde{f} \overset{\mathrm{d}}{=} f$, and similarly for Theorem 2. Consequently, we can replace $([0,1], \mathcal{B}, dx)$ by (T, μ) in Theorems 1 and 2, for any Borel space T with a continuous probability measure μ.

The purpose of this note is to give an elementary proof of an extension of Theorem 1 to functions $f : \Omega^m \to \mathbb{R}$ or $f : \Omega^m \to S$ of several variables, where $m \geq 1$. This is a folklore result, but we do not know any reference, so we provide a detailed statement and proof. Our interest in this problem comes from the following example, with a construction of random graphs used by, for example, Lovász and Szegedy [11] and Bollobás, Janson and Riordan [2] (in different contexts), see further Borgs, Chayes, Lovász, Sós and Vesztergombi [4, 5]; Diaconis and Janson [7] and further references in these papers, and an extension of this example studied in Bollobás, Janson and Riordan [3], where also functions $f : \Omega^m \to [0,1]$ with $m > 2$ are used. We use the notation $[n] := \{1, \dots, n\}$ if $n < \infty$ and $[\infty] := \mathbb{N} := \{1, 2, \dots\}$.

Example 4. Let $f : \Omega^2 \to [0,1]$ be a symmetric measurable function and let $1 \leq n \leq \infty$. Define a random graph on the vertex set $[n]$ by first taking n i.i.d. random variables $(X_i)_1^n$ in Ω with distribution P, and then letting, conditioned on these random variables, the edges ij with $i < j$ appear independently, with the probability of an edge ij equal to $f(X_i, X_j)$.

If we choose another probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ and a symmetric function $\tilde{f} : \tilde{\Omega}^2 \to [0,1]$, we obtain the same distribution of the corresponding two random graphs if the joint distribution of the families $(f(X_i, X_j))_{i<j}$ and $(\tilde{f}(\tilde{X}_i, \tilde{X}_j))_{i<j}$ are equal, where $\tilde{X}_i$ has the distribution $\tilde{P}$ on $\tilde{\Omega}$. This motivates the following definition, where $\mathfrak{S}_\infty$ denotes the set of all permutations of $\mathbb{N}$. (For notational convenience we write $f(X_1, \dots, X_m)$ also when $m = \infty$; this should be interpreted as $f(X_1, X_2, \dots)$.)

Definition. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $f_\alpha : \Omega^{m_\alpha} \to S_\alpha$, $\alpha \in A$, be a collection of measurable functions, where A is any index set,

$m_\alpha \in \mathbb{N}^* := \mathbb{N} \cup \{\infty\}$ are positive integers or ∞, and S_α are measurable spaces. We say that the family $(f_\alpha)_\alpha$ can be *represented on a probability space* $(\tilde\Omega, \tilde{\mathcal{F}}, \tilde P)$ if there exists a collection of measurable functions $\tilde f_\alpha : \tilde\Omega^{m_\alpha} \to S_\alpha$, $\alpha \in A$, such that if $X_1, X_2, \dots$ are i.i.d. random elements of Ω with distribution P, and if $\tilde X_1, \tilde X_2, \dots$ are i.i.d. random elements of $\tilde\Omega$ with distribution $\tilde P$, then the collections $\bigl\{f_\alpha(X_{\sigma(1)}, \dots, X_{\sigma(m_\alpha)}) : \alpha \in A,\ \sigma \in \mathfrak{S}_\infty\bigr\}$ and $\bigl\{\tilde f_\alpha(\tilde X_{\sigma(1)}, \dots, \tilde X_{\sigma(m_\alpha)}) : \alpha \in A,\ \sigma \in \mathfrak{S}_\infty\bigr\}$ have the same distribution.

Equivalently, if we regard each f_α as being defined on Ω^∞ by ignoring all but the m_α first coordinates, and let $\mathfrak{S}_\infty$ act on Ω^∞ in the natural way, then the collections $\bigl\{f_\alpha \circ \sigma : \alpha \in A,\ \sigma \in \mathfrak{S}_\infty\bigr\}$ and $\bigl\{\tilde f_\alpha \circ \sigma : \alpha \in A,\ \sigma \in \mathfrak{S}_\infty\bigr\}$ of random variables defined of Ω^∞ and $\tilde\Omega^\infty$, respectively, have the same distribution.

If we have only a single function $f : \Omega^m \to \mathcal{S}$, it is further equivalent that the collections $f(X_{i_1}, \dots, X_{i_m})$ and $\tilde f(\tilde X_{i_1}, \dots, \tilde X_{i_m})$, indexed by sequences $(i_1, \dots, i_m)$ of distinct integers in $\mathbb{N}$, have the same distribution. In principle, the definition could be stated in this way for families $\{f_\alpha\}$ and $\{\tilde f_\alpha\}$ too, with in general varying (and possibly unbounded) m_α, but we leave it to the reader to find a nice formulation in this form.

Remark 5. We may equip Ω^{m_α} either with the product σ-field $\mathcal{F}^{m_\alpha}$ or with its completion. This does not matter, because if f_α is measurable with respect to the completed σ-field, it can be modified on a set of (product) measure 0 such that it becomes measurable with respect to $\mathcal{F}^{m_\alpha}$, and this does not change any of the distributions above. Similarly, we may assume that $\tilde f_\alpha$ is measurable with respect to $\tilde{\mathcal{F}}^{m_\alpha}$.

