LARGE CLIQUES IN A POWER-LAW RANDOM GRAPH

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Abstract. We study the size of the largest clique $\omega(G(n,\alpha))$ in a random graph $G(n,\alpha)$ on n vertices which has power-law degree distribution with exponent α. We show that for 'flat' degree sequences with $\alpha > 2$ whp the largest clique in $G(n,\alpha)$ is of a constant size, while for the heavy tail distribution, when $0 < \alpha < 2$, $\omega(G(n,\alpha))$ grows as a power of n. Moreover, we show that a natural simple algorithm whp finds in $G(n,\alpha)$ a large clique of size $(1+o(1))\omega(G(n,\alpha))$ in polynomial time.

1. Introduction

Random graphs with finite density and power-law degree sequence have attracted much attention for the last few years (e.g. see Durrett [8] and the references therein). Several models for such graphs has been proposed; in this paper we concentrate on a Poissonian model $G(n,\alpha)$ in which the number of vertices of degree at least i decreases roughly as $ni^{-\alpha}$ (for a precise definition of the model see Section 2 below).

We show (Theorem 1) that there is a major difference in the size of the largest clique $\omega(G(n,\alpha))$ between the cases $\alpha < 2$ and $\alpha > 2$ with an intermediate result for $\alpha = 2$. In the 'light tail case', when $\alpha > 2$ (this is when the asymptotic degree distribution has a finite second moment), the size of the largest clique is either two or three, i.e., it is almost the same as in the standard binomial model of random graph $G(n,p)$ in which the expected average degree is a constant. As opposite to that, in the 'heavy tail case', when $0 < \alpha < 2$, $\omega(G(n,\alpha))$ grows roughly as $n^{1-\alpha/2}$. In the critical case when $\alpha = 2$ we have $\omega(G(n,\alpha)) = O_{\mathrm{p}}(1)$, but the probability that $G(n,\alpha) \geq k$ is bounded away from zero for every k. We also show (Corollary 3) that in each of the above cases there exists a simple algorithm which whp finds in $G(n,\alpha)$ a clique of size $(1-o(1))\omega(G(n,\alpha))$. This is quite different from the binomial case, where it is widely believed that finding large clique is hard (see for instance Frieze and McDiarmid [11]).

Similar but less precise results have been obtained by Bainconi and Marsili [1; 2] for a slightly different model (see Section 6.6 below).

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2. The model and the results

The model we study is a version of the conditionally Poissonian random graph studied by Norros and Reittu [15] (see also Chung and Lu [7] for a related model). For $\alpha > 1$ it is also an example of the 'rank 1 case' of inhomogeneous random graph studied by Bollobás, Janson, and Riordan [5, Section 16.4].

In order to define our model consider a set of n vertices (for convenience labelled $1, \dots, n$). We first assign a *capacity* or *weight* W_i to each vertex i. For definiteness and simplicity, we assume that these are i.i.d. random variables with a distribution with a power-law tail

$$\mathbb{P}(W > x) = ax^{-\alpha}, \qquad x \geq x_0, \tag{2.1}$$

for some constants $a > 0$ and $\alpha > 0$, and some $x_0 > 0$ (here and below we write W for any of the W_i when the index does not matter). Thus, for example, W could have a Pareto distribution, when $x_0 = a^{1/\alpha}$ and $\mathbb{P}(W > x) = 1$ for $x \leq x_0$, but the distribution could be arbitrarily modified for small x. We denote the largest weight by $W_{\max} = \max_i W_i$. Observe that (2.1) implies

$$\mathbb{P}(W_{\max} > tn^{1/\alpha}) \leq n\,\mathbb{P}(W > tn^{1/\alpha}) = O(t^{-\alpha}). \tag{2.2}$$

Note also that $\mathbb{E}\, W^\beta < \infty$ if and only if $\alpha > \beta$; in particular, for the 'heavy tail case' when $\alpha < 2$ we have $\mathbb{E}\, W^2 = \infty$.

Now, conditionally given the weights $\{W_i\}_1^n$, we join each pair $\{i, j\}$ of vertices by E_{ij} parallel edges, where the numbers E_{ij} are independent Poisson distributed random numbers with means

$$\mathbb{E}\, E_{ij} = \lambda_{ij} = b\frac{W_i W_j}{n}, \tag{2.3}$$

where $b > 0$ is another constant. We denote the resulting random (multi)graph by $\hat{G}(n, \alpha)$. For our purposes, parallel edges can be merged into a single edge, so we may alternatively define $G(n, \alpha)$ as the random simple graph where vertices i and j are joined by an edge with probability

$$p_{ij} = 1 - \exp(-\lambda_{ij}), \tag{2.4}$$

independently for all pairs (i, j) with $1 \leq i < j \leq n$.

Then our main result can be stated as follows. Let us recall that an event holds *with high probability* (whp), if it holds with probability tending to 1 as $n \to \infty$. We also use o_{p} and O_{p} in the standard sense (see, for instance, Janson, Łuczak, Ruciński [13]).

