# GENERALIZED STIRLING PERMUTATIONS, FAMILIES OF INCREASING TREES AND URN MODELS

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ABSTRACT. Bona [6] studied the distribution of ascents, plateaux and descents in the class of Stirling permutations, introduced by Gessel and Stanley [14]. Recently, Janson [18] showed the connection between Stirling permutations and plane recursive trees and proved a joint normal law for the parameters considered by Bona. Here we will consider generalized Stirling permutations extending the earlier results of [6], [18], and relate them with certain families of generalized plane recursive trees, and also $(k+1)$-ary increasing trees. We also give two different bijections between certain families of increasing trees, which both give as a special case a bijection between ternary increasing trees and plane recursive trees. In order to describe the (asymptotic) behaviour of the parameters of interests, we study three (generalized) Pólya urn models using various methods.

## 1. INTRODUCTION

Stirling permutations were defined by Gessel and Stanley [14]. A Stirling permutation is a permutation of the multiset $\{1, 1, 2, 2, \dots, n, n\}$ such that for each $i$, $1 \le i \le n$, the elements occuring between the two occurences of $i$ are larger than $i$. The name of these combinatorial objects is due to relations with the Stirling numbers, see [14] for details.

Let $\sigma = a_1 a_2 \cdots a_{2n}$ be a Stirling permutation. Let the index $i$ (or the gap $(i, i+1)$) be called an ascent of $\sigma$ if $i = 0$ or $a_i < a_{i+1}$, let $i$ be called a descent of $\sigma$ if $i = 2n$ or $a_i > a_{i+1}$, and let $i$ be called a plateau of $\sigma$ if $a_i = a_{i+1}$. (It is convenient to define $a_0 = a_{2n+1} = 0$; this takes care of the special cases $i = 0$ and $i = 2n$.) Note that $i$ runs from $0$ to $2n$, so the total number of ascents, descents and plateaux is $2n+1$. Let $\mathcal{Q}_n$ denote the set of Stirling permutation of $\{1, 1, 2, 2, \dots, n, n\}$; we say that these have order $n$. Bona [6] showed that the parameters numbers of ascents, descents and plateaux are equidistributed on $\mathcal{Q}_n$. Moreover, he showed a central limit theorem for the three parameters.

A rooted tree of order $n$ with the vertices labelled $1, 2, \dots, n$, is an increasing tree if the node labelled 1 is distinguished as the root, and for each $2 \le k \le n$, the labels of the nodes in the unique path from the root to the node labelled $k$ form an increasing sequence. We will consider several families of increasing trees. The first one is the family of *increasing*

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*plane trees*, usually called *plane recursive trees*, where the children of a node are ordered (from left to right, say). Note that plane recursive trees also appear in literature under the names plane-oriented recursive trees, heap-ordered trees, and sometimes also as scale-free trees. Further families will be defined later.

Let $\mathcal{T}_n$ denote the set of plane recursive trees with $n$ vertices. It was shown by Janson [18] that plane recursive trees on $n+1$ vertices are in bijection with Stirling permutations on $\{1,1,2,2,\dots,n,n\}$, $\mathcal{T}_{n+1} \cong \mathcal{Q}_n$. Moreover, using this bijective correspondence, he showed that the number of descents in the Stirling permutation corresponds to the number of leaves in the associated plane recursive tree. Furthermore, using an urn model and general theorems, see [15] and also [16], Janson showed the joint normality of the parameters ascent, descent and plateau. The purpose of this work is to extend this connection between Stirling permutations and plane recursive trees in Janson [18], to generalized Stirling permutations. In particular, we give a bijection between Stirling permutations on $\{1^k, 2^k, \dots, n^k\}$, where here and throughout this work $1^l := \underbrace{1,\dots,1}_{l}$, with $l \geq 1$, which we call $k$-Stirling permutations, and $(k+1)$-ary increasing trees; moreover we can also relate $k$-Stirling permutations with a certain family of plane recursive trees, namely $k$-plane recursive trees. Concerning Stirling permutations of the multiset $\{1^k, 2^{k+2}, \dots, n^{k+2}\}$, which we call $k$-bundled Stirling permutations, we obtain a bijection with certain generalized plane recursive trees, namely $k$-bundled increasing trees. We also give two different bijections between certain families of increasing trees, which both give as a special case a bijection between ternary increasing trees and plane oriented increasing trees. Moreover, we will use several different methods, combinatorial and probabilistic, to derive several results in this direction. More precisely, in order to describe the (asymptotic) behaviour of the parameters of interests, we study three (generalized) Pólya urn models.

The parameter $k$ is fixed throughout the paper, and often omitted from the notation. All unspecified limits are as $n \to \infty$. In the results with a.s. convergence, we assume that the random $k$-Stirling permutation grows in the natural way by random addition of new labels; in the other results, this does not matter.

## 2. Preliminaries

2.1. **Generalized Stirling permutations.** A straightforward generalization of Stirling permutations on the multiset $\{1,1,2,2,\dots,n,n\}$ is to consider permutations of a more general multiset $\{1^{k_1}, 2^{k_2}, \dots, n^{k_n}\}$, with $k_i \in \mathbb{N}$ for $1 \leq i \leq n$. We call a permutation of the multiset $\{1^{k_1}, 2^{k_2}, \dots, n^{k_n}\}$ a *generalized Stirling permutation*, if for each $i$, $1 \leq i \leq n$, the elements occurring between two occurrences of $i$ are at least $i$. (In other words, the elements occurring between two consecutive occurrences of $i$ are larger than $i$.) Such permutations have already previously been considered by Brenti [7], [8]. The number of generalized Stirling permutations of $\{1^{k_1}, 2^{k_2}, \dots, n^{k_n}\}$ is

$$\prod_{i=1}^{n-1}(\ell_i + 1) \quad \text{with } \ell_i = \sum_{j=1}^{i} k_j; \tag{1}$$

this is easy to see by induction, since the $k_n$ copies of $n$ have to form a substring, and this substring can be inserted in $\ell_{n-1} + 1$ positions (viz., anywhere, including first or last) in any generalized Stirling permutation of $\{1^{k_1}, 2^{k_2}, \dots, (n-1)^{k_{n-1}}\}$.

We will consider two cases and give them special names: a $k$*-Stirling permutation* of order $n$ is a generalized Stirling permutation of the multiset $\{1^k, 2^k, \dots, n^k\}$, and a $k$*-bundled Stirling permutation* is a generalized Stirling permutation of the multiset $\{1^k, 2^{k+2}, \dots, n^{k+2}\}$. Here $k \geq 1$, but note that 1-Stirling permutations are just ordinary permutations so we will usually consider $k$-Stirling permutations for $k \geq 2$ only; the case $k = 2$ yields the ordinary Stirling permutations defined by Gessel and Stanley [14].

What we call $k$-Stirling permutations was suggested by Gessel and Stanley [14] and has been studied by Park [22, 23, 24] under the name $k$-multipermutations.

In the following, let $\mathcal{Q}_n = \mathcal{Q}_n(k)$ denote the set of $k$-Stirling permutations of order $n$ and let $Q_n = Q_n(k)$ denote the number $|\mathcal{Q}_n(k)|$ of them. By (1),

$$Q_n(k) = |\mathcal{Q}_n(k)| = \prod_{i=1}^{n-1}(ki + 1) = k^n \frac{\Gamma(n + 1/k)}{\Gamma(1/k)}. \tag{2}$$

For $k = 2$ this number is just $Q_n(2) = (2n-1)!!$. In the case $k = 3$, we have for example one permutation of order 1: 111; four permutations of order 2: 111222, 112221, 122211, 222111; etc.

Similarly, let $\overline{\mathcal{Q}}_n = \overline{\mathcal{Q}}_n(k)$ denote the set of $k$-bundled Stirling permutations of order $n$ and let $\overline{Q}_n = \overline{Q}_n(k)$ denote the number of them. We have, by (1),

$$\overline{Q}_n = |\overline{\mathcal{Q}}_n(k)| = \prod_{i=1}^{n-1}(i(k+2) - 1) = (k+2)^{n-1} \frac{\Gamma(n - 1/(k+2))}{\Gamma(1 - 1/(k+2))}. \tag{3}$$

We define *ascents*, *descents* and *plateaux* of a generalized Stirling permutation $\sigma = a_1 a_2 \cdots a_\ell$ of $\{1^{k_1}, 2^{k_2}, \dots, n^{k_n}\}$ (where the length $\ell = \sum_1^n k_i$) as before: we let $a_0 = a_{\ell+1} = 0$ and say that an index $i$, with $0 \leq i \leq \ell$, is an ascent, descent or plateau if $a_i < a_{i+1}$, $a_i > a_{i+1}$ or $a_i = a_{i+1}$, respectively. Note that the total number of them is $\ell + 1$.

We introduce a natural refinement of ascents, descents and plateaux, namely $j$*-ascents*, $j$*-descents*, and $j$*-plateaux*. An index $i$, with $1 \leq i \leq \ell$ is called a $j$-ascent, if $i$ is an ascent and there are exactly $j - 1$ indices $i' < i$ such that $a_{i'} = a_i$; i.e., $a_i$ is the $j$th occurrence of the symbol $a_i$, and similarly for plateaux. For a descent $i$, $a_i$ is always the last occurence of that symbol (just as for an ascent, $a_{i+1}$ is the first of its kind), and we define a $j$-descent as a descent $i < \ell$ such that $a_{i+1}$ is the the $j$th occurrence of that symbol. (Note that we choose not to allow $i = 0$ or $i = \ell$ in these definitions.)

