Zeros of Sections of the Binomial Expansion

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Dedicated to Richard S. Varga, on the occasion of his 80th birthday.

Abstract

We examine the asymptotic behaviour of the zeros of sections of the binomial expansion. That is, we consider the distribution of zeros of $B_{r,n}(z) = \sum_{k=0}^{r} \binom{n}{k} z^k$, where $1 \leq r < n$.

1 Preliminaries

A problem of great interest in the classical Complex Function Theory is the following:

Given a function $f(z) = \sum_{k=0}^{\infty} a_k z^k$, analytic at $z = 0$, determine the asymptotic distribution of the zeros of the *partial sums* $s_n(z) = \sum_{k=0}^{n} a_k z^k$.

Some contributors to this area include Jentzsch [6], who explored the problem for a finite radius of convergence; Szegő [13], who explored the exponential function e^z; Rosenbloom [12], who discussed the angular distribution of zeros using potential theory, and applied his work to sub-class of the confluent hypergeometric functions; Erdős and Turán [4], who used minimization techniques to discuss angular distributions of zeros; Newman and Rivlin [7, 8], who related the work of Szegő to the Central Limit Theorem; Edrei, Saff and

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Varga [3], who gave a thorough analysis for the family of Mittag-Leffler functions; Carpenter, Varga and Waldvogel [2], who refined the work of Szegő; and Norfolk [9, 10], who refined the work of Rosenbloom on the confluent hypergeometric functions and a related set of integral transforms.

In this paper, we will analyze the behaviour of the zeros of sections of the binomial expansion, that is

$$B_{r,n}(z) = \sum_{k=0}^{r} \binom{n}{k} z^k \ , 1 \le r \le n \ . \tag{1.1}$$

This investigation not only fits into the general theme of the works cited, but also arises from matroid theory. Specifically (cf [14]), the *univariate reliability polynomial* for the uniform matroid $U_{r,n}$ is given by

$$\mathrm{Rel}_{r,n}(q) = (1-q)^n B_{r,n}\left(\frac{q}{1-q}\right) = \sum_{k=0}^{r} \binom{n}{k} q^k (1-q)^{n-k} \ , \tag{1.2}$$

which can be written as $\mathrm{Rel}_{r,n}(q) = (1-q)^{n-r} H_{r,n}(q)$, where

$$H_{r,n}(q) = \sum_{k=0}^{r} \binom{n}{k} q^k (1-q)^{r-k} = (1-q)^r B_{r,n}\left(\frac{q}{1-q}\right) \ . \tag{1.3}$$

Some special cases are easy to analyze, and may thus be dispensed with. In particular,

1. $B_{1,n}(z) = 1 + nz$, which has its only zero at $z = -\dfrac{1}{n}$.
2. $B_{n,n}(z) = (1+z)^n$, which clearly has a zero of multiplicity n at $z = -1$.
3. $B_{n-1,n}(z) = (1+z)^n - z^n$. Noting that this polynomial cannot have positive zeros, we obtain the zeros $z = \dfrac{\omega^k}{1-\omega^k}$, for $1 \le k \le n-1$, where $\omega = \exp\left(\dfrac{2\pi i}{n}\right)$ is the principal n-th root of unity, all of which lie on the vertical line $\mathrm{Re}\, z = -\dfrac{1}{2}$.

In what follows, we will therefore focus on the cases $1 \le r < n-1$, and give two collections of results. The first are concerned with bounding regions for the zeros of $B_{r,n}(z)$, the rest with convergence results.

We note that this problem was investigated independently by Ostrovskii [11], who obtained many of the results that we present here. However, those methods involved using a bilinear transformation to convert this problem to an integral formulation. This choice of formulation makes the proofs more involved and requires some additional constraints. By contrast, we claim that our methods given here flow directly from the structure of the problem, and yield additional results, in terms of additional bounds on the zeros, and limiting cases. The paper [11] also gives a result on the spacing of the zeros on the limit curve, using classical potential-theoretic methods. We do not duplicate that result here, but give formulations in terms of specific points on the curve.

The methods used generate a set of constants and related limit curves for $0 < \alpha < 1$, defined by

$$\frac{1}{2} \le K_\alpha = \alpha^\alpha (1-\alpha)^{1-\alpha} < 1 \ , \tag{1.4}$$

$$C_\alpha = \left\{ z \ : \ \frac{|z|^\alpha}{|1+z|} = K_\alpha, \ |z| \le \frac{\alpha}{1-\alpha} \right\} \ , \tag{1.5}$$

and

$$C'_\alpha = \left\{ z \ : \ \frac{|z|^\alpha}{|1+z|} = K_\alpha, \ \frac{\alpha}{1-\alpha} \le |z| \right\} \ . \tag{1.6}$$

The properties of these curves are outlined in Lemma 3.1.

