# On the number of perfect matchings in random lifts

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**Abstract**

Let $G$ be a fixed connected multigraph with no loops. A random $n$-lift of $G$ is obtained by replacing each vertex of $G$ by a set of $n$ vertices (where these sets are pairwise disjoint) and replacing each edge by a randomly chosen perfect matching between the $n$-sets corresponding to the endpoints of the edge. Let $X_G$ be the number of perfect matchings in a random lift of $G$. We study the distribution of $X_G$ in the limit as $n$ tends to infinity. Our aim is to prove a concentration result for $X_G$ using the small subgraph conditioning method.

While we have been unable to prove concentration in general, we present several results including an asymptotic formula for the expectation of $X_G$ when $G$ is $d$-regular, $d \geq 3$. The interaction of perfect matchings with short cycles in random lifts of regular multigraphs is also analysed. Difficulties arise in the calculation of the second moment of $X_G$, where we provide some partial results. Full details are given for two example multigraphs, including the complete graph $K_4$.

To assist in our calculations we provide a theorem for estimating summation over multiple dimensions using Laplace's method. This result is phrased as a summation over lattice points, and may prove useful in future applications.

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# 1 Introduction

Throughout, let $G$ be a fixed connected multigraph with $g$ vertices and no loops. For simplicity we assume that $V(G) = [g] := \{1, \ldots, g\}$. A random $n$-lift of $G$ is a random graph on the vertex set $V_1 \cup V_2 \cup \cdots \cup V_g$, where each $V_i$ is a set of $n$ vertices and these sets are pairwise disjoint, obtained by placing a uniformly chosen random perfect matching between $V_i$ and $V_j$, independently for each edge $e = ij$ of $G$. Denote the resulting random graph by $L_n(G)$. The perfect matching corresponding to the edge $e$ of $G$ is called the *fiber* corresponding to $e$, which we denote by $F_e$. Note that the degree of $v \in V_i$ in $L_n(G)$ is equal to the degree $d_G(i)$ of vertex $i$ in $G$. In particular, if $G$ is $d$-regular, then so is $L_n(G)$. We are interested in asymptotics as $n$ tends to infinity.

This model of sparse random graphs was introduced and studied in a series of papers by Amit, Linial, Matoušek, and Rozenman [1, 2, 3, 8]. Linial and Rozenman [8] studied the existence of a perfect matching in $L_n(G)$ and described a large class of graphs $G$ for which $L_n(G)$ a.a.s. contains a perfect matching (for $n$ even, at least). This class contains all regular graphs and, in turn, is contained in the class of graphs having a fractional perfect matching (see Section 3 for a definition). Observe that if $G$ has a perfect matching then every lift of $G$ has at least one perfect matching.

One of the most interesting questions on random lifts is the problem of existence of a Hamilton cycle. This also seems to be a difficult question. Unlike perfect matchings, the property of having a Hamilton cycle is not carried over from $G$ to its lifts. For instance, it is not difficult to see that a random $n$-lift of a cycle is hamiltonian with probability $1/n$. Hence random lifts of cycles are a.a.s. not hamiltonian, and the same is true for $L_n(G)$ when $G$ equals $K_4$ minus an edge. So for simple graphs, $G = K_4$ is the smallest open case. There is a conjecture (attributed to Linial) that a random lift of $K_4$ is a.a.s. hamiltonian. Indeed, we believe that a.a.s. $L_n(G)$ is hamiltonian for all connected $d$-regular loop-free multigraphs $G$, with $d \geq 3$. (This is known to be true when $G$ is a multigraph with exactly two vertices and at least three edges: see Remark 1.1 below.)

In [4], Burgin et al. showed that a.a.s. $L_n(K_g)$ is hamiltonian when $g$ is large enough (see also [5] for the directed case). The arguments in [4] are combinatorial and utilize the celebrated idea of Pósa. However, for small $g$ an alternative approach is required. We feel that the *small subgraph conditioning method* may be a fruitful line of attack, as it has been very successful for studying Hamilton cycles in random regular graphs (Robinson and Wormald [12, 13], see also [7, Chapter 9]). This method, where it applies, provides a concentration result based on the second moment method conditioned on the number of small cycles. For a concise description of the method, see [7, Theorems 9.12 and 9.13]. Unfortunately, so far we have been unable even to compute the expected number of Hamilton cycles asymptotically, not to mention higher moments. For this reason we have first studied the number of perfect matchings, hoping that this experience will benefit us in future work on Hamilton cycles.

Let $X_G$ be the number of perfect matchings in $L_n(G)$. To apply the small subgraph conditioning method, asymptotic expressions for $\mathbb{E}\, X_G$ and $\mathbb{E}(X_G^2)$ must be found. Then the limit of the ratio $\mathbb{E}(X_G^2)/(\mathbb{E}\, X_G)^2$ is compared against a quantity which depends upon the interaction of perfect matchings and short cycles in $L_n(G)$.

In Sections 3 and 4 we write the first and second moment of $X_G$ as multiple sums of some explicit terms, and then estimate the sums by Laplace's method. This is a standard method for similar moment estimates, and in particular, it has been used in several papers on random regular graphs. (See for example [7, Chapter 9] and the references given there.) However, in the present paper, each summation is over an index set of rather high dimension with a number of side conditions on the indices, while in many previous applications the summations are only over one or two variables. To assist with these calculations, we present a general theorem (Theorem 6.4) that encapsulates Laplace's method for a general situation, with sums over a lattice in a subspace of $\mathbb{R}^N$. We do this both because we think that it clarifies the argument in the present work, and because we hope that it might be useful in future applications. The necessary terminology and notation is introduced in Section 2, while the statement of our results and their proofs can be found in Section 6.

Using this machinery we prove an asymptotic formula for $\mathbb{E}\, X_G$ for any connected regular multigraph $G$ with degree at least three (see Theorem 3.6). However, two difficulties (one algebraic and one analytic) have prevented us from obtaining an asymptotic formula for $\mathbb{E}(X_G^2)$ in the same generality, though we have partial results in Theorem 4.2 and Lemma 4.3. We illustrate these results by calculating $\mathbb{E}(X_G^2)$ for two multigraphs: specifically, for the complete graph $K_4$ and for the multigraph consisting of two vertices and three parallel edges, which we denote by $K_2^3$. These calculations were performed with the aid of `Maple`.

In Section 5 we prove the necessary results relating to short cycles in random lifts (Lemmas 5.1, 5.2 and Corollary 5.4(iii)). As corollaries, using [7, Theorem 9.12] we obtain a concentration result for $X_G$ in our two illustrative examples (see Corollaries 5.7 and 5.8).

**Remark 1.1.** We allow the multigraph $G$ to have multiple edges. The simplest case is when $G$ consists of only two vertices, with $d$ parallel edges between them. The random lift $L_n(G)$ then is a random bipartite (multi)graph obtained by taking the union of $d$ independent random matchings between two sets of $n$ vertices each. Such sums have been studied in [10], where they were shown to be contiguous to random bipartite $d$-regular (multi)graphs. The latter, in turn, is known to be a.a.s. hamiltonian (see [11] for a standard, second moment method proof). Hence for this small multigraph $G$ with $d \geq 3$, the random lift $L_n(G)$ is a.a.s. hamiltonian too.

**Remark 1.2.** Random lifts of multigraphs with loops can also be formed. As in [1], the fiber corresponding to a loop is given by the $n$ edges $i\sigma(i)$ for a random permutation $\sigma$ of $[n]$. This is a random 2-regular (multi)graph, denoted by $\mathbb{P}(n)$ in [7, Remark 9.45]. While we do not allow loops in our current work, for several reasons, we believe that the results here can be extended to multigraphs with loops. A simple and interesting case is when $G$ consists of a single vertex with $d/2$ loops ($d$ even). Then $L_n(G)$ consists of the sum (union) of $d/2$ independent copies of $\mathbb{P}(n)$. Such sums have been shown to be contiguous to random $d$-regular (multi)graphs in [6].

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# 2 Notation and terminology

As mentioned above, $G$ denotes a fixed connected multigraph with $g$ vertices and no loops. For simplicity we assume that $V(G) = [g] := \{1, \dots, g\}$. We denote the number of edges in $G$ by $h$. (Often we assume $G$ to be $d$-regular, and then $h = dg/2$.) Let $A = A_G$ be the $g \times g$ adjacency matrix of $G$ and let $\widehat{A} = \widehat{A}_G$ be the incidence matrix of $G$, with $g$ rows and $h$ columns. Thus

$$\widehat{A}\widehat{A}^T = A + D_G, \tag{2.1}$$

where $D_G$ is the diagonal matrix with entries $d_G(i)$, $i \in V(G)$. Denote the eigenvalues of $A$ by $\alpha_1, \dots, \alpha_g$.

In Section 4 we also need a directed incidence matrix for $G$. Give each edge in $G$ an (arbitrary) direction, and let $\vec{A}_G$ be the corresponding directed incidence matrix. In other words, $\vec{A}_G$ is the $g \times h$ matrix obtained from $\widehat{A}$ by changing the sign of one of the two 1's in each column. Then

$$\vec{A}_G\vec{A}_G^T = D_G - A. \tag{2.2}$$

## 2.1 Lattices

Our version of Laplace's method (Theorem 6.4) involves lattices. A *lattice* is a discrete subgroup of $\mathbb{R}^N$. (Discrete means that the intersection with any bounded set in $\mathbb{R}^N$ is finite.) It is well-known that every lattice $\mathcal{L}$ is isomorphic (as a group) to $\mathbb{Z}^r$ for some $r$ with $0 \le r \le n$. The integer $r$ is called the *rank* of $\mathcal{L}$ and is denoted by $\operatorname{rank}(\mathcal{L})$. In other words, every lattice $\mathcal{L}$ has a *basis*, i.e. a sequence $x_1, \dots, x_r$ of elements of $\mathcal{L}$ such that every element of $\mathcal{L}$ has a unique representation $\sum_{i=1}^r n_i x_i$ with $n_i \in \mathbb{Z}$. Furthermore, the basis elements $x_1, \dots, x_r$ are linearly independent (over $\mathbb{R}$); thus the rank equals the dimension of the linear subspace spanned by $\mathcal{L}$.

The basis is not unique (except in the trivial case $r = 0$); if $\Xi = (\xi_{ij})$ is any $r \times r$ integer matrix such that the determinant $\det(\Xi) = \pm 1$ (which is equivalent to the condition that both $\Xi$ and $\Xi^{-1}$ are integer matrices) and $(x_i)_1^r$ is a basis of $\mathcal{L}$, then $y_i = \sum_j \xi_{ij} x_j$ defines another basis $y_1, \dots, y_r$; conversely, given $(x_i)_1^r$, every basis of $\mathcal{L}$ is obtained in this way by some such matrix $\Xi$.

A *unit cell* of the lattice $\mathcal{L}$ is the set $\{\sum_{i=1}^r t_i x_i : 0 \le t_i < 1\}$ for some basis $(x_i)_i$ of $\mathcal{L}$. If $\mathcal{L} \subset \mathbb{R}^N$ has full rank $N$, and $U$ is any unit cell of $\mathcal{L}$, then $\{x + U\}_{x \in \mathcal{L}}$ is a partition of $\mathbb{R}^N$.

The unit cells of a lattice $\mathcal{L}$ all have the same $r$-dimensional volume (Hausdorff measure), where $r = \operatorname{rank}(\mathcal{L})$; this volume is the *determinant* (or *covolume*) of $\mathcal{L}$, and is denoted by $\det(\mathcal{L})$.

If $(x_i)_{i=1}^r$ is a sequence of vectors in $\mathbb{R}^N$, the symmetric matrix $(\langle x_i, x_j \rangle)_{i,j=1}^r$ of their inner products is called their *Gram matrix*. It is well-known that $x_1, \dots, x_r$ are linearly independent if and only if the Gram matrix is non-singular, i.e., if and only if the *Gram determinant* $\det(\langle x_i, x_j\rangle)_{i,j=1}^r \neq 0$.

The following results are well-known.