Thus, for a generalized Stirling permutation of $\{1^{k_1}, 2^{k_2}, \dots, n^{k_n}\}$, the possible values of $j$ ranges from 1 to $\max_i k_i$ for $j$-ascents and $j$-descents, and from 1 to $\max_i k_i - 1$ for $j$-plateaux. In particular, for $k$-Stirling permutations, $1 \le j \le k$ for $j$-ascents and $j$-descents, and $1 \le j \le k-1$ for $j$-plateaux. Note also that if we reflect a $k$-Stirling permutation, we get a new $k$-Stirling permutation, and $j$-ascents in one of them correspond to $(k+1-j)$-descents in the other.

**Example 1.** Consider the 3-Stirling permutation $\sigma = 112233321$: Index 1 is a 1-plateau, index 2 is a 2-ascent, index 3 is a 1-plateau, index 4 is a 2-ascent, index 5 is a 1-plateau, index 6 is a 2-plateau, index 7 is a 3-descent, and index 8 is a 3-descent. (Indices 0 and 9 are not classified in this way.)

We are interested in the (joint) distributions of the random variables $X_{n,j}$, $Y_{n,j}$ and $Z_{n,j}$, defined as the numbers of $j$-ascents, $j$-descents and $j$-plateaux, respectively, in a random $k$-Stirling permutation (chosen uniformly in $\mathcal{Q}_n(k)$). Note that these trivially are 0 unless $1 \le j \le k$ for $X_{n,j}$ and $Y_{n,j}$, and $1 \le j \le k-1$ for $Z_{n,j}$, and that

$$\sum_{j=1}^{k} (X_{n,j} + Y_{n,j}) + \sum_{j=1}^{k-1} Z_{n,j} = kn - 1$$

We further let $X_n$, $Y_n$ and $Z_n$ denote the total numbers of ascents, descents and plateaux, respectively. Note that, recalling the special definitions at the endpoints,

$$X_n = \sum_{j=1}^{k} X_{n,j} + 1, \tag{4}$$

$$Y_n = \sum_{j=1}^{k} Y_{n,j} + 1, \tag{5}$$

$$Z_n = \sum_{j=1}^{k} Z_{n,j}. \tag{6}$$

It is easy to see that a $j$-ascent with $j < k$ corresponds to a later $(j+1)$-descent, and conversely, so

$$X_{n,j} = Y_{n,j+1}, \qquad 1 \le j \le k-1, \tag{7}$$

see also Theorem 2. However, there is no corresponding relation for $k$-ascents, of for 1-descents, and the total numbers of ascents and descents are typically different, even in the case $k = 2$. Further, since only the last copy of a label can be a descent,

$$X_{n,j} + Z_{n,j} = n, \qquad 1 \le j \le k-1, \tag{8}$$

and, similarly or by (8),

$$Y_{n,j} + Z_{n,j-1} = n, \qquad 2 \le j \le k. \tag{9}$$

Moreover, we are also interested in the distribution of the number of *blocks* in a random $k$-Stirling permutation of order $n$. A block in a generalized Stirling permutation $\sigma = a_1 \cdots a_\ell$ is a substring $a_p \cdots a_q$ with $a_p = a_q$ that is maximal, i.e. not contained in any larger such substring. There is obviously at most one block for every $j = 1, \dots, n$, extending from the first occurrence of $j$ to the last; we say that $j$ forms a block when this substring really is a block, i.e. when it is not contained in a string $i \cdots i$ for some $i < j$. In particular, in a $k$-Stirling permutation, $j$ forms a block if for any $i$ with $1 \le i \le j-1$, there do not exist indices $m_0, \dots m_{k+1}$ , with $1 \le m_0 < \cdots < m_{k+1} \le kn$, such that $\sigma_{m_0} = \sigma_{m_{k+1}} = i$ and $\sigma_{m_1} = \cdots = \sigma_{m_k} = j$. It is easily seen by induction that any generalized Stirling permutation has a unique decomposition as a sequence of its blocks. Note that if we add a string $(n+1)^{k_{n+1}}$ to a generalized Stirling permutation, this string will either be swallowed by one of the existing blocks, or form a block on its own; the latter happens when it is added first, last, or in a gap between two blocks.

**Example 2.** The 3-Stirling permutation $\sigma = 112233321445554666$, has block decomposition $[112233321][445554][666]$.

One may also consider the similar problems for $k$-bundled Stirling permutations; similarly defining random variables $\overline{X}_{n,j}$, $\overline{Y}_{n,j}$ and $\overline{Z}_{n,j}$. However, for most results we restrict ourselves to $k$-Stirling permutations.

2.2. **Generalized plane recursive trees and $d$-ary increasing trees.** In order to relate the $k$-Stirling permutations to families of increasing trees we use a general setting based on earlier considerations of Bergeron et al. [3] and Panholzer and Prodinger [21].

For a given degree-weight sequence $(\varphi_k)_{k\ge 0}$, the corresponding degree-weight generating function $\varphi(t)$ is defined by $\varphi(t) := \sum_{k\ge 0} \varphi_k t^k$. The simple family of increasing trees $\mathcal{T}$ associated with a degree-weight generating function $\varphi(t)$, can be described by the formal recursive equation

$$\mathcal{T} = ① \times \Big(\varphi_0 \cdot \{\epsilon\} \,\dot\cup\, \varphi_1 \cdot \mathcal{T} \,\dot\cup\, \varphi_2 \cdot \mathcal{T} * \mathcal{T} \,\dot\cup\, \varphi_3 \cdot \mathcal{T} * \mathcal{T} * \mathcal{T} \,\dot\cup\, \cdots\Big) = ① \times \varphi(\mathcal{T}), \quad (10)$$

where ① denotes the node labelled by 1, $\times$ the cartesian product, $\dot\cup$ the disjoint union, $*$ the partition product for labelled objects, and $\varphi(\mathcal{T})$ the substituted structure (see e. g., the books [30], [13]). This means that the elements of $\mathcal{T}$ are increasing plane trees, and that a tree with (out-)degrees $d_1, \dots, d_n$ is given weight $\prod_1^n \varphi_{d_i}$. By a random tree of order $n$ from the family $\mathcal{T}$, we mean a tree of order $n$ chosen randomly with probabilities proportional to the weights.

Let $T_n$ be the total weight of all such trees of order $n$. It follows from (10) that the exponential generating function $T(z) := \sum_{n\ge 1} T_n \frac{z^n}{n!}$ of the total weights satisfies the autonomous first order differential equation

$$T'(z) = \varphi\big(T(z)\big), \quad T(0) = 0. \quad (11)$$

The families that we will consider have degree-weights of one of the two following forms, studied by Panholzer and Prodinger [21]:

$$\varphi(t) = \begin{cases} \frac{\varphi_0}{(1+\frac{c_2 t}{\varphi_0})^{-\frac{c_1}{c_2}-1}}, & \text{for } \varphi_0 > 0,\ 0 < -c_2 < c_1, \quad \textit{generalized plane recursive trees,} \\ \varphi_0\Big(1+\frac{c_2 t}{\varphi_0}\Big)^d, & \text{for } \varphi_0, c_2 > 0,\ d := \frac{c_1}{c_2}+1 \in \mathbb{N}\setminus\{1\}, \quad d\textit{-ary increasing trees}. \end{cases} \tag{12}$$

Consequently, by solving (11), we obtain exponential generating function $T(z)$

$$T(z) = \begin{cases} \frac{\varphi_0}{c_2}\Big(\frac{1}{(1-c_1 z)^{\frac{c_2}{c_1}}} - 1\Big), & \text{generalized plane recursive trees,} \\ \frac{\varphi_0}{c_2}\Big(\frac{1}{(1-(d-1)c_2 z)^{\frac{1}{d-1}}} - 1\Big), & d\text{-ary increasing trees,} \end{cases} \tag{13}$$

and the total weights $T_n$,

$$T_n = \varphi_0 c_1^{n-1}(n-1)!\binom{n-1+\frac{c_2}{c_1}}{n-1}. \tag{14}$$

Note that changing $\varphi_k$ to $ab^k\varphi_k$ for some positive constants $a$ and $b$ will affect the weights of all trees of a given order $n$ by the same factor $a^n b^{n-1}$, which does not affect the distribution of a random tree from the family. Hence, when considering random trees from these two classes, $\varphi_0$ is irrelevant and $c_1$ and $c_2$ are relevant only through the ratio $c_1/c_2$. (We may thus, if we like, normalize $\varphi_0 = 1$ and either $c_1$ or $|c_2|$, but not both.)

As shown by Panholzer and Prodinger [21], random trees in the two classes of families given in (12) can be grown as an evolution process in the following way. The process, evolving in discrete time, starts with the root labelled by 1. At step $i+1$ the node with label $i+1$ is attached to any previous node $v$ (with out-degree $d(v)$) of the already grown tree of order $i$ with probabilities $p(v)$ given by

$$p(v) = \begin{cases} \frac{d(v)+\alpha}{(\alpha+1)i-1} & \text{with } \alpha := -1-\frac{c_1}{c_2} > 0, \quad \text{generalized plane recursive trees,} \\ \frac{d-d(v)}{(d-1)i+1}, & d\text{-ary increasing trees.} \end{cases}$$

Moreover, Panholzer and Prodinger [21] showed that there are only three classes of simple families that can be grown in this way (for suitable $p(v)$): the two classes given in (12) and the recursive trees given by $\varphi(t) = \varphi_0 e^{c_1 t/\varphi_0}$ with $\varphi_0, c_1 > 0$ (which can be regarded as a limiting case of any of the two classes above, letting $c_2 \to 0$.)

**Example 3.** *Plane recursive trees* are plane increasing trees such that all node degrees are allowed, with all trees having weight 1. Thus $\varphi_k = 1$ and the degree-weight generating function is $\varphi(t) = \frac{1}{1-t}$, which is of the form in (12) with $\varphi_0 = 1$, $c_1 = 2$ and $c_2 = -1$. We have

$$T(z) = 1 - \sqrt{1-2z}, \quad \text{and} \quad T_n = 1\cdot 3\cdot 5\cdots(2n-3) = (2n-3)!!, \ \text{ for } n \geq 1.$$

Furthermore, $\alpha = -1-\frac{c_1}{c_2} = 1$, and consequently, the probability attaching to node $v$ at step $i+1$ is given by $p(v) = \frac{d(v)+1}{2i-1}$.