2 Main Results

As discussed above, we begin with a theorem on bounds of the zeros of $B_{r,n}(z)$, and follow with results on convergence of those zeros.

Theorem 2.1 *Let r, n be positive integers, with $1 \le r < n-1$, and let z^* be any zero of $B_{r,n}(z) = \sum_{k=0}^{r} \binom{n}{k} z^k$.*

Then, z^* lies in a region defined by the intersection of two circles and a plane closed curve, to the right of a vertical line. Specifically,

$$|z^*| \le \frac{r}{n+1-r} \ , \tag{2.1}$$

$$\left| z^* - \frac{\gamma^2}{1-\gamma^2} \right| \le \frac{\gamma}{(1-\gamma^2)} \ , \ \text{where } \gamma = \frac{r}{n-1} \ , \tag{2.2}$$

$$\text{Re } z^* > -\frac{1}{2} \ , \tag{2.3}$$

and z^* lies exterior to the curve $C_{r/n}$, as defined in (1.4, 1.5).

Proof. We begin by considering the ratio of coefficients

$$\frac{\binom{n}{k}}{\binom{n}{k-1}} = \frac{n-k+1}{k} \ , \tag{2.4}$$

which is decreasing in k.

Hence, writing $B_{r,n}\left(\frac{r}{n-r+1}z\right) = \sum_{k=0}^{r} a_k z^k$, we have that

$$\frac{a_k}{a_{k-1}} = \frac{n-k+1}{k} \cdot \frac{r}{n-r+1} \ge 1 \ .$$

That is, $\{a_k\}_{k=0}^{r}$ is non-decreasing, so by the Eneström-Kakeya Theorem ([5], p. 462), the zeros of this polynomial satisfy $|z| \le 1$. Hence, the zeros of $B_{r,n}(z)$ satisfy $|z| \le \frac{r}{n-r+1}$.

For the second bounding circle, we defer to Wagner [14], where it is shown, again using the Eneström-Kakeya Theorem, that the zeros of $H_{r,n}(q)$ as given in (1.3), lie in the annulus

$$\frac{1}{n-r} \le |q| \le \frac{r}{n-1} \ .$$

Since $z = -1$ is clearly not a zero of $B_{r,n}(z)$ for $r < n$, we may make the substitution $z = \frac{q}{1-q}$ (or equivalently $q = \frac{z}{1+z}$) in (1.3), which shows immediately that $H_{r,n}(q) = (1+z)^{-r} B_{r,n}(z)$, from which

$$\left| \frac{z}{1+z} \right| \le \frac{r}{n-1} =: \gamma \ . \tag{2.5}$$

Writing this last inequality in terms of the real and imaginary parts of z yields the claimed result.

Noting that (2.5) implies that $\left|\frac{z}{1+z}\right| < 1$, yields the half-plane Re $z > -\frac{1}{2}$, as claimed.

For the final bound, we mimic the analysis of Buckholtz [1] on the partial sums of e^z, and write

$$(1+z)^{-n} B_{r,n}(z) = 1 - \frac{z^r}{(1+z)^n} \cdot R_{r,n}(z) \ , \tag{2.6}$$

where

$$R_{r,n}(z) = \sum_{k=r+1}^{n} \binom{n}{k} z^{k-r} \ = z^{n-r} B_{n-r-1,n}\left(\frac{1}{z}\right) . \tag{2.7}$$

For clarity, we set $\beta = \frac{r}{n}$. Inside and on the curve C_β (1.4,1.5), we have $|z| < \frac{\beta}{1-\beta}$ and $\left|\frac{z^r}{(1+z)^n}\right| \le K_\beta^n$, where K_β is defined in (1.4). This, with the upper bound of Lemma 3.3 yields

$$\left|(1+z)^{-n} B_{r,n}(z)\right| \ge 1 - \left|\frac{z^r}{(1+z)^n}\right| \cdot |R_{r,n}(z)| \ > 1 - K_\beta^n \cdot K_\beta^{-n} = 0 \ , \tag{2.8}$$

which is the desired result. *Q.E.D.*

Note that the second bounding circle of this result, namely

$$\left|z - \frac{\alpha^2}{1-\alpha^2}\right| = \frac{\alpha}{1-\alpha^2} \ ,$$

intersects the negative real axis at $z = -\frac{\alpha}{1+\alpha}$. This circle is contained in the first, namely $|z| = \frac{\alpha}{1-\alpha}$, and both meet at the common point $z = \frac{\alpha}{1-\alpha}$.