**Lemma 2.1.** *If $(x_i)_{i=1}^r$ is a basis of a lattice $\mathcal{L}$ in $\mathbb{R}^N$, then*

$$\det(\langle x_i, x_j\rangle)_{i,j=1}^r = \det(\mathcal{L})^2. \tag{2.3}$$

**Lemma 2.2.** *If $\mathcal{L}_1 \subseteq \mathcal{L}_2$ are two lattices of the same rank, then $\mathcal{L}_2/\mathcal{L}_1$ is a finite group of order $\det(\mathcal{L}_1)/\det(\mathcal{L}_2)$.*

## 2.2 The Hessian

The *Hessian* or second derivative $D^2\phi(x_0)$ of a function $\phi$ at a point $x_0 \in \mathbb{R}^N$ is an $N \times N$ matrix; it is also naturally regarded as a bilinear form on $\mathbb{R}^N$. In general, if $B$ is a bilinear form on $\mathbb{R}^N$, it corresponds to the matrix $(\langle Be_i, e_j\rangle)_{i,j=1}^N$, where $(e_i)_{i=1}^N$ is the standard basis. We define the determinant $\det(B)$ as $\det(\langle Be_i, e_j\rangle)_{i,j=1}^N$, and note that if $z_1, \ldots, z_N$ is any basis in $\mathbb{R}^N$, then

$$\det(B) = \frac{\det(\langle Bz_i, z_j\rangle)_{i,j=1}^N}{\det(\langle z_i, z_j\rangle)_{i,j=1}^N}. \tag{2.4}$$

(Proof: If $Z$ is the matrix with column vectors $z_1, \ldots, z_N$ and $\bar{B} = (\langle Be_i, e_j\rangle)_{i,j}$, then $z_i = Ze_i$ and $(\langle Bz_j, z_i\rangle)_{i,j} = Z^T\bar{B}Z$ and $(\langle z_j, z_i\rangle)_{i,j} = Z^TZ$, and the result follows.)

We are interested in the restriction to a subspace. If $B$ is a bilinear form on $\mathbb{R}^N$ and $V \subseteq \mathbb{R}^N$ is a subspace, we let $\det(B|_V)$ denote the determinant of $B$ regarded as a bilinear form on $V$. By (2.4), this can be computed as

$$\det(B|_V) = \frac{\det(\langle Bz_i, z_j\rangle)_{i,j=1}^r}{\det(\langle z_i, z_j\rangle)_{i,j=1}^r}. \tag{2.5}$$

for any basis $z_1, \ldots, z_r$ of $V$.

# 3 Expected number of perfect matchings

A *fractional perfect matching* of the multigraph $G$ is a function $f : E(G) \to [0, 1]$ such that

$$\sum_{e \ni v} f(e) = 1 \text{ for all } v \in V(G).$$

Note that every $d$-regular multigraph has a trivial fractional perfect matching obtained by giving each edge weight $1/d$. We often treat $f$ as a vector $(f(e))_{e\in E(G)}$.

First, note that if there is a perfect matching at all in a lift $L_n(G)$ of $G$, then there exists a fractional perfect matching $f$ of $G$ such that $nf(e)$ is an integer for each $e$. Indeed, suppose that $M$ is a perfect matching of a lift of $G$. Let $\ell_e$ be the number of edges from the fiber $F_e$ in $M$, for each edge $e \in E(G)$. Then the function $f : E(G) \to [0, 1]$

defined by $f(e) = \ell_e/n$ is a fractional perfect matching of $G$. Conversely, suppose that there exists a fractional perfect matching $z = (z_e)_e$ in $G$ such that $nz_e$ is an integer for each $e$. We may construct an $n$-lift of $G$ that contains a perfect matching as follows: First take $nz_e$ edges above each edge $e \in E(G)$, with all their endpoints disjoint. This yields $n$ endpoints above each vertex $i \in G$, so we have constructed the sets $V_i$, and a perfect matching. Extend this perfect matching to an $n$-lift by adding further edges between $V_i$ and $V_j$ for all edges $e = ij$. Consequently, $L_n(G)$ has a perfect matching with positive probability if and only if there exists a fractional perfect matching $z$ with $nz$ integer-valued. We will in the sequel, for a given graph $G$, consider only $n$ such that this holds, since otherwise trivially $X_G = 0$.

**Remark 3.1.** It seems to be an interesting problem to characterize the set of such $n$ for a given graph, but this is outside the scope of the present paper, and we note only the following examples: If $G$ itself has a perfect matching then every $n$ is allowed. On the other hand, if $g$ is odd, then only even $n$ are possible. If $G$ is of odd order and hamiltonian, then the set of allowed $n$ is exactly the set of positive even integers. If $G$ is $d$-regular, then $(1/d, \dots, 1/d)$ is a fractional perfect matching, so every multiple of $d$ is an allowed $n$ (but there might be others too). The result by Linial and Rozenman [8] implies that for a large class of graphs defined there, every large even $n$ is allowed. Note finally that if $n_1$ and $n_2$ are allowed, then so is $n_1 + n_2$. Hence the set of allowed $n$ is always infinite, unless it is empty, so it makes sense to talk about asymptotic results.

Thus, suppose that there exists a fractional perfect matching $z = (z_e)_e$ in $G$ with $nz$ an integer vector. If a perfect matching in $L_n(G)$ has $\ell_e$ edges in the fiber $F_e$ over $e$, then $\sum_{e \ni v} \ell_e = n = n \sum_{e \ni v} z_e$ for every $e$, so $(\ell_e)_e - nz$ belongs to the lattice $\mathcal{L}_G^{(1)}$ in $\mathbb{R}^{E(G)}$ defined by

$$\begin{aligned}\mathcal{L}_G^{(1)} &:= \Big\{ (\nu_e)_e \in \mathbb{Z}^{E(G)} : \sum_{e \ni v} \nu_e = 0 \text{ for every } v \in V(G) \Big\} \\ &= \{ \nu \in \mathbb{Z}^{E(G)} : \widehat{A}\nu = 0 \}.\end{aligned}$$

(The superscript 1 denotes the first moment.) Here, and elsewhere when convenient, we think of the vectors as column vectors although we write them as row vectors for typographical reasons. Conversely, if $\ell = (\ell_e)_e$ is a vector such that $\ell - nz \in \mathcal{L}_G^{(1)}$, then $\ell$ is an integer vector and $\sum_{e \ni v} \ell_e = \sum_{e \ni v} nz_e = n$ for every $v$.

Given such an integer vector $(\ell_e)_e \in \mathcal{L}_G^{(1)} + nz$, let us compute the expected number of perfect matchings in $L_n(G)$ with $\ell_e$ edges in the fiber $F_e$. Clearly this number is zero unless $0 \le \ell_e \le n$ for all $e$. Then the endpoints of the edges in the matching may be chosen in

$$\prod_{v \in V(G)} \frac{n!}{\prod_{e \ni v} \ell_e!} = n!^g \prod_e (\ell_e!)^{-2}$$

ways, and for each choice, there are $\ell_e!(n - \ell_e)!$ possibilities for the fiber $F_e$, with probability $1/n!$ each. Hence, defining $K = [0, 1]^{E(G)}$ we have

$$\mathbb{E}(X_G) = \sum_{\ell \in (\mathcal{L}_G^{(1)} + nz) \cap nK} a_n(\ell) \tag{3.1}$$

where

$$a_n(\ell) := n!^{g-h} \prod_e \frac{(n-\ell_e)!}{\ell_e!}.$$

We wish to evaluate the sum (3.1) asymptotically by Laplace's method: more precisely, by applying Theorem 6.4. We use Stirling's formula in the following form, valid for all $n \geq 0$, where $x \vee y := \max(x, y)$,

$$\ln(n!) = n \ln n - n + \tfrac{1}{2}\ln(n \vee 1) + \tfrac{1}{2}\ln 2\pi + O(1/(n+1)). \tag{3.2}$$

Let $x_e = \ell_e/n$ for all $e \in E(G)$. Applying (3.2) we obtain, uniformly for $\ell \in (\mathcal{L}_G^{(1)} + nz) \cap nK$,

$$\begin{aligned}
\ln(a_n(\ell)) &= (g-h)\ln(n!) + \sum_{e \in E(G)} \Big(\ln((n-\ell_e)!) - \ln(\ell_e!)\Big) \\
&= (g-h)\left(n(\ln(n)-1) + \tfrac{1}{2}\ln(n) + \tfrac{1}{2}\ln(2\pi) + O(1/n)\right) + \sum_{e \in E(G)} (n - 2\ell_e)(\ln(n)-1) \\
&\qquad + n \sum_{e \in E(G)} \Big((1-x_e)\ln(1-x_e) - x_e \ln(x_e)\Big) \\
&\quad + \frac{1}{2}\sum_{e \in E(G)} \left(\ln((1-x_e) \vee n^{-1}) - \ln(x_e \vee n^{-1})\right) + \sum_{e \in E(G)} O\left(\frac{1}{\ell_e + 1} + \frac{1}{n - \ell_e + 1}\right).
\end{aligned}$$

Since

$$\sum_{e \in E(G)} \ell_e = \tfrac{1}{2}\sum_v \sum_{e \ni v} \ell_e = \tfrac{1}{2}\sum_v n = \tfrac{1}{2}gn,$$

after cancellation, $a_n(\ell)$ can be expressed as

$$a_n(\ell) = b_n\, \psi(\ell/n)\, \exp\left(n\phi(\ell/n)\right) \left(1 + O\Big(\frac{1}{\min \ell_e + 1}\Big) + O\Big(\frac{1}{n - \max \ell_e + 1}\Big)\right)$$

where, for $x \in \mathbb{R}^{E(G)}$,

$$b_n := (2\pi n)^{(g-h)/2}, \tag{3.3}$$

$$\phi(x) := \sum_e \Big((1-x_e)\ln(1-x_e) - x_e \ln(x_e)\Big), \tag{3.4}$$

$$\psi(x) := \prod_e \left(\frac{1-x_e}{x_e}\right)^{1/2}, \tag{3.5}$$

except that if some $x_e$ or $1 - x_e$ is 0, we replace it by $1/n$ in (3.5). This implies that $a_n(\ell)$ satisfies condition (6.10) of Theorem 6.4 with the above $b_n$, $\phi$, and $\psi$. We will now check all the remaining assumptions of Theorem 6.4. Let

$$W := \Big\{x = (x_e) \in \mathbb{R}^{E(G)} : \sum_{e \ni v} x_e = 1 \text{ for every } v \in V(G)\Big\} = \big\{x : \widehat{A}x = (1, \dots, 1)\big\}.$$

As is well-known, and described in Section 6 in detail, the sum (3.1) is dominated by the terms where $\phi(\ell/n)$ is close to its maximum. In order to find the maximum, we restrict ourselves to regular multigraphs, where the result is simple. (The method applies to other graphs as well, provided one can find the maximum point(s) of $\phi$.)