**Example 4.** For an integer $d \geq 2$, *$d$-ary increasing trees* are increasing trees where each node has $d$ (labelled) positions for children. Thus, only outdegrees $0, \ldots, d$ are allowed; moreover, for a node with $k$ children in given order, there is thus $\binom{d}{k}$ ways to attach them. Hence, this family is given by vertex weights $\varphi_k = \binom{d}{k}$ and thus the degree-weight generating function $\varphi(t) = 1 + t^d$, which is of the form in (12) with $\varphi_0 = 1$, $c_1 = d - 1$ and $c_2 = 1$. By (13),

$$T(z) = \big(1 - (d-1)z\big)^{-1/(d-1)} - 1.$$

## 3. Increasing trees associated to generalized Stirling permutations

3.1. **$(k+1)$-ary increasing trees, $k$-plane recursive trees and $k$-Stirling permutations.** Recall from Example 4 that, for $k \geq 1$, the degree-weight generating function of $(k+1)$-ary increasing trees is given by $\varphi(t) = (1+t)^{k+1}$, i.e. $\varphi_0 = 1$, $c_1 = k$ and $c_2 = 1$. Consequently, the generating function $T(z)$ and the numbers $T_n$ of $(k+1)$-ary trees of order $n$ are given by

$$T(z) = \frac{1}{(1-kz)^{\frac{1}{k}}} - 1, \qquad T_n = \prod_{l=1}^{n}(k(l-1)+1), \quad n \geq 1,$$

and the the probability of attaching to node $v$ at step $i+1$ is given by $p(v) = \frac{k+1-d(v)}{ki+1}$.

Note that $T_n = Q_n$, the number of $k$-Stirling permutation, which makes the following theorem reasonable.

**Theorem 1** (Gessel)**.** *Let $k \geq 1$. The family $\mathcal{A}_n = \mathcal{A}_n(k+1)$ of $(k+1)$-ary increasing trees of order $n$ is in a natural bijection with $k$-Stirling permutations, $\mathcal{A}_n(k+1) \cong \mathcal{Q}_n(k)$.*

**Remark 1.** The authors independently derived the result above, and later discovered the work of Park [22], in which Gessel's result was mentioned but the proof only sketched. The result of Gessel never appeared in print except this mentioning in Park [22], to the best of the authors' knowledge. We will give a detailed proof of the result above, which has interesting consequences regarding the (refined) parameters ascents, descents and plataeux, and also number of blocks, which we will state in Theorem 2.

**Remark 2.** For $k = 1$ we obtain a bijection between 1-Stirling permutations (ordinary permutations) and binary increasing trees, which is very well known.

*Proof.* We use a slightly modified bijection to the one given by Janson in [18] for Stirling permutation and plane recursive tree, and use a depth-first walk. The depth-first walk of a rooted (plane) tree starts at the root, goes first to the leftmost child of the root, explores that branch (recursively, using the same rules), returns to the root, and continues with the next child of the root, until there are no more children left. We think of $(k+1)$-ary increasing trees, where the empty places are represented by "exterior nodes". Hence, at any time, any (interior) node has $k+1$ children, some of which may be exterior nodes. Between these $k$ edges going out from a node labelled $v$, we place $k$ integers $v$. (Exterior nodes have no children and no labels.) Now we perform the depth-first walk and code the $(k+1)$-ary increasing tree by the sequence of the labels visited as we go around the tree (one may think of actually going around the tree like drawing the contour). In other words, we add label $v$ to the code the $k$ first times we return to node $v$, but not the first time we arrive there or the last time we return. A $(k+1)$-ary increasing tree of order 1 is encoded by $1^k$. A $(k+1)$-ary increasing tree of order $n$ is encoded by a string of $k \cdot n$ integers, where each of the labels $1, \dots, n$ appears exactly $k$ times. In other words, the code is a permutation of the multiset $\{1^k, 2^k, \dots, n^k\}$. Note that for each $i$, $1 \le i \le n$, the elements occurring between the two occurrences of $i$ are larger than $i$, since we can only visit nodes with higher labels. Hence the code is a $k$-Stirling permutation. Moreover, adding a new node $n+1$ at one of the $kn+1$ free positions (i.e., the positions occupied by exterior nodes) corresponds to inserting the $k$-tuple $(n+1)^k$ in the code at one of $kn+1$ gaps; note (e.g., by induction) that there is a bijection between exterior nodes in the tree and gaps in the code. This shows that the code determines the $(k+1)$-ary increasing tree uniquely and that the coding is a bijection. See Figure 1 for an illustration.

The inverse, starting with a $k$-Stirling permutation $\sigma$ of order $n$ and constructing the corresponding $(k+1)$-ary increasing tree can be described as follows. We proceed recursively starting at step one by decomposing the permutation as $\sigma = \sigma_1 1 \sigma_2 1 \dots \sigma_k 1 \sigma_{k+1}$, where (after a proper relabelling) the $\sigma_i$'s are again $k$-Stirling permutations. Now the smallest label in each $\sigma_i$ is attached to the root node labelled 1. We recursively apply this procedure to each $\sigma_i$ to obtain the tree representation. $\square$

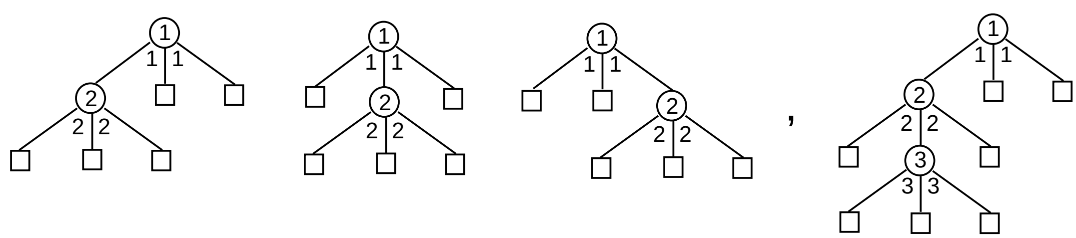

FIGURE 1. The three ternary trees of order 2 encoded by 2211, 1221 and 1122; an order 3 ternary increasing tree encoded by the sequence 233211.

Now we relate the distribution of $j$-ascents, $j$-descents and $j$-plateaux in $k$-Stirling permutations with certain parameters in $(k+1)$-ary increasing trees. In order to do so we introduce two kinds of parameters. The parameter $D_{n,j}$, standing for "$j$th children" , counts the number of nodes in a random $(k+1)$-ary increasing tree of order $n$ that are the $j$th children of their respective parents, going from left to right, with $1 \le j \le k+1$. Similarly, the parameter $L_{n,j}$, standing for "leaves" of type $j$, counts the number of exterior nodes that are $j$th children of their parents, $1 \le j \le k+1$. We thus have

$$L_{n,j} = n - D_{n,j}, \qquad 1 \le j \le k+1, \tag{15}$$

and, counting the total numbers of interior and exterior children,

$$\sum_{j=1}^{k+1} D_{n,j} = n - 1, \qquad \sum_{j=1}^{k+1} L_{n,j} = kn + 1. \tag{16}$$

Concerning the number of blocks in $k$-Stirling permutation, we introduce one more parameter in $(k+1)$-ary increasing trees. Let $LR_n$ denote the number of (interior) nodes that have the property that the path to the root consists exclusively of the leftmost or rightmost possible edge at each node, i.e., the edge in position 1 or $k+1$, and no other "inner" edges. Subsequently, we will call such nodes *left-right nodes*. The root is trivially a left-right node.

**Theorem 2.** *Let $k \geq 1$. Under the bijection in Theorem 1, the numbers of $j$-ascents $X_{n,j}$, $j$-descents $Y_{n,j}$ and $j$-plateaux $Z_{n,j}$ in a $k$-Stirling permutation of order $n$ coincide with the (shifted) numbers of $j$-children $D_{n,j}$, and $j$-leaves $L_{n,j}$ in a $(k+1)$-ary increasing tree of order $n$ by the formulas*

$$\begin{aligned} X_{n,j} &= D_{n,j+1}, && 1 \leq j \leq k, \\ Y_{n,j} &= D_{n,j}, && 1 \leq j \leq k, \\ Z_{n,j} &= L_{n,j+1} = n - D_{n,j+1}, && 1 \leq j \leq k-1. \end{aligned}$$

*As a consequence, for the numbers of ascents, descents and plateaux,*

$$\begin{aligned} X_n &= n - D_{n,1} = L_{n,1}, \\ Y_n &= n - D_{n,k+1} = L_{n,k+1}, \\ Z_n &= \sum_{j=2}^{k} L_{n,j}. \end{aligned}$$

*Furthermore, the number of blocks $S_n$ in a $k$-Stirling permutations of order $n$ coincides with the number of left-right nodes in the corresponding $(k+1)$-ary increasing trees of order $n$,*

$$S_n = LR_n. \tag{17}$$

*Proof.* Using the stated bijection we observe that a $(j+1)$-child, $1 \leq j \leq k$, corresponds to a $j$-ascent, since the step from the parent node $v$ to the $(j+1)$-child $u$ corresponds to having recorded $j$ times the label of the parent $v$ and then another label $w$, with $w \geq u$. Similar considerations prove the results for $j$-descents and $j$-plateaux. The results for $X_n$, $Y_n$, $Z_n$ then follow from (4)–(6) and (16).