The limiting case $|z| = \frac{\alpha}{1-\alpha}$ of the first bounding circle, and the bounding half-plane Re $z > -1/2$ both appear in [11], with proofs that require significantly more detailed derivations.

We now use these results, and the bounds in the proof, to discuss some convergence results.

Theorem 2.2 *Suppose that $1 \leq r_j < n_j - 1$ for all j, that $\lim_{j\to\infty} n_j = \infty$, and that*

$$\lim_{j\to\infty} \frac{r_j}{n_j} = \alpha,\ 0 < \alpha < 1 \ .$$

Then, the zeros of $\{B_{r_j,n_j}(z)\}_{j=1}^{\infty}$ converge uniformly to the points of C_α.

Proof. For clarity, we let $\beta = \dfrac{r}{n}$, dispense with the subscript j, and write $B_{r,n}(z)$ using (2.6). The zeros of $R_{r,n}(z)$ then satisfy

$$\frac{z^r}{(1+z)^n} \cdot R_{r,n}(z) = 1 \ .$$

Given that the polynomial can have no positive real zeros, we may take roots with a cut along the positive real axis, and write the equation as

$$\left(\frac{z^\beta}{1+z} \cdot R_{r,n}^{1/n}(z)\right)^n = 1 \ . \tag{2.9}$$

Using Theorem 2.1, Lemma 3.1 and Lemma 3.3, these zeros lie outside the curve C_β, and thus satisfy $\nu\beta < X_\beta \leq |z| \leq \dfrac{\beta}{1-\beta}$. Hence,

$$\frac{\nu r}{n(r+1)} \leq \frac{K_\beta^n R_{r,n}(z)}{\sum_{k=r+1}^n \binom{n}{k} \beta^k (1-\beta)^{n-k}} \leq 1 \ ,$$

for this region.

Consequently, $R_{r,n}^{1/n}(z) \to K_\beta^{-1}$ uniformly, from which the desired zeros are asymptotically the solutions to

$$\left(\frac{z^\beta}{1+z} \cdot K_\beta^{-1}\right)^n = 1 \ . \tag{2.10}$$

As described in Lemma 3.1, the function $w = K_\beta^{-1} \dfrac{z^\beta}{1+z}$ maps the curve C_β (with the singular point $z_\beta = \frac{\beta}{1-\beta}$ deleted), onto the arc $0 < \text{Arg w} < 2\pi\beta$. Thus, for n large, there exist points ζ_m, $1 \leq m < r$, for which

$$\frac{\zeta_m}{1+\zeta_m} \cdot K_\beta^{-1} = \exp\left(\frac{2\pi m i}{n}\right) \ . \tag{2.11}$$

These points are clearly solutions to (2.10), are asymptotically dense on the curve C_β, and approximate the desired zeros of the polynomial. Taking limits as $\beta \to \alpha$ thus yields the desired result. *Q.E.D.*

We note that, thanks to (2.7), the non-trivial zeros of $R_{r,n}(z)$ converge uniformly to all points which lie on the curve C'_α, as defined in (1.6).

This result also appears in [11], using more elaborate asymptotics. The analysis presented requires a deletion of a neighbourhood of the singular point $z_\alpha = \dfrac{\alpha}{1-\alpha}$. Comparison with the results of Lemma 3.3 shows that this is not necessary with our methods.

The remaining results presented here do not appear in the literature.

The asymptotic expansions in the proof of Theorem 2.2 immediately give the following result on the rate of convergence. We note that, as shown in [2] in the case of the exponential function, this rate is best possible.

Theorem 2.3 *Suppose that r, n are large, and $0 < \delta < \dfrac{r}{n} < 1 - \delta$. Then, given any zero z^* of $B_{r,n}(z)$, there exists a constant c such that*

$$|z^* - \zeta| \leq \frac{c}{|z^* - \frac{r}{n-r}|} \cdot \frac{\ln n}{n} \ .$$

Additionally, proximity to the singular point $z_{r/n} = \dfrac{r}{n-r}$ is of order $O\left(\frac{1}{\sqrt{n}}\right)$.

Proof. As before, we set $\beta = \dfrac{r}{n}$. As described in the proof of Theorem 2.2, the points ζ_m, as defined in (2.11), uniformly approximate the zeros of the polynomial. Hence, given a zero z^* of $B_{r,n}(z)$, we may choose k so that $|z^* - \zeta_k| = \min |z^* - \zeta_j|$ is small.

