THRESHOLD GRAPH LIMITS AND RANDOM THRESHOLD GRAPHS

PERSI DIACONIS, SUSAN HOLMES, AND SVANTE JANSON

ABSTRACT. We study the limit theory of large threshold graphs and apply this to a variety of models for random threshold graphs. The results give a nice set of examples for the emerging theory of graph limits.

1. INTRODUCTION

Threshold Graphs. Graphs have important applications in modern systems biology and social sciences. Edges are created between interacting genes or people who know each other. However graphs are not objects which are naturally amenable to simple statistical analyses, there is no natural average graph for instance. Being able to predict or replace a graph by hidden (statisticians call them latent) real variables has many advantages. This paper studies such a class of graphs, that sits within the larger class of interval graphs [27], itself a subset of intersection graphs [11]; see also [6].

Consider the following properties of a simple graph G on $[n] := \{1, 2, \dots, n\}$.

(1.1) There are real weights w_i and a threshold value t such that there is an edge from i to j if and only if $w_i + w_j > t$. Thus "*the rich people always know each other*".

(1.2) G can be built sequentially from the empty graph by adding vertices one at a time, where each new vertex, is either isolated (non-adjacent to all the previous) or dominating (connected to all the previous).

(1.3) The graph is uniquely determined (as a labeled graph) by its degree sequence.

(1.4) Any induced subgraph has either an isolated or a dominating vertex.

(1.5) There is no induced subgraph $2K_2$, P_4 or C_4. (Equivalently, there is no *alternating 4-cycle*, i.e., four distinct vertices x, y, z, w with edges xy and zw but no edges yz and xw; the diagonals xz and yw may or may not exist.)

These properties are equivalent and define the class of *threshold graphs*. The book by Mahadev and Peled [25] contains proofs and several other seemingly different characterizations. Note that the complement of a threshold graph is a threshold graph (by any of (1.1)–(1.5)). By (1.2), a threshold graph is either connected (if the last vertex is dominating) or has an isolated vertex

Date: August 12, 2009.

(if the last vertex is isolated); clearly these two possibilities exclude each other when $n > 1$.

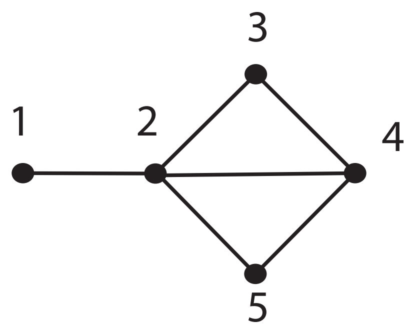


FIGURE 1. A threshold graph

Example 1.1. The graph in Figure 1 is a threshold graph, from (1.1) by taking weights 1,5,2,3,2 on vertices 1–5 with $t = 4.5$, or from (1.2) by adding vertices 3, 5 (isolated), 4 (dominating), 1 (isolated) and 2 (dominating).

While many familiar graphs are threshold graphs (stars or complete graphs for example), many are not (e.g. paths or cycles of length 4 or more). For example, of the 64 labeled graphs on 4 vertices, 46 are threshold graphs; the other 18 are paths P_4, cycles C_4, and pairs of edges $2K_2$ (which is the complement of C_4). Considering unlabeled graphs, there are 11 graphs on 4 vertices, and 8 of them are threshold graphs.

Random Threshold Graphs. It is natural to study random threshold graphs. There are several different natural random constructions; we will in particular consider the following three:

(1.6) From (1.1) by choosing $\{w_i\}_{1\le i\le n}$ as independent and identically distributed (i.i.d.) random variables from some probability distribution. (We also choose some fixed t; we may assume $t = 0$ by replacing w_i by $w_i - t/2$.)

(1.7) From (1.2) by ordering the vertices randomly and adding the vertices one by one, each time choosing at random between the qualifiers 'dominating' or 'isolated' with probabilities p_i and $1-p_i$, respectively, $1 \le i \le n$. This is a simple random attachment model in a similar vein as those in [30]. We mainly consider the case when all p_i are equal to a single parameter $p \in [0, 1]$.

(1.8) The uniform distribution on the set of threshold graphs.

Example 1.2. Figure 2 shows a random threshold graph constructed by (1.6) with w_i chosen independently from the standardized Normal distribution and $t = 3$. About half of the vertices are isolated, most of those with negative weights.

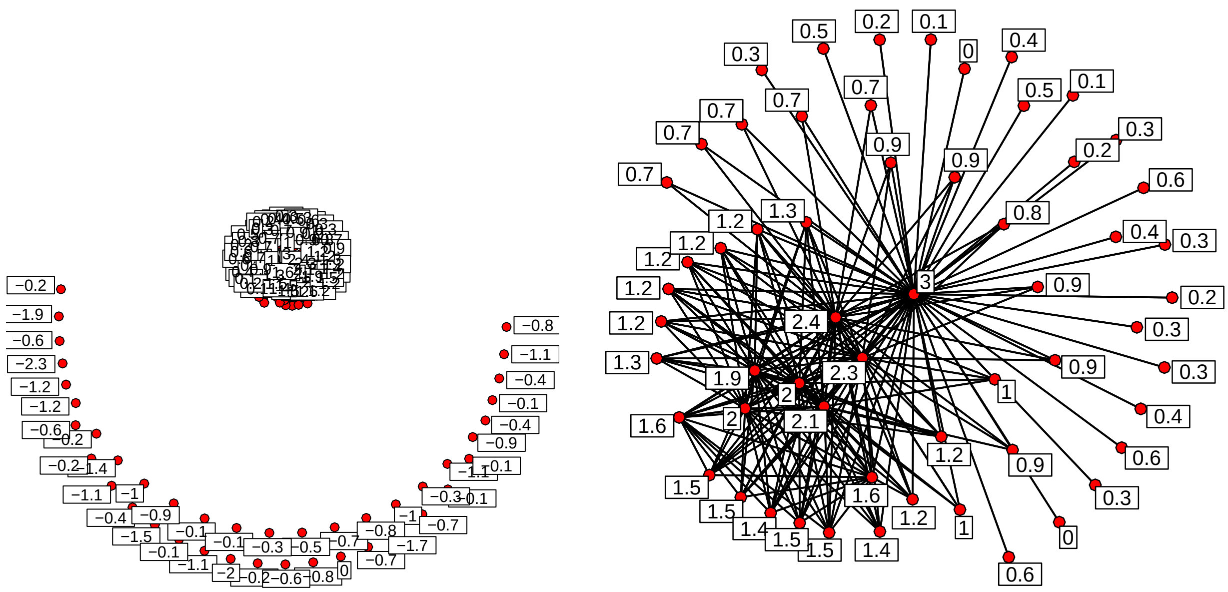


FIGURE 2. A whole threshold graph with isolates (left) and with only the connected part expanded (right); the labels are the rounded weights w_i.

Example 1.3. Figure 3 shows a random threshold graph constructed by (1.6) with w_i chosen as i.i.d. uniform random variables on $[0, 1]$ and $t = 1$. This instance is connected; this happens if and only if the maximum and minimum of the w_i's add to more than 1 (then there is a dominating vertex); in this example this has probability $1/2$.

We show below (Corollaries 6.5 and 6.6) that this uniform weight model is equivalent to adding isolated or dominating nodes as in (1.7) with probability $p = 1/2$, independently and in random order. It follows that this same distribution appears as the stationary distribution of a Markov chain on threshold graphs which picks a vertex at random and changes it to dominating or isolated with probability $1/2$ (this walk is analysed in [7]). Furthermore, it follows from Subsection 2.1 that these models yield a uniform distribution on the set of unlabeled threshold graphs of order n.

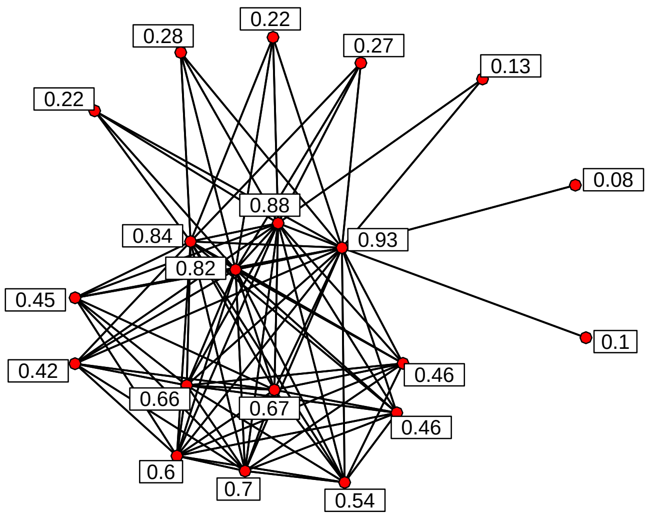


FIGURE 3. A threshold graph with $n = 20$ and uniform w_i. It turns out that this instance had no isolates. The labels are the rounded weights w_i.

Bipartite Threshold Graphs. We also study the parallel case of bipartite threshold graphs (difference graphs), both for its own sake and because one of the main theorems is proved by first considering the bipartite case.

By a *bipartite graph*, we mean a graph with an explicit bipartition of the vertex set; it can thus be written as (V_1, V_2, E) where the edge set $E \subseteq V_1 \times V_2$. These following properties of a bipartite graph are equivalent and define the class of *bipartite threshold graphs*. (See [25] for further characterizations.)

(1.9) There are real weights w'_i, $i \in V_1$ and w''_j, $j \in V_2$, and a threshold value t such that there is an edge from i to j if and only if $w'_i + w''_j > t$.

(1.10) G can be built sequentially starting from n_1 white vertices and n_2 black vertices in some fixed total order. Proceeding in this order, make each white vertex dominate or isolated from all the black vertices that precede it and each black vertex dominate or isolated from all earlier white vertices.

(1.11) Any induced subgraph has either an isolated vertex or a vertex dominating every vertex in the other part.
(1.12) There is no induced subgraph $2K_2$.

Remarks. 1. Threshold graphs were defined by Chvátal and Hammer [8]. Bipartite threshold graphs were studied by Hammer, Peled and Sun [17] under the name *difference graphs* because they can equivalently be characterized as the graphs (V, E) for which there exist weights w_v, $v \in V$, and a real number t such that $|w_v| < t$ for every v and $uv \in E \iff |w_u - w_v| > t$; it is easily seen that every such graph is bipartite with $V_1 = \{v : w_v \geq 0\}$ and $V_2 = \{v : w_v < 0\}$ and that is satisfies the definition above (e.g., with $w'_v = w_v$ and $w''_v = -w_v$), and conversely. We will use the name bipartite threshold graph to emphasize that we consider these graphs equipped with a given bipartition. The same graphs were called *chain graphs* by Yannakakis [33] because each partition can be linearly ordered for the inclusion of the neighborhoods of its elements.