Concerning the connection between blocks and left-right nodes we make the following observation. Starting with a $(k+1)$-ary increasing tree, and inserting nodes one after another, we note that only a leftmost or rightmost child leads to a new block in the corresponding $k$-Stirling permutation. Hence, the number of left-right nodes is equal to the number of blocks, since we start with a single block $1^k$ and a single left-right node (the root). $\square$

**Remark 3.** Note that the number of leaves in $(k+1)$-ary increasing trees of order $n \geq 2$ corresponds to the number of locally maximal substrings $l^k = l \cdots l$, i.e. substrings $il^k j$, with $0 \leq i, j < l$, for $2 \leq l \leq n$, in $k$-Stirling permutations of order $n$, which can also be seen from the bijection.

**Remark 4.** In the case $k = 2$ we thus have the symmetric situation that $X_n = L_{n,1}$, $Y_n = L_{n,3}$ and $Z_n = L_{n,2}$, which by Theorem 8 below gives a new proof that $X_n$, $Y_n$ and $Z_n$ have the same distribution, and further are exchangeable, as shown by [6] and [18]. We see also that this will not hold for larger $k$, see for example Theorem 9.

For $k = 2$, Theorem 1 gives a bijection between Stirling permutation and ternary increasing trees, while Janson [18] gives a bijection with plane recursive trees of order $n+1$. (These are related by a bijection given in Section 4.) Next we will show that also for $k > 2$, there is a suitable family of generalized plane recursive trees that is closely related to $k$-Stirling permutations.

**Definition 1.** For $k \geq 2$, the family of *$k$-plane recursive trees* is specified by the degree-weight generating function $\varphi(t) = (1-(k-1)t)^{-\frac{1}{k-1}}$, i.e. it is the family of generalized plane recursive trees with $\varphi_0 = 1$, $c_1 = k$ and $c_2 = -(k-1)$. Explicitly, $\varphi_d = \frac{1}{d!}\prod_{l=1}^{d}\big((k-1)(l-1)+1\big)$. Consequently, by (13) and (14), the generating function $T(z)$ and the total weight $T_n$ are given by

$$T(z) = \frac{1}{k-1}\Big(1-(1-kz)^{\frac{k-1}{k}}\Big), \qquad T_{n+1} = \prod_{l=1}^{n}(k(l-1)+1), \text{ with } T_1 = T_2 = 1,$$

$\alpha = \frac{1}{k-1}$, and the probability of attaching to node $v$ at step $i+1$ is given by $p(v) = \frac{d(v)+\frac{1}{k-1}}{\frac{ki}{k-1}-1}$.

For $k = 2$, these are the plane recursive trees in Example 3.

**Remark 5.** We did not succeed in finding a bijective correspondence between $k$-Stirling permutations and $k$-plane recursive trees, in the case of $k > 2$, generalizing the bijection in [18] for $k = 2$, since for $k > 2$ it seems difficult to obtain a combinatorial interpretation of the weights of the trees. We leave this as an open problem. However, the distribution of the leaves still coincides with the distribution of the number of ascents or descents.

**Theorem 3.** *The number (total weight) of $k$-plane recursive trees of order $n+1$ equals the number of $k$-Stirling permutations of order $n$, $T_{n+1} = Q_n$. Moreover, the distribution of the number $\tilde{L}_{n+1}$ of leaves of $k$-plane recursive trees of order $n+1$ coincides with the distribution of the number $X_n$ of ascents (descents) of $k$-Stirling permutations of order $n$.*

*Proof.* The first part is already shown.

The second part is trivial for $n = 1$, with one leaf and one ascent. We proceed by induction, and suppose that the relation is true for $n$: $\tilde{L}_{n+1} \overset{\mathrm{d}}{=} X_n$. We observe that adding the new node labelled $n+2$ to a leaf does not change the number of leaves, whereas adding the new node at any other place gives rise to a new leaf. Further, by the formula for $p(v)$ above with d(v)=0, the probability of adding node $n+2$ to a given leaf in a tree of order $n+1$ is $p(v) = \frac{\frac{1}{k-1}}{\frac{k(n+1)}{k-1}-1} = \frac{1}{kn+1}$. Hence, conditioned on the number of leaves $\tilde{L}_{n+1}$ being $m$, we have $\tilde{L}_{n+2} = m$ or $m+1$ with

$$\mathbb{P}(\tilde{L}_{n+2} = m \mid \tilde{L}_{n+1} = m) = \frac{m}{kn+1}. \tag{18}$$

Similarly, when adding a string $(n+1)^k$ to a $k$-Stirling permutation of order $n$, we will always create a new ascent, and we will destroy one if and only if we add the string at an ascent. Since there are $kn+1$ gaps where the new string can be added, conditioned on the number $X_n$ of ascents being $m$, we have $X_{n+1} = m$ or $m+1$ with $\mathbb{P}(X_{n+2} = m \mid X_{n+1} = m) = \frac{m}{kn+1}$. This is the same relation as (18), and thus $\tilde{L}_{n+2} \overset{\mathrm{d}}{=} X_{n+1}$, which verifies the induction step. $\square$

**Remark 6.** The distribution of the number of leaves is fairly well studied. Let $T(z,v) = \sum_{n\geq 1} T_{n,m}\frac{z^n}{n!}v^m$ denote the bivariate generating function of the number of $k$-plane recursive trees having exactly $m$ leaves, also encoding the number $k$-Stirling permutations of order $n-1$ having $m$ descents. Bergeron et al. [3] determined the generating function $T(z,v)$ by the implicit equation

$$\int_0^T \frac{dt}{(v-1)\varphi_0 + \varphi(t)} = z,$$

Note that the implicit equation is true for a much larger class of increasing trees; moreover one may derive the normal limit of the number of leaves from the implicit equation above, see [3].

### 3.2. $k$-bundled increasing trees and $k$-bundled Stirling permutations.

**Definition 2.** For $k \geq 0$, the family of $(k+1)$-*bundled increasing trees* is specified by the degree-weight generating function $\varphi(t) = \frac{1}{(1-t)^{k+1}}$, i.e. it is the family of generalized plane recursive trees with $\varphi_0 = 1$, $c_1 = k+2$ and $c_2 = -1$. Explicitly, $\varphi_j = \binom{k+j}{j}$. Consequently, by (13) and (14), the generating function $T(z)$ and the total weight $T_n$ are given by

$$T(z) = 1 - (1-(k+2)z)^{\frac{1}{k+2}}, \qquad T_n = \prod_{l=1}^{n-1}(l(k+2)-1),$$

$\alpha = k+1$, and the the probability attaching to node $v$ at step $i+1$ is given by $p(v) = \frac{d(v)+k+1}{(k+2)i-1}$.

**Remark 7.** One may think of $(k+1)$-bundled increasing trees of order $n$ as consisting of a root node labelled 1 which has $k+1$ positions, with a (possibly empty) sequence of labelled $(k+1)$-bundled increasing trees attached to each position (with disjoint sets of labels, forming a partition of $\{2,\ldots,n\}$). Equivalently, one may think of each node as having $k$ separation walls, which can be regarded as a special type of edges.

Note that the 1-bundled increasing trees are just ordinary plane recursive trees, cf. Example 3, and that the bijection stated below also holds for this case, which corresponds to the result of [18] that $\mathcal{B}_n(1) = \mathcal{T}_n \cong \mathcal{Q}_{n-1}(2)$, since obviously $\overline{\mathcal{Q}}_n(0) \cong \mathcal{Q}_{n-1}(2)$ by relabelling.

**Theorem 4.** *The family $\mathcal{B}_n = \mathcal{B}_n(k+1)$ of $(k+1)$-bundled increasing trees of order $n$ is in a natural bijection with $k$-bundled Stirling permutations, $\mathcal{B}_n(k+1) \cong \overline{\mathcal{Q}}_n(k)$.*

*Proof.* We proceed as before using a depth-first walk. We label each auxilliary separation wall of a node labelled $v$ by the label of the node $v$. Moreover, we label any (proper) edge by the label of the child. Hence, at any time, any node has at least $k$ outgoing edges, thinking of the walls as a special type of edges. Now we perform the depth-first walk and code the $k$-bundled increasing tree by the sequence of the labels visited on the edges, under the additional rule that a label on a separation wall only contributes once. Since every proper edge is traversed twice, and every label except 1 occurs on exactly one proper edge, a $(k+1)$-bundled increasing tree of order $n$ is encoded by a string of $(k+2)(n-1)+k$ integers, where each of the labels $2, \dots, n$ appears exactly $k+2$ times and label 1 appears $k$ times. In other words, the code is a permutation of the multiset $\{1^k, 2^{k+2}, \dots, n^{k+2}\}$. Note that for each $i$, $1 \le i \le n$, the elements occurring between the two occurrences of $i$ are larger than $i$, since we can only visit nodes with higher labels. Hence the code is a $k$-bundled Stirling permutation. Moreover, adding a new node $n+1$ at one of the $(k+2)(n-1)+k+1$ possible places corresponds to inserting the $(k+2)$-tuple $(n+1)^{k+2}$ in the code, at one of $(k+2)(n-1)+k+1$ possible places. This shows that the code determines the $(k+1)$-bundled increasing tree uniquely and that the coding is a bijection. See Figure 2 for an illustration. □

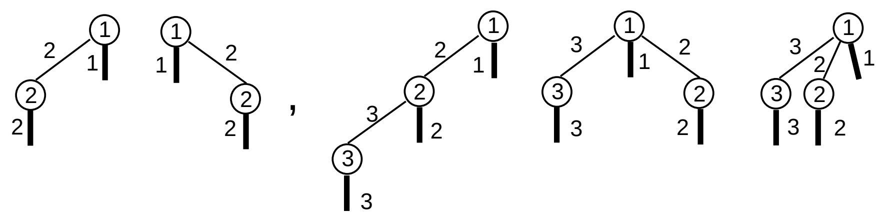


FIGURE 2. The two 2-bundled increasing trees of order 2 encoded by 2221, 1222; Three 2-bundled increasing trees of order 3 encoded by the sequences 2333221, 3331222 and 3332221.