Using the results of Lemma 3.3, we have the approximation

$$R_{r,n}^{1/n}(z) \cdot K_{r/n}^{-1} = 1 + G(z) \cdot \left(\frac{\ln n}{n}\right) \ .$$

where $G(z)$ is uniformly bounded in the region containing the zeros.

Expanding, using the equations (2.11) and (2.9) yields

$$z^* - \zeta_k \approx \frac{\zeta_k(1+\zeta_k)}{\beta-(1-\beta)\zeta_k} \cdot G(z^*)\left(\frac{\ln n}{n}\right) , \tag{2.12}$$

to first order. This not only gives the desired result, but shows that, as expected, the rate of convergence is worst for those points closest to the singular point $z_\beta = \dfrac{\beta}{1-\beta}$.

To discuss the convergence at the singular point, we take an approach similar to that used for the exponential function in [7, 8] and for the Mittag-Leffler functions in [3]. For convenience, we set $\beta = \dfrac{r}{n}$, $\mu = n\beta = r$, and $\sigma^2 = n\beta(1-\beta)$. Then,

$$f_{r,n}(w) = (1-\beta)^n B_{r,n}\left(\frac{\beta e^{w/\sigma}}{1-\beta}\right) = \sum_{k=0}^{r}\binom{n}{k}\beta^k(1-\beta)^{n-k}e^{kw/\sigma} ,$$

which is a truncated moment generating function for a binomial distribution with mean μ and variance σ. Using the Central Limit Theorem,

$$f_{r,n}(w) \approx \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\mu} e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)+\frac{tw}{\sigma}}dt .$$

Making the substitution $s = \dfrac{t-\mu-\sigma w}{\sqrt{2}\sigma}$ yields

$$e^{-\mu w/\sigma - w^2/2} f_{r,n}(w) \approx \frac{1}{\sqrt{\pi}}\int_{-\infty}^{-w/\sqrt{2}} e^{-s^2}ds = \frac{1}{2}\mathrm{erfc}\left(\frac{w}{\sqrt{2}}\right) ,$$

the complementary error function. Thus, given the zero χ of erfc(z) which is closest to the origin, there must exist a zero z^* of $B_{r,n}(z)$ for which

$$z^* \approx \frac{\beta e^{\sqrt{2}\chi/\sigma}}{1-\beta} \approx \frac{\beta}{1-\beta} + \sqrt{\frac{2\beta}{(1-\beta)^3}} \cdot \frac{\chi}{\sqrt{n}} ,$$

the desired result. *Q.E.D.*

The figures 1 and 2 show the zeros, bounding curve and bounding circles for the cases $r = 10, n = 30$ and $r = 30, n = 90$ respectively. Since the ratio r/n

is the same in both cases, they serve to illustrate both the rate of convergence of the zeros to the limit curve, and the rate of convergence of the bounding circles.

Figure 3 shows the zeros for the case $r = 40, n = 80$, as well as the curve $C_{1/2}$ and the points ζ_j of the proof of Theorem 2.3.

It should be noted at this point that, due to the structure of the coefficients of these polynomials, direct computation of the zeros for significantly higher degrees suffers due to numerical instability.

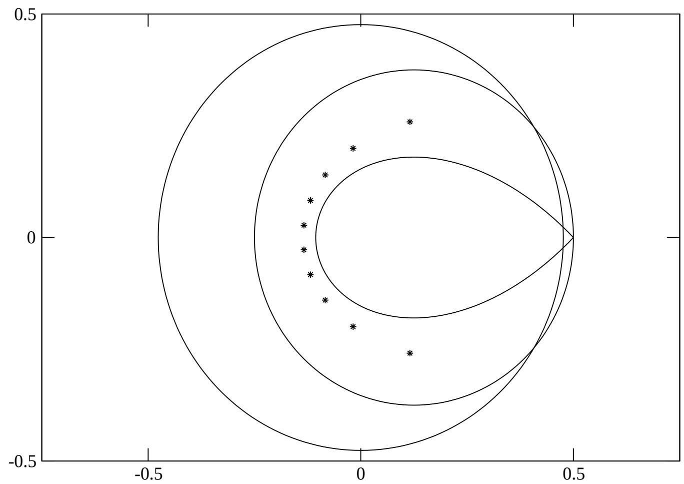

Figure 1: The bounding curves and zeros for $r = 10$, $n = 30$

We conclude by considering the limiting cases $\alpha = 0$ and $\alpha = 1$. The trivial result for $\alpha = 0$, given the radius $\dfrac{r}{n+1-r}$ of the bounding circle, is that all zeros converge uniformly to 0 in this case. However, a slight modification gives a much more interesting result.