2. A suite of programs for working with threshold graphs appears in [15] with further developments in [21, 26].

3. The most natural class of graphs built from a coordinate system are commonly called geometric graphs [31] or geographical graphs [21, 26]. Threshold graphs are a special case of these. Their recognition and manipulation in a statistical context relies on useful measures on such graphs. We will start by defining such measures and developing a limit theory.

Overview of the Paper. The purpose of this paper is to study the limiting properties of large threshold graphs in the spirit of the theory of graph limits developed by Lovász and Szegedy [22] and Borgs, Chayes, Lovász, Sós, Vesztergombi [5] (and in further papers by these authors and others). As explained below, the limiting objects are not graphs, but can rather be represented by symmetric functions $W(x, y)$ from $[0, 1]^2$ to $[0, 1]$; any sequence of graphs that converges in the appropriate way has such a limit. Conversely, such a function W may be used to form a random graph G_n by choosing independent random points U_i in $[0, 1]$, and then for each pair (i, j) with $1 \leq i < j \leq n$ flipping a biased coin with heads probability $W(U_i, U_j)$, putting an edge from i to j if the coin comes up heads. The resulting sequence of random graphs is (almost surely) an example of a sequence of graphs converging to W. For Example 1.3, letting $n \to \infty$, there is (as we show in greater generality in Section 6) a limit W that may be pictured as in Figure 4.

One of our main results (Theorems 5.3) shows that graph limits of threshold graphs have unique representations by increasing symmetric zero-one valued functions W. Furthermore, there is a one-to-one correspondence between these limiting objects and a certain type of 'symmetric' probability distributions P_W on $[0, 1]$. A threshold graphs is characterized by its degree sequence; normalizing this to be a probability distribution, say $\nu(G_n)$, we show (Theorem 5.5) that a sequence of threshold graphs converges to W

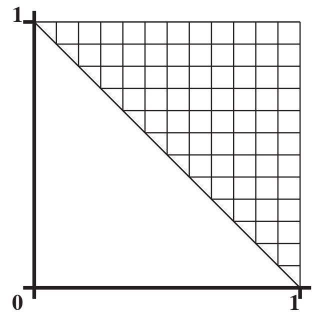


FIGURE 4. The function $W(x, y)$ for Example 1.3. Hashed values have $W(x, y) = 1$, unhashed $W(x, y) = 0$.

when $n \to \infty$ if and only if $\nu(G_n)$ converges to P_W. (Hence, P_W can be regarded as the degree distribution of the limit. The result that a limit of threshold graphs is determined by its degree distribution is a natural analogue for the limit objects of the fact that an unlabeled threshold graph is uniquely determined by its degree distribution.)

Figure 5 and Figure 7 show simulations of these results. In Figure 5, 10,000 graphs with $n = 50$ were generated from (1.6) with uniform weights as in Example 1.3.

In the bipartite case, there is a similar 1–1 correspondence between the limit objects and probability distributions on [0,1]; now all probability distributions on [0,1] appear in the representation of the limits (Theorem 5.1).

Section 2 discusses uniform random threshold graphs (both labeled and unlabeled) and methods to generate them. Section 3 gives a succint review of notation and graph limits. Section 4 develops the limit theory of degree sequences; this is not restricted to threshold graphs. Section 5 develops the limit theory for threshold graphs both deterministic and random. Section 6 treats examples of random threshold graphs and their limits, and Section 8 gives corresponding examples and results for random bipartite threshold graphs. Section 9 treats the spectrum of the Laplacian of threshold graphs.

We denote the vertex and edge sets of a graph G by $V(G)$ and $E(G)$, and the numbers of vertices and edges by $v(G) := |V(G)|$ and $e(G) := |E(G)|$. For a bipartite graph we similarly use $V_j(G)$ and $v_j(G)$, $j = 1, 2$.

Throughout the paper, 'increasing' and 'decreasing' should be interpreted in the weak sense (non-decreasing and non-increasing). Unspecified limits are as $n \to \infty$.

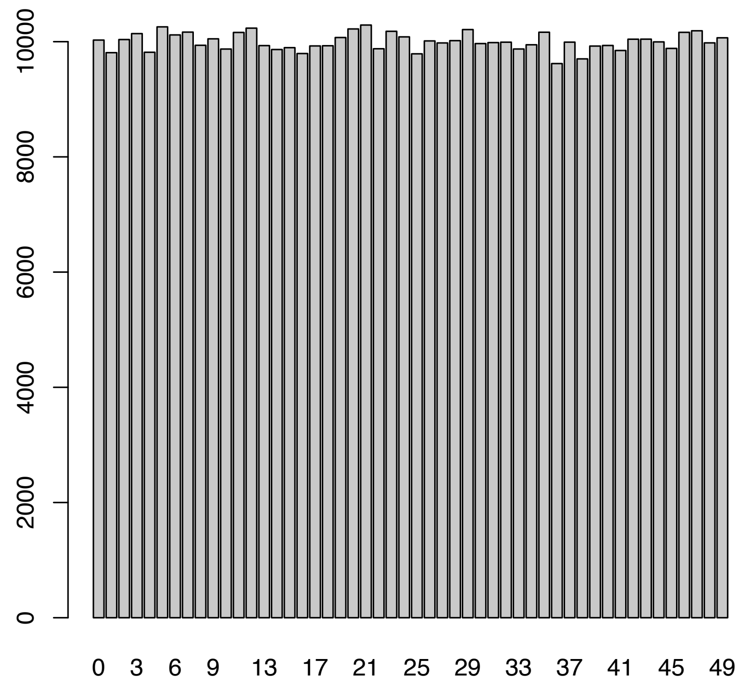


FIGURE 5. Threshold graphs were generated with $n = 50$ as in Example 1.3 with uniform w_i and $t = 1$; this is the degree histogram for a sample of 10,000 random graphs.

2. Generating threshold graphs uniformly

This section gives algorithms for generating uniformly distributed threshold graphs. Both in the labeled case and in the unlabeled case. The algorithms are used here for simulation and in Sections 6 and 7 to prove limit theorems.

Let $\mathcal{T}_n$ and $\mathcal{LT}_n$ be the sets of unlabeled and labeled threshold graphs on n vertices. These are different objects, $\mathcal{T}_n$ is a quotient of $\mathcal{LT}_n$, and we treat counting and uniform generation separately for the two cases. We assume in this section that $n \geq 2$.

2.1. **Unlabeled threshold graphs.** We can code an unlabeled threshold graph on n vertices by a binary code $\alpha_2 \cdots \alpha_n$ of length $n-1$: Given a code $\alpha_2 \cdots \alpha_n$, we construct G by (1.2) adding vertex i as a dominating vertex if and only if $\alpha_i = 1$ ($i \geq 2$). Conversely, given G of order $n \geq 2$, let $\alpha_n = 1$ if there is a dominating vertex (G is connected) and $\alpha_n = 0$ if there is an

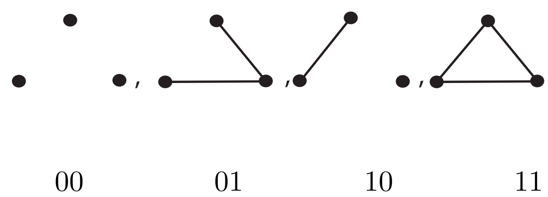


FIGURE 6. The four graphs in $\mathcal{T}_3$ and their codes.

isolated vertex (G is disconnected); we then remove one such dominating or isolated vertex and continue recursively to define $\alpha_{n-1}, \dots, \alpha_2$.

Since all dominating (isolated) vertices are equivalent to each other, this coding gives a bijection between $\mathcal{T}_n$ and $\{0,1\}^{n-1}$. In particular,

$$|\mathcal{T}_n| = 2^{n-1}, \qquad n \geq 1.$$

See Figure 6 for an example.

This leads to a simple algorithm to generate a uniformly distributed random unlabeled threshold graph: we construct a random code by making $n-1$ coin flips. In other words:

Algorithm 2.1. *Algorithm for generating uniform random unlabeled threshold graphs of a given order n.*

> **Step 1:** Add n vertices by (1.2), each time randomly choosing 'isolated' or 'dominating' with probability 1/2.

This is thus the same as the second method in Example 1.3, so Corollary 6.5 shows that the first method in Example 1.3 also yields uniform random unlabeled threshold graphs (if we forget the labels).

The following notation is used to define two further algorithms (Subsection 2.3) and for proof of the limiting results in Section 7.

Define the *extended binary code* of a threshold graph to be the binary code with the first binary digit repeated; it is thus $\alpha_1\alpha_2\alpha_3 \cdots \alpha_n$ with $\alpha_1 := \alpha_2$. The runs of 0's and 1's in the extended binary code then correspond to blocks of vertices that can be added together in (1.2) as either isolated or dominating vertices, with the blocks alternating between isolated and dominating. The vertices in each block are equivalent and have, in particular, the same vertex degrees, while vertices in different blocks can be seen to have different degrees. (The degree increases strictly from one dominating block to the next and decreases strictly from one isolated block to the next, with every dominating block having higher degree than every isolated block; cf. Example 2.3 below.) The number of different vertex degrees thus equals the number of blocks.

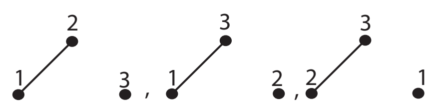
If the lengths of the blocks are $b_1, b_2, \dots, b_\tau$, then the number of automorphisms of G is thus $\prod_{j=1}^{\tau} b_j!$, since the vertices in each block may be permuted arbitrarily.

Note that if $b_1, \dots, b_\tau$ are the lengths of the blocks then

$$b_1 \geq 2, \qquad b_k \geq 1 \ (k \geq 2), \qquad \sum_{k=1}^{\tau} b_k = n. \tag{2.1}$$

Since the blocks are alternatingly dominating or isolated, and the first block may be either, each sequence $b_1, \dots, b_\tau$ satisfying (2.1) corresponds to exactly 2 unlabeled threshold graphs of order n. (These graphs are the complements of each other. One has isolated blocks where the other has dominating blocks.)

2.2. **Labeled threshold graphs.** The situation is different for labeled threshold graphs. For example, all of the $2^{\binom{3}{2}} = 8$ labeled graphs with $n = 3$ turn out to be threshold graphs and for instance



are distinguished. Hence the distribution of a uniform random labeled threshold graph differs from the distribution of a uniform unlabeled threshold graph (even if we forget the labels). In particular, Example 1.3 does *not* produce uniform random labeled threshold graphs.

Let G be an unlabeled threshold graph with an extended code having block lengths (runs) $b_1, \dots, b_\tau$. Then the number of labeled threshold graphs corresponding to G is $n!/\prod_1^\tau b_j!$, since every such graph corresponds to a unique assignment of the labels $1, \dots, n$ to the τ blocks, with b_i labels to block i. (Alternatively and equivalently, this follows from the number $\prod_1^\tau b_j!$ of automorphisms given above.)