**Lemma 3.2.** *Suppose that $G$ is $d$-regular, where $d \geq 3$. Then $\phi$ defined by* (3.4) *has a unique maximum on $K \cap W = \{x \in K : \widehat{A}x = (1, \dots, 1)\}$, attained at the point $x^0 = (1/d, \dots, 1/d)$. The maximum value is*

$$\phi(x^0) = \frac{g}{2} \ln \left( \frac{(d-1)^{d-1}}{d^{d-2}} \right),$$

*and, for $\psi$ in* (3.5) *and the Hessian $D^2\phi$,*

$$\psi(x^0) = (d-1)^{h/2}, \qquad D^2\phi(x^0) = -\frac{d(d-2)}{d-1} I.$$

*Proof.* We write $\phi = \frac{1}{2} \sum_{v \in V(G)} \phi_v$, where

$$\phi_v(x_e : e \ni v) = \sum_{e \ni v} \big( (1 - x_e) \ln(1 - x_e) - x_e \ln(x_e) \big). \tag{3.6}$$

Fix a vertex $v \in V(G)$. We rename the variables $x_e$, $e \ni v$, by $x_1, \dots, x_d$, for convenience. Since $\phi_v$ is continuous, it has a maximum over the compact set

$$\Sigma_d := \Big\{ (x_i)_i \in [0,1]^d : \sum_1^d x_i = 1 \Big\}.$$

Let $x^v \in \Sigma_d$ be a maximum point of $\phi_v$. Assume first that $x^v$ is an interior point, i.e., that $x^v \in (0,1)^d$. Then the function $f(y) = \phi_v(x_1^v + y, x_2^v - y, x_3^v, \dots, x_d^v)$ achieves a maximum at $y = 0$. Therefore, $f'(0) = 0$ and by the chain rule,

$$\frac{\partial \phi_v(x)}{\partial x_1}(x^v) = \frac{\partial \phi_v(x)}{\partial x_2}(x^v).$$

By the same argument (or by the general Lagrange multiplier method), we have that for some constant $c_v > 0$

$$\frac{\partial \phi_v(x)}{\partial x_i}(x^v) = c_v, \text{ for } i = 1, \dots, d.$$

But

$$\frac{\partial \phi_v(x)}{\partial x_i}(x^v) = -\ln(1 - x_i) - \ln x_i - 2,$$

so

$$x_i^v(1 - x_i^v) = \exp\{-c_v - 2\} \text{ for all } i = 1, \dots, d.$$

This implies that the $x_i^v$'s are all at the same distance from $1/2$. That is, for some constant $c'_v \geq 0$ we have $x_i^v = 1/2 \pm c'_v$ for $i = 1, \dots, d$. Since $\sum_i x_i^v = 1$ and $d \geq 3$, we have to choose the minus sign for all $i$, and thus all $x_i^v$ are equal. Since $x^v \in \Sigma_d$ we conclude that $x_i^v = 1/d$ for $i = 1, \dots, d$.

We also have to consider the boundary of $\Sigma_d$. If, say, $x_1^v = 0$ and $0 < x_2^v < 1$, then $f$ above is defined for small positive $y$ with $f'(0+) = +\infty$, so $x^v$ cannot be a maximum point on $\Sigma_d$. The only remaining points are those with all $x_i \in \{0, 1\}$, but then $\phi_v(x) = 0$, while $\phi_v(1/d, \dots, 1/d) > 0$, so these too cannot be (global) maximum points. Hence $x^v$ is the unique maximum point for $\phi_v$ on $\Sigma_d$.

Setting $x^0 = (1/d, \dots, 1/d) \in \mathbb{R}^g$, we have for all $x \in K \cap W$,

$$\phi(x) \leq \tfrac{1}{2} \sum_v \phi_v(x^v) = \phi(x^0).$$

Moreover, the inequality is strict for all $x \neq x^0$. This proves that $x^0$ is a unique maximum point of $\phi$ in $K \cap W$. Clearly, $x^0$ belongs to the interior of $K$. Moreover, $\phi(x^0)$ and $\psi(x^0)$ are given by the formulas stated in Lemma 3.2.

Finally, the Hessian $D^2\phi(x)$ is diagonal with entries $(1 - x_e)^{-1} - x_e^{-1}$. Hence, at $x^0$ we have $D^2\phi(x^0) = -\frac{d(d-2)}{d-1} I$. □

We have verified all assumptions of Theorem 6.4, for any neigbourhood $K_1$ of $x^0$ with $\overline{K_1} \subset K^\circ$. To apply formula (6.11), we still need to compute the rank of the lattice $\mathcal{L}_G^{(1)}$ and its determinant $\det(\mathcal{L}_G^{(1)})$.

**Lemma 3.3.** (i) *If $G$ is non-bipartite then the lattice $\mathcal{L}_G^{(1)}$ has rank $h - g$ and determinant* $\det(\mathcal{L}_G^{(1)}) = \frac{1}{2} \det(A + D_G)^{1/2}$.

(ii) *If $G$ is bipartite then the lattice $\mathcal{L}_G^{(1)}$ has rank $h - g + 1$ and determinant* $\det(\mathcal{L}_G^{(1)}) = \det(A' + D'_G)^{1/2}$, *where the matrix $A'$ (respectively, $D'_G$) is obtained by deleting the last row and column of $A$ (respectively, $D_G$).*

*Proof.* For $v \in V(G)$ define the vector $x^v = \big(\mathbf{1}[v \in e], e \in E(G)\big)$ given by the row of the incidence matrix $\widehat{A}$ corresponding to $v$. For convenience, rename these vectors $x_1, \dots, x_g$. Then, by (2.1), the Gram matrix of $x_1, \dots, x_g$ is $\widehat{A}\widehat{A}^T = A + D_G$. This matrix is singular if and only if there exists a non-zero vector $y = (y_v) \in \mathbb{R}^{V(G)}$ with $y\widehat{A} = 0$. This is equivalent to $y_i = -y_j$ for every edge $ij$, and it is easily seen that, when $G$ is connected, such a non-zero vector $y$ exists only if $G$ is bipartite, and that if $G$ is connected and bipartite, there is a one-dimensional space of such solutions $y$.

Consequently, in the non-bipartite case (i), the vectors $x_1, \dots, x_g$ are linearly independent. We apply Lemma 6.2 with $N = h$, $m = g$ and using the vectors $x_1, \dots, x_g$. Let $\mathcal{L}$, $\mathcal{L}^\perp$ and $\mathcal{L}_0$ be as in Lemma 6.2. Then $\mathcal{L}_G^{(1)} = \mathcal{L}^\perp$, and thus $\mathcal{L}_G^{(1)}$ has rank $h - g$, by Lemma 6.2. Furthermore, by Lemma 2.1 and (2.1),

$$\det(\mathcal{L}_0) = \big(\det(\langle x_i, x_j \rangle)_{i,j=1}^g\big)^{1/2} = \det(A + D_G)^{1/2}.$$

Moreover, $(t_v,\, v \in V(G))$ solves (6.1) if and only if $t_v \equiv -t_w \pmod 1$ for every edge $vw$. Going around an odd cycle, we see that $t_v \equiv 0$ or $t_v \equiv 1/2$ for every vertex on the cycle. Since $G$ is connected, it follows that there are exactly two solutions to (6.1): $t_v \equiv 0$ for every $v$ and $t_v \equiv 1/2$ for every $v$. Hence $q = 2$ in Lemma 6.2, and the result follows.

Now suppose that $G$ is bipartite. Then the vectors $x_1, \ldots, x_{g-1}$ are linearly independent and $x_g$ can be written as a $\{\pm 1\}$-combination of $x_1, \ldots, x_{g-1}$, since the sum of vectors $x^v$ over all vertices $v$ on either side of the vertex bipartition gives the vector $(1, 1, \ldots, 1)$. We apply Lemma 6.2 with $N = h$, $m = g - 1$ and using the vectors $x_1, \ldots, x_{g-1}$. The lemma asserts that $\mathcal{L}_G^{(1)} = \mathcal{L}^\perp$ has rank $h - g + 1$, and

$$\det(\mathcal{L}_0) = \left(\det(\langle x_i, x_j \rangle)_{i,j=1}^{g-1}\right)^{1/2} = \det(A' + D'_G)^{1/2}.$$

Finally, let $w \in V(G)$ correspond to $x_g$. If $(t_v,\, v \in V(G) \setminus \{w\})$ solves (6.1) then $t_u = 0$ for every neighbour $u$ of $w$. In turn this implies that $t_u = 0$ for every vertex $u$ at distance 2 from $w$, and iterating this shows that $t_u = 0$ for all vertices $u$ in the connected graph $G$. Therefore $q = 1$ in Lemma 6.2 and the proof is complete. $\square$

**Example 3.4.** When $G = K_4$,

$$\det(A + D_G) = \begin{vmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{vmatrix} = 48.$$

Thus Lemma 3.3(i) says that $\mathcal{L}_G^{(1)}$ has rank 2 and

$$\det(\mathcal{L}_G^{(1)}) = \frac{\sqrt{48}}{2} = \sqrt{12}.$$

**Example 3.5.** Let $G = K_2^3$ be the multigraph with two vertices and three parallel edges. Then $A + D_G = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}$ and deleting one row and column gives the $1 \times 1$ matrix $(3)$. Hence $\mathcal{L}_G^{(1)}$ has rank 2 and $\det(\mathcal{L}_G^{(1)}) = \sqrt{3}$, using Lemma 3.3(ii).

We are ready to apply formula (6.11) of Theorem 6.4.

**Theorem 3.6.** *Suppose that $G$ is $d$-regular, where $d \geq 3$.*

(i) *If $G$ is non-bipartite then*

$$\begin{aligned} \mathbb{E}\, X_G &\sim \frac{2(d-1)^{dg/4}}{\sqrt{\det(A + dI)}} \left(\frac{d-1}{d(d-2)}\right)^{dg/4 - g/2} \left(\frac{(d-1)^{d-1}}{d^{d-2}}\right)^{gn/2} \\ &= \frac{2(d-1)^{(d-1)g/2}}{(d(d-2))^{dg/4 - g/2} \sqrt{\det(A + dI)}} \left(\frac{(d-1)^{d-1}}{d^{d-2}}\right)^{gn/2}. \end{aligned}$$

(ii) *If $G$ is bipartite then*

$$\mathbb{E}\, X_G \sim \frac{(d-1)^{dg/4}}{\sqrt{\det(A'+dI)}} \left(\frac{d-1}{d(d-2)}\right)^{dg/4-g/2+1/2} (2\pi n)^{1/2} \left(\frac{(d-1)^{d-1}}{d^{d-2}}\right)^{gn/2}$$
$$= \frac{(d-1)^{(d-1)g/2+1/2}}{(d(d-2))^{dg/4-g/2+1/2}\sqrt{\det(A'+dI)}} (2\pi n)^{1/2} \left(\frac{(d-1)^{d-1}}{d^{d-2}}\right)^{gn/2}$$

*where $A'$ is obtained by deleting the last row and column of $A$.*

*Proof.* Let $r$ be the rank of $\mathcal{L}_G^{(1)}$, and recall that the Hessian $H = D^2\phi(x^0)$ is diagonal and equals $-\frac{d(d-2)}{d-1}I$ by Lemma 3.2. Thus $H|_V = -\frac{d(d-2)}{d-1}I$ too, and $\det(-H|_V) = \left(\frac{d(d-2)}{d-1}\right)^r$. Hence the result follows from (3.1) and Theorem 6.4, using Lemmas 3.2 and 3.3, and the fact that $h = dg/2$. □

**Example 3.7.** For $G = K_4$, $d = 3$, $g = 4$ and thus, using Example 3.4,

$$\mathbb{E}\, X_G \sim \frac{2\cdot 2^4}{3\sqrt{48}} \left(\frac{4}{3}\right)^{2n} = \frac{8}{3\sqrt{3}} \left(\frac{4}{3}\right)^{2n}.$$

**Example 3.8.** For the bipartite multigraph $G = K_2^3$ with two vertices and three parallel edges we have $d = 3$, $g = 2$ and by Example 3.5,

$$\mathbb{E}\, X_G \sim \frac{8}{3\sqrt{3}} \sqrt{\pi n} \left(\frac{4}{3}\right)^{n}.$$

# 4 The second moment of $X_G$

We now work towards an asymptotic expression for the second moment of $X_G$, using the same approach as in the previous section. To simplify our calculations we consider only regular multigraphs $G$ of degree at least three.

Given a pair $(M_1, M_2)$ of perfect matchings in $L_n(G)$, let for a vertex $i \in V(G)$ and two (possibly equal) edges $e, f \ni i$, $\ell_{ief}$ be the number of vertices in $V_i$ whose incident edges in $M_1$ and $M_2$ lie, respectively, in the fibers $F_e$ and $F_f$. Form these numbers into the $gd^2$-dimensional vector $\ell = \ell(M_1, M_2) = \big(\ell_{ief} : i \in [g],\, e, f \ni i\big)$.