Theorem 2.4 *Suppose that* $\lim_{j\to\infty} r_j = \infty$ *and that* $\lim_{j\to\infty} \dfrac{r_j}{n_j} = 0$.
Then, the limit points of the zeros of $\left\{ B_{r_j,n_j}\left(\dfrac{r_j z}{n_j - r_j}\right)\right\}_{j=1}^{\infty}$ *are precisely the points of the Szegő curve* $|ze^{1-z}| = 1$, $|z| \le 1$.

Proof. As in the above, we dispense with the subscript j for clarity. With the normalization, the results of Theorem 2.1 yield that the zeros of the normalized polynomial above satisfy

$$1 = \left(\frac{r}{n-r}\right)^r K_{r/n}^{-n} \frac{z^r}{\left(1+\frac{rz}{n-r}\right)^n} h(z) \text{ and } |z| \le 1\ , \tag{2.13}$$

where

$$h(z) = \sum_{k=r+1}^{n} \binom{n}{k} \left(\frac{r}{n}\right)^k \left(1-\frac{r}{n}\right)^{n-k} z^{k-r}. \tag{2.14}$$

Noting that

$$\left(\frac{r}{n-r}\right)^r K_{r/n}^{-n} = \left(1-\frac{r}{n}\right)^{-n}\quad ,$$

we may use standard expansions to convert (2.13) to the form

$$1 = (ze^{1-z+g(z)})^r h(z)\ , \tag{2.15}$$

where $|g(z)| \le \dfrac{3r}{n}$ uniformly in the unit disk.

Considering points inside and on the curve $|ze^{1-z}| = e^{-3r/n}$, and noting that $|h(z)| \le h(1) < 1$ on the unit disk, we may repeat the analysis of (2.8) to deduce that the zeros are uniformly bounded away from zero by $|z| \ge \eta > 0$. This implies that we may repeat the bounding process of Lemma 3.3 to deduce that $h^{1/r}(z) \to 1$ uniformly in $\eta \le |z| \le 1$, defining the roots by a cut along the positive real axis. This establishes the desired result. *Q.E.D.*

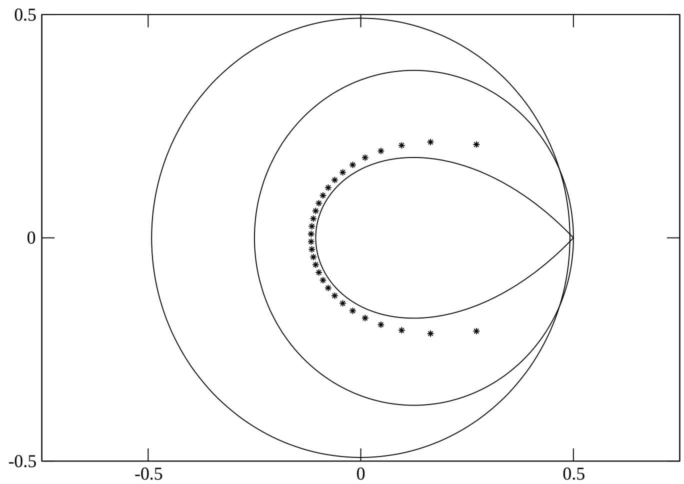

Figure 2: The bounding curves and zeros for $r = 30$, $n = 90$

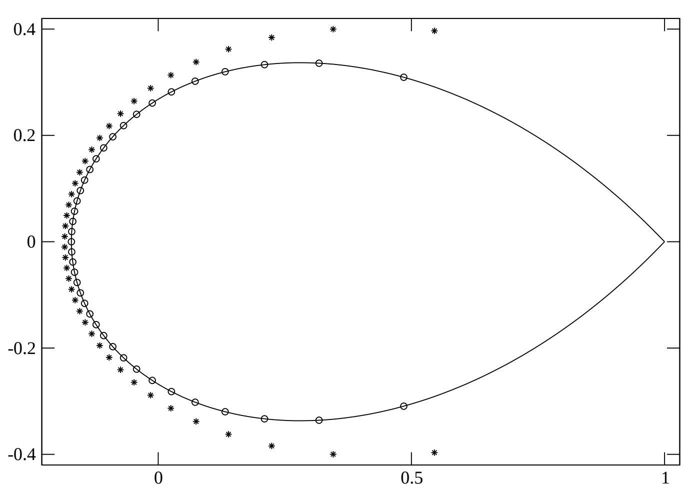

Figure 3: The curve and zeros for $r = 40$, $n = 80$

Finally, we consider the other limiting case.