Theorem 1. (i) *If* $0 < \alpha < 2$, *then*

$$\omega(G(n, \alpha)) = (c + o_{\mathrm{p}}(1))n^{1-\alpha/2}(\log n)^{-\alpha/2},$$

where

$$c = ab^{\alpha/2}(1 - \alpha/2)^{-\alpha/2}. \tag{2.5}$$

(ii) *If* $\alpha = 2$, *then* $\omega(G(n,\alpha)) = O_p(1)$; *that is, for every* $\varepsilon > 0$ *there exists a constant* C_ε *such that* $\mathbb{P}(\omega(G(n,\alpha)) > C_\varepsilon) \le \varepsilon$ *for every* n. *However, there is no fixed finite bound* C *such that* $\omega(G(n,\alpha)) \le C$ *whp.*

(iii) *If* $\alpha > 2$, *then* $\omega(G(n,\alpha)) \in \{2,3\}$ *whp. Moreover, the probabilities of each of the events* $\omega(G(n,\alpha)) = 2$ *and* $\omega(G(n,\alpha)) = 3$ *tend to positive limits, given by* (5.10).

A question which naturally emerges when studying the size of the largest clique in a class of graphs is whether one can find a large clique in such graph in a polynomial time. By Theorem 1, whp one can find $\omega(G(n,\alpha))$ in a polynomial time for $\alpha > 2$, and, with some extra effort, the same can be accomplished for $\alpha = 2$ (see Corollary 3). Thus let us concentrate for the case $\alpha < 2$, when the large clique is of polynomial size. Let us suppose that we know the vertex weights W_i defined in Section 2 and, for simplicity, that these are distinct (otherwise we resolve ties randomly; we omit the details). Since vertices with larger weights tend to have higher degrees, they are more likely to be in a large clique, so it is natural to try to find a large clique by looking at the vertices with largest weights. One simple way is the greedy algorithm which checks the vertices in order of decreasing weights and selects every vertex that is joined to every other vertex already selected. This evidently yields a clique, which we call the *greedy clique* and denote by $\mathcal{K}_{gr}$. Thus

$$\mathcal{K}_{gr} = \{i : i \sim j \text{ for all } j \text{ with } W_j > W_i \text{ and } j \in \mathcal{K}_{gr}\}.$$

A simplified algorithm is to select every vertex that is joined to every vertex with higher weight, regardless of whether these already are selected or not. This gives the *quasi top clique* studied by Norros [14], which we denote by $\mathcal{K}_{qt}$. Thus

$$\mathcal{K}_{qt} = \{i : i \sim j \text{ for all } j \text{ with } W_j > W_i\}.$$

Obviously, $\mathcal{K}_{qt} \subseteq \mathcal{K}_{gr}$. The difference between the two cliques is that if we, while checking vertices in order of decreasing weights, reject a vertex, then that vertex is ignored for future tests when constructing $\mathcal{K}_{gr}$, but not for $\mathcal{K}_{qt}$. A more drastic approach is to stop at the first failure; we define the *full top clique* $\mathcal{K}_{ft}$ as the result, i.e.

$$\mathcal{K}_{ft} = \{i : j \sim k \text{ for all distinct } j,k \text{ with } W_j, W_k \ge W_i\}.$$

Thus $\mathcal{K}_{ft}$ is the largest clique consisting of all vertices with weights in some interval $[w,\infty)$. Clearly, $\mathcal{K}_{ft} \subseteq \mathcal{K}_{qt} \subseteq \mathcal{K}_{gr}$. Finally, by $\mathcal{K}_{max}$ we denote the largest clique (chosen at random, say, if there is a tie). Thus

$$|\mathcal{K}_{ft}| \le |\mathcal{K}_{qt}| \le |\mathcal{K}_{gr}| \le |\mathcal{K}_{max}| = \omega(G(n,\alpha)). \tag{2.6}$$

The following theorem shows that the last two inequalities in (2.6) are asymptotic equalities, but not the first one. Here we use $\xrightarrow{p}$ for convergence in probability, and all unspecified limits are as $n \to \infty$.

Theorem 2. *If $0 < \alpha < 2$, then $G(n,\alpha)$, $\mathcal{K}_{\rm gr}$ and $\mathcal{K}_{\rm qt}$ both have size $(1+o_{\rm p}(1))\omega(G(n,\alpha))$; in other words*

$$|\mathcal{K}_{\rm gr}|/|\mathcal{K}_{\max}| \xrightarrow{\rm p} 1 \quad \textit{and} \quad |\mathcal{K}_{\rm qt}|/|\mathcal{K}_{\max}| \xrightarrow{\rm p} 1.$$

On the other hand,

$$|\mathcal{K}_{\rm ft}|/|\mathcal{K}_{\max}| \xrightarrow{\rm p} 2^{-\alpha/2}.$$

Thus, whp $\mathcal{K}_{\rm gr}$ and $\mathcal{K}_{\rm qt}$ almost attain the maximum size of a clique, while $\mathcal{K}_{\rm ft}$ falls short by a constant factor. As a simple corollary of the above result one can get the following.

