Corrigendum to: "A central limit theorem for random ordered factorizations of integers"
[H.-K. Hwang and S. Janson, *Electron. J. Probab.* **16** (2011), 347–361]

Dr. Ian Morris (University of Surrey) kindly pointed out that our application of Delange's Tauberian theorem contains a gap, which arises from the fact that $D_3(s)$ (and thus $D_1(s)$) has a branch-type singularity at ρ; see (1) below. Thus the function G (pp. 350–351 in our paper [3]) in the statement of Delange's Tauberian theorem fails to be analytic at ρ. This gap can be readily filled by the following arguments.

We first show that D_1 has a branch singularity at $s=\rho$. Let $k=2\ell-1$, $\ell\geqslant 1$. Consider (same notations as in [3])

$$\begin{aligned} D_1(s) &:= \sum_{n\geqslant 1} n^{-s}\sum_{m\geqslant 0} a_m(n)\Big((m-\mu\log n)^k + (\log n)^{k/2}\Big)^2, \\ D_2(s) &:= \sum_{n\geqslant 1} n^{-s}\sum_{m\geqslant 0} a_m(n)\Big((m-\mu\log n)^{2k} + (\log n)^k\Big) \\ &= \mathcal{M}_{2k}(s) + (-1)^k\mathcal{A}^{(k)}(s), \end{aligned}$$

and

$$D_3(s) := \tfrac{1}{2}(D_1(s)-D_2(s)) = (-1)^\ell \pi^{-1/2}\int_0^\infty \mathcal{M}_k^{(\ell)}(s+t)t^{-1/2}\,\mathrm{d}t.$$

By induction using the recurrence (Eq. (2.11) in [3])

$$\mathcal{M}_k(s) = \frac{1}{1-\mathcal{P}(s)}\sum_{0\leqslant j<k}\binom{k}{j}\mathcal{M}_j(s)\mathcal{B}_{k-j}(s) \qquad (k\geqslant 1)$$

with $\mathcal{M}_0(s) = 1/(1-\mathcal{P}(s))$, where $\mathcal{B}_k(s) := \sum_{0\leqslant \ell\leqslant k}\binom{k}{\ell}\mu^\ell\mathcal{P}^{(\ell)}(s)$, we deduce the local expansion

$$\mathcal{M}_k(s) = \sum_{1\leqslant j\leqslant k+1} c_j(s-\rho)^{-j} + H_\rho(s),$$

for some coefficients c_j, where the generic symbol $H_c(s)$ represents an analytic function for $\Re(s)\geqslant c$, not necessarily the same at each occurrence. This in turn yields

$$D_3(s) = (-1)^\ell \sum_{1\leqslant j\leqslant k+1}\frac{c_j\Gamma(j-1/2)}{(j-1)!}(s-\rho)^{-j+1/2} + H_\rho(s). \tag{1}$$

Now

$$\begin{aligned} D_1(s) &= D_2(s) + 2D_3(s) \\ &= \sum_{1\leqslant j\leqslant k+1}\bar{c}_j(s-\rho)^{-j} + 2(-1)^\ell\sum_{1\leqslant j\leqslant k+1}\frac{c_j\Gamma(j-1/2)}{(j-1)!}(s-\rho)^{-j+1/2} + H_\rho(s), \end{aligned} \tag{2}$$

for some coefficients $\bar{c}_j$.

Thus, due to the presence of the branch singularity at $s=\rho$, we cannot apply the Tauberian theorem as that stated in [3]. However, as pointed out to us by Dr. Morris, we can apply the more general version of Delange's Tauberian theorem (also due to Delange; see [1, Theorem III] or [2, Theorem A]).

Let $F(s) := \sum_{n\geqslant 1} \alpha(n) n^{-s}$ *be a Dirichlet series with nonnegative coefficients and convergent for* $\Re(s) > \varrho > 0$*. Assume (i)* $F(s)$ *is analytic for all points on* $\Re(s) = \varrho$ *except at* $s = \varrho$*; (ii) for* $s \sim \varrho$, $\Re(s) > \varrho$,

$$F(s) = \frac{G(s)}{(s-\varrho)^{\beta}} + \sum_{1\leqslant j\leqslant m} (s-\varrho)^{-\beta_j} G_j(s) + H(s) \qquad (\beta > 0),$$

where $m \geqslant 0$, $\Re(\beta_j) < \beta$ *and* G, H *and the* G_j*'s are analytic at* $s = \varrho$ *with* $G(\varrho) \neq 0$*. Then*

$$\sum_{n\leqslant N} \alpha(n) \sim \frac{G(\varrho)}{\varrho\Gamma(\beta)} N^{\varrho} (\log N)^{\beta-1},$$

as $N \to \infty$*.*

An alternative approach to fill the gap, still relying on the Tauberian theorem stated in [3], is to subtract from D_1 suitable functions having the same local expansion near ρ. More precisely, define

$$Z_\alpha(s) := \sum_{n\geqslant 2} n^{-s} (\log n)^{\alpha} \qquad (\alpha > 0).$$

Then ($m := \lfloor \alpha \rfloor$ and $\theta := \{\alpha\}$)

$$Z_{m+\theta}(s) = \frac{(-1)^{m+1}}{\Gamma(1-\theta)} \int_0^\infty \zeta^{(m+1)}(s+t) t^{-\theta} \, \mathrm{d}t,$$

where ζ denotes Riemann's zeta function. Note that

$$\zeta(s) = \frac{1}{s-1} + \text{entire function},$$

so that

$$\begin{aligned} Z_\alpha(s) &= \frac{(m+1)!}{\Gamma(1-\theta)} (s-1)^{-1-m-\theta} \int_0^\infty x^{-\theta} (1+x)^{-m-2} \, \mathrm{d}x + H_1(s) \\ &= \Gamma(1+\alpha) \, (s-1)^{-1-\alpha} + H_1(s). \end{aligned}$$

Now let

$$D_4(s) := -2(-1)^{\ell} \sum_{1\leqslant j\leqslant k+1} \frac{c_j}{(j-1)!} Z_{j-3/2}(s+1-\rho) + C Z_k(s+1-\rho),$$

where C is chosen so large that D_4 has only nonnegative coefficients. Consider now

$$D_1(s) + D_4(s).$$

Then, by (2), Delange's Tauberian theorem (in the form stated in [3]) applies.

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References

[1] H. Delange, Généralisation du théorème de Ikehara, *Ann. Sci. Éc. Norm. Supér.* **71** (1954), 213–242.

[2] H. Delange, Théorèmes taubériens et applications arithmétiques, in *Théorie des Nombres, Séminaire Delange-Pisot* **4** (1962/63), No. 16, 17 pages (1967).

[3] H.-K. Hwang and S. Janson, A central limit theorem for random ordered factorizations of integers, *Electron. J. Probab.* **16** (2011), 347–361.