Theorem 2.5 *Suppose that* $\lim_{j\to\infty} r_j = \infty$ *and* $\lim_{j\to\infty} \frac{r_j}{n_j} = 1$.
Then, the limit points of the zeros of the polynomials $\left\{B_{r_j,n_j}(z)\right\}_{j=1}^{\infty}$ *are precisely the points of the line* $\mathrm{Re}\ z = -1/2$.

Proof. As in the previous proofs, we dispense with the subscript j, and write the equation for the zeros as.

$$1 = \frac{z^r}{(1+z)^n} R_{r,n}(z)\ .$$

We again use the bounds of Lemma 3.3 and obtain the desired result, using the fact that $\lim_{\alpha\to 1^-} K_\alpha = 1$. *Q.E.D.*

3 Technical Results

Here we give the properties and inequalities necessary for the main results, beginning with the properties of the bounding curves.

Lemma 3.1 *Fix* $0 < \alpha < 1$, *and let*

$$K_\alpha = \alpha^\alpha (1-\alpha)^{1-\alpha} \tag{3.1}$$

and

$$C_\alpha = \left\{ z \ :\ \frac{|z|^\alpha}{|1+z|} = K_\alpha,\ |z| \le \frac{\alpha}{1-\alpha} \right\}\ . \tag{3.2}$$

Then,

1. $\frac{1}{2} \le K_\alpha < 1$, $\lim_{\alpha\to 0^+} K_\alpha = 1$, $\lim_{\alpha\to 1^-} K_\alpha = 1$.
2. C_α *is a simple, smooth closed curve, symmetric with respect to the real axis, starlike with respect to* $z = 0$, *which passes through* $z = \frac{\alpha}{1-\alpha}$.
3. *The intersection of* C_α *with the negative real axis occurs at* $z = -X_\alpha$, *where* $\frac{1}{2} > X_\alpha \ge \nu\alpha$ *and* $\nu = 0.278\cdots$ *is the unique positive root of* $xe^{1+x} = 1$.

4. $|z| \geq X_\alpha$ *for any* $z \in C_\alpha$*.*

Proof.

1. A simple calculation gives the limits. Taking derivatives yields

$$\frac{dK_\alpha}{d\alpha} = K_\alpha \ln\left(\frac{\alpha}{1-\alpha}\right) \ ,$$

   which shows that K_α is decreasing on $\left(0, \frac{1}{2}\right)$ and increasing on $\left(\frac{1}{2}, 1\right)$. Calculating $K_{1/2}$ directly gives the equality.

2. Clearly, the definition shows that C_α is closed and symmetric, and direct calculation shows that it passes through the point $z = \dfrac{\alpha}{1-\alpha}$.

   We write $z = re^{i\theta}$, and set

$$c_\theta(r) = \frac{|z|^\alpha}{|1+z|} = \frac{r^\alpha}{\sqrt{1 + 2r\cos\theta + r^2}} \ . \tag{3.3}$$

   Clearly, $c_\theta(0) = 0$ and $\lim_{r\to\infty} c_\theta(r) = 0$.

   For $\theta = 0$, we have

$$c_0'(r) = \frac{r^{\alpha-1}}{(1+r)^2}[\alpha - (1-\alpha)r] \ ,$$

   which shows that the given point is the only positive real value satisfying the equation.

   For $0 < \theta < \pi$, we have

$$c_\theta'(r) = r^{\alpha-1}(1 + 2r\cos\theta + r^2)^{-3/2}[(\alpha-1)r^2 + (2\alpha-1)r\cos\theta + \alpha] \ .$$

   Since $\alpha - 1 < 0$, this derivative has exactly one positive root, which is a maximum of the function. Further, a simple calculation shows that

$$c_\theta\left(\frac{\alpha}{1-\alpha}\right) > K_\alpha \ ,$$

from which each such ray yields exactly one point on the curve, inside the bounding circle, $|z| = \dfrac{\alpha}{1-\alpha}$. Considering the defining function, this value of r is clearly decreasing in $0 \le \theta < \pi$. Hence, the curve is simple and starlike with respect to 0.

Finally, for $\theta = \pi$, we have that

$$c'_\pi(r) = \frac{r^{\alpha-1}}{(1-r)^2}[\alpha + (1-\alpha)r] > 0$$

for $0 < r < 1$, and $\lim_{r \to 1^-} c_\pi(r) = \infty$, which gives exactly one solution in this range.

That these points are the only solutions within the bounding circle can be deduced from the fact that $z \in C_\alpha$ if and only if $\dfrac{1}{z} \in C'_{1-\alpha}$.