Next we relate the distribution of ascents, descents and plateaux in $k$-bundled Stirling permutations with certain parameters in $(k+1)$-bundled increasing trees. In order to do so we introduce three parameters for a $(k+1)$-bundled increasing tree $\tau$. The parameter $B_A$ counts the number of ascents in the bundles of $\tau$, plus the number of non-empty bundles, plus 1 if the first bundle of the root is empty, where an ascent in a bundle occurs if the root of a subtree is smaller then the root of the next subtree, going from left to right. The parameter $B_D$ counts the number descents in the bundles of $\tau$, plus the number of non-empty bundles, plus 1 if the last bundle of the root is empty, where a descent in a bundle occurs if the root of subtree is larger then its neighbour. The number $B_E$ counts the number of empty bundles of the nodes with labels larger than one plus the number of empty inner bundles of the root. With these definitions, the following correspondences are straightforward.

**Theorem 5.** *Under the bijection in Theorem 4, the numbers of ascents, descents and plateaux in a $k$-bundled Stirling permutation of order $n$ coincide with the parameters $B_A$, $B_D$ and $B_E$ in a $(k+1)$-bundled increasing tree of order $n$.*

**Remark 8.** Note that the number of leaves in $(k+1)$-bundled increasing trees of order $n$ corresponds to the number of sequences of the form $l^{k+2} = l \cdots l$, with $2 \leq l \leq n$, in $k$-bundled Stirling permutations of order $n$, as for $k$-ary increasing trees. Moreover, the parameter "number of descendants of node $j$" in a $(k+1)$-bundled increasing tree of order $n$, with $2 \leq j \leq n$, counts the number of different entries $l$ with $j < l \leq n$ between the first and the last occurrence of $j$ in the corresponding $k$-bundled Stirling permutation of order $n$.

## 4. Further bijections

The bijections of Theorem 1 (with $k = 2$) and [18] (or Theorem 4 with $k = 0$) imply a bijection between ordinary plane recursive trees of order $n+1$ and ternary increasing trees of order $n$, using the connections to 2-Stirling permutations. In the following we will give two direct bijections, which both encompass this bijection between plane recursive trees and ternary increasing trees.

First we give a bijection between sequences of $k$-bundled increasing trees and $(k+2)$-ary increasing trees, which for $k = 1$ just gives the desired bijection.

Let $\text{SEQ}(\mathcal{B})_n = \text{SEQ}(\mathcal{B})_n(k)$ denote the family of sequences of $k$-bundled increasing trees with total order $n$, labelled with disjoint sets of labels forming a partition of $\{1, \dots, n\}$. (Note that our notation slightly abuses the common sequence notation SEQ of combinatorial objects, since we also assume properly distributed labels.)

**Remark 9.** By introducing a new root labelled 0, connecting all roots of the sequence with the new root, and performing a proper relabelling, $\text{SEQ}(\mathcal{B})_n$ is in bijection with the family of increasing plane trees of order $n+1$ where each node except the root is $k$-bundled as in Definition 2. (Equivalently, $\text{SEQ}(\mathcal{B})_n$ is in bijection with the family of $k$-bundled increasing trees of order $n+1$ where the root has only the first bundle non-empty.)

**Theorem 6.** *The family $\text{SEQ}(\mathcal{B})_n = \text{SEQ}(\mathcal{B})_n(k)$ of sequences of $k$-bundled increasing trees of total order $n$ is in bijection with $\mathcal{A}_n(k+2)$, the family of $(k+2)$-ary increasing trees of order $n$: $\text{SEQ}(\mathcal{B})_n(k) \cong \mathcal{A}_n(k+2)$.*

**Remark 10.** Recall that 1-bundled increasing trees are exactly plane recursive trees. Moreover, in the case of $k = 1$, the bijection in Remark 9 is the standard bijection between sequences of plane recursive trees of total order $n$ and plane recursive trees of order $n+1$; hence $\text{SEQ}(\mathcal{B})_n(1) \cong \mathcal{T}_{n+1}$. See Figure 3 for an illustration. It is easily seen that, for $k = 1$, the bijection $\mathcal{T}_{n+1} \cong \mathrm{SEQ}(\mathcal{B})_n(1) \cong \mathcal{A}_n(3)$ constructed in the proof below yields the correspondence between the two bijections $\mathcal{A}_n(3) \cong \mathcal{Q}_n(2)$ in Theorem 1 and $\mathcal{T}_{n+1} \cong \mathcal{Q}_n(2)$ in [18] or $\mathcal{B}_{n+1}(1) \cong \overline{\mathcal{Q}}_{n+1}(0) \cong \mathcal{Q}_n(2)$ in Theorem 4.

*Proof.* We use a recursive construction, see Figure 3. For a given sequence of $k$-bundled increasing trees, we choose in the first step the tree of the sequence with node labelled 1: this node is going be the root of the $(k+2)$-ary increasing tree. Since a $(k+2)$-ary increasing trees has $k+2$ (possibly empty) subtrees $S_1, \dots, S_{k+2}$, going from left to right, we proceed as follows. The sequence of $k$-bundled increasing trees to the left of the tree with root $1$ forms (recursively) the subtree $S_1$, conversely the sequence of $k$-bundled increasing trees to the right of the tree with root $1$ forms the subtree $S_{k+2}$. The $k$ bundles, possibly empty, attached to the tree with root labelled $1$, form the subtrees $S_2, \dots, S_{k+1}$ of the $(k+2)$-ary increasing tree. Now we can proceed recursively, since the bundles are themselves just sequences of $k$-bundled increasing trees.

Conversely, starting with a $(k+2)$-ary increasing tree of order $n$, we recursively build a sequence of $k$-bundled increasing trees as follows. In the first step we build a tree with root node labelled 1. The sequence to the left of the tree with root labelled 1 is built from the subtree $S_1$ of the $(k+2)$-ary increasing tree of order $n$, the sequence on the right from the subtree $S_{k+2}$, and the $k$ bundles are built from the subtrees $S_2, \dots, S_{k+1}$. We proceed recursively until the sequence is constructed. Note that during this process, we connect any leftmost or rightmost child of a node $v$ to the same parent as $v$. $\square$

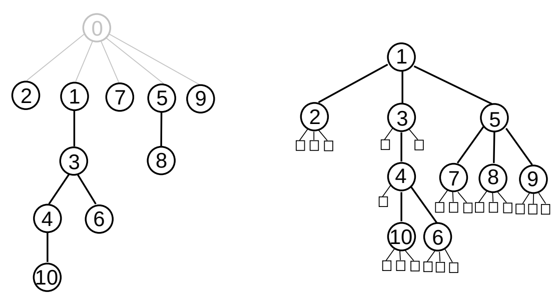


FIGURE 3. A sequence of 1-bundled increasing trees of order 10, or equivalently a plane recursive tree of order 11, and the corresponding ternary increasing tree of order 10.

Next we consider a bijection between $k$-bundled increasing trees and so-called $F_{k,k+2}$-increasing trees. The family of $F_{k,k+2}$-increasing trees consists of modified $(k+2)$-ary increasing trees: any node except the root of a $F_{k,k+2}$-increasing tree has $k+2$ labelled positions where children may be attached, whereas the root has only $k$ positions (and thus outdegree bounded by $k$). Note that for $k=1$, the root has a single child and that chopping off the root yields a simple bijection between $F_{1,3}$-increasing trees of order $n+1$ and ternary increasing trees of order $n$. Thus the statement below implies for $k=1$ a bijection between ternary increasing trees and plane recursive trees, $\mathcal{A}_n(3) \cong \mathcal{B}_{n+1}(1) = \mathcal{T}_{n+1}$, which is just the bijection discussed in Remark 10.

**Theorem 7.** *The family $\mathcal{F}_n = \mathcal{F}_n(k)$ of $F_{k,k+2}$-increasing trees of order $n$, is in bijection with the family of $k$-bundled increasing trees of order $n$, $\mathcal{F}_n(k) \cong \mathcal{B}_n(k)$, $k \geq 1$.*

*Proof.* For a given $k$-bundled increasing tree of order $n$, we simply apply $k$ times the bijection between sequences of $k$-bundled increasing trees and $(k+2)$-ary increasing trees to the $k$ bundles attached to the root and the $k$ positions of the root of the $F_{k,k+2}$-increasing tree. $\square$

**Remark 11.** To give an overview, we have provided the following bijections in Theorems 1, 4, 6 and 7.

$$\mathcal{A}_n(k+1) \cong \begin{cases} \mathcal{Q}_n(k), \\ \mathrm{SEQ}(\mathcal{B})_n(k-1), \end{cases} \qquad \mathcal{B}_n(k+1) \cong \begin{cases} \overline{\mathcal{Q}}_n(k), \\ \mathcal{F}_n(k+1). \end{cases}$$

It is also possibly to give bijections $\mathcal{Q}_n(k) \cong \mathrm{SEQ}(\mathcal{B})_n(k-1)$ and $\overline{\mathcal{Q}}_n(k) \cong \mathcal{F}_n(k+1)$, by simple modifications of the stated bijections.

**Remark 12.** The families $\mathrm{SEQ}(\mathcal{B})(k)$ of sequences $k$-bundled increasing trees and $\mathcal{F}(k)$ of $F_{k,k+2}$-increasing trees are non-standard in the sense that they are not part of the characterization given by Panholzer and Prodinger [21]. However, the counting problem concerning such tree families can be treated in a general manner, which will be discussed elsewhere.

## 5. The distribution of $j$-ascents, $j$-descents and $j$-plateaux

We are interested in the joint asymptotic distribution of $j$-ascents $X_{n,j}$, $j$-descents $Y_{n,j}$ and $j$-plateaux $Z_{n,j}$ in a $k$-Stirling permutations of order $n$, or equivalently in the joint distribution of $j$-children $D_{n,j}$ and $j$-leaves $L_{n,j+1}$ in $(k+1)$-ary increasing trees of order $n$. Following Janson [18] we use a (generalized) Pólya urn model, see [15].