The number $t(n) := |\mathcal{LT}_n|$ of labeled threshold graphs [32, A005840] has been studied by Beissinger and Peled [2]. Among other things, they show that

$$\sum_{n=0}^{\infty} t(n) \frac{x^n}{n!} = \frac{(1-x)e^x}{2-e^x} \tag{2.2}$$

so, by Taylor expansion,

n	1	2	3	4	5	6	7	8	9	10
$t(n)$	1	2	8	46	332	2874	29024	334982	4349492	62749906

and by expanding the singularities (cf. [12, Chapter IV]) the exact formula

$$\frac{t(n)}{n!} = \sum_{k=-\infty}^{\infty} \Bigl(\frac{1}{\log 2 + 2\pi \mathrm{i} k} - 1\Bigr)\Bigl(\frac{1}{\log 2 + 2\pi \mathrm{i} k}\Bigr)^n, \qquad n \geq 2, \tag{2.3}$$

where the leading term is the one with $k = 0$, and thus the asymptotics

$$\frac{t(n)}{n!} = \Bigl(\frac{1}{\log 2} - 1\Bigr)\Bigl(\frac{1}{\log 2}\Bigr)^n + \epsilon(n), \qquad |\epsilon(n)| \leq \frac{2\zeta(n)}{(2\pi)^n}, \tag{2.4}$$

where $\zeta(n)$ is the zeta function and thus $\zeta(n) \to 1$; furthermore,

$$t(n) = 2R_n - 2nR_{n-1},\ n \geq 2,\ \text{with } R_n = \sum_{k=1}^{n} k! S(n,k) = \sum_{\ell=0}^{\infty} \frac{\ell^n}{2^{\ell+1}}, \tag{2.5}$$

where $S(n, k)$ are Stirling numbers; R_n is the number of preferential arrangements of n labeled elements, or number of weak orders on n labeled elements [32, A000670], also called surjection numbers [12, II.3]. (This is easily seen using the blocks above; the number of labeled threshold graphs with a given sequence of blocks is twice (since the first block may be either isolated or dominating) the number of preferential arrangements with the same block sizes; if we did not require $b_1 \geq 2$, this would yield $2R_n$, but we have to subtract twice the number of preferential arrangements with $b_1 = 1$, which is $2nR_{n-1}$.) We note for future use the generating function [12, (II.15)]

$$\sum_{n=0}^{\infty} R_n \frac{x^n}{n!} = \frac{1}{2 - e^x}. \tag{2.6}$$

Let $t(n, j)$ be the number of labeled threshold graphs with j isolated points. Then, as also shown in [2] (and easily seen), for $n \geq 2$,

$$\begin{aligned} t(n,0) &= t(n)/2, \\ t(n,j) &= \begin{cases} \binom{n}{j} t(n-j,0) = \frac{1}{2}\binom{n}{j} t(n-j), & 0 \leq j \leq n-2, \\ 0, & j = n-1, \\ 1, & j = n. \end{cases} \end{aligned} \tag{2.7}$$

Thus knowledge of $t(n)$ provides $t(n, j)$.

These ingredients allow us to give an algorithm for choosing uniformly in $\mathcal{LT}_n$.

Algorithm 2.2. *Algorithm for generating uniform random labeled threshold graphs of a given order n.*

- **Step 0:** Make a list of $t(k)$ for k between 1 and n. Make lists of $t(k, j)$ for $k = 1, \dots, n$ and $j = 0, \dots, k$.
- **Step 1:** Choose an integer j_0 in $\{0, \dots, n\}$ with probability that $j_0 = j$ given by $t(n, j)/t(n)$. Choose (at random) a subset of j_0 points in $\{1, \dots, n\}$. These are the isolated vertices in the graph. Let $n' := n - j_0$ be the number of remaining points. If $n' = 0$ then stop.

Step 2: Choose an integer j_1 in $\{1, \dots, n'\}$ with probability that $j_1 = j$ given by $t(n', j)/(t(n') - t(n', 0)) = 2t(n', j)/t(n')$ and choose (at random) j_1 points of those remaining; these will dominate all further points, so add edges between these vertices and from them to all remaining points. Update n' to $n' - j_1$, the number of remaining points. If $n' = 0$ then stop.

Step 3: Choose an integer j_2 in $\{1, \dots, n'\}$ with probability that $j_2 = j$ given by $2t(n', j)/t(n')$ and choose (at random) j_2 points of those remaining; these will be isolated among the remaining points, so no further edges are added. Update n' to $n' - j_1$, the number of remaining points. If $n' = 0$ then stop.

Step 4: Repeat from Step 2 with the remaining n' points.

Alternatively, instead of selecting the subsets in Steps 1 and 2 at random, we may choose them in any way, provided the algorithm begins or ends with a random permutation of the points.

The algorithm works because of a characterization of threshold graphs by Chvátal and Hammer [8], cf. (1.4): A graph is a threshold graph iff any subset S of vertices contains at least one isolate or one dominating vertex (within the graph induced by S). Thus in step 2, since there are no isolates among the n' vertices left there must be at least one dominating vertex. (Note that j_0 may be zero, but not $j_1, j_2 \dots$.) The probability distribution for the number of dominating vertices follows the same law as that of the isolates because the complement of a threshold graph is a threshold graph (or because of the interchangeability of 0's and 1's in the binary coding given earlier in this section).

Note that this algorithm treats vertices in the reverse of the order in (1.2) where we add vertices instead of peeling them off as here. It follows that we obtain the extended binary code of the graph by taking runs of j_0 0's, j_1 1's, j_2 0's, and so on, and then reversing the order. Hence, in the notation used above, the sequence (b_k) equals (j_k) in reverse order, ignoring j_0 if $j_0 = 0$. (In particular note that the last $j_k \geq 2$, since $t(n', n' - 1) = 0$ for $n' \geq 2$, which corresponds to the first block $b_1 \geq 2$.)

Example 2.3. A sequence of js generated for a threshold graph of size 20 is `0 2 3 1 1 1 3 1 1 3 1 1 2`, which yields the sequence `d d i i i d i d i i i d i d d d i d i i` of dominating and isolated vertices. A random permutation of $\{1, \dots, 20\}$ was generated and we obtain

```
13 2 11 15 8 20 6 12 16 4 18 7 10 9 14 17 1 19 5 3
 d d  i  i i  d i  d  i i  i d  i d  d  d i  d i i
```

where `d` signifies that the vertex is connected to all later vertices in this list. The degree sequence is thus, taking the vertices in this order: 19, 19, 2, 2, 2, 16, 3, 15, 4, 4, 4, 12, 5, 11, 11, 11, 8, 10, 9, 9. The extended binary code 00101110100010100011 is obtained by translating `i` to 0 and `d` to 1, and reversing the order.

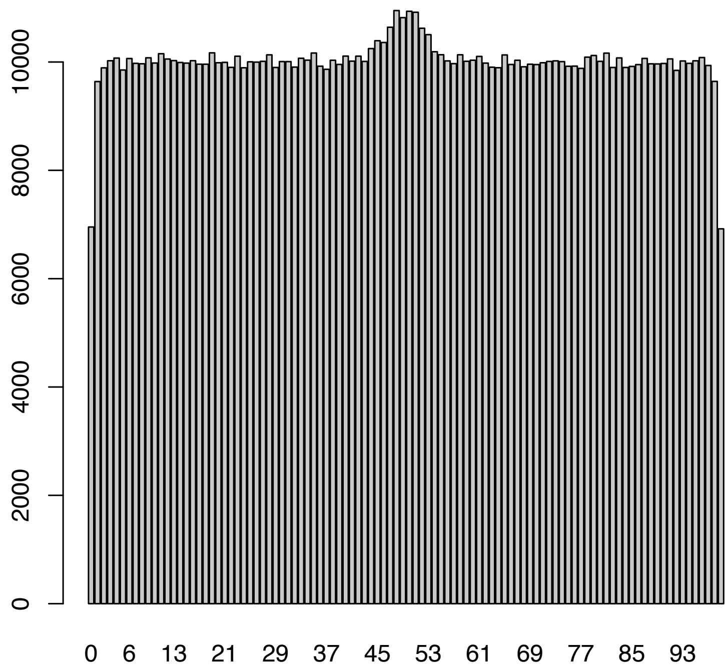


FIGURE 7. Threshold graphs were generated according to the algorithm of this section, this is the degree histogram.

In Figure 7, 10,000 graphs were generated with $n = 100$ according to the uniform distribution over all labeled threshold graphs. We discuss the central 'bump' and other features of Figure 7 in Theorem 7.4.

2.3. **The distribution of block lengths.** We have seen in Subsection 2.1 that if $b_1, \dots, b_\tau$ are the lengths of the blocks of isolated or dominating vertices added to the graph when building it as in (1.2), then (2.1) holds. Consider now a sequence of independent integer random variables $B_1, B_2, \dots$ with $B_1 \geq 2$ and $B_j \geq 1$ for $j \geq 2$, and let $S_k := \sum_{j=1}^{k} B_j$ be the partial sums. If some $S_\tau = n$, then stop and output the sequence $(B_1, \dots, B_\tau)$. Conditioning on the event that $S_\tau = n$ for some τ, this yields a random sequence $b_1, \dots, b_\tau$ satisfying (2.1), and the probability that we obtain a given sequence $(b_j)_1^\tau$ equals $c \prod_{j=1}^{\tau} \mathbb{P}(B_j = b_j)$ for some normalizing constant c. We now specialize to the case when $B_1 \overset{\mathrm{d}}{=} (B^* \mid B^* \geq 2)$ and $B_j \overset{\mathrm{d}}{=} (B^* \mid B^* \geq 1)$ for $j \geq 2$, for some given random variable B^*. Then the (conditional) probability of obtaining a given $b_1, \dots, b_\tau$ satisfying (2.1) can be written

$$c' \prod_{j=1}^{\tau} \frac{\mathbb{P}(B^* = b_j)}{\mathbb{P}(B^* \geq 1)} \tag{2.8}$$

(with $c' = c\,\mathbb{P}(B^* \geq 1)/\,\mathbb{P}(B^* \geq 2)$).

There are two important cases. First, if we take $B^* \sim \mathrm{Ge}(1/2)$, then $\mathbb{P}(B^* = b_j)/\,\mathbb{P}(B^* \geq 1) = 2^{-b_j}$, and thus (2.8) yields $c' 2^{-\sum_j b_j} = c' 2^{-n}$, so the probability is the same for all allowed sequences. Hence, in this case the distribution of the constructed sequence is uniform on the set of sequences satisfying (2.1), so it equals the distribution of block lengths for a random unlabeled threshold graph of size n.