Corollary 3. *For every $\alpha > 0$ there exists an algorithm which whp finds in $G(n,\alpha)$ a clique of size $(1+o(1))\omega(G(n,\alpha))$ in a polynomial time.*

3. The proof for the case $\alpha < 2$ (no second moment)

We begin with a simple lemma giving an upper bound for the clique number of the Erdős–Rényi random graph $G(n,p)$ (for much more precise results see, for instance, Janson, Łuczak, Ruciński [13]).

Lemma 4. *For any $p = p(n)$, whp*

$$\omega(G(n,p)) \le \frac{2\log n}{1-p}.$$

Proof. Denote by X_k the number of cliques of order k in $G(n,p)$. For the expected number of such cliques we have

$$\mathbb{E}\, X_k = \binom{n}{k} p^{\binom{k}{2}} \le \Bigl(\frac{ne}{k} p^{(k-1)/2}\Bigr)^k.$$

If we set $k \ge \lfloor 2\log(n)/(1-p)\rfloor$, then

$$p^{(k-1)/2} = \bigl(1-(1-p)\bigr)^{(k-1)/2} \le e^{-(1-p)(k-1)/2} \le e/n\,.$$

Consequently, we arrive at

$$\mathbb{P}(\omega(G(n,p)) \ge k) = \mathbb{P}(X_k \ge 1) \le \mathbb{E}\, X_k \le \Bigl(\frac{e^2}{k}\Bigr)^k \to 0,$$

since $k \ge \lfloor 2\log n\rfloor \to \infty$. □

Proof of Theorems 1(i) *and 2.* For $s > 0$, let us partition the vertex set $V = \{1,\dots,n\}$ of $G_n = G(n,\alpha)$ into

$$V_s^- = \{i : W_i \le s\sqrt{n\log n}\} \qquad \text{and} \qquad V_s^+ = \{i : W_i > s\sqrt{n\log n}\};$$

we may think of elements of V_s^- and V_s^+ as 'light' and 'heavy' vertices, respectively. By (2.1),

$$\mathbb{E}\,|V_s^+| = n\,\mathbb{P}(W > s\sqrt{n\log n}) = as^{-\alpha}n^{1-\alpha/2}\log^{-\alpha/2} n. \tag{3.1}$$

Moreover, $|V_s^+| \sim \mathrm{Bin}(n, \mathbb{P}(W > s\sqrt{n\log n}))$, and Chebyshev's inequality (or the sharper Chernoff bounds [13, Section 2.1]) implies that whp

$$|V_s^+| = (1+o(1))\,\mathbb{E}\,|V_s^+| = (1+o(1))as^{-\alpha}n^{1-\alpha/2}\log^{-\alpha/2} n. \tag{3.2}$$

We now condition on the sequence of weights $\{W_k\}$. We will repeatedly use the fact that if i and j are vertices with weights $W_i = x\sqrt{n \log n}$ and $W_j = y\sqrt{n \log n}$, then by (2.3)–(2.4), $\lambda_{ij} = bxy \log n$ and

$$p_{ij} = 1 - e^{-\lambda_{ij}} = 1 - e^{-bxy \log n} = 1 - n^{-bxy}. \tag{3.3}$$

In particular,

$$p_{ij} \le 1 - n^{-bs^2}, \qquad \text{if } i, j \in V_s^-, \tag{3.4}$$

$$p_{ij} > 1 - n^{-bs^2}, \qquad \text{if } i, j \in V_s^+. \tag{3.5}$$

Consider, still conditioning on $\{W_k\}$, for an s that will be chosen later, the induced subgraph $G_n[V_s^-]$ of $G(n, \alpha)$ with vertex set V_s^-. This graph has at most n vertices and, by (3.4), all edge probabilities are at most $1 - n^{-bs^2}$, so we may regard $G_n[V_s^-]$ as a subgraph of $G(n,p)$ with $p = 1 - n^{-bs^2}$. Hence, Lemma 4 implies that whp

$$\omega(G_n[V_s^-]) \le \frac{2 \log n}{n^{-bs^2}} = 2n^{bs^2} \log n. \tag{3.6}$$

If $\mathcal{K}$ is any clique in $G(n, \alpha)$, then $\mathcal{K} \cap V_s^-$ is a clique in $G_n[V_s^-]$, and thus $|\mathcal{K} \cap V_s^-| \le \omega(G_n[V_s^-])$; further, trivially, $|\mathcal{K} \cap V_s^+| \le |V_s^+|$. Hence, $|\mathcal{K}| \le \omega(G_n[V_s^-]) + |V_s^+|$, and thus

$$\omega(G(n, \alpha)) \le \omega(G_n[V_s^-]) + |V_s^+|. \tag{3.7}$$

We choose, for a given $\varepsilon > 0$, $s = (1 - \varepsilon) b^{-1/2} (1 - \alpha/2)^{1/2}$ so that the exponents of n in (3.6) and (3.2) almost match; we then obtain from (3.7), (3.2), and (3.6), that whp

$$\begin{aligned} \omega(G(n, \alpha)) &\le (1 + o(1)) a s^{-\alpha} n^{1-\alpha/2} \log^{-\alpha/2} n \\ &= (1 + o(1))(1 - \varepsilon)^{-\alpha} c n^{1-\alpha/2} \log^{-\alpha/2} n, \end{aligned} \tag{3.8}$$

with c defined as in (2.5).