### 5.1. An urn model for the exterior leaves.

Since we already know from (15) that $n - D_{n,j} = L_{n,j}$, we can restrict ourselves to the study of the exterior nodes. We will use the following urn model.

**Urn I.** Consider an urn with balls of $k+1$ colours, and let $(L_{n,1}, \ldots, L_{n,k+1})$ be the number of balls of each colour at time $n$. At each time step, draw one ball at random from the urn, discard it, and add one new ball of each colour. Start with $(L_{1,1}, \ldots, L_{1,k+1}) = (1, 1, \ldots, 1)$. Note that the vector $(L_{n,1}, \ldots, L_{n,k+1})$ *exactly* coincides (in distribution) with the numbers of the exterior nodes of types $1, \ldots, k+1$ in a random $(k+1)$-ary increasing tree, see Section 3.1.

Urn I is completely symmetric in the $k+1$ colours, and we thus immediately see the following.

**Theorem 8.** *For each $n \geq 1$, the distribution of $(L_{n,1}, \dots, L_{n,k+1})$ is exchangeable, i.e., invariant under any permutation of the $k+1$ variables.*

It is customary and convenient to formulate generalized Pólya urns using drawings with replacement. In the case of Urn I, we thus restate the description above and say instead that we draw a ball and replace it together with one ball each of the $k$ other colours. In other words, Urn I is described by the $(k+1) \times (k+1)$ replacement matrix

$$A = (1 - \delta_{i,j})_{1 \leq i,j \leq k+1} = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 0 & 1 & \ddots & \ddots & \ddots & 1 \\ 1 & 1 & 0 & \ddots & \ddots & \ddots & 1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1 & \ddots & \ddots & \ddots & 0 & 1 & 1 \\ 1 & \ddots & \ddots & \ddots & 1 & 0 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & 0 \end{pmatrix},$$

where $\delta_{i,j}$ denotes the Kronecker delta.

5.2. **Means.** By Theorem 8, the variables $L_{n,j}$, $j = 1, \dots, k+1$, have the same mean, and since their sum is $kn+1$ by (16), we see that they each have mean $\frac{kn+1}{k+1}$. By (15) and Theorem 2, we obtain the following exact formulas for the means.

**Theorem 9.** *The following hold, for $n \geq 1$ and $k \geq 1$:*

$$\begin{aligned} \mathbb{E}L_{n,j} &= \frac{kn+1}{k+1}, && 1 \leq j \leq k+1, \\ \mathbb{E}D_{n,j} &= \frac{n-1}{k+1}, && 1 \leq j \leq k+1, \\ \mathbb{E}X_{n,j} = \mathbb{E}Y_{n,j} &= \frac{n-1}{k+1}, && 1 \leq j \leq k, \\ \mathbb{E}Z_{n,j} &= \frac{kn+1}{k+1}, && 1 \leq j \leq k-1, \\ \mathbb{E}X_n = \mathbb{E}Y_n &= \frac{kn+1}{k+1}, \\ \mathbb{E}Z_n &= (k-1)\frac{kn+1}{k+1}. \end{aligned}$$

5.3. **Asymptotic distribution of $j$-ascents, $j$-descents and $j$-plateaux.** We use the urn model Urn I to obtain asymptotic normality. We begin with a general result.

**Theorem 10.** *Consider an urn with balls of $q \geq 2$ colours, where at each step one ball is drawn at random and discarded, and one ball of each colour is added. If $N_{n,j}$ is the number of balls of colour $j$ after $n$ steps, then, for any initial values $N_{0,1}, \dots, N_{0,q}$,*

$$\frac{N_{n,j} - \frac{q-1}{q}n}{\sqrt{n}} \xrightarrow{(d)} \zeta_j,$$

*jointly for* $j = 1, \dots, q$*, where* $\zeta_j$ *are jointly normal random variables with means* $0$ *and (co)variances*

$$\operatorname{Cov}(\zeta_i, \zeta_j) = \frac{q-1}{q^2(q+1)}(q\delta_{i,j} - 1), \qquad 1 \le i, j \le q.$$

Note that $\sum_1^q \zeta_j = 0$, for example because $\sum_j N_{n,j}$ is deterministic.

*Proof.* This urn has replacement matrix $A = (1 - \delta_{i,j})_{i,j=1}^q = (1)_{i,j=1}^q - I$. Since the rank 1 matrix $(1)_{i,j=1}^q$ has one eigenvalue $q$ and $q-1$ eigenvalues 0, $A$ has largest eigenvalue $\lambda_1 = q - 1$ and $q - 1$ eigenvalues $-1$. Theorem 3.22 in [15] applies, with $v_1 = (\frac{1}{q}, \dots, \frac{1}{q})$, and shows joint convergence (in distribution) to normal variables $\zeta_j$ with mean 0 and a certain covariance matrix $\Sigma$. The formula for $\Sigma$ in [15, Theorem 3.22] is complicated, so we use [15, Lemma 5.4], with $a = (a_i)_1^q = (1)_1^q$ and $m = \lambda_1 = q - 1$, which yields $\Sigma = m\Sigma_I$, with $\Sigma_I$ defined in [15, (2.15)]. Here $\xi_i = (\xi_{i,j})_{j=1}^q = (1 - \delta_{i,j})_{j=1}^q$, $B_i = (\xi_{i,j}\xi_{i,l})_{j,l=1}^q$, $v_{1i} = 1/q$ and, using the symmetry, $B = (b_{ij})_{i,j=1}^q$ with $b_{i,i} = \frac{q-1}{q}$ and $b_{i,j} = \frac{q-2}{q}$, $i \ne j$; hence $B = \frac{q-2}{q}A + \frac{q-1}{q}I$. Further, $P_I$ is the projection onto the eigenspace of $A$ for the eigenvalue $-1$, and thus $P_I = \frac{(q-1)I - A}{q}$. Further, on this eigenspace $A = -I$ and thus $B = \frac{1}{q}I$, and [15, (2.15)] yields, noting that all involved matrices are symmetric and commute,

$$\begin{aligned}\Sigma_I &= \int_0^\infty P_I e^{sA} B e^{sA'} P_I' e^{-\lambda_1 s}\, ds = \frac{1}{q} P_I \int_0^\infty e^{-s-s-(q-1)s}\, ds = \frac{1}{q(q+1)} P_I \\ &= \frac{(q-1)I - A}{q^2(q+1)} = \left(\frac{q\delta_{i,j} - 1}{q^2(q+1)}\right)_{i,j=1}^q.\end{aligned}$$

Recalling that $\Sigma = m\Sigma_I = (q-1)\Sigma_I$, we obtain the result. $\square$

**Remark 13.** Similar calculations show, more generally, that if we at each step add a fixed number $s_i$ balls of colour $i$, $i = 1, \dots, q$, independently of the colour of the drawn and discarded ball, then $n^{-1/2}\big(N_{n,i} - \frac{\sum_l s_l - 1}{\sum_l s_l} s_i n\big) \xrightarrow{(d)} \zeta_i$, jointly, where $\zeta_i$ are jointly normal variables with means 0 and

$$\operatorname{Cov}(\zeta_i, \zeta_j) = \frac{\sum_l s_l - 1}{\sum_l s_l + 1}\left(\frac{s_i}{\sum_l s_l}\delta_{i,j} - \frac{s_i s_j}{(\sum_l s_l)^2}\right).$$

As an example, the numbers $X_n$, $Y_n$ and $Z_n$ of ascents, descents and plateaux in a random $k$-Stirling permutation can be seen as such an urn with replacement vector $(1, 1, k-1)$, which yields an alternative proof of the limit distribution found above for them.

We apply Theorem 10, with $q = k + 1$, to Urn I and obtain using (15) and Theorem 2 the following.

**Theorem 11.** *Let* $k \ge 1$ *and let* $\zeta_j$, $j = 1, \dots, k+1$*, be jointly normal random variables with means* $0$ *and (co)variances*

$$\operatorname{Cov}(\zeta_i, \zeta_j) = \frac{k}{(k+1)^2(k+2)}((k+1)\delta_{i,j} - 1), \qquad 1 \le i, j \le k+1,$$

*in particular* $\mathrm{Var}(\zeta_j) = \frac{k^2}{(k+1)^2(k+2)}$. *Note that this implies* $\sum_{j=1}^{k+1} \zeta_j = 0$. *Then, the following holds, jointly for all variables,*

$$\begin{aligned}
\frac{L_{n,j} - \frac{k}{k+1}n}{\sqrt{n}} &\xrightarrow{(d)} \zeta_j, && 1 \le j \le k+1,\\
\frac{D_{n,j} - \frac{1}{k+1}n}{\sqrt{n}} &\xrightarrow{(d)} -\zeta_j, && 1 \le j \le k+1,\\
\frac{X_{n,j} - \frac{1}{k+1}n}{\sqrt{n}} &\xrightarrow{(d)} \xi_j := -\zeta_{j+1}, && 1 \le j \le k,\\
\frac{Y_{n,j} - \frac{1}{k+1}n}{\sqrt{n}} &\xrightarrow{(d)} \eta_j := -\zeta_j, && 1 \le j \le k,\\
\frac{Z_{n,j} - \frac{k}{k+1}n}{\sqrt{n}} &\xrightarrow{(d)} \zeta_{j+1}, && 1 \le j \le k-1,\\
\frac{X_n - \frac{k}{k+1}n}{\sqrt{n}} &\xrightarrow{(d)} \xi := \zeta_1,\\
\frac{Y_n - \frac{k}{k+1}n}{\sqrt{n}} &\xrightarrow{(d)} \eta := \zeta_{k+1},\\
\frac{Z_n - \frac{k(k-1)}{k+1}n}{\sqrt{n}} &\xrightarrow{(d)} \zeta := \sum_{j=2}^{k} \zeta_j = -\xi - \eta.
\end{aligned}$$

A simple calculation shows that the covariance matrix of $(\xi, \eta, \zeta)$ is (cf. Remark 13)

$$\begin{pmatrix}
\frac{k^2}{(k+1)^2(k+2)} & -\frac{k}{(k+1)^2(k+2)} & -\frac{k(k-1)}{(k+1)^2(k+2)}\\
-\frac{k}{(k+1)^2(k+2)} & \frac{k^2}{(k+1)^2(k+2)} & -\frac{k(k-1)}{(k+1)^2(k+2)}\\
-\frac{k(k-1)}{(k+1)^2(k+2)} & -\frac{k(k-1)}{(k+1)^2(k+2)} & \frac{2k(k-1)}{(k+1)^2(k+2)}
\end{pmatrix}.$$

For $k = 2$, this yields the univariate limit theorems by Bona [6] and the multivariate limit theorem by Janson [18] for $X_n, Y_n, Z_n$.