Examining the function $w = K_\alpha^{-1}\dfrac{z^\alpha}{1+z}$ using arguments in the range $(0, 2\pi)$ shows that C_α maps onto the approriate arc of the unit circle in the w-plane. This mapping is also one-to-one along the arc $0 <$ Arg w $< 2\pi\alpha$, since $w' \neq 0$ on the cut plane. This fact is implicitly used in the calculation of the rate of convergence.

3. The solution on the negative real axis is $-t = -X_\alpha$, and satisfies

$$\frac{t^\alpha}{1-t} = K_\alpha \ ,$$

which we write as

$$f(t) = t^\alpha + \alpha^\alpha(1-\alpha)^{1-\alpha}(t-1) = 0 \ . \tag{3.4}$$

Now, $f(t)$ is increasing, with $f(0) < 0$, $f(X_\alpha) = 0$, and

$$f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^\alpha - \frac{1}{2}K_\alpha > \frac{1}{2}(1 - K_\alpha) > 0 \ ,$$

from which $X_\alpha < \dfrac{1}{2}$ follows immediately.

To show that $\nu\alpha < X_\alpha$, we consider

$$f(\nu\alpha) = \alpha^\alpha(\nu^\alpha - (1-\nu\alpha)(1-\alpha)^{1-\alpha}) \ . \tag{3.5}$$

and set

$$g(\alpha) = \ln((1-\nu\alpha)(1-\alpha)^{1-\alpha}) \ , \tag{3.6}$$

which satisfies $g(0) = 0$, $g'(0) = -\nu - 1$ and

$$g''(\alpha) = \frac{(\nu\alpha)^2 + (\nu-2)(\nu\alpha) + 1 - \nu^2}{(1-\alpha)(1-\nu\alpha)^2} > 0 \ . \tag{3.7}$$

The last inequality follows since the quadratic in the numerator has discriminant $\nu^3(5\nu - 4) < 0$, from Lemma 3.1, and so has no real zeros.

Hence,

$$e^{g(\alpha)} > e^{-(\nu+1)\alpha} = e^{\alpha \ln \nu} = \nu^\alpha \ ,$$

and thus, by (3.5), $f(\nu\alpha) < 0$ for $0 < \alpha < 1$, as desired. *Q.E.D.*

We continue with a lemma required for one of the bounds.

Lemma 3.2 *Let* $f(z) = \sum_{k=0}^{\infty} b_k z^k$ *satisfy*

$$b_0 > b_1 \geq 0, \ \ b_k \geq 0, \ \ b_1 b_{k-1} - b_0 b_k \geq 0 \text{ for } k \geq 1 \ . \tag{3.8}$$

Then, $|f(z)| \geq \dfrac{b_0 - b_1}{b_0 + b_1} f(1)$ *for* $|z| \leq 1$.

Proof. The conditions given imply that $\{b_k\}$ is strictly decreasing, unless $b_k = 0$ for $k \geq K$. Let $r = \dfrac{b_1}{b_0} < 1$. Then, the conditions given show that $b_k \leq r b_{k-1}$ for $k \geq 1$. Hence, $f(z)$ is analytic for $|z| < \dfrac{1}{r}$, and in particular in the closed unit disk. Applying the Eneström-Kakaya Theorem to the partial sums $p_n(z) = \sum_{k=0}^{n} b_k z^k$ shows that all have their zeros in the region $|z| > 1$, hence, by Hurwitz' Theorem, $f(z)$ cannot have any zeros inside the unit disk. Thus, applying the Minimum Modulus Theorem, the minimum value of $|f(z)|$ for $|z| \leq 1$ must occur on the boundary.

For $|z| = 1$, we have

$$
\begin{array}{rcl}
|(b_0 - b_1 z) f(z)| & = & \left| b_0^2 + \sum_{k=1}^{\infty} (b_0 b_k - b_1 b_{k-1}) z^k \right| \\
& \geq & b_0^2 - \sum_{k=1}^{\infty} |(b_1 b_{k-1} - b_0 b_k)| \\
& = & b_0^2 - \sum_{k=1}^{\infty} b_1 b_{k-1} + \sum_{k=1}^{\infty} b_0 b_k \\
& = & b_0^2 - b_1 f(1) + b_0 (f(1) - b_0) \\
& = & (b_0 - b_1) f(1) \ .
\end{array}
\tag{3.9}
$$

Hence, we have

$$
|f(z)| \geq \frac{(b_0 - b_1) f(1)}{|b_0 - b_1 z|} \geq \frac{(b_0 - b_1) f(1)}{b_0 + b_1} \ , \tag{3.10}
$$

the desired result. *Q.E.D.*

Finally, we have the estimates of the remainder term.