The other case is $B^* \sim \mathrm{Po}(\log 2)$. Then $\mathbb{P}(B^* \geq 1) = 1 - e^{-\log 2} = 1/2$, and $\mathbb{P}(B^* = b_j)/\,\mathbb{P}(B^* \geq 1) = (\log 2)^{b_j}/b_j!$. Thus, (2.8) yields the probability $c'(\log 2)^n/\prod_j b_j!$, which is proportional to the number $2 \cdot n!/\prod_j b_j!$ of labeled threshold graphs with the block lengths $b_1, \dots, b_\tau$. Hence, in this case the distribution of the constructed sequence equals the distribution of block lengths for a random labeled threshold graph of size n.

We have shown the following result.

Theorem 2.4. *Construct a random sequence $B_1, \dots, B_\tau$ as above, based on a random variable B^*, stopping when $\sum_1^\tau B_j \geq n$ and conditioning on $\sum_1^\tau B_j = n$.*

(i) *If $B^* \sim \mathrm{Ge}(1/2)$, then $(B_1, \dots, B_\tau)$ has the same distribution as the block lengths in a random unlabeled threshold graph of order n.*

(ii) *If $B^* \sim \mathrm{Po}(\log 2)$, then $(B_1, \dots, B_\tau)$ has the same distribution as the block lengths in a random labeled threshold graph of order n.*

It follows that the length of a typical (for example a random) block converges in distribution to $(B^* \mid B^* \geq 1)$. Theorem 2.4 also leads to another algorithm to construct uniform random threshold graphs.

Algorithm 2.5. *Algorithm for generating uniform unlabeled or labeled threshold graphs of a given order n.*

Step 1: In the unlabeled case, let $B^* \sim \mathrm{Ge}(1/2)$. In the labeled case, let $B^* \sim \mathrm{Po}(\log 2)$.

Step 2: Choose independent random numbers $B_1, B_2, \dots, B_\tau$, with $B_1 \overset{\mathrm{d}}{=} (B^* \mid B^* \geq 2)$ and $B_j \overset{\mathrm{d}}{=} (B^* \mid B^* \geq 1)$, $j \geq 2$, until the sum $\sum_1^\tau B_j \geq n$.

Step 3: If $\sum_1^\tau B_j > n$, start again with Step 2.

Step 4: We have found $B_1, \dots, B_\tau$ with $\sum_1^\tau B_j = n$. Toss a coin to decide whether the first block is isolated or dominating; the following blocks alternate. Construct a threshold graph by adding vertices as in (1.2), block by block.

Step 5: In the labeled case, make a random labeling of the graph.

By standard renewal theory, the probability that $\sum_1^\tau B_j$ is exactly n is asymptotically $1/\,\mathbb{E}(B^* \mid B^* \geq 1) = \mathbb{P}(B^* \geq 1)/\,\mathbb{E}\,B^*$, which is $1/2$ in the unlabeled case and $1/(2\log 2) \approx 0.72$ in the labeled case, so we do not have to do very many restarts in Step 3.

3. Graph limits

This section reviews needed tools from the emerging field of graph limits.

3.1. **Graph limits.** Here we review briefly the theory of graph limits as described in Lovász and Szegedy [22], Borgs, Chayes, Lovász, Sós and Vesztergombi [5] and Diaconis and Janson [10].

If F and G are two graphs, let $t(F,G)$ be the probability that a random mapping $\phi : V(F) \to V(G)$ defines a graph homomorphism, i.e., that $\phi(v)\phi(w) \in E(G)$ when $vw \in E(F)$. (By a random mapping we mean a mapping uniformly chosen among all $v(G)^{v(F)}$ possible ones; the images of the vertices in F are thus independent and uniformly distributed over $V(G)$, i.e., they are obtained by random sampling with replacement.)

The basic definition is that a sequence G_n of (generally unlabeled) graphs converges if $t(F,G_n)$ converges for every graph F; as in [10] we will further assume $v(G_n) \to \infty$. More precisely, the (countable and discrete) set $\mathcal{U}$ of all unlabeled graphs can be embedded in a compact metric space $\overline{\mathcal{U}}$ such that a sequence $G_n \in \mathcal{U}$ of graphs with $v(G_n) \to \infty$ converges in $\overline{\mathcal{U}}$ to some limit $\Gamma \in \overline{\mathcal{U}}$ if and only if $t(F,G_n)$ converges for every graph F (see [22], [5], [10]). Let $\mathcal{U}_\infty := \overline{\mathcal{U}} \setminus \mathcal{U}$ be the set of proper limit elements; we call the elements of $\mathcal{U}_\infty$ *graph limits*. The functionals $t(F,\cdot)$ extend to continuous functions on $\overline{\mathcal{U}}$, so $G_n \to \Gamma \in \mathcal{U}_\infty$ if and only if $v(G_n) \to \infty$ and $t(F,G_n) \to t(F,\Gamma)$ for every graph F.

Let $\mathcal{W}$ be the set of all measurable functions $W : [0,1]^2 \to [0,1]$ and let $\mathcal{W}_{\mathsf{s}}$ be the subset of symmetric functions. The main result of Lovász and Szegedy [22] is that every element of $\mathcal{U}_\infty$ can be represented by a (non-unique) function $W \in \mathcal{W}_{\mathsf{s}}$. We let $\Gamma_W \in \mathcal{U}_\infty$ denote the graph limit defined by W. (We sometimes use the notation $\Gamma(W)$ for readability.) Then, for every graph F,

$$t(F,\Gamma_W) = \int_{[0,1]^{v(F)}} \prod_{ij \in E(F)} W(x_i,x_j)\,\mathrm{d}x_1 \cdots \mathrm{d}x_{v(F)}. \tag{3.1}$$

Moreover, define, for every $n \geq 1$, a random graph $G(n,W)$ as follows: first choose a sequence $X_1, X_2, \dots, X_n$ of i.i.d. random variables uniformly distributed on $[0,1]$, and then, given this sequence, for each pair (i,j) with $i < j$ draw an edge ij with probability $W(X_i,X_j)$, independently for all pairs (i,j) with $i < j$. Then the random graph $G(n,W)$ converges to Γ_W a.s. as $n \to \infty$.

If G is a graph, with $V(G) = \{1,\dots,v(G)\}$ for simplicity, we define a function $W_G \in \mathcal{W}_{\mathsf{s}}$ by partitioning $[0,1]$ into $v(G)$ intervals I_i, $i = 1,\dots,v(G)$, and letting W_G be the indicator $\mathbf{1}[ij \in E(G)]$ on $I_i \times I_j$. (In other words, W_G is a step function corresponding to the adjacency matrix of G.) We let $\pi(G) := \Gamma(W_G)$ denote the corresponding object in $\mathcal{U}_\infty$. It follows easily from (3.1) that $t(F, \pi(G)) = t(F, G)$ for every graph F. In particular, if G_n is a sequence of graphs with $v(G_n) \to \infty$, then G_n converges to some graph limit Γ if and only if $\pi(G_n) \to \Gamma$ in $\mathcal{U}_\infty$. (Unlike [22] and [5] we distinguish between graphs and limit objects and we do not identify G and $\pi(G)$, see [10].)

3.2. **Bipartite graphs and their limits.** In the bipartite case, there are analoguous definitions and results (see [10] for further details). We define a bipartite graph to be a graph G with an explicit bipartition $V(G) = V_1(G) \cup V_2(G)$ of the vertex set, such that the edge set $E(G) \subseteq V_1(G) \times V_2(G)$. Then we define $t(F, G)$ in the same way as above but now for bipartite graphs F, by letting $\phi = (\phi_1, \phi_2)$ be a pair of random mappings $\phi_j : V_j(F) \to V_j(G)$. We let $\mathcal{B}$ be the set of all unlabeled bipartite graphs and embed $\mathcal{B}$ in a compact metric space $\overline{\mathcal{B}}$. A sequence (G_n) of bipartite graphs with $v_1(G_n), v_2(G_n) \to \infty$ converges in $\overline{\mathcal{B}}$ if and only if $t(F, G_n)$ converges for every bipartite graph F. Let $\mathcal{B}_{\infty\infty}$ be the (compact) set of all such limits; we call the elements of $\mathcal{B}_{\infty\infty}$ *bipartite graph limits.* Every element of $\mathcal{B}_{\infty\infty}$ can be represented by a (non-unique) function $W \in \mathcal{W}$. We let $\Gamma''_W \in \mathcal{B}_{\infty\infty}$ denote the element represented by W and have, for every bipartite F

$$t(F, \Gamma''_W) = \int_{[0,1]^{v_1(F)+v_2(F)}} \prod_{ij \in E(F)} W(x_i, y_j) \, \mathrm{d}x_1 \cdots \mathrm{d}x_{v_1(F)} \, \mathrm{d}y_1 \cdots \mathrm{d}y_{v_2(F)}. \tag{3.2}$$

Given $W \in \mathcal{W}$ and $n_1, n_2 \geq 1$, we define a random bipartite graph $G(n_1, n_2, W)$ by an analogue of the construction in Subsection 3.1: first choose two sequences $X_1, X_2, \dots, X_{n_1}$ and $Y_1, Y_2, \dots, Y_{n_2}$ of i.i.d. random variables uniformly distributed on $[0, 1]$, and then, given thess sequences, for each pair (i, j) draw an edge ij with probability $W(X_i, Y_j)$, independently for all pairs $(i, j) \in [n_1 \times [n_2]$.

If G is a bipartite graph we define $W_G \in \mathcal{W}$ similarly as above (in general with different numbers of steps in the two variables; note that W_G now in general is not symmetric) and let $\pi(G) := \Gamma''(W_G)$. Then, by (3.2), $t(F, \pi(G)) = t(F, G)$ for every bipartite graph F. Hence, if G_n is a sequence of bipartite graphs with $v_1(G_n), v_2(G_n) \to \infty$, then G_n converges to some bipartite graph limit Γ if and only if $\pi(G_n) \to \Gamma$ in $\mathcal{B}_{\infty\infty}$.

3.3. **Cut-distance.** Borgs, Chayes, Lovász, Sós and Vesztergombi [5, Section 3.4] define a (pseudo-)metric $\delta_\square$ on $\mathcal{W}_\mathsf{s}$ called the *cut-distance.* This is only a pseudo-metric since two different functions in $\mathcal{W}_\mathsf{s}$ may have cut-distance 0 (for example, if one is obtained by a measure preserving transformation of the other, see further [4] and [10]), and it is shown in [5] that, in fact, $\delta_\square(W_1, W_2) = 0$ if and only if $t(F, W_1) = t(F, W_2)$ for every graph F, i.e., if and only if $\Gamma_{W_1} = \Gamma_{W_2}$ in $\mathcal{U}_\infty$. Moreover, the quotient space $\mathcal{W}_\mathsf{s}/\delta_\square$, where we identify elements of $\mathcal{W}_\mathsf{s}$ with cut-distance 0, is a compact metric space and the mapping $W \mapsto \Gamma_W$ is a homeomorphism of $\mathcal{W}_\mathsf{s}/\delta_\square$ onto $\mathcal{U}_\infty$.