To obtain a matching lower bound, we consider the quasi top clique $\mathcal{K}_{\text{qt}}$. Let, again, s be fixed and condition on the weights $\{W_k\}$. If $i, j \in V_s^+$, then by (3.5), the probability that i is not joined to j is less than n^{-bs^2}. Hence, conditioned on the weights $\{W_k\}$, the probability that a given vertex $i \in V_s^+$ is not joined to every other $j \in V_s^+$ is at most $|V_s^+| n^{-bs^2}$, which by (3.2) whp is at most $2as^{-\alpha} n^{1-\alpha/2-bs^2} \log^{-\alpha/2} n$. We now choose $s = (1 + \varepsilon) b^{-1/2} (1 - \alpha/2)^{1/2}$ with $\varepsilon > 0$. Then, for some constant $C < \infty$, whp

$$\mathbb{P}\big(i \notin \mathcal{K}_{\text{qt}} \mid \{W_k\}\big) \le C n^{-2\varepsilon(1-\alpha/2)}$$

and thus

$$\mathbb{E}\big(|V_s^+ \setminus \mathcal{K}_{\text{qt}}| \mid \{W_k\}\big) \le C n^{-2\varepsilon(1-\alpha/2)} |V_s^+|.$$

Hence, by Markov's inequality, whp

$$|V_s^+ \setminus \mathcal{K}_{\text{qt}}| \le C n^{-\varepsilon(1-\alpha/2)} |V_s^+|.$$

Thus, using (3.2), whp

$$\begin{aligned}\omega(G(n,\alpha)) \ge |\mathcal{K}_{\mathrm{gr}}| \ge |\mathcal{K}_{\mathrm{qt}}| \ge |V_s^+| - |V_s^+ \setminus \mathcal{K}_{\mathrm{qt}}| &\ge (1-o(1))|V_s^+| \\ &\ge (1-o(1))as^{-\alpha}n^{1-\alpha/2}\log^{-\alpha/2} n \\ &= (1+o(1))(1+\varepsilon)^{-\alpha}cn^{1-\alpha/2}\log^{-\alpha/2} n. \qquad (3.9)\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, (3.8) and (3.9) imply Theorem 1(i) and the first part of Theorem 2.

In order to complete the proof of Theorem 2, it remains to consider $\mathcal{K}_{\mathrm{ft}}$. Define $\overline{G_n}$ as the complement of $G(n,\alpha)$. Then, using (3.5) and conditioned on $\{W_i\}$, we infer that the expected number of edges of $\overline{G_n}$ with both endpoints in V_s^+ is at most $n^{-bs^2}|V_s^+|^2$. If we choose $s = b^{-1/2}(2-\alpha)^{1/2}$, then (3.2) implies that this is whp $o(1)$; hence whp V_s^+ contains no edges of $\overline{G_n}$, i.e., $\mathcal{K}_{\mathrm{ft}} \supseteq V_s^+$.

On the other hand, let $0 < \varepsilon < 1/2$ and define, still with $s = b^{-1/2}(2-\alpha)^{1/2}$, $V' = V^+_{(1-2\varepsilon)s} \cap V^-_{(1-\varepsilon)s}$. Then, conditioned on $\{W_i\}$, the probability of having no edges of $\overline{G_n}$ in V' is, by (3.4),

$$\prod_{i,j\in V'} p_{ij} \le \bigl(1-n^{-b(1-\varepsilon)^2s^2}\bigr)^{\binom{|V'|}{2}} \le \exp\bigl(-n^{-(1-\varepsilon)^2(2-\alpha)}(|V'|-1)^2/2\bigr). \tag{3.10}$$

By (3.2), whp

$$\begin{aligned}|V'|-1 &= |V^+_{(1-2\varepsilon)s}| - |V^+_{(1-\varepsilon)s}| - 1 \\ &= (1+o(1))a\bigl((1-2\varepsilon)^{-\alpha} - (1-\varepsilon)^{-\alpha}\bigr)s^{-\alpha}n^{1-\alpha/2}\log^{-\alpha/2} n,\end{aligned}$$

and it follows from (3.10) that

$$\mathbb{P}(\mathcal{K}_{\mathrm{ft}} \supseteq V^+_{(1-2\varepsilon)s}) \le \prod_{i,j\in V'} p_{ij} \to 0.$$

Hence, whp $\mathcal{K}_{\mathrm{ft}} \subset V^+_{(1-2\varepsilon)s}$.