For $k = 1$, the result for $X_n$ or $Y_n$ reduces to the classical result on the asymptotics of the number of ascents or descents in a random permutation. (In this case $Z_n = 0$.)

## 6. The distribution of the number of blocks

The number $S_n$ of blocks in a random $k$-Stirling permutation is described by another urn model.

**Urn II.** This urn has balls of two colours, black and white. At each time step, draw a ball at random from the urn, replace it and add $k$ further balls: if the drawn ball was black, add $k$ black balls; if the drawn ball was white, add 1 white ball and $k-1$ black. Let $B_n$ and $W_n$ be the numbers of black and white balls in the urn at time $n$, and start with $W_n = 2$, $B_n = k-1$.

We thus have $B_n + W_n = kn + 1$ balls in the urn at time $n$, and it is easily seen that the number of white balls can be interpreted as the number of gaps between the blocks, or first or last, in a random $k$-Stirling permutation of order $n$, i.e. as the number of gaps where addition of a string $(n+1)^k$ create a new block. This is one more than the number of blocks, and thus we have the equality in distribution

$$S_n \stackrel{\mathrm{d}}{=} W_n - 1. \tag{19}$$

Urn II is thus a $2 \times 2$ generalized Pólya urn with ball replacement matrix $M = \left(\begin{smallmatrix} k & 0 \\ k-1 & 1 \end{smallmatrix}\right)$. This urn model is a special case of the triangular $2 \times 2$ urn models analysed in detail by Janson [17], where the asymptotic distribution is given. The special case of balanced triangular $2 \times 2$ urn models was also studied by Flajolet et al. [12]. (An urn is called balanced if the total number of added balls is constant, independently of the observed color.) For the special case treated here we can add the exact distribution using the tree representation, the moments of $S_n$, and almost sure convergence.

**Theorem 12.** *The probability mass function of the random variable $S_n$ counting the number of blocks in a random $k$-Stirling permutation of order $n$ is given by*

$$\mathbb{P}\{S_n = m\} = \sum_{\ell=0}^{m} \binom{m}{l} (-1)^\ell \frac{\binom{n-\frac{\ell}{k}-1}{n}}{\binom{n+\frac{1}{k}-1}{n}}, \qquad 1 \le m \le n.$$

*The binomial moments $\mathbb{E}\binom{S_n+r}{r}$ are given by the explicit formula*

$$\mathbb{E}\binom{S_n+r}{r} = \frac{\binom{n-1+\frac{r+1}{k}}{n}}{\binom{n-1+\frac{1}{k}}{n}} = (r+1)\frac{\binom{n-1+\frac{r+1}{k}}{n-1}}{\binom{n-1+\frac{1}{k}}{n-1}}, \quad r = 1, 2, \dots$$

*The random variable $\mathcal{S}_n := \frac{\binom{n-1+\frac{1}{k}}{n-1}}{\binom{n-1+\frac{2}{k}}{n-1}}(S_n+1)$ is a positive martingale and converges almost surely to a limit $\tilde{\zeta}$, i.e. $\mathcal{S}_n \xrightarrow{(a.s.)} \tilde{\zeta}$, further*

$$n^{-1/k} S_n \xrightarrow{(a.s.)} \zeta = \frac{\Gamma(1+\frac{1}{k})}{\Gamma(1+\frac{2}{k})}\tilde{\zeta}.$$

*The limits $\tilde{\zeta}$ and $\zeta$ can be specified by the moments*

$$\mathbb{E}(\zeta^r) = (r+1)!\frac{\Gamma(1+\frac{1}{k})}{\Gamma(1+\frac{r+1}{k})}, \quad r \ge 0.$$

*Further, $\zeta$ has a density function $f(x)$ that can be written as $f(x) = \Gamma(\frac{1}{k})x^{-k}g(x^{-k})$, $x > 0$, where $g$ is the density function of a positive $\frac{1}{k}$-stable distribution with Laplace transform $e^{-\lambda^{1/k}}$; it is thus given by the series expansion*

$$f(x) = \frac{\Gamma(\frac{1}{k})}{\pi} \sum_{j=1}^{\infty} (-1)^{j-1} \frac{\Gamma(\frac{j}{k}+1)\sin\frac{j\pi}{k}}{j!} x^j, \qquad x > 0.$$

**Remark 14.** The simple structure of the binomial moments and the almost sure convergence is actually true for all balanced triangular urns of the form $M = \left(\begin{smallmatrix} \alpha & 0 \\ \beta-\alpha & \beta \end{smallmatrix}\right)$, $0 < \alpha < \beta$, which is easily seen to be true by extending the martingale arguments to thie general case.

*Proof.* We use three different approaches to study the block structure $S_n$ in $k$-Stirling permutations or equivalently the number of left-right edges $LR_n$ in $(k+1)$-ary increasing trees, see (17). In order to obtain the explicit results for the probability distribution of $S_n$, we analyze $LR_n$. We can use the tree decomposition (10) in order to obtain the differential equation

$$\frac{\partial}{\partial z} T(z, v) = v(1 + T(z, v))^2 (1 + T(z))^{k-1}, \quad T(0, v) = 0,$$

where $T(z, v) = \sum_{n\geq 1} \sum_{m\geq 1} \mathbb{P}\{S_n = m\} T_n \frac{z^n}{n!} v^m$ denotes the bivariate generating function of the numbers $\mathbb{P}\{S_n = m\} T_n$, and $T(z) = T(z, 1)$ is the generating function of total weights of $(k+1)$-ary increasing trees. By Example 4, $1 + T(z) = (1 - kz)^{-1/k}$. Solving the differential equation and adapting to the initial condition gives the solution

$$T(z, v) = \frac{1}{1 - v\big(1 - (1 - kz)^{1/k}\big)} - 1.$$

Extraction of coefficients gives then the stated result for the probability mass function. Moreover, the stated binomial moments may be obtained from the generating function by extracting coefficients,

$$\mathbb{E}\binom{S_n + r}{r} = \frac{n!}{T_n} [z^n w^r] \frac{1}{1 - w} T\Big(z, \frac{1}{1 - w}\Big).$$

For the almost sure convergence we proceed as follows. Let $W_n = S_n + 1$ be the number of gaps between blocks, or, equivalently, the number of white balls in Urn II, see (19). Let $\mathcal{F}_n$ denote the $\sigma$-field generated by the first $n$ steps. Moreover denote by $\Delta_n = W_n - W_{n-1} = S_n - S_{n-1} \in \{0, 1\}$ the increment at step $n$. We have

$$\mathbb{E}(W_n \mid \mathcal{F}_{n-1}) = \mathbb{E}(W_{n-1} + \Delta_n \mid \mathcal{F}_{n-1}) = W_{n-1} + \mathbb{E}(\Delta_n \mid \mathcal{F}_{n-1}).$$

Since the probability that a new white ball is generated at step $n$ is proportional to the number of existing white balls (at step $n - 1$), we obtain further

$$\mathbb{E}(W_n \mid \mathcal{F}_{n-1}) = W_{n-1} + \frac{W_{n-1}}{k(n-1)+1} = \frac{k(n-1)+2}{k(n-1)+1} W_{n-1}, \quad n \geq 2.$$

Hence,

$$\mathbb{E}(\mathcal{S}_n \mid \mathcal{F}_{n-1}) = \frac{\binom{n-1+\frac{1}{k}}{n-1}}{\binom{n-1+\frac{2}{k}}{n-1}} \frac{k(n-1)+2}{k(n-1)+1} W_{n-1} = \mathcal{S}_{n-1}, \quad n \geq 2.$$

Hence, $\mathcal{S}_n$ is a martingale. Since it is a positive martingale, it converges almost surely to a limit $\tilde{\zeta}$. By the well-known asymptotic formula $\binom{n+a}{n} \sim n^a/\Gamma(a + 1)$, for any fixed real $a$, $\mathcal{S}_n \sim \frac{\Gamma(1+\frac{2}{k})}{\Gamma(1+\frac{1}{k})} n^{-1/k} S_n$ (provided $S_n \to \infty$), and thus $\mathcal{S}_n \xrightarrow{(a.s.)} \tilde{\zeta}$ can also be written $n^{-1/k} S_n \xrightarrow{(a.s.)} \zeta$.

More generally, we similarly have for any positive integer $r$

$$\mathbb{E}\left(\binom{S_n+r}{r} \,\middle|\, \mathcal{F}_{n-1}\right) = \binom{S_{n-1}+r}{r} + \binom{S_{n-1}+r}{r-1}\frac{S_{n-1}+1}{k(n-1)+1}$$
$$= \binom{S_{n-1}+r}{r}\frac{n-1+\frac{r+1}{k}}{n-1+\frac{1}{k}}.$$

Hence, $\binom{S_n+r}{r}\frac{\binom{n-1+\frac{1}{k}}{n-1}}{\binom{n-1+\frac{r+1}{k}}{n-1}}$ is a martingale, which also leads to the stated result for the moments in an alternative way, using the recurrence relation for the unconditional expectation.