This extends to the bipartite case. In this case, we define $\delta''_\square$ on $\mathcal{W}$ as $\delta_\square$ is defined in [5, Section 3.4], but allowing different measure preserving mappings for the two coordinates. Then, if we identify elements in $\mathcal{W}$ with cut-distance 0, $W \mapsto \Gamma''_W$ becomes a homeomorphism of $\mathcal{W}/\delta''_\square$ onto $\mathcal{B}_{\infty\infty}$. Instead of repeating and modifying the complicated proofs from [5], one can use their result in the symmetric case and define an embedding $W \mapsto \widetilde{W}$ of $\mathcal{W}$ into $\mathcal{W}_{\mathsf{s}}$ by

$$\widetilde{W}(x,y) = \begin{cases} 0, & x < 1/2,\ y < 1/2; \\ 1, & x > 1/2,\ y > 1/2; \\ \frac14 + \frac12 W(2x-1, 2y), & x > 1/2,\ y < 1/2; \\ \frac14 + \frac12 W(2y-1, 2x), & x < 1/2,\ y > 1/2. \end{cases}$$

It is easily seen that $\delta''_\square(W_1, W_2)$ and $\delta_\square(\widetilde{W}_1, \widetilde{W}_2)$ are equal within some constant factors, for $W_1, W_2 \in \mathcal{W}$, and that for each graph F, $t(F, \widetilde{W})$ is a linear combination of $t(F_i, W)$ for a family of bipartite graphs F (obtained by partitioning $V(F)$ and erasing edges within the two parts). This and the results in [5], together with the simple fact that $W \mapsto t(F, W)$ is continuous for $\delta''_\square$ for every bipartite graph F, imply easily the result claimed.

3.4. **A reflection involution.** If G is a bipartite graph, let $G^\dagger$ be the graph obtained by interchanging the order of the two vertex sets; thus, $V_j(G^\dagger) = V_{3-j}(G)$ and $E(G^\dagger) = \{uv : vu \in E(G)\}$. We say that $G^\dagger$ is the *reflection* of G. Obviously, $t(F, G^\dagger) = t(F^\dagger, G)$ for any bipartite graphs F and G. It follows that if $G_n \to \Gamma \in \overline{\mathcal{B}}$, then $G_n^\dagger \to \Gamma^\dagger$ for some $\Gamma^\dagger \in \overline{\mathcal{B}}$, and this defines a continuous map of $\overline{\mathcal{B}}$ onto itself which extends the map just defined for bipartite graphs. We have, by continuity,

$$t(F, \Gamma^\dagger) = t(F^\dagger, \Gamma), \qquad F \in \mathcal{B},\ G \in \overline{\mathcal{B}}. \tag{3.3}$$

Furthermore, $\Gamma^{\dagger\dagger} = \Gamma$, so the map is an involution, and it maps $\mathcal{B}_{\infty\infty}$ onto itself.

For a function W on $[0,1]^2$, let $W^\dagger(x,y) := W(y,x)$ be its reflection in the main diagonal. It follows from (3.2) and (3.3) that $\Gamma''(W^\dagger) = \Gamma''(W)^\dagger$.

3.5. **Threshold graph limits.** Let $\mathcal{T} := \bigcup_{n=1}^\infty \mathcal{T}_n$ be the family of all (unlabeled) threshold graphs. Thus $\mathcal{T}$ is a subset of the family $\mathcal{U}$ of all unlabeled graphs, and we define $\overline{\mathcal{T}}$ as the closure of $\mathcal{T}$ in $\overline{\mathcal{U}}$, and $\mathcal{T}_\infty := \overline{\mathcal{T}} \setminus \mathcal{T} = \overline{\mathcal{T}} \cap \mathcal{U}_\infty$, i.e., the set of proper limits of sequences of threshold graphs; we call these *threshold graph limits.*

In the bipartite case, we similarly consider the set $\mathcal{T}'' := \bigcup_{n_1,n_2 \ge 1} \mathcal{T}_{n_1,n_2} \subset \mathcal{B}$ of all bipartite threshold graphs, and let $\overline{\mathcal{T}''} \subset \overline{\mathcal{B}}$ be its closure in $\overline{\mathcal{B}}$ and $\mathcal{T}''_{\infty,\infty} := \overline{\mathcal{T}''} \cap \mathcal{B}_{\infty\infty}$ the set of proper limits of sequences of bipartite threshold graphs; we call these *bipartite threshold graph limits.*

Note that $\overline{\mathcal{T}}$, $\mathcal{T}_\infty$, $\overline{\mathcal{T}''}$, $\mathcal{T}''_{\infty,\infty}$ are compact metric spaces, since they are closed subsets of $\overline{\mathcal{U}}$ or $\overline{\mathcal{B}}$.

We will give concrete representations of the threshold graph limits in Section 5. Here we only give a more abstract characterization.

Recall that $t(F,G)$ is defined as the proportion of maps $V(F) \to V(G)$ that are graph homomorphisms. Since we only are interested in limits with $v(G) \to \infty$, it is equivalent to consider injective maps only. By inclusion-exclusion, it is further equivalent to consider $t_{\mathrm{ind}}(F,G)$, defined as the probability that a random injective map $V(F) \to V(G)$ maps F isomorphically onto an induced copy of F in G; in other words, $t_{\mathrm{ind}}(F,G)$ equals the number of labeled induced copies of F in G divided by the falling factorial $v(G)\cdots(v(G)-v(F)+1)$. Then $t_{\mathrm{ind}}(F,\cdot)$ extends by continuity to $\overline{\mathcal{U}}$, and by inclusion-exclusion, for graph limits $\Gamma \in \mathcal{U}_\infty$, $t_{\mathrm{ind}}(F,\Gamma)$ can be written as a linear combination of $t(F_i,\Gamma)$ for subgraphs $F_i \subseteq F$. We can define t_{ind} for bipartite graphs in the same way; further details are in [5] and [10].

Theorem 3.1. (i) *Let $\Gamma \in \mathcal{U}_\infty$; i.e., Γ is a graph limit. Then $\Gamma \in \mathcal{T}_\infty$ if and only if $t_{\mathrm{ind}}(P_4,\Gamma) = t_{\mathrm{ind}}(C_4,\Gamma) = t_{\mathrm{ind}}(2K_2,\Gamma) = 0$.*

(ii) *Let $\Gamma \in \mathcal{B}_{\infty\infty}$; i.e., Γ is a bipartite graph limit. Then $\Gamma \in \mathcal{T}''_{\infty,\infty}$ if and only if $t_{\mathrm{ind}}(2K_2,\Gamma) = 0$.*

In view of (1.5) and (1.12), this is a special case of the following simple general statement.

Theorem 3.2. *Let $\mathcal{F} = \{F_1, F_2, \dots\}$ be a finite or infinite family of graphs, and let $\mathcal{U}_\mathcal{F} \subseteq \mathcal{U}$ be the set of all graphs that do not contain any graph from $\mathcal{F}$ as an induced subgraph, i.e.,*

$$\mathcal{U}_\mathcal{F} := \{G \in \mathcal{U} : t_{\mathrm{ind}}(F,G) = 0 \text{ for } F \in \mathcal{F}\}.$$

Let $\overline{\mathcal{U}}_\mathcal{F}$ be the closure of $\mathcal{U}_\mathcal{F}$ in $\overline{\mathcal{U}}$. Then

$$\overline{\mathcal{U}}_\mathcal{F} := \{\Gamma \in \overline{\mathcal{U}} : t_{\mathrm{ind}}(F,\Gamma) = 0 \text{ for } F \in \mathcal{F}\}.$$

In other words, if $\Gamma \in \mathcal{U}_\infty$ is a graph limit, then Γ is a limit of a sequence of graphs in $\mathcal{U}_\mathcal{F}$ if and only if $t_{\mathrm{ind}}(F,\Gamma) = 0$ for $F \in \mathcal{F}$.

Conversely, if $\Gamma \in \overline{\mathcal{U}}_\mathcal{F} \cap \mathcal{U}_\infty$ is represented by a function W, then the random graph $G(n,W) \in \mathcal{U}_\mathcal{F}$ (almost surely).

The same results hold in the bipartite case.

Proof. If $G_n \to \Gamma$ with $G \in \mathcal{U}_\mathcal{F}$, then $t(F,\Gamma) = \lim_{n\to\infty} t(F,G_n) = 0$ for every $F \in \mathcal{F}$, by the continuity of $t(F,\cdot)$.

Conversely, suppose that $\Gamma \in \mathcal{U}_\infty$ and $t(F,\Gamma) = 0$ for $F \in \mathcal{F}$, and let Γ be represented by a function W. It follows from (3.1) that if $F \in \mathcal{F}$ then $\mathbb{E}\, t(F, G(n,W)) = t(F,\Gamma) = 0$, and thus $t(F, G(n,W)) = 0$ a.s.; consequently $G(n,W) \in \mathcal{U}_\mathcal{F}$ a.s. This proves the second statement. Since $G(n,W) \to \Gamma$ a.s., it also shows that Γ is the limit of a sequence in $\mathcal{U}_\mathcal{F}$, and thus $\Gamma \in \overline{\mathcal{U}}_\mathcal{F}$, which completes the proof of the first part. ∎

4. Degree distributions

The results in this section hold for general graphs, they are applied to threshold graphs in section Section 5.

Let $\mathcal{P}$ be the set of probability measures on $[0,1]$, equipped with the standard topology of weak convergence, which makes $\mathcal{P}$ a compact metric space (see e.g. Billingsley [3]).

If G is a graph, let $d(v) = d_G(v)$ denote the degree of vertex $v \in V(G)$, and let D_G denote the random variable defined as the degree $d_G(v)$ of a randomly chosen vertex v (with the uniform distribution on $V(G)$). Thus $0 \le D_G \le v(G)-1$. For a bipartite graph we similarly define $D_{G;j}$ as the degree $d_G(v)$ of a randomly chosen vertex $v \in V_j(G)$, $j = 1, 2$. Note that $0 \le D_{G;1} \le v_2(G)$ and $0 \le D_{G;2} \le v_1(G)$. Since we are interested in dense graphs, we will normalize these random degrees to $D_G/v(G)$ and, in the bipartite case, $D_{G;1}/v_2(G)$ and $D_{G;2}/v_1(G)$; these are random variables in [0,1]. The distribution of $D_G/v(G)$ will be called the (normalized) degree distribution of G and denoted by $\nu(G) \in \mathcal{P}$; in other words, $\nu(G)$ is the empirical distribution function of $\{d_G(v)/v(G) : v \in V(G)\}$. In the bipartite case we similarly have two (normalized) degree distributions: $\nu_1(G)$ for $V_1(G)$ and $\nu_2(G)$ for $V_2(G)$.

The moments of the degree distribution(s) are given by the functional $t(F, \cdot)$ for stars F, as stated in the following lemma. We omit the proof, which is a straightforward consequence of the definitions.