We have shown that, for any $\varepsilon \in (0,1/2)$, whp $V_s^+ \subseteq \mathcal{K}_{\mathrm{ft}} \subset V^+_{(1-2\varepsilon)s}$, and it follows by (3.2) and (2.5) (by letting $\varepsilon \to 0$) that whp

$$\begin{aligned}|\mathcal{K}_{\mathrm{ft}}| &= (1+o(1))|V_s^+| = (1+o(1))as^{-\alpha}n^{1-\alpha/2}\log^{-\alpha/2} n \\ &= (1+o(1))2^{-\alpha/2}cn^{1-\alpha/2}\log^{-\alpha/2} n = (1+o(1))2^{-\alpha/2}\omega(G(n,\alpha)),\end{aligned}$$

where the last equality follows from Theorem 1. □

4. The case $\alpha = 2$ (still no second moment)

Proof of Theorem 1(ii) *and Corollary 3.* Given the weights W_i, the probability that four vertices i,j,k,l form a clique is, by (2.4) and (2.3),

$$p_{ij}p_{ik}p_{il}p_{jk}p_{jl}p_{kl} \le \lambda_{ij}\lambda_{ik}\lambda_{il}\lambda_{jk}\lambda_{jl}\lambda_{kl} = b^6\frac{W_i^3W_j^3W_k^3W_l^3}{n^6}.$$

Thus, if X_m is the number of cliques of size m in $G(n,\alpha)$, then the conditional expectation of X_4 is

$$\mathbb{E}(X_4 \mid \{W_i\}_1^n) \le b^6 n^{-6} \sum_{i<j<k<l} W_i^3 W_j^3 W_k^3 W_l^3 \le b^6 \Big(n^{-3/2} \sum_i W_i^3 \Big)^4. \quad (4.1)$$

To show that the number of such quadruples is bounded in probability, we shall calculate a truncated expectation of $\sum_i W_i^3$. Using (2.1), for any constant $A > 0$, we get

$$\begin{aligned} \mathbb{E}\Big(\sum_i W_i^3;\ W_{\max} \le An^{1/2}\Big) &\le \mathbb{E}\Big(\sum_i \min(W_i, An^{1/2})^3\Big) \\ &= n\,\mathbb{E}\min(W, An^{1/2})^3 \\ &= n \int_0^{An^{1/2}} 3x^2\,\mathbb{P}(W > x)\,\mathrm{d}x \\ &= O(nAn^{1/2}), \end{aligned} \quad (4.2)$$

and thus, using (2.2) and Markov's inequality, for every $t > 0$ and some constant C independent of A, t and n, we arrive at

$$\begin{aligned} &\mathbb{P}\Big(n^{-3/2} \sum_i W_i^3 > t\Big) \\ &\qquad \le t^{-1}\,\mathbb{E}\Big(n^{-3/2} \sum_i W_i^3;\ W_{\max} \le An^{1/2}\Big) + \mathbb{P}(W_{\max} > An^{1/2}) \\ &\qquad \le CAt^{-1} + CA^{-2}. \end{aligned} \quad (4.3)$$

Given $t > 0$, we choose $A = t^{1/3}$ and find $\mathbb{P}\Big(n^{-3/2} \sum_i W_i^3 > t\Big) = O(t^{-2/3})$. Hence, $n^{-3/2} \sum_i W_i^3 = O_{\mathrm{p}}(1)$, and it follows by (4.1) and Markov's inequality that $X_4 = O_{\mathrm{p}}(1)$.

Finally, we observe that, for any $m \ge 4$,

$$\mathbb{P}(\omega(G(n,\alpha)) \ge m) \le \mathbb{P}\Big(X_4 \ge \binom{m}{4}\Big), \quad (4.4)$$

which thus can be made arbitrarily small (uniformly in n) by choosing m large enough. Hence, $\omega(G(n,\alpha)) = O_{\mathrm{p}}(1)$.

To complete the proof of Theorem 1(ii) let us note that for any fixed $m \le n$, the probability that there are at least m vertices with weights $W_i > n^{1/2}$ is larger than $c_1 > 0$ for some absolute constant $c_1 > 0$, and conditioned on this event, the probability that the m first of these vertices form a clique is larger than c_2 for some absolute constants c_1, c_2 not depending on n.

Finally, we remark that all cliques of size four can clearly be found in time $O(n^4)$. The number of such cliques is whp at most $\log\log n$, say, so there exists an algorithm which whp finds the largest clique in a polynomial time (for example by crudely checking all sets of cliques of size 4). □

5. $\alpha > 2$ (finite second moment)

Proof of Theorem 1(iii). Choose ν such that $1/2 > \nu > 1/\alpha$. Then (2.2) (or (2.1) directly) implies that whp $W_{\max} \le n^\nu$. Furthermore, in analogy to (4.2) and (4.3),

$$\mathbb{E}\Bigl(\sum_i W_i^3;\ W_{\max} \le n^\nu\Bigr) \le \mathbb{E}\Bigl(\sum_i \min(W_i, n^\nu)^3\Bigr) = n\int_0^{n^\nu} 3x^2\,\mathbb{P}(W > x)\,\mathrm{d}x = O(nn^\nu) = o(n^{3/2}), \quad (5.1)$$

and thus

$$\mathbb{P}\Bigl(n^{-3/2}\sum_i W_i^3 > t\Bigr) \le t^{-1}\,\mathbb{E}\Bigl(n^{-3/2}\sum_i W_i^3;\ W_{\max} \le n^\nu\Bigr) + \mathbb{P}(W_{\max} > n^\nu) = o(1). \quad (5.2)$$