Let

$$V^* := \Big\{ \big(z_{ief} : i \in [g],\, e, f \ni i\big) \in \mathbb{R}^{gd^2} : \text{for every } e \in E(G) \text{ with endpoints } i \text{ and } j,$$
$$z_{iee} = z_{jee}, \qquad \sum_{f \ni i} z_{ief} = \sum_{f \ni j} z_{jef}, \qquad \sum_{f \ni i} z_{ife} = \sum_{f \ni j} z_{jfe} \Big\}.$$

Then the vector $\ell$ belongs to the set

$$Q := \Big\{ (z_{ief}) \in V^* \cap \mathbb{Z}^{gd^2} : \sum_{e,f \ni i} z_{ief} = n \text{ for } i \in [g] \Big\}.$$

(The three conditions in $V^*$ follow from consideration of the edges in $M_1 \cap M_2$, $M_1$ and $M_2$, respectively.) Fix a particular vector $z$ with $nz \in Q$. (By our assumption that there is a perfect matching in $L_n(G)$, it follows that at least one such vector exists.) Then $Q = \mathcal{L}_G^{(2)} + nz$, where $\mathcal{L}_G^{(2)}$ is the lattice defined by

$$\mathcal{L}_G^{(2)} := \Big\{ (\nu_{ief}) \in V^* \cap \mathbb{Z}^{gd^2} : \sum_{e,f \ni i} \nu_{ief} = 0 \text{ for } i \in [g] \Big\}.$$

(The superscript 2 denotes the second moment.)

Given a pair $(M_1, M_2)$ of perfect matchings and thus a vector $\ell \in Q$, we further define, for an edge $e \in E(G)$ and an endpoint $i$ of $e$,

$$s_e = s_{ie}(\ell) = \sum_{f \ni i,\, f \neq e} \ell_{ief}, \quad t_e = t_{ie}(\ell) = \sum_{f \ni i,\, f \neq e} \ell_{ife}, \quad u_e = u_{ie}(\ell) = \sum_{f,f' \ni i;\, f,f' \neq e} \ell_{iff'};$$

these are the numbers of edges in the fiber $F_e$ that belong to $M_1 \setminus M_2$, $M_2 \setminus M_1$ and $(M_1 \cup M_2)^c$, respectively, so they do not depend on the choice of endpoint $i$ of $e$. We have, for every edge $e$ and endpoint $i$,

$$s_e + t_e + u_e + \ell_{iee} = n.$$

We now calculate the expected number of pairs of perfect matchings $(M_1, M_2)$ in $L_n(G)$ corresponding to a given nonnegative integer vector $\ell = (\ell_{ief}) \in \mathcal{L}_G^{(2)} + nz$. First, partition each $V_i$ into $d^2$ subsets of sizes $(\ell_{ief})_{e,f \ni i}$; this can be done in

$$\prod_{i=1}^{g} \frac{n!}{\prod_{e,f \ni i} \ell_{ief}!} = n!^g \prod_{i=1}^{g} \prod_{e,f \ni i} (\ell_{ief}!)^{-1}$$

ways. Given these partitions there are

$$s_e!\, t_e!\, u_e!\, \ell_{iee}!$$

possibilities for the fiber $F_e$ (where $i$ is an endpoint of $e$), with probability $1/n!$ each. Hence the expected number of pairs $(M_1, M_2)$ of perfect matchings in $L_n(G)$ which correspond to the vector $\ell$ is given by

$$a_n(\ell) = n!^{g - dg/2} \prod_{i \in [g]} \left( \prod_{e \ni i} \left( \frac{s_e!\, t_e!\, u_e!}{\ell_{iee}!} \right)^{1/2} \prod_{f \ni i,\, f \neq e} \frac{1}{\ell_{ief}!} \right).$$

Thus we can write

$$\mathbb{E}(X_G^2) = \sum_{\ell \in (\mathcal{L}_G^{(2)} + nz) \cap nK} a_n(\ell) \tag{4.1}$$

where $K = [0,1]^{gd^2}$. This will allow us to apply the same arguments as used in Section 3.

We now switch to continuous variables $x \in \mathbb{R}^{gd^2}$, where $x_{ief}$ corresponds to $\ell_{ief}/n$. Define the functions $\sigma_{ie} = \sigma_{ie}(x)$, $\tau_{ie} = \tau_{ie}(x)$ and $\gamma_{ie} = \gamma_{ie}(x)$ to be continuous scaled analogues of $s_{ie}$, $t_{ie}$ and $u_{ie}$ respectively. That is,

$$\sigma_{ie} = \sum_{f\ni i,\, f\neq e} x_{ief}, \qquad \tau_{ie} = \sum_{f\ni i,\, f\neq e} x_{ife}, \qquad \gamma_{ie} = \sum_{f,f'\ni i;\, f,f'\neq e} x_{iff'},$$

so that $\sigma_{ie}(\ell/n) = s_{ie}(\ell)/n$ and so on. Then, applying (3.2), it follows that $a_n(\ell)$ satisfies condition (6.10) of Theorem 6.4 with

$$\begin{aligned} b_n &= (2\pi n)^{g/2+3h/2-d^2g/2}, \\ \psi(x) &= \prod_{i\in[g]} \prod_{e\ni i} \left( \frac{\sigma_{ie}\tau_{ie}\gamma_{ie}}{x_{iee}} \right)^{1/4} \prod_{f\ni i,\, f\neq e} x_{ief}^{-1/2}, \\ \phi(x) &= \tfrac{1}{2} \sum_{i\in[g]} \sum_{e\ni i} \Bigg( \sigma_{ie}\ln\sigma_{ie} + \tau_{ie}\ln\tau_{ie} + \gamma_{ie}\ln\gamma_{ie} - x_{iee}\ln x_{iee} \\ &\qquad\qquad - 2 \sum_{f\ni i,\, f\neq e} x_{ief}\ln x_{ief} \Bigg). \end{aligned} \tag{4.2}$$

(Again, if some $x_{ief}$, $\sigma_{ie}$, $\tau_{ie}$ or $\gamma_{ie}$ is 0, then we replace it by $1/n$ in the definition of $\psi(x)$.)

Let $W$ be the domain defined by

$$W := \Big\{ (x_{ief}) \in V^* : \sum_{e,f\ni i} x_{ief} = 1 \text{ for } i \in [g] \Big\}.$$

We conjecture that for all connected $d$-regular multigraphs $G$ with no loops, the function $\phi$ has a unique maximum on $K \cap W$, attained at the point

$$x^0 = (1/d^2, \ldots, 1/d^2).$$

Unfortunately, we have been unable to prove this, and have only been able to verify this computationally for $d = 3$. For future reference, note that

$$\psi(x^0) = \big((d-1)d^{d-2}\big)^{dg}, \qquad \phi(x^0) = g \ln\left( \frac{(d-1)^{d-1}}{d^{d-2}} \right). \tag{4.3}$$

One approach to finding the maximum of $\phi$ is to mimic the proof of Lemma 3.2. The function $\phi$ can be written as the sum over $i = 1, \ldots, g$ of functions $\phi_i$, where the sets of variables appearing in different $\phi_i$ are disjoint. For convenience we drop the index $i$ and rename all variables corresponding to vertex $i$ as $x_{ef} := x_{ief}$, and let $\sigma_e := \sigma_{ie}$, $\tau_e := \tau_{ie}$, $\gamma_e := \gamma_{ie}$. Then

$$\phi_i(x) = \tfrac{1}{2} \sum_{e\ni i} \Big\{ \sigma_e\ln\sigma_e + \tau_e\ln\tau_e + \gamma_e\ln\gamma_e - x_{ee}\ln x_{ee} - 2 \sum_{f\ni i,\, f\neq e} x_{ef}\ln x_{ef} \Big\}.$$

Since $G$ is $d$-regular and $\phi_i$ depends only on the degree of $i$ in $G$, all the functions $\phi_i$ are equivalent under relabelling of variables.

Now define the domain

$$\Sigma_{d^2} = \Big\{ (x_{ef})_{e,f\ni i} \in [0,1]^{d^2} : \sum_{e,f\ni i} x_{ef} = 1 \Big\}.$$

It suffices to prove that $\phi_i$ has a unique maximum on $\Sigma_{d^2}$ attained at the point $(1/d^2, \dots, 1/d^2)$. Applying the Lagrange multiplier method to $\Sigma_{d^2}$, we see that at an interior maximum point, all partial derivatives of $\phi_i$ must be equal. This gives $d^2-1$ (non-linear) equations (together with $\sum_{e,f} x_{ef} = 1$) to be solved for $d^2$ variables. We tried to solve this system using `Maple`. Unfortunately, `Maple` seems unable to handle the computations for $d \geq 4$. Hence we only have the desired result for $d = 3$.

**Lemma 4.1.** *If $G$ is 3-regular then the function $\phi$ defined by* (4.2) *has a unique maximum on $K \cap W$ attained at the point* $(1/9, \dots, 1/9) \in \mathbb{R}^{9g}$.

*Proof.* As explained above, we consider only the function $\phi_i$ for a fixed vertex $i$. Using `Maple`, we solved for points in $\big\{(x_{ef})_{e,f} : \sum_{e,f} x_{e,f} = 1\big\}$ where all the 9 partial derivatives of $\phi_i$ are equal. Exactly four solutions were found, of which only one lies in $[0,1]^9$, giving the point $x^0 = (1/9, \dots, 1/9) \in \Sigma_9$. (The other three solutions each contain both positive and negative entries.) We have $\phi(x^0) = \ln(4/3)$.

It remains to consider the boundary, where one or several $x_{ef} = 0$. If $x_{ee} = 0$ and $\gamma_f > 0$ for $f \neq e$, then $\frac{\partial}{\partial x_{ee}} \phi(x) = +\infty$, and thus $x$ is not a maximum point. Similarly, $x$ cannot be a maximum point if $x_{ef} = 0$, where $e \neq f$ and at most one of $\sigma_e$, $\tau_f$ and $\gamma_{f'}$ (where $f'$ is the third index) vanishes. It is easily seen that the only remaining cases are when the only non-zero variables (after relabelling the indices as $1, 2, 3$ in some order) are $\{x_{12}, x_{21}\}$, $\{x_{11}, x_{22}, x_{33}\}$ or $\{x_{11}, x_{12}, x_{13}\}$, or a subset of one of these. In the first case we have $\phi = 0$. In the two latter cases, $\phi_i$ equals, after relabelling, $\frac{1}{2}\phi_v$ defined in (3.6) (at the corresponding step of the first moment calculation), and thus the maximum over one of these sets is $\frac{1}{2}\ln(4/3) < \phi(x_0)$. (We omit the details.) Hence, there is no global maximum on the boundary.

Consequently, $x^0$ is the unique maximum point of $\phi_i$ on $\Sigma_9$. Arguing as in Lemma 3.2 completes the proof. □

Let $V = W - z$ be the subspace spanned by $\mathcal{L}_G^{(2)}$, i.e.,

$$V := \Big\{ (x_{ief}) \in V^* : \sum_{e,f\ni i} x_{ief} = 0 \text{ for } i \in [g] \Big\}.$$

**Theorem 4.2.** *Suppose that $G$ is $d$-regular, where $d \geq 3$. If the function $\phi$ defined in* (4.2) *has a unique maximum on $K \cap W$ at $x^0 = (1/d^2, \dots, 1/d^2)$, then*

$$\mathbb{E}(X_G^2) \sim \frac{\big((d-1)d^{d-2}\big)^{dg}}{\det(\mathcal{L}_G^{(2)}) \det(-H|_V)^{1/2}} (2\pi n)^{r/2+g/2+3dg/4-d^2g/2} \left( \frac{(d-1)^{d-1}}{d^{d-2}} \right)^{gn},$$

*where $r$ is the rank of $\mathcal{L}_G^{(2)}$ and $H = D^2\phi(x^0)$ is the Hessian of $\phi$ at $x^0$, provided the determinant in the denominator is non-zero. In particular, this expression holds for all 3-regular connected graphs $G$.*

*Proof.* This is now an immediate consequence of Theorem 6.4, using (4.1) and (4.3). The final statement follows from Lemma 4.1. □

It remains to calculate the determinants of $\mathcal{L}_G^{(2)}$ and $-H|_V$, and the rank $r$. In the non-bipartite case, part of this is covered by the next lemma.