Lemma 4.1. *The moments of $\nu(G)$ are given by*

$$\int_0^1 t^k \,\mathrm{d}\nu(G)(t) = t(K_{1,k}, G), \qquad k \ge 1, \tag{4.1}$$

where $K_{1,k}$ is a star with k edges.

In the bipartite case, similarly, for $k \ge 1$,

$$\int_0^1 t^k \,\mathrm{d}\nu_1(G)(t) = t(K_{1,k}, G), \qquad \int_0^1 t^k \,\mathrm{d}\nu_2(G)(t) = t(K_{k,1}, G). \tag{4.2}$$

This enables us to extend the definition of the (normalized) degree distribution to the limit objects by continuity.

Theorem 4.2. *If G_n are graphs with $v(G_n) \to \infty$ and $G_n \to \Gamma$ for some $\Gamma \in \overline{\mathcal{U}}$ as $n \to \infty$, then $\nu(G_n) \to \nu(\Gamma)$ for some distribution $\nu(\Gamma) \in \mathcal{P}$. This defines the 'degree distribution' $\nu(\Gamma)$ (uniquely) for every graph limit $\Gamma \in \mathcal{U}_\infty$, and $\Gamma \mapsto \nu(\Gamma)$ is a continuous map $\mathcal{U}_\infty \to \mathcal{P}$. Furthermore,* (4.1) *holds for all $G \in \overline{\mathcal{U}}$.*

Similarly, in the bipartite case, ν_1 and ν_2 extend to continuous maps $\overline{\mathcal{B}} \to \mathcal{P}$ such that (4.2) *holds for all $G \in \overline{\mathcal{B}}$. Furthermore, $\nu_2(\Gamma) = \nu_1(\Gamma^\dagger)$ for $\Gamma \in \overline{\mathcal{B}}$.*

Proof. An immediate consequence of Lemma 4.1 and the method of moments. The last sentence follows from (4.2) and (3.3). □

Remark. Theorem 4.2 says that the degree distribution ν is a *testable* graph parameter in the sense of Borgs, Chayes, Lovász, Sós and Vesztergombi [5], see in particular [5, Section 6]. (Except that ν takes values in $\mathcal{P}$ instead of $\mathbb{R}$.)

If Γ is represented by a function W on $[0,1]^2$, we can easily find its degree distribution from W.

Theorem 4.3. *If $W \in \mathcal{W}_{\mathsf{s}}$, then $\nu(\Gamma_W)$ equals the distribution of $\int_0^1 W(U,y)\,\mathrm{d}y$, where $U \sim U(0,1)$.*

Similarly, in the bipartite case, if $W \in \mathcal{W}$, then $\nu_1(\Gamma''_W)$ equals the distribution of $\int_0^1 W(U,y)\,\mathrm{d}y$ and $\nu_2(\Gamma''_W)$ equals the distribution of $\int_0^1 W(x,U)\,\mathrm{d}x$.

Proof. By (4.1) and (3.1),

$$\begin{aligned}\int_0^1 t^k \,\mathrm{d}\nu(\Gamma_W)(t) = t(K_{1,k},\Gamma_W) &= \int_{[0,1]} \left(\int_{[0,1]} W(x,y)\,\mathrm{d}y \right)^k \mathrm{d}x \\ &= \mathbb{E}\left(\int_{[0,1]} W(U,y)\,\mathrm{d}y \right)^k\end{aligned}$$

for every $k \geq 1$, and the result follows. The bipartite case is similar, using (3.2). □

If a graph G has n vertices, its number of edges is

$$|E(G)| = \frac{1}{2} \sum_{v \in V(G)} d(v) = \frac{n}{2}\,\mathbb{E}\,D_G = \frac{n^2}{2}\,\mathbb{E}(D_G/n) = \frac{n^2}{2} \int_0^1 t\,\mathrm{d}\nu(G)(t).$$

Hence, the edge density of G is

$$|E(G)|/\binom{n}{2} = \frac{n}{n-1} \int_0^1 t\,\mathrm{d}\nu(G)(t). \tag{4.3}$$

If (G_n) is a sequence of graphs with $v(G_n) \to \infty$ and $G_n \to \Gamma \in \mathcal{U}_\infty$, we see from (4.3) and Theorem 4.2 that the graph densities converge to $\int_0^1 t\,\mathrm{d}\nu(\Gamma)(t)$, the mean of the distribution $\nu(\Gamma)$, which thus may be called the (edge) density of $\Gamma \in \mathcal{U}_\infty$.

If Γ is represented by a function W on $[0,1]^2$, Theorem 4.3 yields the following.

Corollary 4.4. *Γ_W has edge density $\iint_{[0,1]^2} W(x,y)\,\mathrm{d}x\,\mathrm{d}y$ for every $W \in \mathcal{W}_{\mathsf{s}}$.*

Proof. By Theorem 4.3, the mean of $\mu(\Gamma_W)$ equals

$$\mathbb{E} \int_0^1 W(U,y)\,\mathrm{d}y = \int_0^1 \int_0^1 W(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$ □

5. Limits of threshold graphs

Recall from Subsection 3.5 that $\mathcal{T}_\infty$ is the set of limits of threshold graphs, and $\mathcal{T}''_{\infty,\infty}$ is the set of limits of bipartite threshold graphs. Our purpose in this section is to characterize the threshold graph limits, i.e. the elements of $\mathcal{T}_\infty$ and $\mathcal{T}''_{\infty,\infty}$, and give simple criteria for the convergence of a sequence of threshold graphs to one of these limits. We begin with some definitions.

A function $W : [0,1]^2 \to \mathbb{R}$ is *increasing* if $W(x,y) \le W(x',y')$ whenever $0 \le x \le x' \le 1$ and $0 \le y \le y' \le 1$. A set $S \subseteq [0,1]^2$ is increasing if its indicator $\mathbf{1}_S$ is an increasing function on $[0,1]^2$, i.e., if $(x,y) \in S$ implies $(x',y') \in S$ whenever $0 \le x \le x' \le 1$ and $0 \le y \le y' \le 1$.

If $\mu \in \mathcal{P}$, let F_μ be its distribution function $F_\mu(x) := \mu([0,x])$, and let $F_\mu(x-) := \mu([0,x))$ be its left-continuous version. Thus $F_\mu(0-) = 0 \le F_\mu(0)$ and $F_\mu(1-) \le 1 = F_\mu(1)$. Further, let $F_\mu^{-1} : [0,1] \to [0,1]$ be the right-continuous inverse defined by

$$F_\mu^{-1}(x) := \sup\{t \le 1 : F_\mu(t) \le x\}. \tag{5.1}$$

Note that $F_\mu^{-1}(0) \ge 0$ and $F_\mu^{-1}(1) = 1$. Finally, define

$$S_\mu := \bigl\{(x,y) \in [0,1]^2 : x \ge F_\mu\bigl((1-y)-\bigr)\bigr\}. \tag{5.2}$$

It is easily seen that S_μ is a closed increasing subset of $[0,1]^2$ and that it contains the upper and right edges $\{(x,1)\}$ and $\{(1,y)\}$. Since $x \ge F_\mu\bigl((1-y)-\bigr) \iff F_\mu^{-1}(x) \ge 1-y$, we also have

$$S_\mu = \bigl\{(x,y) \in [0,1]^2 : F_\mu^{-1}(x) + y \ge 1\bigr\}. \tag{5.3}$$

We further write $W_\mu := \mathbf{1}_{S_\mu}$ and let $\Gamma''_\mu := \Gamma''(W_\mu)$ and, when W is symmetric, $\Gamma_\mu := \Gamma(W_\mu)$. We denote the interior of a set S by S°. It is easily verified from (5.2) that

$$S_\mu^\circ = \bigl\{(x,y) \in (0,1)^2 : x > F_\mu(1-y)\bigr\}. \tag{5.4}$$

Recall that the *Hausdorff distance* between two non-empty compact subsets K_1 and K_2 of some metric space $\mathcal{S}$ is defined by

$$d_H(K_1,K_2) := \max\bigl(\max_{x\in K_1} d(x,K_2),\ \max_{y\in K_2} d(y,K_1)\bigr). \tag{5.5}$$

This defines a metric on the set of all non-empty compact subsets of $\mathcal{S}$. If $\mathcal{S}$ is compact, the resulting topology on the set of compact subsets of $\mathcal{S}$ (with the empty set as an isolated point) is compact and equals the *Fell topology* (see e.g. [19, Appendix A.2]) on the set of all closed subsets of $\mathcal{S}$.

Let λ_d denote the Lebesgue measure in $\mathbb{R}^d$. For measurable subsets S_1, S_2 of $[0,1]^2$, we also consider their *measure distance* $\lambda_2(S_1 \Delta S_2)$. This equals the L^1-distance of their indicator functions, and is thus a metric modulo null sets.

For functions in $\mathcal{W}$ we also use two different metrics: the L^1*-distance* $\int_{[0,1]^2} |W_1(x,y) - W_2(x,y)| \,\mathrm{d}x\,\mathrm{d}y$ and, in the symmetric case, the *cut-distance* $\delta_\square$ defined by Borgs, Chayes, Lovász, Sós and Vesztergombi [5], and in the bipartite case its analogue $\delta''_\square$, see Section 3. Note that the cut-distance is only a pseudo-metric, since the distance of two different functions may be 0. Note further that the cut-distance is less than or equal to the L^1-distance.

We can now prove one of our main results, giving several related characterizations of threshold graph limits. There are two versions, since we treat the bipartite case in parallel.

The bipartite case. It is convenient to begin with the bipartite case.