Letting $n\to\infty$ in the moment formula yields

$$\mathbb{E}\binom{S_n+r}{r} = (r+1)\frac{\binom{n-1+\frac{r+1}{k}}{n-1}}{\binom{n-1+\frac{1}{k}}{n-1}} \sim (r+1)\frac{\Gamma(1+\frac{1}{k})}{\Gamma(1+\frac{r+1}{k})}n^{r/k}$$

which leads to $\mathbb{E}S_n^r \sim (r+1)!\frac{\Gamma(1+\frac{1}{k})}{\Gamma(1+\frac{r+1}{k})}n^{r/k}$. Hence all moments of $n^{-1/k}S_n$ converge, and the limits must be the moments of $\zeta$. Letting $r\to\infty$, we see that the moments do not grow too fast so that the moment generating function $\mathbb{E}e^{t\zeta}$ is finite for all $t$, and thus the distribution is determined by the moments.

Finally, we use the general results for urn models in Janson [17]. First, [17, Theorem 1.3(v)] yields the convergence $W_n \xrightarrow{(d)} \zeta$ in distribution, and [17, Theorem 1.7] yields the moments of $\zeta$ that we just derived in a different way; note however that [17, Theorem 1.7] yields the formula above also for non-integer $r \geq 0$, with the standard interpretation $(r+1)! = \Gamma(r+2)$. Furthermore, [17, Theorem 1.8] shows that $\zeta$ has a density and gives the explicit formulas stated above. $\square$

## 7. The sizes of the blocks

Recall that every block in a $k$-Stirling permutation begins and ends with the same label, which we can regard as a label of the block. We order the blocks in the block decomposition as $\widetilde{\mathcal{K}}_1,\dots,\widetilde{\mathcal{K}}_s$ according to this label (where $s$ is the number of blocks); thus $\widetilde{\mathcal{K}}_1$ is the block extending from the first 1 to the last, $\widetilde{\mathcal{K}}_2$ is the block formed by the smallest label not in $\widetilde{\mathcal{K}}_1$, and so on. We also let $\widetilde{K}_i := |\widetilde{\mathcal{K}}_i|$ denote the size of the $i$th block in this order, and put $\widetilde{\mathcal{K}}_i=\emptyset$, $\widetilde{K}_i = 0$ for $i > s$.

Alternatively, we may order the blocks according to decreasing size. We let $K_1 \geq K_2 \geq \dots$ be the sizes of the blocks in this order, again with $K_i = 0$ for $i > s$. Thus, $(K_i)_1^\infty$ is the decreasing rearrangement of $(\widetilde{K}_i)_1^\infty$.

For a random $k$-Stirling permutation of order $n$, we use the notations $\widetilde{\mathcal{K}}_{n,i}$, $\widetilde{K}_{n,i}$ and $K_{n,i}$. Note that $\sum_i \widetilde{K}_{n,i} = \sum_i K_{n,i} = kn$.

To study these sizes we introduce another urn model. Consider first an urn with balls of two colours, $\widetilde{K}_{n,1}-1$ white balls representing the gaps inside the block $\widetilde{\mathcal{K}}_{n,1}$ and $nk+2-\widetilde{K}_{n,1}$

black balls representing the gaps outside. Adding the string $(n+1)^k$ at one of the gaps inside $\widetilde{\mathcal{K}}_{n,1}$ means increasing $\widetilde{K}_{n,1}$ by $k$, and adding it outside means keeping $\widetilde{K}_{n,1}$ unchanged; hence this is a Pólya urn of the original type considered by Eggenberger and Pólya [11], [28], where we draw a ball at random and replace it together with $k$ balls of the same colour. We start with $\widetilde{K}_{1,1} = k$, and thus $k-1$ white and 2 black balls.

Next, let us study the second block, $\widetilde{\mathcal{K}}_{n,2}$. At the first $n$ where this is non-empty, we have $k+2$ gaps outside the first block $\widetilde{\mathcal{K}}_{n,1}$, $k-1$ of them in $\widetilde{\mathcal{K}}_{n,2}$ and 3 outside both blocks. Let us now ignore the first block and consider an urn with $\widetilde{K}_{n,2}-1$ white balls representing the gaps in $\widetilde{\mathcal{K}}_{n,2}$ and black balls representing the gaps outside both $\widetilde{\mathcal{K}}_{n,1}$ and $\widetilde{\mathcal{K}}_{n,2}$. The balls in this urn are drawn at random times (when we do not add to a gap in $\widetilde{\mathcal{K}}_{n,1}$), but when they are drawn, the urn behaves exactly as for $\widetilde{\mathcal{K}}_{n,1}$: we replace the drawn ball together with $k$ of the same colour.

The same argument applies to $\widetilde{\mathcal{K}}_{n,m}$ for any $m \geq 2$; if we ignore the preceding blocks and additions to them, we have the same Pólya urn again, but now started with $m+1$ black balls, representing the gaps outside the first $m$ blocks. We hence make the following definition.

**Urn III.** This is the standard Pólya urn with balls of two colours and where each drawn ball is replaced together with $k$ balls of the same colour. Let Urn $\mathrm{III}_m$ be the version where we start with $k-1$ white and $m+1$ black balls, and let $W_{N,m}$ and $B_{N,m}$ denote the numbers of white and black balls after $N-1$ draws, when the urn contains $W_{N,m} + B_{N,m} = kN + m$ balls.

We can thus identify (with the urns Urn $\mathrm{III}_1$, Urn $\mathrm{III}_2$, ... independent), recalling that the balls in urn Urn $\mathrm{III}_{m+1}$ correspond to the black balls in urn Urn $\mathrm{III}_m$,

$$\begin{aligned}
\widetilde{K}_{n,1} &= W_{n,1} + 1, \\
\widetilde{K}_{n,2} &= W_{N_2,2} + 1, \quad \text{with} \quad kN_2 + 2 = B_{n,1}, \\
\widetilde{K}_{n,3} &= W_{N_3,3} + 1, \quad \text{with} \quad kN_3 + 3 = B_{N_2,1},
\end{aligned}$$

and so on.

**Theorem 13.** *There exists a sequence of independent beta distributed random variables $\beta_m \sim \mathrm{Beta}(\frac{k-1}{k}, \frac{m+1}{k})$ such that*

$$\frac{1}{kn}(\widetilde{K}_{n,1}, \widetilde{K}_{n,2}, \dots) \xrightarrow{(a.s.)} (\beta_1, (1-\beta_1)\beta_2, (1-\beta_1)(1-\beta_2)\beta_3, \dots). \tag{20}$$

*Proof.* The basic limit theorem for Pólya urns says that, as $N \to \infty$,

$$\frac{W_{N,m}}{kN} \xrightarrow{(a.s.)} \beta_m \sim \mathrm{Beta}\Big(\frac{k-1}{k}, \frac{m+1}{k}\Big)$$

and thus $\frac{B_{N,m}}{kN} \xrightarrow{(a.s.)} 1-\beta_m$. (This is already in Pólya [28] for convergence in distribution. See, for example, [19] or [17, Section 11].) Consequently,

$$\frac{\widetilde{K}_{n,1}}{kn} = \frac{W_{n,1}+1}{kn} \xrightarrow{(a.s.)} \beta_1,$$
$$\frac{\widetilde{K}_{n,2}}{kn} = \frac{B_{n,1}}{kn} \cdot \frac{W_{N_2,2}+1}{kN_2+2} \xrightarrow{(a.s.)} (1-\beta_1)\beta_2,$$

and so on. □

Note that both sides of (20) are elements of $\mathcal{P}$, the space of non-negative sequences $(p_i)_1^\infty$ with $\sum_i p_i = 1$; $\mathcal{P}$ can be seen as the space of probability distributions on $\mathbb{N}$. The convergence in the proof above was componentwise, i.e. in the product topology, but it is well-known (and easy to verify) that on $\mathcal{P}$, this topology is equivalent to the $\ell^1$-topology with the metric $d((p_i),(p'_i)) = \sum_i |p_i - p'_i|$, and also to the usual weak topology of probability distributions; hence the theorem holds for any of these topologies.

Let $\widetilde{V}_i = \beta_i \prod_{j=1}^{i-1}(1-\beta_j)$ be the elements of the limit sequence in (20), and let $(V_i)_1^\infty$ denote the decreasing rearrangements of them. The distribution of this random element of $\mathcal{P}$ is denoted $\mathrm{PD}(\frac{1}{k},\frac{1}{k})$, see Pitman and Yor [27] or Bertoin [4].

Taking the decreasing rearrangement is a continuous operation on $\mathcal{P}$, and thus we immediately obtain from Theorem 13 the following.

**Theorem 14.**
$$\frac{1}{kn}(K_{n,1},K_{n,2},\dots) \xrightarrow{(a.s.)} (V_1,V_2,\dots) \sim \mathrm{PD}\Big(\frac{1}{k},\frac{1}{k}\Big). \tag{21}$$

**Corollary 1.** *The largest block size has the limit*
$$\frac{K_{n,1}}{kn} \xrightarrow{(a.s.)} V_1 = \max_{i\ge 1} \widetilde{V}_i.$$

**Remark 15.** These results can be compared with the classical result that the lengths of the cycles in a random permutation, arranged in decreasing order and divided by the size of the permutation, converge (in distribution) to $\mathrm{PD}(1) = \mathrm{PD}(0,1)$, see e.g. [2, Sections 5.5 and 5.7].

**Remark 16.** For $k = 2$ we obtain in Theorem 14 the limit distribution $\mathrm{PD}(\frac{1}{2},\frac{1}{2})$ which arises in other contexts too: it is the distribution of the sequence of excursion lengths in a Brownian bridge [26], [25], [1], [27] (for a related characterization for $k > 2$ see [27]) and it is the asymptotic distribution of the sizes of the tree components in a random mapping, see [29] and [1]. It is an interesting problem to see whether there are more direct relations with these objects.

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