**Lemma 4.3.** *Suppose that $G$ is non-bipartite and $d$-regular, where $d \geq 3$. Recall that $h$ denotes the number of edges in $G$, so $h = dg/2$. Then the lattice $\mathcal{L}_G^{(2)}$ has rank $d^2g - (g + 3h) = d^2g - g - 3dg/2$ and determinant*

$$\det(\mathcal{L}_G^{(2)}) = 2^{3h/2-3g/2-2}\big(d(d-2)\big)^{h/2-g/2}\det(dI + A)\det(d(2d-3)I - A)^{1/2}$$
$$= 2^{3h/2-3g/2-2}\big(d(d-2)\big)^{h/2-g/2}\prod_{i=1}^{g}(d + \alpha_i)(d(2d-3) - \alpha_i)^{1/2},$$

*where $\alpha_1, \ldots, \alpha_g$ are the eigenvalues of $A$.*

*Proof.* The linear space $V$ spanned by $\mathcal{L}_G^{(2)}$ is the subspace of $\mathbb{R}^{gd^2}$ orthogonal to the following $g + 3h$ vectors:

- one vector $x^{0j}$ for every $j \in V(G)$, with $x^{0j}_{ief} = \mathbf{1}[i = j]$.
- one vector $x^{1\varepsilon}$ for every $\varepsilon \in E(G)$, with $x^{1\varepsilon}_{ief} = \vec{a}_{i\varepsilon}\mathbf{1}[e = f = \varepsilon]$.
- one vector $x^{2\varepsilon}$ for every $\varepsilon \in E(G)$, with $x^{2\varepsilon}_{ief} = \vec{a}_{i\varepsilon}\mathbf{1}[e = \varepsilon \neq f]$.
- one vector $x^{3\varepsilon}$ for every $\varepsilon \in E(G)$, with $x^{3\varepsilon}_{ief} = \vec{a}_{i\varepsilon}\mathbf{1}[e \neq \varepsilon = f]$.

Relabel these vectors (in this order) as $x_1, \ldots, x_{g+3h}$. Then their Gram matrix $\Gamma$ can be written in block form, with blocks of dimensions $g, h, h, h$:

$$\Gamma = \begin{pmatrix} d^2I & \vec{A} & (d-1)\vec{A} & (d-1)\vec{A} \\ \vec{A}^T & 2I & 0 & 0 \\ (d-1)\vec{A}^T & 0 & 2(d-1)I & \vec{A}^T\vec{A} - 2I \\ (d-1)\vec{A}^T & 0 & \vec{A}^T\vec{A} - 2I & 2(d-1)I \end{pmatrix}.$$

In order to evaluate the Gram determinant $\det(\Gamma)$, we may make an orthogonal change of basis in the first component $\mathbb{R}^g$, and another orthogonal change of basis in each of the components $\mathbb{R}^h$ (we choose the same change in all three). It is well-known that we can make such changes of basis such that any given $g \times h$ matrix $B$ obtains the form of a diagonal $g \times g$ matrix $D_s$ with $h - g$ additional columns of 0's; this is known as the singular value decomposition of $B$, and is easily seen by choosing an orthonormal basis $z_1, \ldots, z_h$ in $\mathbb{R}^h$ such that $B^TB$ is diagonal, and then choosing an orthonormal basis in

$\mathbb{R}^g$ containing the vectors $Bz_i/\|Bz_i\|$, for all $i$ such that $Bz_i \neq 0$. We choose such bases for $B = \vec{A}$. The diagonal entries $s_1 \ldots, s_g$ of $D_s$ can be assumed to be non-negative, and they are identified by the fact that the eigenvalues of $BB^T = \vec{A}\vec{A}^T$ are $\{s_i^2\}$. By (2.2), we thus have

$$s_i^2 = d - \alpha_i. \tag{4.4}$$

Hence, with $\tilde{D}_s = (D_s,\ 0)$ a $g \times h$ matrix with non-zero elements given by (4.4),

$$\det \Gamma = \begin{vmatrix} d^2 I & \tilde{D}_s & (d-1)\tilde{D}_s & (d-1)\tilde{D}_s \\ \tilde{D}_s^T & 2I & 0 & 0 \\ (d-1)\tilde{D}_s^T & 0 & 2(d-1)I & \tilde{D}_s^T\tilde{D}_s - 2I \\ (d-1)\tilde{D}_s^T & 0 & \tilde{D}_s^T\tilde{D}_s - 2I & 2(d-1)I \end{vmatrix}. \tag{4.5}$$

Since $D_s$ is a diagonal matrix, we can reorder the rows and columns in (4.5) so that we obtain a block diagonal matrix with $g$ $4 \times 4$ blocks

$$\Gamma_i := \begin{pmatrix} d^2 & s_i & (d-1)s_i & (d-1)s_i \\ s_i & 2 & 0 & 0 \\ (d-1)s_i & 0 & 2(d-1) & s_i^2 - 2 \\ (d-1)s_i & 0 & s_i^2 - 2 & 2(d-1) \end{pmatrix} \tag{4.6}$$

and $h - g$ identical $3 \times 3$ blocks

$$\Gamma_0 := \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2(d-1) & -2 \\ 0 & -2 & 2(d-1) \end{pmatrix}. \tag{4.7}$$

Hence, by straightforward calculations,

$$\begin{aligned} \det(\Gamma) &= \det(\Gamma_0)^{h-g} \prod_{i=1}^{g} \det(\Gamma_i) \\ &= \big(8d(d-2)\big)^{h-g} \prod_{i=1}^{g} (2d - s_i^2)^2 \big(2d^2 - 4d + s_i^2\big) \\ &= \big(8d(d-2)\big)^{h-g} \prod_{i=1}^{g} (d + \alpha_i)^2 (d(2d-3) - \alpha_i) \end{aligned} \tag{4.8}$$

Since $G$ is non-bipartite, $-d < \alpha_i \leq d$ for every $i$, and thus (4.8) shows that $\det(\Gamma) \neq 0$. Hence, the vectors $x_1, \ldots, x_{g+3h}$, or in different notation

$$\{x^{0j} : j \in V(G)\} \cup \{x^{1\varepsilon}, x^{2\varepsilon}, x^{3\varepsilon} : \varepsilon \in E(G)\}, \tag{4.9}$$

are linearly independent, so they form a basis in $V^\perp$.

We apply Lemma 6.2, with $N = d^2 g$, $m = g + 3h = g + 3dg/2$, and using the vectors $x_1, \ldots, x_{g+3h}$ in (4.9). Then $\mathcal{L}_G^{(2)} = \mathcal{L}^\perp$. Hence, $\operatorname{rank}(\mathcal{L}_G^{(2)}) = N - m = d^2 g - g - 3h$. We have $\det(\mathcal{L}_0) = \det(\Gamma)^{1/2}$ by Lemma 2.1. Finally, we claim that there are 4 solutions (mod 1) to (6.1): if we let $t_{0j}$ denote the coefficient of $x^{0j}$, and so on, the solutions have $t_{0j} = t_0$ for all $j$ and $t_{1\varepsilon} = t_1$, $t_{2\varepsilon} = t_2$, $t_{3\varepsilon} = t_3$ for all $\varepsilon$, where $(t_0, t_1, t_2, t_3) = (0, 0, 0, 0)$, $(0, 0, \frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0)$, or $(\frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2})$. (To prove this, first consider the equations in (6.1) which correspond to variables $x_{iee}$, and use the existence of an odd cycle. This gives the possible values of $t_0$ and $t_1$. The rest of the proof follows by considering the equations in (6.1) which correspond to variables $x_{ief}$ for a given vertex $i$, with $e \neq f$.)

Hence $q = 4$, and Lemma 6.2 yields

$$\det(\mathcal{L}_G^{(2)}) = \det(\mathcal{L}^\perp) = \det(\Gamma)^{1/2}/4.$$

The result follows by (4.8). □

**Example 4.4.** For $G = K_4$, we have $d = 3$, $g = 4$, $h = 6$, and $A$ has the eigenvalues $3, -1, -1, -1$. Hence Lemma 4.3 yields $\det(\mathcal{L}_G^{(2)}) = 2^7\, 3^{5/2}\, 5^{3/2}$.

We believe that there is a similar result for regular bipartite graphs, but we have not explored it. (Presumably, the rank is then $d^2 g - g - 3h + 2$).

Unfortunately, we have not been able to find a similar general formula for $\det(-H|_V)$ in Theorem 4.2. However, this quantity can be calculated directly for a particular graph $G$, once a basis for $\mathcal{L}_G^{(2)}$ is known.

**Example 4.5.** When $G = K_4$, using `Maple` we found a basis $\{z_1, \dots, z_{14}\}$ of $V$ and then calculated $\det(-H|_V) = 2^{-22}\, 3^{28}\, 5^{-1}\, 11^3$ using (2.5). Hence by Theorem 4.2 and Example 4.4,

$$\mathbb{E}(X_G^2) \sim 2^{16}\, 3^{-9/2}\, 5^{-1}\, 11^{-3/2} \left(\frac{4}{3}\right)^{4n}.$$

**Example 4.6.** When $G = K_2^3$ is the multigraph with two vertices and three parallel edges, `Maple` computations confirmed that $\mathcal{L}_G^{(2)}$ has rank 9 and gave $\det(\mathcal{L}_G^{(2)}) = 2^4\, 3^{3/2}$ and $\det(-H|_V) = 2^{-16}\, 3^{18}\, 5^2$. Hence by Theorem 4.2,

$$\mathbb{E}(X_G^2) \sim 2^{11}\, 3^{-9/2}\, 5^{-1}\, \pi n \left(\frac{4}{3}\right)^{2n}.$$

# 5 Short cycles in random lifts

Let $Z_k$ denote the number of cycles of length $k$ in $L_n(G)$, for $k \geq 2$. (Note that $Z_2$ is zero unless there are multiple edges in $G$.) To apply the small subgraph conditioning method to $X_G$, we must understand the distribution of short cycles in random lifts, as well as their interaction with perfect matchings. This will enable us to verify conditions (A1) – (A3) of [7, Theorem 9.12], with their $Y_n$ given by our $X_G$ (the index $n$ is suppressed), and with their $X_{kn}$ given by our $Z_k$.

To compute the limiting distributions in (A1) and (A2) of [7, Theorem 9.12], we will use the method of moments. Moreover, for (A2) we will be guided by [7, Lemma 9.17 and Remark 9.18], which tell us that we need only compute asymptotically

$$\mathbb{E}(X_G\,(Z_2)_{j_2} \cdots (Z_m)_{j_m}) / \,\mathbb{E}\, X_G,$$

for integer constants $m \geq 0$ and $j_2, \ldots, j_m \geq 0$.

Let $k$ be a fixed positive integer. It is more convenient to count rooted oriented $k$-cycles, which introduces a factor of $2k$ into the calculations. A $k$-cycle in $L_n(G)$ can be then thought of as a lift of a *legal* closed $k$-walk in $G$, which is a walk $i_0 e_1 i_1 e_2 \ldots i_{k-1} e_k$ in $G$ such that $e_j$ is an edge of $G$ with endpoints $\{i_j, i_{j+1}\}$ and $e_j \neq e_{j-1}$, for $1 \leq j \leq k$. (Here and throughout this section, arithmetic on indices in $k$-walks is performed modulo $k$.) Note that if $G$ is simple then any three consecutive vertices on the walk must all be distinct.