Theorem 5.1. *There are bijections between the set* $\mathcal{T}''_{\infty,\infty}$ *of graph limits of bipartite threshold graphs and each of the following sets.*

(i) *The set* $\mathcal{P}$ *of probability distributions on* $[0,1]$.
(ii) *The set* $\mathcal{C}_{\mathcal{B}}$ *of increasing closed sets* $S \subseteq [0,1]^2$ *that contain the upper and right edges* $[0,1]\times\{1\}\cup\{1\}\times[0,1]$.
(iii) *The set* $\mathcal{O}_{\mathcal{B}}$ *of increasing open sets* $S \subseteq (0,1)^2$.
(iv) *The set* $\mathcal{W}_{\mathcal{B}}$ *of increasing* 0–1 *valued functions* $W : [0,1]^2 \to \{0,1\}$ *modulo a.e. equality.*

More precisely, there are commuting bijections between these sets given by the following mappings and their compositions:

$$\begin{aligned}
\iota_{\mathcal{BP}} &: \mathcal{T}''_{\infty,\infty} \to \mathcal{P}, & \iota_{\mathcal{BP}}(\Gamma) &:= \nu_1(\Gamma);\\
\iota_{\mathcal{PC}} &: \mathcal{P} \to \mathcal{C}_{\mathcal{B}}, & \iota_{\mathcal{PC}}(\mu) &:= S_\mu;\\
\iota_{\mathcal{CO}} &: \mathcal{C}_{\mathcal{B}} \to \mathcal{O}_{\mathcal{B}}, & \iota_{\mathcal{CO}}(S) &:= S^\circ;\\
\iota_{\mathcal{CW}} &: \mathcal{C}_{\mathcal{B}} \to \mathcal{W}_{\mathcal{B}}, & \iota_{\mathcal{CW}}(S) &:= \mathbf{1}_S;\\
\iota_{\mathcal{OW}} &: \mathcal{O}_{\mathcal{B}} \to \mathcal{W}_{\mathcal{B}}, & \iota_{\mathcal{OW}}(S) &:= \mathbf{1}_S;\\
\iota_{\mathcal{WB}} &: \mathcal{W}_{\mathcal{B}} \to \mathcal{T}''_{\infty,\infty}, & \iota_{\mathcal{WB}}(W) &:= \Gamma''_W.
\end{aligned} \tag{5.6}$$

$$\begin{tikzcd}
\mathcal{T}''_{\infty,\infty} \arrow[r, "\iota_{\mathcal{BP}}"] & \mathcal{P} \arrow[r, "\iota_{\mathcal{PC}}"] & \mathcal{C}_{\mathcal{B}} \arrow[dl, "\iota_{\mathcal{CW}}"'] \arrow[d, "\iota_{\mathcal{CO}}"] \\
& \mathcal{W}_{\mathcal{B}} \arrow[ul, "\iota_{\mathcal{WB}}"] & \mathcal{O}_{\mathcal{B}} \arrow[l, "\iota_{\mathcal{OW}}"]
\end{tikzcd}$$

In particular, a probability distribution $\mu \in \mathcal{P}$ *corresponds to* $\Gamma''_\mu \in \mathcal{T}''_{\infty,\infty}$ *and to* $S_\mu \in \mathcal{C}_{\mathcal{B}}$, $S^\circ_\mu \in \mathcal{O}_{\mathcal{B}}$, *and* $W_\mu \in \mathcal{W}_{\mathcal{B}}$. *Conversely,* $\Gamma \in \mathcal{T}''_{\infty,\infty}$ *corresponds to* $\nu_1(\Gamma) \in \mathcal{P}$. *Thus, the mappings* $\Gamma \mapsto \nu_1(\Gamma)$ *and* $\mu \mapsto \Gamma''_\mu$ *are the inverses of each other.*

Moreover, these bijections are homeomorphisms, with any of the following topologies or metrics: the standard (weak) topology on $\mathcal{P}$*; the Hausdorff metric, or the Fell topology, or the measure distance on* $\mathcal{C}_{\mathcal{B}}$*; the measure distance on* $\mathcal{O}_{\mathcal{B}}$*; the* L^1*-distance or the cut-distance on the set* $\mathcal{W}_{\mathcal{B}}$.

Proof. The mappings in (5.6) are all well-defined, except that we do not yet know that $\iota_{\mathcal{WB}}$ maps $\mathcal{W}_{\mathcal{B}}$ into $\mathcal{T}''_{\infty,\infty}$. We thus regard $\iota_{\mathcal{WB}}$ as a map $\mathcal{W}_{\mathcal{B}} \to \mathcal{B}_{\infty\infty}$ and let $\widetilde{\mathcal{B}} := \iota_{\mathcal{WB}}(\mathcal{W}_{\mathcal{B}})$ be its image; we will identify this as $\mathcal{T}''_{\infty,\infty}$ later. For the time being we also regard $\iota_{\mathcal{BP}}$ as defined on $\widetilde{\mathcal{B}}$ (or on all of $\mathcal{B}_{\infty\infty}$).

Consider first $\iota_{\mathcal{PC}} : \mathcal{P} \to \mathcal{C}_{\mathcal{B}}$. By (5.2), S_μ determines F_μ at all continuity points, and thus it determines μ. Consequently, $\iota_{\mathcal{PC}}$ is injective.

If $S \in \mathcal{C}_{\mathcal{B}}$ and $y \in [0,1]$, then $\{x : (x,y) \in S\}$ is a closed subinterval of $[0,1]$ that contains 1, and thus $S = \{(x,y) \in [0,1]^2 : x \geq g(y)\}$ for some function $g : [0,1] \to [0,1]$. Moreover, $g(1) = 0$, g is decreasing, i.e. $g(y_2) \leq g(y_1)$ if $y_1 \leq y_2$, and, since S is closed, g is right-continuous. Thus $g(1-x)$ is increasing and left-continuous, and hence there exists a probability measure $\mu \in \mathcal{P}$ such that $F_\mu(x-) = g(1-x)$, $x \in [0,1]$. By (5.2), then

$$\iota_{\mathcal{PC}}(\mu) = S_\mu = \{(x,y) \in [0,1]^2 : x \geq g(y)\} = S.$$

Hence $\iota_{\mathcal{PC}}$ is onto. Consequently, $\iota_{\mathcal{PC}}$ is a bijection of $\mathcal{P}$ onto $\mathcal{C}_{\mathcal{B}}$.

If S_1 and S_2 are two different sets in $\mathcal{C}_{\mathcal{B}}$, then there exists a point $(x,y) \in S_1 \setminus S_2$, say. There is a small open disc with center in (x,y) that does not intersect S_2, and since S_1 is increasing, at least a quarter of the disc is contained in $S_1 \setminus S_2$. Hence, $\lambda_2(S_1 \Delta S_2) > 0$. Similarly, if S_1 and S_2 are two different sets in $\mathcal{O}_{\mathcal{B}}$ and $(x,y) \in S_1 \setminus S_2$, then there is a small open disc with center in (x,y) that is contained in S_1, and since S_2 is increasing, at least a quarter of the disc is contained in $S_1 \setminus S_2$, whence $\lambda_2(S_1 \Delta S_2) > 0$. This shows that the measure distance is a metric on $\mathcal{C}_{\mathcal{B}}$ and on $\mathcal{O}_{\mathcal{B}}$, and that the mappings $\iota_{\mathcal{CW}}$ and $\iota_{\mathcal{OW}}$ into $\mathcal{W}_{\mathcal{B}}$ are injective (remember that a.e. equal functions are identified in $\mathcal{W}_{\mathcal{B}}$).

Next, let $S \subseteq [0,1]^2$ be increasing. If $(x,y) \in \overline{S}$ with $x < 1$ and $y < 1$, it is easily seen that $(x, x+\delta) \times (y, y+\delta) \subseteq S$ for $\delta = \min\{1-x, 1-y\}$, and thus $(x, x+\delta) \times (y, y+\delta) \subseteq S^\circ$. It follows that, for any real a, the intersection of the boundary $\partial S := \overline{S} \setminus S^\circ$ with the diagonal line $L_a := \{(x, x+a) : x \in \mathbb{R}\}$ consists of at most two points (of which one is on the boundary of $[0,1]^2$). In particular, $\lambda_1(\partial S \cap L_a) = 0$ and thus

$$\lambda_2(\partial S) = 2^{-1/2} \int_{-1}^{1} \lambda_1(\partial S \cap L_a) \, \mathrm{d}a = 0. \tag{5.7}$$

Consequently, ∂S is a null set for every increasing S. Among other things, this shows that if $S \in \mathcal{C}_{\mathcal{B}}$, then $\iota_{\mathcal{OW}} \iota_{\mathcal{CO}}(S) = \mathbf{1}_{S^\circ} = \mathbf{1}_S$ a.e. Since elements of $\mathcal{W}_{\mathcal{B}}$ are defined modulo a.e. equality, this shows that $\iota_{\mathcal{OW}} \iota_{\mathcal{CO}} = \iota_{\mathcal{CW}} : \mathcal{C}_{\mathcal{B}} \to \mathcal{W}_{\mathcal{B}}$.

If $W \in \mathcal{W}_{\mathcal{B}}$, and thus $W = \mathbf{1}_S$ for some increasing $S \subseteq [0,1]^2$, let

$$\widetilde{S} := \overline{S \cup [0,1] \times \{1\} \cup \{1\} \times [0,1]}. \tag{5.8}$$

Then $\widetilde{S} \in \mathcal{C}_{\mathcal{B}}$ and (5.7) implies that $\iota_{\mathcal{CW}}(\widetilde{S}) = \mathbf{1}_{\widetilde{S}} = \mathbf{1}_S = W$ a.e. Similarly, $S^\circ \in \mathcal{O}_{\mathcal{B}}$ and $\iota_{\mathcal{OW}}(S^\circ) = \mathbf{1}_S = W$ a.e. Consequently, $\iota_{\mathcal{CW}}$ and $\iota_{\mathcal{OW}}$ are onto, and thus bijections. Similarly (or as a consequence), $\iota_{\mathcal{CO}}$ is a bijection of $\mathcal{C}_{\mathcal{B}}$ onto $\mathcal{O}_{\mathcal{B}}$, with inverse $S \mapsto \widetilde{S}$ given by (5.8).

Note that the composition $\iota_{\mathcal{CW}} \iota_{\mathcal{PC}}$ maps $\mu \mapsto \mathbf{1}_{S_\mu} = W_\mu$, and let $\iota_{\mathcal{PB}}$ be the composition $\iota_{\mathcal{WB}} \iota_{\mathcal{CW}} \iota_{\mathcal{PC}} : \mu \mapsto \Gamma''(W_\mu) = \Gamma''_\mu$ mapping $\mathcal{P}$ into $\mathcal{B}_{\infty\infty}$. Since $\iota_{\mathcal{PC}}$ and $\iota_{\mathcal{CW}}$ are bijections, its image $\iota_{\mathcal{BP}}(\mathcal{P}) = \iota_{\mathcal{WB}}(\mathcal{W}_{\mathcal{B}}) = \widetilde{\mathcal{B}} \subseteq \mathcal{B}_{\infty\infty}$.

If $\mu \in \mathcal{P}$, then the composition $\iota_{\mathcal{BP}}\iota_{\mathcal{PB}}(\mu) = \nu_1(\Gamma''_\mu)$ equals by Theorem 4.3 and (5.3) the distribution of

$$\int_0^1 \mathbf{1}_{S_\mu}(U,y)\,\mathrm{d}y = F_\mu^{-1}(U). \tag{5.9}$$

As is well-known, and easy to verify using (5.1), this distribution equals μ. Hence, the composition $\iota_{\mathcal{BP}}\iota_{\mathcal{PB}}$ is the identity. It follows that $\iota_{\mathcal{PB}}$ is injective and thus a bijection of $\mathcal{P}$ onto its image $\widetilde{\mathcal{B}}$, and that $\iota_{\mathcal{BP}}$ (restricted to $\widetilde{\mathcal{B}}$) is its inverse.