Remark 6. As is well-known, the representing functions $\tilde f_\alpha$ are in general far from unique. See, for example, [10, Theorem 7.28] for a general result on equivalence of representations.

The main result is that every countable collection of functions on any probability space, with values in a Borel space, can be represented on $([0,1], \mathcal{B}, dx)$.

Theorem 7. *Let $f_i : \Omega^{m_i} \to S_i$, $i = 1, 2, \dots$, be a finite or countable family of measurable functions, where $m_i \in \mathbb{N}^*$ and S_i are Borel spaces. Then $(f_i)_i$ can be represented on $([0,1], \mathcal{B}, dx)$.*

Moreover, if every f_i is symmetric, then we can choose the representing functions on $([0,1], \mathcal{B}, dx)$ to be symmetric too.

Proof. First, by the argument in the proof of Theorem 2, it suffices to consider the case $S_i = \mathbb{R}$ for all i. Further, we may by Remark 5 assume that each f_i is measurable with respect to $\mathcal{F}^{m_\alpha}$, without completion.

Let $\mathcal{A} := \{\mathcal{C} \subseteq \mathcal{F} : |\mathcal{C}| \le \aleph_0\}$ be the collection of all countable families of measurable sets in Ω. Since a countable union $\bigcup_j \mathcal{C}_j$ of families $\mathcal{C}_j \in \mathcal{A}$

also belongs to $\mathcal{A}$, it is easily seen that $\bigcup_{\mathcal{C}\in\mathcal{A}} \sigma(\mathcal{C})^m$ is a σ-field on Ω^m, for every $m \le \infty$; since further, by the definition of product σ-fields, every set in $\mathcal{F}^m$ belongs to a sub-σ-field generated by a countable number of cylinder sets, and thus to some $\sigma(\mathcal{C})^m$, it follows that $\mathcal{F}^m = \bigcup_{\mathcal{C}\in\mathcal{A}} \sigma(\mathcal{C})^m$. Moreover, every f_i is measurable with respect to the σ-field generated by the countable family $\{\omega' \in \Omega^{m_\alpha} : f_i(\omega') < r\}$, $r \in \mathbb{Q}$, and each of these thus belongs to the product σ-field $\mathcal{F}_{i,r}^{m_\alpha}$ defined by some countably generated sub-σ-field $\mathcal{F}_{i,r}$ of $\mathcal{F}$. Consequently, there exists a countably generated sub-σ-field $\mathcal{F}_0$ of $\mathcal{F}$ such that each f_i is $\mathcal{F}_0^{m_i}$ measurable.

Next, let $A_1, A_2, \dots$ be a sequence of subsets of Ω that generate $\mathcal{F}_0$. Let $h : \Omega \to \mathcal{D} := \{0,1\}^\infty$ be defined by $h(x) = (\mathbf{1}[x \in A_i])_i$. ($\mathcal{D}$ is, topologically, the Cantor set.) Then $\mathcal{F}_0$ equals the σ-field generated on Ω by h, and thus $\mathcal{F}_0^m$ equals the σ-field on Ω^m generated by $h^m := (h, \dots, h) : \Omega^m \to \mathcal{D}^m$. Since f_i is $\mathcal{F}_0^{m_i}$-measurable, it follows that $f_i = g_i \circ h^{m_i}$ for some measurable function $g_i : \mathcal{D}^{m_i} \to \mathbb{R}$. If we let $\tilde{\Omega} = \mathcal{D}$, let $\tilde{\mathcal{F}}$ be the Borel σ-field on $\mathcal{D}$, and let $\tilde{P}$ be the probability measure induced by h, it follows immediately that the family $(g_i)_i$ gives a representation of $(f_i)_i$ on $(\mathcal{D}, \tilde{P})$.

Applying Theorem 2 to the identity function $\iota : (\mathcal{D}, \tilde{P}) \to (\mathcal{D}, \tilde{P})$, we see that there exists a function $\tilde{\iota} : [0,1] \to \mathcal{D}$ mapping the Lebesgue measure dx to $\tilde{P}$. Consequently, the mappings $g_i \circ \tilde{\iota}^{m_i} \to \mathbb{R}$ represent $(f_i)_i$ on $([0,1], \mathcal{B}, dx)$.

It is clear that g_i is symmetric if every f_i is. $\square$

The symmetry statement in Theorem 7 can be extended to include partial symmetries, antisymmetries, ...; we leave this to the reader.

Remark 8. Theorem 7 is related to the theory for exchangeable arrays by Aldous [1] and Hoover [8], see Kallenberg [10, Chapter 7] for a detailed exposition. Consider for simplicity a single function $f : \Omega^2 \to \mathcal{S}$. It is obvious that the random variables $f(X_i, X_j)$, $i \ne j$, form an exchangeable family. Conversely, the theorem by Aldous–Hoover [10, Theorem 7.22] says that every exchangeable family can be represented on $([0,1], \mathcal{B}, dx)$, but in the more general form $g(X_\emptyset, X_i, X_j, X_{ij})$. It seems likely that a further study of the representations can lead to a proof that we may in this case take g independent of $X_\emptyset$ and X_{ij}, which would yield Theorem 7 in this case. (And presumably this could be extended to the general case.) Nevertheless, such a proof would be highly technical, and hardly shorter than the elementary proof given above, so we have not pursued this.

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Department of Mathematics, Uppsala University, PO Box 480, SE-751 06 Uppsala, Sweden

E-mail address: `svante.janson@math.uu.se`

URL: `http://www.math.uu.se/~svante/`