We can denote a directed edge of $G$ by $(e, i, j)$ where $e \in E(G)$ is incident to $i, j \in V(G)$, and $i \neq j$; this denotes $e$ directed from $i$ to $j$. Now let $R$ be the $dg \times dg$ matrix with rows and columns indexed by directed edges of $G$, and

$$R_{(e,i,j),(f,p,q)} = \begin{cases} 1 & \text{if } p = j \text{ and } f \neq e, \\ 0 & \text{otherwise.} \end{cases}$$

(Here $R$ is the adjacency matrix of a version of the directed line graph of $G$, where $U$-turns are forbidden.) Then $\mathrm{Tr}(R^k)$ counts the number of legal closed $k$-walks in $G$. Let

$$w_k := \mathrm{Tr}(R^k) = \theta_1^k + \cdots + \theta_{dg}^k, \tag{5.1}$$

where $\theta_1, \ldots, \theta_{dg}$ are the eigenvalues of $R$. Note that $d-1$ is an eigenvalue of $R$ with eigenvector $(1, 1, \ldots, 1)^T$; since $R$ has non-negative entries, this is the eigenvalue with largest modulus.

The following lemma shows that condition (A1) of [7, Theorem 9.12] holds.

**Lemma 5.1.** *Let $\lambda_k = w_k/(2k)$ for all $k \geq 2$, where $w_k$ is as in* (5.1)*. Then $Z_k \sim \mathrm{Po}(\lambda_k)$, jointly for all $k \geq 2$.*

*Proof.* Fix a legal closed $k$-walk $C = i_0 e_1 i_1 \cdots i_{k-1} e_k$ in $G$. The (oriented) $k$-cycle $C' = f_1 f_2 \cdots f_k$ in $L_n(G)$ is a lift of $C$ if $f_j \in F_{e_j}$ for $j = 1, \ldots, k$. Hence the number of possible lifts $C'$ of $C$ is $(1 + o(1))n^k$, and each will appear in $L_n(G)$ with probability $(1 + o(1))n^{-k}$. It follows that

$$\mathbb{E}\, Z_k = \sum_C \sum_{C'} \mathbb{P}(C' \subset L_n(G)) = \frac{w_k}{2k} + o(1).$$

Similar arguments hold for higher joint factorial moments, completing the proof. □

For the remainder of this section we restrict our attention to $d$-regular multigraphs with $d \geq 3$. Next we verify condition (A2) of [7, Theorem 9.12] using the approach suggested in [7, Remark 9.18].

**Lemma 5.2.** *Suppose that $G$ is $d$-regular with $d \geq 3$, and for $k \geq 2$, let*

$$\mu_k = \left(1 + \left(\frac{-1}{d-1}\right)^k\right) \lambda_k.$$

*Then for any integer $m \geq 2$ and non-negative integers $j_2, \ldots, j_m$,*

$$\frac{\mathbb{E}(X_G\,(Z_2)_{j_2}\cdots(Z_m)_{j_m})}{\mathbb{E}\,X_G} \longrightarrow \prod_{i=2}^{m} \mu_i^{j_i} \ \textit{as } n \to \infty.$$

*Proof.* For ease of notation, throughout this proof we write $\mathbb{P}(M) := \mathbb{P}(M \in L_n(G))$, $\mathbb{P}(M, C') := \mathbb{P}(M \in L_n(G), C' \in L_n(G))$, and so on. First we estimate $\mathbb{E}(X_G\,Z_k)$. We write

$$\mathbb{E}(X_G\,Z_k) = \sum_M \sum_C \sum_{C'} \mathbb{P}(M, C') = \sum_M \mathbb{P}(M) \sum_C \sum_{C'} \mathbb{P}(C'|M),$$

where the sums extend over all possible perfect matchings $M$ in $L_n(G)$, all legal closed $k$-walks $C$ in $G$, and all their possible lifts $C'$, respectively.

To calculate the inner double sum, we fix a perfect matching $M_0$ and condition on its presence in $L_n(G)$. Let $C = i_0 e_1 i_1 \ldots i_{k-1} e_k$ be a given legal closed $k$-walk in $G$. For a lift $C'$ of $C$ with edges $f_1 f_2 \cdots f_k$, let

$$\xi_j(C') = \begin{cases} 1 \text{ if } f_j \in M_0, \\ 0 \text{ otherwise,} \end{cases} \qquad \text{for } 1 \leq j \leq k.$$

To estimate the expected number of lifts of $C$ given $M_0$, we break the sum over all $C'$ according to the vector $\xi(C')$:

$$\sum_{C'} \mathbb{P}(C'|M_0) = \sum_{u \in \{0,1\}^k} \ \sum_{C' : \xi(C') = u} \mathbb{P}(C'|M_0).$$

Let $\ell_e$ be the number of edges of $M_0$ in the fiber $F_e$, and say that $M_0$ is *good* if

$$|\ell_e - n/d| \leq n^{2/3} \text{ for every } e.$$

We may assume that $M_0$ is good, since the calculations for the expectation in Section 3 show that the contribution from other matchings is negligible. (Specifically, this follows from the proof of Lemma 6.3: in particular the fact that $S_2 = o(1)$, $S_3 = o(1)$, using notation from that proof.)

Hence, for a given $u = (u_1, u_2, \ldots, u_k) \in \{0,1\}^k$,

$$\mathbb{P}(C'|M_0) \sim \left(\frac{1}{n - n/d}\right)^{k - \sum_i u_i}.$$

Let $t_{00}(u)$ and $t_{01}(u)$ be the numbers of substrings 00 and 01 in $u$, respectively. Next we prove that the number of lifts $C' = f_1 \cdots f_k$ of $C$ such that $\xi(C') = u$ is asymptotically equal to

$$\left(n - \frac{2n}{d}\right)^{t_{00}(u)} \left(\frac{n}{d}\right)^{t_{01}(u)}.$$

Indeed, let $V_{ie}$ be the set of endpoints in $V_i$ of the $\ell_e$ edges in $M_0 \cap F_e$, for $i$ incident to $e \in E(G)$. If, say, $u_1 = u_2 = 0$, which means that both, $f_1$ and $f_2$, are not in $M_0$, then we can choose the end of $f_1$ in $V_{i_1}$ from $V_{i_1} \setminus (V_{i_1e_1} \cup V_{i_1e_2})$, and $|V_{i_1} \setminus (V_{i_1e_1} \cup V_{i_1e_2})| \sim n - 2n/d$ since we assume that $M_0$ is good. Similarly, if $u_1 = 0$ and $u_2 = 1$, which means that $f_1 \notin M_0$ but $f_2 \in M_0$, then we have to choose the end of $f_1$ from $V_{i_1e_2}$, a set of size $\sim n/d$. Note also that if $u_1 = 1$ then we must have $u_2 = 0$, and if we have already selected the end $w$ of $f_1$ in $V_{i_0}$, then the other end of $f_1$ is completely determined as the partner of $w$ in $M_0$.

Multiplying these two expressions together yields that

$$\sum_{C':\xi(C')=u} \mathbb{P}(C'|M_0) = b_{u_1u_2} \cdots b_{u_{k-1}u_k} b_{u_ku_1} + o(1),$$

where $b_{00}, b_{01}, b_{10}, b_{11}$ form the matrix

$$B = \begin{pmatrix} \frac{d-2}{d-1} & \frac{1}{d-1} \\ 1 & 0 \end{pmatrix}.$$

Note that $B$ has eigenvalues 1 and $-1/(d-1)$. Summing over all $u = (u_1, \ldots, u_k)$, we find that the conditional expected number of lifts of $C$ is

$$\sum_{C'} \mathbb{P}(C'|M_0) = \mathrm{Tr}(B^k) + o(1) = 1 + \left(\frac{-1}{d-1}\right)^k + o(1).$$

Hence the expected number of $k$-cycles in $L_n(G)$, conditioned on the existence of a given good perfect matching $M_0$, is asymptotically equal to

$$\sum_C \sum_{C'} \mathbb{P}(C'|M_0) \sim \mu_k := \left(1 + \left(\frac{-1}{d-1}\right)^k\right) \frac{w_k}{2k} = \left(1 + \left(\frac{-1}{d-1}\right)^k\right) \lambda_k.$$

Finally,

$$\mathbb{E}(X_G\, Z_k) \sim \sum_M \mathbb{P}(M) \mu_k = \mu_k\, \mathbb{E}\, X_G.$$

All the above calculations work similarly for higher factorial moments and yield the desired result. □

Now for $k \geq 2$,

$$\mu_k = (1 + \delta_k)\lambda_k, \text{ where } \delta_k = \left(\frac{-1}{d-1}\right)^k > -1.$$

Therefore the quantity $\sum_k \lambda_k \delta_k^2$ in condition (A3) of [7, Theorem 9.12] is

$$\begin{aligned} \sum_k \lambda_k \delta_k^2 &= \sum_{k \geq 1} \frac{w_k}{2k\,(d-1)^{2k}} = \sum_{k \geq 1} \frac{1}{2k} \sum_{t=1}^{dg} \left(\frac{\theta_t}{(d-1)^2}\right)^k \\ &= -\frac{1}{2} \sum_{t=1}^{dg} \ln\left(1 - \frac{\theta_t}{(d-1)^2}\right), \end{aligned}$$

which is finite as required. Furthermore,

$$\exp\left(\sum_k \lambda_k \delta_k^2\right) = (d-1)^{dg}\left(\prod_{t=1}^{dg}((d-1)^2-\theta_t)\right)^{-1/2} = (d-1)^{dg}\det\big((d-1)^2 I - R\big)^{-1/2}. \tag{5.2}$$

In order to facilitate the verification of condition (A4) from [7, Theorem 9.12], we will rewrite (5.2) in terms of the adjacency matrix $A$ of $G$. To this end, we now express the eigenvalues $\theta_1, \ldots, \theta_{dg}$ of $R$ in terms of the eigenvalues $\alpha_1, \ldots, \alpha_g$ of $A$.

**Lemma 5.3.** *Suppose that $G$ is $d$-regular with $d \geq 3$. For $i = 1, \ldots, g$ denote the roots of the quadratic $x^2 - \alpha_i x + d - 1 = 0$ by $\beta_i^+$ and $\beta_i^-$. That is,*

$$\beta_i^+ = \tfrac{1}{2}\alpha_i + \sqrt{\tfrac{1}{4}\alpha_i^2 - (d-1)}, \quad \beta_i^- = \tfrac{1}{2}\alpha_i - \sqrt{\tfrac{1}{4}\alpha_i^2 - (d-1)}.$$

*Then the eigenvalues of $R$ are $\beta_i^+$, $\beta_i^-$ for $i = 1, \ldots, g$, together with $1$ and $-1$, the latter two repeated $g(d-2)/2$ times each.*

Note that there may be repetitions among $\beta_i^+, \beta_i^-$, and some of these may coincide with $\pm 1$. Hence the multiplicities of these eigenvalues may not be exactly 1 or $g(d-2)/2$: see Example 5.5 below.

*Proof.* Define the generating function $W(x) = \sum_{k=1}^{\infty} w_k x^k$. (As usual, $x$ can be regarded as a formal variable or as a small complex number.) By (5.1),

$$W(x) = \sum_{k=1}^{dg} \frac{\theta_i x}{1-\theta_i x} = \sum_{k=1}^{dg} \frac{1}{1-\theta_i x} - dg. \tag{5.3}$$

Let $b_{ij,k}$ be the number of legal $k$-walks in $G$ from vertex $i$ to vertex $j$. Furthermore, let $B_{ij} = \sum_{k=1}^{\infty} b_{ij,k} x^k$ and define $B(x) = (B_{ij}(x))_{i,j=1}^g$. We have $b_{ij,1} = a_{ij}$, and

$$\begin{aligned}\sum_l b_{il,k} a_{lj} &= b_{ij,k+1} + (d-1) b_{ij,k-1}, \qquad k \geq 2,\\ \sum_l b_{il,1} a_{lj} &= b_{ij,2} + d\delta_{i,j},\end{aligned}$$

where $\delta_{i,j}$ is the Kronecker delta. Hence,

$$(I + B(x))Ax = B(x) + (d-1)x^2 B(x) + dx^2 I$$

and so,

$$B(x) = (xA - dx^2 I)(I - xA + (d-1)x^2 I)^{-1}.$$

By the Spectral Mapping Theorem, $B(x)$ has eigenvalues

$$\frac{\alpha_i x - dx^2}{1 - \alpha_i x + (d-1)x^2}$$

for $i = 1, \dots, g$. Thus

$$F(x) := \operatorname{Tr}(B(x)) = \sum_{i=1}^{g} \frac{\alpha_i x - dx^2}{1 - \alpha_i x + (d-1)x^2}.$$

The coefficients

$$f_k := [x^k]F(x) = \sum_{i=1}^{g} b_{ii,k}$$

count the number of *almost-legal* closed $k$-walks in $G$. They are not necessarily legal, because they may start and end with the same edge in opposite directions causing a (single) backtracking edge. We define the *tail* of an almost-legal closed walk as the longest sequence of edges at the beginning which are repeated in opposite order and opposite direction at the end of the walk. Then an almost-legal closed $k$-walk consists of a tail of length $j$, for some $j \geq 0$, attached to a truly legal closed walk of length $k - 2j$.