Hence, $n^{-3/2}\sum_i W_i^3 \xrightarrow{\mathrm{p}} 0$, and it follows from (4.1) that

$$\mathbb{P}(\omega(G(n,\alpha)) \ge 4) = \mathbb{P}(X_4 \ge 1) \le \mathbb{E}\bigl(\min\bigl(1, \mathbb{E}(X_4 \mid \{W_i\}_1^n)\bigr)\bigr) \to 0.$$

Consequently, whp $\omega(G(n,\alpha)) \le 3$.

Moreover, we can similarly estimate

$$\mathbb{E}\,X_3 \le \mathbb{E}\sum_{i<j<k}\lambda_{ij}\lambda_{ik}\lambda_{jk} = \mathbb{E}\Bigl(b^3 n^{-3}\sum_{i<j<k} W_i^2W_j^2W_k^2\Bigr) \le \tfrac16 b^3\bigl(\mathbb{E}\,W^2\bigr)^3; \quad (5.3)$$

note that $\mathbb{E}\,W^2 < \infty$ by (2.1) and the assumption $\alpha > 2$. Hence, the number of K_3 in $G(n,\alpha)$ is $O_{\mathrm{p}}(1)$. To obtain the limit distribution, it is convenient to truncate the distribution, as we have done it in the previous section. We let A be a fixed large constant, and let X_3^A be the number of K_3 in $G(n,\alpha)$ such that all three vertices have weights at most A, and let X_3^{A*} be the number of the remaining triangles. Arguing as in (5.3), we see easily that

$$\mathbb{E}\,X_3^A \le \tfrac16 b^3\bigl(\mathbb{E}(W^2;\ W \le A)\bigr)^3 \quad (5.4)$$

$$\mathbb{E}\,X_3^{A*} \le b^3\bigl(\mathbb{E}(W^2)\bigr)^2\,\mathbb{E}(W^2;\ W > A). \quad (5.5)$$

Moreover, if $W_i, W_j \le A$, then $\lambda_{ij} = O(1/n)$, and thus $p_{ij} \sim \lambda_{ij}$, and it is easily seen that (5.4) can be sharpened to

$$\mathbb{E}\,X_3^A \to \mu_A = \tfrac16 b^3\bigl(\mathbb{E}(W^2;\ W \le A)\bigr)^3. \quad (5.6)$$

Furthermore, we may calculate fractional moments $\mathbb{E}(X_3^A)_m$ by the same method, and it follows easily by a standard argument (see, for instance, [13, Theorem 3.19] for $G(n,p)$) that $\mathbb{E}(X_3^A)_m \to \mu_A^m$ for every $m \ge 1$, and thus by the method of moments [13, Corollary 6.8]

$$X_3^A \xrightarrow{\mathrm{d}} \mathrm{Po}(\mu_A) \quad (5.7)$$

as $n \to \infty$, for every fixed A, where $\xrightarrow{\mathrm{d}}$ denotes the convergence in distribution.

Finally, we note that the right hand side of (5.5) can be made arbitrarily small by choosing A large enough, and hence

$$\lim_{A\to\infty} \limsup_{n\to\infty} \mathbb{P}(X_3^{A*} \neq 0) = 0, \tag{5.8}$$

and that $\mu_A \to \mu = \frac{1}{6}(b\,\mathbb{E}(W^2))^3$ as $A \to \infty$. It follows by a standard argument (see Billingsley [3, Theorem 4.2]) that we can let $A \to \infty$ in (5.7) and obtain

$$X_3 \xrightarrow{\mathrm{d}} \mathrm{Po}(\mu). \tag{5.9}$$

In particular, $\mathbb{P}(X_3 = 0) \to e^{-\mu}$, which yields the following result:

$$\begin{aligned} &\mathbb{P}(\omega(G(n,\alpha)) = 2) \to e^{-\mu} = e^{-\frac{1}{6}(b\,\mathbb{E}(W^2))^3}, \\ &\mathbb{P}(\omega(G(n,\alpha)) = 3) \to 1 - e^{-\mu} = 1 - e^{-\frac{1}{6}(b\,\mathbb{E}(W^2))^3}. \end{aligned} \tag{5.10}$$

Finally, note that $G(n,\alpha)$ whp contains cliques K_2, i.e. edges, so clearly $\omega(G(n,\alpha)) \geq 2$. □

6. Final remarks

In this section we make some comments on other models of power-law random graphs as well as some remarks on possible variants of our results. We omit detailed proofs.

Let us remark first that, for convenience and to facilitate comparisons with other papers, in the definition of $G(n,\alpha)$ we used two scale parameters a and b above, besides the important exponent α. By rescaling $W_i \mapsto tW_i$ for some fixed $t > 0$, we obtain the same $G(n,\alpha)$ for the parameters at^α and bt^{-2}; hence only the combination $ab^{\alpha/2}$ matters, and we could fix either a or b as 1 without loss of generality.