Lemma 3.3 *Given integers $1 \leq r < n$, we set $\beta = \dfrac{r}{n}$, and consider the remainder term*

$$
R_{r,n}(z) = \sum_{k=r+1}^{n} \binom{n}{k} z^{k-r} \ . \tag{3.11}
$$

Then, for $|z| \leq \dfrac{\beta}{1-\beta}$, we have

$$
|R_{r,n}(z)| \leq K_\beta^{-n} \sum_{k=r+1}^{n} \binom{n}{k} \beta^k (1-\beta)^{n-k} \leq K_\beta^{-n} \tag{3.12}
$$

and

$$
|R_{r,n}(z)| \geq \frac{|z|}{r+1} K_\beta^{-n} \sum_{k=r+1}^{n} \binom{n}{k} \beta^k (1-\beta)^{n-k} \ . \tag{3.13}
$$

Proof. Given that all coefficients are positive, we use the value of K_β from (1.4) and the bound on $|z|$ to deduce that

$$
|R_{r,n}(z)| \leq R_{r,n}\left(\frac{\beta}{1-\beta}\right) = K_\beta^{-n} \sum_{k=r+1}^{n} \binom{n}{k} \beta^k (1-\beta)^{n-k} \ .
$$

The latter sum is clearly bounded by 1, using the binomial expansion. In fact, using the Central Limit Theorem, it is asymptotically 1/2 for large $r, n-r$.

For the lower bound, we consider

$$g(z) = \left(\frac{1-\beta}{\beta z}\right) R_{r,n}\left(\frac{\beta z}{1-\beta}\right) = \sum_{k=0}^{n-r-1} b_k z^k \ ,$$

where

$$b_k = \binom{n}{k+r+1}\left(\frac{\beta}{1-\beta}\right)^k \ .$$

It is simple to show that $g(z)$ satisfies the conditions of Lemma 3.2, that

$$\frac{b_0 - b_1}{b_0 + b_1} = \frac{2n-r}{2r(n-r)+(2n-3r)} \geq \frac{1}{r+1} \ ,$$

and finally that

$$g(1) = \left(\frac{1-\beta}{\beta}\right) K_\beta^{-n} \sum_{k=r+1}^{n} \binom{n}{k} \beta^k (1-\beta)^{n-k} \ .$$

Rewriting $R_{r,n}(z)$ in terms of $g(z)$ yields the result. *Q.E.D.*

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References

[1] J. D. Buckholtz. A characterization of the exponential series. *American Math. Monthly.* (73), Part II:121–123, 1966.

[2] A. J. Carpenter, R. S. Varga and J. Waldvogel. Asymptotics for the Zeros of the Partial Sums of e^z. I. *Rocky Mountain Journal of Mathematics.* (1), 21:99–120, Winter 1991.

[3] A. Edrei, E. B. Saff and R. S. Varga. Zeros of Sections of Power Series. Springer-Verlag, 1983.

[4] P. Erdős and P. Turán. On the distribution of roots of polynomials. *Ann. of Math.* (2), 51:105–119, 1950.

[5] P. Henrici. *Applied and Computational Complex Analysis,* Vol.I. John Wiley & Sons, New York, 1974.

[6] R. Jentzsch. Untersuchungen zur Theorie der Folgen analytischer Funktionen. *Acta Math.*, 41:219–251, 1917.

[7] D. J. Newman and T. J. Rivlin. The zeros of the partial sums of the exponential function. *J. Approx Theory*, 5:405–412, 1972.

[8] D. J. Newman and T. J. Rivlin. Correction : The zeros of the partial sums of the exponential function. *J. Approx Theory*, 16:299–300, 1976.

[9] T. S. Norfolk. On the Zeros of the Partial Sums to ${}_1F_1(1; b; z)$. *J. of Mathematical Analysis and Applications*, 218:421–438, 1998.

[10] T. S. Norfolk. Asymptotics of the partial sums of a set of integral transforms. *Numerical Algorithms*, 25:279–291, 2000.

[11] I. V. Ostrovskii. On a Problem of A. Eremenko. *Computational Methods and Function Theory* 2, 4:275–282, 2004.

[12] P. C. Rosenbloom. Distribution of zeros of polynomials. In *Lectures on Functions of a Complex Variable*, (W. Kaplan, editor), pages 265–285. University of Michigan Press, 1955.

[13] G. Szegő. Über eine Eigenschaft der Exponentialreihe, *Sitzungsber. Berl. Math. Ges.*, 23:50–64, 1924.

[14] D. G. Wagner. Zeros of reliability polynomials and f-vectors of matroids. *Combin. Probab. Comput.*, 9:167–190, 2000.