Now, for each legal closed walk, the number of possible tails of length $j$ is $(d-2)(d-1)^{j-1}$ for $j \geq 1$, together with one empty tail of length zero. Thus,

$$F(x) = W(x) \left( 1 + \sum_{j=1}^{\infty} (d-2)(d-1)^{j-1} x^{2j} \right) = W(x) \frac{1 - x^2}{1 - (d-1)x^2}.$$

Consequently,

$$W(x) = \frac{1 - (d-1)x^2}{1 - x^2} F(x) = \frac{1 - (d-1)x^2}{1 - x^2} \sum_{i=1}^{g} \frac{\alpha_i x - dx^2}{1 - \alpha_i x + (d-1)x^2}.$$

It is easily verified that this rational function has the partial fraction expansion

$$W(x) = \sum_{i=1}^{g} \left( \frac{1}{1 - \beta_i^+ x} + \frac{1}{1 - \beta_i^- x} \right) + \tfrac{1}{2} g(d-2) \left( \frac{1}{1-x} + \frac{1}{1+x} \right) - gd.$$

The result follows by comparison with (5.3). □

We now derive three consequences of Lemma 5.3. The first two may be of independent interest, while the third will be used in our applications of [7, Theorem 9.12].

**Corollary 5.4.** *Suppose that $G$ with $d$-regular, with $d \geq 3$.*

(i) *For $k \geq 2$, the number $w_k$ of legal closed $k$-walks in $G$ is given by*

$$w_k = \tfrac{1}{2} g(d-2) \left( 1 + (-1)^k \right) + \sum_{i=1}^{g} \left( (\beta_i^+)^k + (\beta_i^-)^k \right),$$

*with $\beta_i^{\pm}$ as in Lemma 5.3.*

(ii) *The characteristic polynomial of $R$ is given by*

$$\det(\lambda I - R) = (\lambda^2 - 1)^{(d-2)g/2} \det\big((\lambda^2 + d - 1)I - \lambda A\big).$$

(iii) *The expression in* (5.2) *can be written as*

$$\begin{aligned}
&\exp\left(\sum_k \lambda_k \delta_k^2\right) \\
&\quad = (d-1)^{dg-g/2}((d-1)^4-1)^{-(d-2)g/4} \det\big(((d-1)^3+1)I - (d-1)A\big)^{-1/2} \\
&\quad = (d-1)^{dg-g/2}((d-1)^4-1)^{-(d-2)g/4} \prod_{i=1}^{g} \big((d-1)^3 + 1 - (d-1)\alpha_i\big)^{-1/2}.
\end{aligned}$$

*Proof.* Part (i) follows immediately from (5.1) and Lemma 5.3.

For (ii), by Lemma 5.3 we have

$$\begin{aligned}
\det(\lambda I - R) &= \prod_{i=1}^{dg}(\lambda - \theta_i) = (\lambda-1)^{(d-2)g/2}(\lambda+1)^{(d-2)g/2}\prod_{i=1}^{g}(\lambda - \beta_i^+)(\lambda - \beta_i^-) \\
&= (\lambda^2-1)^{(d-2)g/2}\prod_{i=1}^{g}(\lambda^2 - \alpha_i\lambda + d - 1).
\end{aligned}$$

Part (iii) follows from part (ii) by (5.2). □

**Example 5.5.** When $G = K_4$ the eigenvalues of $A$ are $\alpha_1 = 3, \alpha_2 = \alpha_3 = \alpha_4 = -1$. By Lemma 5.3, the eigenvalues of $R$ are 2, 1 (three times), $-1$ (twice), and $\frac{1}{2}(-1 \pm \sqrt{7}i)$ (three times each). Hence by Corollary 5.4(i), the number of legal closed $k$-walks in $K_4$ is

$$w_k = 2^k + 3 + 2(-1)^k + 3\Big(\frac{-1+\sqrt{7}i}{2}\Big)^k + 3\Big(\frac{-1-\sqrt{7}i}{2}\Big)^k.$$

Furthermore, by Corollary 5.4(iii),

$$\exp\left(\sum_k \lambda_k \delta_k^2\right) = 2^{10}\,15^{-1}\det(9I - 2A)^{-1/2} = 2^{10}\,3^{-3/2}\,5^{-1}\,11^{-3/2}.$$

**Example 5.6.** The multigraph with two vertices connected by $d$ parallel edges has adjacency matrix

$$A = \begin{pmatrix} 0 & d \\ d & 0 \end{pmatrix}.$$

We have $\beta_1^{\pm}, \beta_2^{\pm} = \pm(d-1), \pm 1$ and by Lemma 5.3, $R$ has eigenvalues $\pm(d-1)$ and $\pm 1$, the latter with mulitiplicities $d-1$. Corollary 5.4(i) yields $w_k = 2(d-1)^k + 2(d-1)$ if $k \geq 2$ is even, and $w_k = 0$ if $k$ is odd. Corollary 5.4(iii) yields, after some algebra,

$$\exp\left(\sum_k \lambda_k \delta_k^2\right) = (d-1)^{2d-1} d^{-d/2}(d-2)^{-d/2}(d^2-2d+2)^{-d/2+1/2}.$$

For example, when $d = 3$ this is $2^5 3^{-3/2} 5^{-1}$, while for $d = 4$ it is $2^{-15/2} 3^7 5^{-3/2}$.

To complete this section, we prove a concentration result for the number of perfect matchings in $L_n(G)$ when $G = K_4$ and when $G$ is the multigraph $K_2^3$ with 2 vertices and 3 parallel edges. We conjecture that the analogous result is true for any connected $d$-regular multigraph $G$ with no loops, where $d \geq 3$, with $\delta_k = -(1/(d-1))^k$.

**Corollary 5.7.** *For $k \geq 3$ let $w_k$ be the number of legal closed walks of length $k$ in $K_4$, and define $\lambda_k = w_k/2k$. Further, let $Y_k$ be a Poisson random variable with expectation $\lambda_k$, with $\{Y_k\}_k$ independent, and define $\delta_k = (-1/2)^k$. Then with $G = K_4$,*

$$\frac{X_G}{\mathbb{E}\, X_G} \xrightarrow{\text{d}} W := \prod_{i=3}^{\infty} (1+\delta_i)^{Y_i}\, e^{-\lambda_i \delta_i}.$$

*Proof.* Let $X = X_{K_4}$. It follows from Examples 3.7 and 4.5 that

$$\frac{\mathbb{E}(X^2)}{(\mathbb{E}\, X)^2} \sim 2^{10}\, 3^{-3/2}\, 5^{-1}\, 11^{-3/2}.$$

By comparing with Example 5.5, we find that (A4) of [7, Theorem 9.12] is satisfied: that is,

$$\frac{\mathbb{E}\, X^2}{(\mathbb{E}\, X)^2} \to \exp\left(\sum_k \lambda_k \delta_k^2\right) \qquad \text{as } n \to \infty.$$

The other conditions of [7, Theorem 9.12] hold, as follows from Lemmas 5.1 and 5.2. Applying [7, Theorem 9.12] completes the proof. □

The same argument applies for the multigraph with two vertices and three parallel edges, this time using Examples 3.8, 4.6 and 5.6, leading to the following.

**Corollary 5.8.** *Recall that $K_2^3$ denotes the multigraph with two vertices and three parallel edges. For $k \geq 2$ let $w_k$ be the number of legal closed walks of length $k$, and define $\lambda_k = w_k/2k$. Further, let $Y_k$ be a Poisson random variable with expectation $\lambda_k$, with $\{Y_k\}_k$ independent, and define $\delta_k = (-1/2)^k$. Then with $G = K_2^3$,*

$$\frac{X_G}{\mathbb{E}\, X_G} \xrightarrow{\text{d}} W := \prod_{i=1}^{\infty} (1+\delta_{2i})^{Y_{2i}}\, e^{-\lambda_{2i} \delta_{2i}}.$$

It is immediate that the limiting distribution $W$ satisfies $W > 0$ (with probability 1) in both Corollary 5.7 and 5.8. Hence $L_n(G)$ a.a.s. has a perfect matching, for both $G = K_4$ and $G = K_2^3$. This also follows from [8].

# 6 Summation by Laplace's method

In this section we provide our main approximation tool, Theorem 6.4, which performs a summation over lattice points. We will require a little more theory about lattices. The following surprising duality was proved by McMullen [9]. (See also [14].)

**Lemma 6.1.** *Let $V$ be a subspace of $\mathbb{R}^N$ and let $V^\perp$ be its orthogonal complement. Let $\mathcal{L}$ and $\mathcal{L}^\perp$ be the lattices $V \cap \mathbb{Z}^N$ and $V^\perp \cap \mathbb{Z}^N$, and assume that the rank of $\mathcal{L}$ equals the dimension of $V$ (i.e., that $\mathcal{L}$ spans $V$). Then $\mathcal{L}^\perp$ has rank $\dim(V^\perp) = N - \dim(V)$ and*

$$\det(\mathcal{L}^\perp) = \det(\mathcal{L}).$$

For our purposes we need a simple extension.

**Lemma 6.2.** *Let $0 \le m \le N$. Let $x_1, \dots, x_m$ be linearly independent vectors in $\mathbb{Z}^N$. Let $V$ be the subspace of $\mathbb{R}^N$ spanned by $x_1, \dots, x_m$ and let $V^\perp$ be its orthogonal complement; thus*

$$V^\perp = \{y \in \mathbb{R}^N : \langle y, x_i \rangle = 0 \text{ for } i = 1, \dots, m\}.$$

*Let $\mathcal{L}$ and $\mathcal{L}^\perp$ be the lattices $V \cap \mathbb{Z}^N$ and $V^\perp \cap \mathbb{Z}^N$, and let $\mathcal{L}_0$ be the lattice spanned by $x_1, \dots, x_m$ (i.e., the set $\{\sum_1^m n_i x_i : n_i \in \mathbb{Z}\}$ of integer combinations). Then $\mathcal{L}^\perp$ has rank $N - m$ and*

$$\det(\mathcal{L}^\perp) = \det(\mathcal{L}) = \det(\mathcal{L}_0)/q,$$

*where $q$ is the order of the finite group $\mathcal{L}/\mathcal{L}_0$. Explicitly, $q$ is the number of solutions $(t_1, \dots, t_m)$ in $(\mathbb{R}/\mathbb{Z})^m$ (or $(\mathbb{Q}/\mathbb{Z})^m$) of the system, with $x_i = (x_{ij})_{j=1}^N$,*

$$\sum_i x_{ij} t_i \equiv 0 \pmod 1, \qquad j = 1, \dots, N. \tag{6.1}$$

*Proof.* Since $\operatorname{rank}(\mathcal{L}) = m = \dim(V)$, we can apply Lemma 6.1 and conclude that $\operatorname{rank}(\mathcal{L}^\perp) = N - m$ and $\det(\mathcal{L}^\perp) = \det(\mathcal{L})$.