6.1. **Algorithms based on degrees.** As for the algorithmic result Theorem 2, it remains true if we search for large cliques examing the vertices one by one in order not by their weights but by their degrees and modify the definition of $\mathcal{K}_{\mathrm{gr}}$, $\mathcal{K}_{\mathrm{qt}}$, and $\mathcal{K}_{\mathrm{ft}}$ accordingly. (This holds both if we take the degrees in the multigraph $\hat{G}(n,\alpha)$, or if we consider the corresponding simple graph.) The reason is that, for the vertices of large weight that we are interested in, the degrees are whp all almost proportinal to the weights, and thus the two orders do not differ very much. This enables us to find an almost maximal clique in polynomial time, even without knowing the weights.

6.2. **More general weight distributions.** Observe that Theorems 1 and 2 remain true (and can be shown by basically the same argument), provided only that the power law holds asymptotically for large weights, i.e., (2.1) may be relaxed to

$$\mathbb{P}(W > x) \sim ax^{-\alpha} \qquad \text{as } x \to \infty. \tag{6.1}$$

6.3. **Deterministic weights.** Instead of choosing weights independently according to the distribution W we may as well take a suitable deterministic sequence W_i of weights (as in Chung and Lu [7]), for example

$$W_i = a^{1/\alpha} \frac{n^{1/\alpha}}{i^{1/\alpha}}, \qquad i = 1, \dots, n. \tag{6.2}$$

All our results remain true also in this setting; in fact the proofs are slightly simpler for this model. A particularly interesting special case for this model (see Bollobás, Janson, and Riordan [5, Section 16.2] and Riordan [17]) is when $\alpha = 2$, where (2.3) and (6.2) combine to yield

$$\lambda_{ij} = \frac{ab}{\sqrt{ij}}.$$

6.4. **Poisson number of vertices.** We may also let the number of vertices be random with a Poisson $\mathrm{Po}(n)$ distribution (as in, e.g., Norros [14]). Then the set of weights $\{W_i\}_1^n$ can be regarded as a Poisson process on $[0, \infty)$ with intensity measure $n\,\mathrm{d}\mu$, where μ is the distribution of the random variable W in (2.1). Note that now n can be any positive real number.

6.5. **Different normalization.** A slightly different power-law random graph model emerges when instead of (2.3) we define the intensities λ_{ij} by

$$\lambda_{ij} = \frac{W_i W_j}{\sum_{k=1}^n W_k} \tag{6.3}$$

(see for instance Chung and Lu [7] and Norros and Reittu [15]). Let us call this model $\widetilde{G}(n, \alpha)$. In the case $\alpha > 1$, when the mean $\mathbb{E}\, W < \infty$, the results for $\widetilde{G}(n, \alpha)$ and $G(n, \alpha)$ are not much different. In fact, by the law of large numbers, $\sum_1^n W_k/n \to \mathbb{E}\, W$ a.s., so we may for any $\varepsilon > 0$ couple $\widetilde{G}(n, \alpha)$ constructed by this model with $G(n, \alpha)^\pm$ constructed as above, using (2.3) with $b = 1/(\mathbb{E}\, W \mp \varepsilon)$, such that whp $G(n, \alpha)^- \subseteq \widetilde{G}(n, \alpha) \subseteq G(n, \alpha)^+$, and it follows that we have the same asymptotic results as in our theorems if we let $b = 1/\mathbb{E}\, W$.

On the other hand, for $\alpha = 1$, $\sum_1^n W_k = (a + o_{\mathrm{p}}(1)) n \log n$, and for $0 < \alpha < 1$, $\sum_1^n W_k / n^{1/\alpha} \xrightarrow{\mathrm{d}} Y$, where Y is a stable distribution with exponent α (e.g.see Feller [10, Section XVII.5]). It follows, arguing as in Section 3, that for $\alpha = 2$, the largest clique in $\widetilde{G}(n, \alpha)$ has

$$(1 + o_{\mathrm{p}}(1)) \sqrt{2an} \log^{-1} n$$

vertices, while for $0 < \alpha < 1$ the size of the largest clique is always close to $\sqrt{n}$; more precisely,

$$\frac{\omega(\widetilde{G}(n, \alpha))}{\sqrt{n} \log^{-\alpha/2} n} \xrightarrow{\mathrm{d}} Z = a 2^{\alpha/2} Y^{-\alpha/2},$$

where Z is an absolutely continuous random variable whose distribution has the entire positive real axis as support. (The square Z^2 has, apart from a scale factor, a Mittag-Leffler distribution with parameter α, see Bingham, Goldie, Teugels [4, Section 8.0.5].) Thus, for $\alpha < 1$, $\omega(\widetilde{G}(n,\alpha))$ is not sharply concentrated around its median; this is caused by the fact that the normalizing factor $\sum_i W_i$ is determined by its first terms which, clearly, are not sharply concentrated around their medians as well. Interestingly enough, since in the proof of Theorem 2 we dealt mostly with the probability space where we conditioned on W_i, the analogue of Theorem 2 holds for this model as well. Thus, for instance, despite of the fact that neither the largest clique nor the full top clique are sharply concentrated in this model, one can show the sharp concentration result for the ratio of these two variables.