Next, $\mathcal{L}_0 \subseteq V \cap \mathbb{Z}^N = \mathcal{L}$; moreover, $\mathcal{L}_0$ and $\mathcal{L}$ both span $V$ and have thus the same rank. Hence Lemma 2.2 shows that $\mathcal{L}/\mathcal{L}_0$ is finite and $\det(\mathcal{L}) = \det(\mathcal{L}_0)/q$. Note further that $\mathcal{L} \subseteq V = \{\sum_i t_i x_i : t_i \in \mathbb{R}\}$ and thus

$$q = |\mathcal{L}/\mathcal{L}_0| = \Bigl|\Bigl\{(t_i) \in [0,1)^m : \sum_i t_i x_i \in \mathcal{L}\Bigr\}\Bigr|.$$

Furthermore,

$$\sum_i t_i x_i \in \mathcal{L} \iff \sum_i t_i x_i \in \mathbb{Z}^N \iff \sum_i x_{ij} t_i \equiv 0 \pmod 1 \text{ for } j = 1, \dots, J,$$

and the characterization of $q$ follows. □

The proof of Theorem 6.4 involves reduction to a special case, which we prove first.

**Lemma 6.3.** *Suppose the following, for $n \ge 1$.*

(i) *$\mathcal{L} \subset \mathbb{R}^r$ is a lattice with full rank $r$.*

(ii) *$K \subset \mathbb{R}^r$ is a compact convex set with non-empty interior $K^\circ$.*

(iii) *$\phi : K \to \mathbb{R}$ is a continuous function with a unique maximum at some interior point $x_0 \in K^\circ$.*

(iv) *$\phi$ is twice continuously differentiable in a neighbourhood of $x_0$ and the Hessian $H := D^2\phi(x_0)$ is strictly negative definite.*

(v) *$\psi : K_1 \to \mathbb{R}$ is a continuous function on some neighbourhood $K_1 \subseteq K$ of $x_0$ with $\psi(x_0) > 0$.*

(vi) *$\ell_n$ is a vector in $\mathbb{R}^r$.*

(vii) *$a_n(\ell)$, $\ell \in (\mathcal{L} + \ell_n) \cap nK$, are real numbers such that as $n \to \infty$, for some $b_n > 0$ and uniformly for $\ell$ in the indicated sets,*

$$a_n(\ell) = O\big(b_n e^{n\phi(\ell/n)+o(n)}\big), \qquad \ell \in (\mathcal{L} + \ell_n) \cap nK, \tag{6.2}$$

*and*

$$a_n(\ell) = b_n\big(\psi(\ell/n) + o(1)\big)e^{n\phi(\ell/n)}, \qquad \ell \in (\mathcal{L} + \ell_n) \cap nK_1. \tag{6.3}$$

*Then, as $n \to \infty$,*

$$\sum_{\ell \in (\mathcal{L}+\ell_n)\cap nK} a_n(\ell) \sim \frac{(2\pi)^{r/2}\psi(x_0)}{\det(\mathcal{L})\det\big(-H\big)^{1/2}} b_n n^{r/2} e^{n\phi(x_0)}. \tag{6.4}$$

*Proof.* We begin with a few simplifications. We may obviously assume $b_n = 1$. Furthermore, by subtracting $\phi(x_0)$ from $\phi$, and dividing $a_n(\ell)$ by $e^{n\phi(x_0)}$, we may suppose that $\phi(x_0) = 0$.

Since $x_0$ is an interior maximum point, the gradient $D\phi(x_0)$ vanishes, and a Taylor expansion at $x_0$ shows that, using (iv), as $|x - x_0| \to 0$,

$$\begin{aligned} \phi(x) &= \tfrac{1}{2}\langle x - x_0, D^2\phi(x_0)(x - x_0)\rangle + o(|x - x_0|^2) \\ &\le -c_1|x - x_0|^2 + o(|x - x_0|^2). \end{aligned} \tag{6.5}$$

Consequently, there exists $\delta > 0$ such that the neighbourhood $\{x : |x - x_0| \le \delta\} \subset K_1$ and

$$\phi(x) \le -c_2|x - x_0|^2, \qquad |x - x_0| < \delta. \tag{6.6}$$

We divide the sum in (6.4) into three parts:

$$S_1 := \sum_{|\ell/n - x_0| < n^{-1/3}}, \qquad S_2 := \sum_{n^{-1/3} \le |\ell/n - x_0| < \delta}, \qquad S_3 := \sum_{|\ell/n - x_0| \ge \delta}.$$

In the sum $S_2$ we use (6.3) and (6.6); thus each term is

$$a_n(\ell) = O(e^{n\phi(\ell/n)}) = O(e^{-c_2 n^{1/3}}).$$

Since the number of terms is $O(n^r)$, we obtain $S_2 = o(1)$.

Similarly, by compactness, if $|x - x_0| \geq \delta$, then $\phi(x) \leq -c_3$. Consequently, for large $n$, (6.2) shows that each term in $S_3$ is

$$a_n(\ell) = O(e^{n\phi(\ell/n)+c_3n/2}) = O(e^{-c_3n/2}).$$

Again, the number of terms is $O(n^r)$ and we obtain $S_3 = o(1)$.

We convert the sum $S_1$ into an integral by picking a unit cell $U$ of the lattice $\mathcal{L}$ and defining $a_n(y) := a_n(\ell)$ for $y \in U + \ell$, $\ell \in \mathcal{L} + \ell_n$. Let $Q_n := \bigcup_{|\ell/n - x_0| < n^{-1/3}} (U + \ell)$, and let $\widetilde{Q}_n := \{z : nx_0 + \sqrt{n}z \in Q_n\}$. Then

$$S_1 = \det(\mathcal{L})^{-1} \int_{Q_n} a_n(y)\,\mathrm{d}y = \det(\mathcal{L})^{-1} n^{r/2} \int_{\widetilde{Q}_n} a_n\bigl(nx_0 + \sqrt{n}z\bigr)\,\mathrm{d}z. \tag{6.7}$$

Note that $Q_n$ is roughly a ball of radius $n^{2/3}$ centered at $nx_0$, and $\widetilde{Q}_n$ is roughly a ball of radius $n^{1/6}$ centered at 0.

If $y \in Q_n$, then $|y/n - x_0| \leq n^{-1/3} + O(n^{-1})$. Since the gradient $D\phi(x_0) = 0$, (iv) implies that for $x \in Q_n/n$,

$$|D\phi(x)| = O(|x - x_0|) = O(n^{-1/3}). \tag{6.8}$$

If $y \in U + \ell \subset Q_n$, then $|y/n - \ell/n| = O(1/n)$ and (6.8) implies

$$n\phi(y/n) - n\phi(\ell/n) = O\bigl(nn^{-1/3}n^{-1}\bigr) = O\bigl(n^{-1/3}\bigr),$$

and thus (6.3) implies, uniformly for $y \in Q_n$,

$$a_n(y) = a_n(\ell) = \bigl(\psi(y/n) + o(1)\bigr)e^{n\phi(y/n)}. \tag{6.9}$$

For every fixed $z \in \mathbb{R}^r$, this and the Taylor expansion (6.5) show that, as $n \to \infty$, using the continuity of $\psi$,

$$a_n(nx_0 + \sqrt{n}z) \to \psi(x_0)e^{\frac{1}{2}\langle z, D^2\phi(x_0)z\rangle}.$$

Moreover, (6.6) and (6.9) provide a uniform bound, for all $z \in \mathbb{R}^r$,

$$|a_n(nx_0 + \sqrt{n}z)\mathbf{1}_{\widetilde{Q}_n}(z)| \leq C_1 e^{-c_2|z|^2}.$$

Further, $\mathbf{1}_{\widetilde{Q}_n}(z) \to 1$ for every $z$. Hence, dominated convergence shows that

$$\begin{aligned}\int_{\widetilde{Q}_n} a_n(nx_0 + \sqrt{n}z)\,\mathrm{d}z &\to \int_{\mathbb{R}^r} \psi(x_0)e^{\frac{1}{2}\langle z, D^2\phi(x_0)z\rangle}\,\mathrm{d}z \\ &= \psi(x_0)(2\pi)^{r/2}\det\bigl(-D^2\phi(x_0)\bigr)^{-1/2}.\end{aligned}$$

The result follows from this and (6.7), together with the estimates $S_2 = o(1)$ and $S_3 = o(1)$ above. □

**Theorem 6.4.** *Suppose the following.*

(i) $\mathcal{L} \subset \mathbb{R}^N$ *is a lattice with rank* $r \le N$.

(ii) $V \subseteq \mathbb{R}^N$ *is the* $r$*-dimensional subspace spanned by* $\mathcal{L}$.

(iii) $W = V + w$ *is an affine subspace parallel to* $V$, *for some* $w \in \mathbb{R}^N$.

(iv) $K \subset \mathbb{R}^N$ *is a compact convex set with non-empty interior* $K^\circ$.

(v) $\phi : K \to \mathbb{R}$ *is a continuous function and the restriction of* $\phi$ *to* $K \cap W$ *has a unique maximum at some point* $x_0 \in K^\circ \cap W$.

(vi) $\phi$ *is twice continuously differentiable in a neighbourhood of* $x_0$ *and* $H := D^2\phi(x_0)$ *is its Hessian at* $x_0$.

(vii) $\psi : K_1 \to \mathbb{R}$ *is a continuous function on some neighbourhood* $K_1 \subseteq K$ *of* $x_0$ *with* $\psi(x_0) > 0$.

(viii) $\ell_n$ *is a vector in* $\mathbb{R}^N$ *with* $\ell_n/n \in W$,

(ix) $a_n(\ell)$, $\ell \in \mathcal{L} \cap nK$, *are real numbers such that as* $n \to \infty$, *for some* $b_n > 0$ *and uniformly for* $\ell$ *in the indicated sets,*

$$a_n(\ell) = O\big(b_n e^{n\phi(\ell/n)+o(n)}\big), \qquad \ell \in \mathcal{L} \cap nK, \tag{6.10}$$

*and*

$$a_n(\ell) = b_n\big(\psi(\ell/n) + o(1)\big)e^{n\phi(\ell/n)}, \qquad \ell \in \mathcal{L} \cap nK_1.$$

*Then, as* $n \to \infty$, *provided* $\det(-H|_V) \neq 0$,

$$\sum_{\ell \in (\mathcal{L}+\ell_n)\cap nK} a_n(\ell) \sim \frac{(2\pi)^{r/2}\psi(x_0)}{\det(\mathcal{L})\det(-H|_V)^{1/2}} b_n n^{r/2} e^{n\phi(x_0)}. \tag{6.11}$$

*Proof.* First, replacing $K$ by $K - w$, $a_n(\ell)$ by $a'_n(\ell) := a_n(\ell + nw)$, $\ell_n$ by $\ell_n - nw$, and translating $\phi$ and $\psi$, we reduce to the case $w = 0$ and thus $W = V$ and $\ell_n \in V$.

Choose a lattice basis $\{z_1, \dots, z_r\}$ of $\mathcal{L}$. Consider the mapping $T : \mathbb{R}^r \to V \subseteq \mathbb{R}^N$ given by $(y_1, \dots, y_r) \mapsto \sum_{i=1}^r y_i z_i$, which thus maps $\mathbb{Z}^r$ onto $\mathcal{L}$. We apply Lemma 6.3 to $\mathcal{L}' := \mathbb{Z}^r$, $K' := T^{-1}(K)$, $\phi \circ T$, $\psi \circ T$, $\ell'_n := T^{-1}(\ell_n)$, and $a_n(T(k))$, $k \in (\mathcal{L}' + \ell'_n) \cap nK'$. The Hessian $D^2(\phi \circ T)(T^{-1}x_0)$ equals $(\langle Hz_i, z_j\rangle)_{i,j=1}^r$, and its negative has determinant, by (2.5) and (2.3),

$$\det(\langle Hz_i, z_j\rangle)_{i,j=1}^r = \det(-H|_V)\det(\langle z_i, z_j\rangle)_{i,j=1}^r = \det(-H|_V)\det(\mathcal{L})^2. \tag{6.12}$$

Hence, (6.11) follows from Lemma 6.3. Note that the Hessian $D^2(\phi\circ T)(T^{-1}x_0)$ is always negative semi-definite, because $x_0$ is a maximum point. Hence, it is negative definite unless its determinant is zero, which is ruled out by (6.12) and the assumption that $\det(-H|_V) \neq 0$. □

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