6.6. **The model** $\min(\lambda_{ij}, 1)$**.** For small λ_{ij}, (2.4) implies $p_{ij} \approx \lambda_{ij}$. In most works on inhomogeneous random graphs, it does not matter whether we use (2.4) or, for example, $p_{ij} = \min(\lambda_{ij}, 1)$ or $p_{ij} = \lambda_{ij}/(1 + \lambda_{ij})$ (as in Britton, Deijfen, and Martin-Löf [6]), see Bollobás, Janson, Riordan [5]. For the cliques studied here, however, what matters is mainly the probabilities p_{ij} that are close to 1, and the precise size of $1 - p_{ij}$ for them is important; thus it is important that we use (2.4) (cf. Bianconi and Marsili [1; 2] where a cutoff is introduced). For instance, a common version (see e.g. [5]) of $G(n,\alpha)$ replaces (2.4) by

$$p_{ij} = \min(\lambda_{ij}, 1). \tag{6.4}$$

This makes very little difference when λ_{ij} is small, which is the case for most i and j, and for many asymptotical properties the two versions are equivalent (see again [5]). In the case $\sum_i W_i^3 = o_{\mathrm{p}}(n^{3/2})$, which in our case with W_i governed by (2.1) holds for $\alpha > 2$ as a consequence of (5.2), a strong general form of asymptotic equivalence is proved in Janson [12]; in the case $\alpha = 2$, when $\sum_i W_i^3 = o_{\mathrm{p}}(n^{3/2})$ by (4.2), a somewhat weaker form of equivalence (known as contiguity) holds provided also, say, $\max_{ij} \lambda_{ij} \le 0.9$, see again [12]. In our case we do not need these general equivalence results; the proofs above for the cases $\alpha \ge 2$ hold for this model too, so Theorem 1(ii)(iii) hold without changes.

If $\alpha < 2$, however, the results are different. In fact, (6.4) implies that all vertices with $W_i \ge b^{-1/2} n^{1/2}$ are joined to each other, and thus form a clique; conversely, if we now define $V^- = \{i : W_i \le (b+\varepsilon)^{-1/2} n^{1/2}\}$, then $p_{ij} = \lambda_{ij} \le b/(b+\varepsilon)$ for $i, j \in V^-$, and thus $\omega(G(n,\alpha)[V^-]) = O(\log n)$ whp by Lemma 4. Consequently, arguing as in Section 3,

$$\omega(G(n,\alpha)) = (1 + o_{\mathrm{p}}(1)) n\, \mathbb{P}(W > b^{-1/2} n^{1/2}) = (1 + o_{\mathrm{p}}(1)) a b^{\alpha/2} n^{1-\alpha/2},$$

so the logarithmic factor in Theorem 1(i) disappears.

6.7. **The model** $\lambda_{ij}/(1+\lambda_{ij})$**.** Another version of $G(n,\alpha)$ replaces (2.4) by

$$p_{ij} = \frac{\lambda_{ij}}{1 + \lambda_{ij}}. \tag{6.5}$$

This version has the interesting feature that conditioned on the vertex degrees, the distribution is uniform over all graphs with that degree sequence, see Britton, Deijfen, and Martin-Löf [6].

In this version, for large λ_{ij}, $1 - p_{ij} = 1/(1 + \lambda_{ij})$ is considerably larger than for (2.4) (or (6.4)), and as a consequence, the clique number is smaller. For $\alpha \geq 2$, stochastic domination (or a repetition of the proofs above) shows that Theorem 1(ii)(iii) hold without changes.

For $\alpha < 2$, there is a significant difference. Arguing as in Section 3, we find that, for some constants c and C depending on a, b and α, whp

$$cn^{(2-\alpha)/(2+\alpha)} \leq \omega(G(n,\alpha)) \leq Cn^{(2-\alpha)/(2+\alpha)}(\log n)^{\alpha/(2+\alpha)}.$$

Although this only determines the clique number up to a logarithmic factor, note that the exponent of n is $\frac{2-\alpha}{2+\alpha}$, which is strictly less than the exponent $\frac{2-\alpha}{2}$ in Theorem 1.

6.8. **Preferential attachment.** Finally, let us observe that not all power-law random graph models contain large cliques. Indeed, one of the most popular types of models of such graphs are preferential attachment graphs in which the graph grows by acquiring new vertices, where each new vertex v is joined to some number k_v of 'old' vertices according to some random rule (which usually depends on the structure of the graph we have constructed so far), see, for instance, Durrett [8]. Clearly, such a graph on n vertices cannot have cliques larger than $X_n = \max_{v \leq n} k_v + 1$, and since for most of the models X_n is bounded from above by an absolute constant or grows very slowly with n, typically the size of the largest clique in preferential attachment random graphs is small.

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