We have shown that all mappings in (5.6) are bijections, except that we have not yet shown that $\widetilde{\mathcal{B}} = \mathcal{T}''_{\infty,\infty}$. We next show that the mappings are homeomorphisms.

Recall that the topology on $\mathcal{P}$ can be defined by the Lévy metric defined by (see e.g. [13, Problem 5.25])

$$d_L(\mu_1,\mu_2) := \inf\{\varepsilon > 0 : F_{\mu_1}(x-\varepsilon)-\varepsilon \le F_{\mu_2}(x) \le F_{\mu_1}(x+\varepsilon)+\varepsilon \text{ for all } x\}. \tag{5.10}$$

If $\mu_1,\mu_2 \in \mathcal{P}$ with $d_L(\mu_1,\mu_2) < \varepsilon$, it follows from (5.2) and (5.10) that if $(x,y) \in S_{\mu_1}$ and $x,y < 1-\varepsilon$, then

$$F_{\mu_2}\bigl((1-y-\varepsilon)-\bigr) \le F_{\mu_1}\bigl((1-y)-\bigr) + \varepsilon \le x+\varepsilon$$

and thus $(x+\varepsilon, y+\varepsilon) \in S_{\mu_2}$. Considering also the simple cases $x \in [1-\varepsilon,1]$ and $y \in [1-\varepsilon,1]$, it follows that if $(x,y) \in S_{\mu_1}$, then $d\bigl((x,y),S_{\mu_2}\bigr) \le \sqrt{2}\,\varepsilon$. Consequently, by (5.5) and symmetry,

$$d_H(S_{\mu_1},S_{\mu_2}) \le \sqrt{2}\,d_L(\mu_1,\mu_2),$$

which shows that $\iota_{\mathcal{PC}}$ is continuous if $\mathcal{C}_{\mathcal{B}}$ is given the topology given by the Hausdorff metric.

The same argument shows that for any (x_0,y_0), the intersection of the difference $S_{\mu_1}\Delta S_{\mu_2}$ with the diagonal line L_a defined above is an interval of length at most $\sqrt{2}\,d_L(\mu_1,\mu_2)$, and thus, by integration over a as in (5.7),

$$\lambda_2\bigl(S_{\mu_1}\Delta S_{\mu_2}\bigr) \le 2d_L(\mu_1,\mu_2).$$

Hence, $\iota_{\mathcal{PC}}$ is continuous also if $\mathcal{C}_{\mathcal{B}}$ is given the topology given by the measure distance.

Since $\mathcal{P}$ is compact and $\iota_{\mathcal{PC}}$ is a bijection, it follows that $\iota_{\mathcal{PC}}$ is a homeomorphism for both these topologies on $\mathcal{C}_{\mathcal{B}}$. In particular, these topologies coincide on $\mathcal{C}_{\mathcal{B}}$. As remarked before the theorem, since $[0,1]^2$ is compact, also the Fell topology coincide with these on $\mathcal{C}_{\mathcal{B}}$.

The bijections $\iota_{\mathcal{CO}}$, $\iota_{\mathcal{CW}}$ and $\iota_{\mathcal{OW}}$ are isometries for the measure distance on $\mathcal{C}_{\mathcal{B}}$ and $\mathcal{O}_{\mathcal{B}}$ and the L^1-distance on $\mathcal{W}_{\mathcal{B}}$, and thus homeomorphisms. Furthermore, still using the L^1-distance on $\mathcal{W}_{\mathcal{B}}$, it is easily seen from (3.2), as for the symmetric case in [22], [5], that for every fixed bipartite graph F, the mapping $W \mapsto t(F,\Gamma''_W)$ is continuous, which by definition of the topology in $\mathcal{B}_{\infty\infty}$ means that $\iota_{\mathcal{WB}} : W \mapsto \Gamma''_W$ is continuous. Hence, the bijection $\iota_{\mathcal{WB}}$ is a homeomorphism of the compact space $\mathcal{W}_{\mathcal{B}}$ onto its image $\widetilde{\mathcal{B}}$.

As said above, the cut-distance is only a pseudo-metric on $\mathcal{W}$. But two functions in $\mathcal{W}$ with cut-distance 0 are mapped onto the same element in $\mathcal{B}_{\infty\infty}$, and since we have shown that $\iota_{\mathcal{WB}}$ is injective on $\mathcal{W}_{\mathcal{B}}$, it follows that the restriction of the cut-distance to $\mathcal{W}_{\mathcal{B}}$ is a metric. Moreover, the identity map on $\mathcal{W}_{\mathcal{B}}$ is continuous from the L^1-metric to the cut-metric, and since the space is compact under the former metric, the two metrics are equivalent on $\mathcal{W}_{\mathcal{B}}$.

We have shown that all mappings are homeomorphisms. It remains only to show that $\widetilde{\mathcal{B}} = \mathcal{T}''_{\infty,\infty}$. To do this, observe first that if G is a bipartite threshold graph, and we order its vertices in each of the two vertex sets with increasing vertex degrees, then the function W_G defined in Section 3 is increasing and belongs thus to $\mathcal{W}_{\mathcal{B}}$. Consequently, $\pi(G) = \iota_{\mathcal{WB}}(W_G) \in \widetilde{\mathcal{B}}$. If $\Gamma \in \mathcal{T}''_{\infty,\infty}$, then by definition there exists a sequence G_n of bipartite threshold graphs with $v_1(G_n), v_2(G_n) \to \infty$ such that $G_n \to \Gamma$ in $\overline{\mathcal{B}}$. This implies that $\pi(G_n) \to \Gamma$ in $\mathcal{B}_{\infty\infty}$, and since $\pi(G_n) \in \widetilde{\mathcal{B}}$ and $\widetilde{\mathcal{B}}$ is compact and thus a closed subset of $\mathcal{B}_{\infty\infty}$, we find $\Gamma \in \widetilde{\mathcal{B}}$.

Conversely, if $\Gamma \in \widetilde{\mathcal{B}}$, then $\Gamma = \iota_{\mathcal{WB}}\iota_{\mathcal{CW}}(S)$ for some set $S \in \mathcal{C}_{\mathcal{B}}$. For each n, partition $[0,1]^2$ into n^2 closed squares Q_{ij} of side $1/n$, and let S_n be the union of all Q_{ij} that intersect S. Then $S_n \in \mathcal{C}_{\mathcal{B}}$, $S \subseteq S_n$ and $d_H(S_n, S) \le \sqrt{2}/n$. Let $W_n := \mathbf{1}_{S_n} = \iota_{\mathcal{CW}}(S_n)$ and let $\Gamma_n := \iota_{\mathcal{WB}}(W_n) \in \widetilde{\mathcal{B}}$. Since $\iota_{\mathcal{CW}}$ and $\iota_{\mathcal{WB}}$ are continuous, $W_n \to W := \mathbf{1}_S$ in $\mathcal{W}_{\mathcal{B}}$ and $\Gamma_n \to \iota_{\mathcal{WB}}(W) = \Gamma$ in $\widetilde{\mathcal{B}} \subset \mathcal{B}_{\infty\infty}$. However, W_n is a step function of the form $W(G_n)$ for some bipartite graph G_n with $v_1(G_n) = v_2(G_n) = n$, and thus $\pi(G_n) = \Gamma''_{W_n} = \Gamma_n$. Moreover, each S_n and thus each W_n is increasing, and hence G_n is a bipartite threshold graph. Since $\pi(G_n) = \Gamma_n \to \Gamma$ in $\mathcal{B}_{\infty\infty}$, it follows that $G_n \to \Gamma$ in $\overline{\mathcal{B}}$, and thus $\Gamma \in \mathcal{T}''_{\infty,\infty}$.

Consequently, $\widetilde{\mathcal{B}} = \mathcal{T}''_{\infty,\infty}$, which completes the proof. $\square$

Remark 5.1. Another unique representation by increasing closed sets is given by the family $\mathcal{C}'_{\mathcal{B}}$ of closed increasing subsets S of $[0,1]^2$ that satisfy $S = \overline{S^\circ}$; there are bijections $\mathcal{C}'_{\mathcal{B}} \to \mathcal{O}_{\mathcal{B}}$ and $\mathcal{O}_{\mathcal{B}} \to \mathcal{C}'_{\mathcal{B}}$ given by $S \mapsto S^\circ$ and $S \mapsto \overline{S}$. We can, again, use the measure distance on $\mathcal{C}'_{\mathcal{B}}$, but not the Hausdorff distance. (For example, $[0,1] \times [1-\varepsilon, 1] \to \emptyset$ in $\mathcal{C}'_{\mathcal{B}}$ as $\varepsilon \to 0$.)

Corollary 5.2. *The degree distribution yields a homeomorphism $\Gamma \mapsto \nu_1(\Gamma)$ of $\mathcal{T}''_{\infty,\infty}$ onto $\mathcal{P}$.*

Of course, $\Gamma \mapsto \nu_2(\Gamma) = \nu_1(\Gamma^\dagger)$ yields another homeomorphism of $\mathcal{T}''_{\infty,\infty}$ onto $\mathcal{P}$. To see the connection between these, and (more importantly) to prepare for the corresponding result in the non-bipartite case, we investigate further the reflection involution.

If $S \subseteq [0,1]^2$, let $S^\dagger := \{(x,y) : (y,x) \in S\}$ be the set S reflected in the main diagonal. Thus $\mathbf{1}_{S^\dagger} = \mathbf{1}_S^\dagger$. We have defined the reflection map $\dagger$ for bipartite graphs and graph limits, and for the sets and functions in Theorem 5.1(ii)(iii)(iv), and it is easily seen that these correspond to each other by the bijections in Theorem 5.1. Consequently, there is a corresponding map (involution) $\mu \mapsto \mu^\dagger$ of $\mathcal{P}$ onto itself too. This map is less intuitive than the others; to find it explicitly, we find from (5.2), (5.3) and $S_{\mu^\dagger} = S_\mu^\dagger$ that

$$x \geq F_{\mu^\dagger}\big((1-y)-\big) \iff (y,x) \in S_\mu \iff F_\mu^{-1}(y) + x \geq 1$$

and thus $F_{\mu^\dagger}\big((1-y)-\big) = 1 - F_\mu^{-1}(y)$ and

$$F_{\mu^\dagger}(t) = 1 - F_\mu^{-1}\big((1-t)-\big), \qquad 0 \leq t \leq 1. \tag{5.11}$$

This means that the graph of the distribution function is reflected about the diagonal between $(0,1)$ and $(1,0)$ (and adjusted at the jumps).

The map $\dagger$ is continuous on $\mathcal{P}$, by Theorem 5.1 and the obvious fact that $S \mapsto S^\dagger$ is continuous on, for example, $\mathcal{C}_\mathcal{B}$.

We let $\mathcal{P}_\mathsf{s} := \{\mu \in \mathcal{P} : \mu^\dagger = \mu\} = \{\mu \in \mathcal{P} : S_\mu = S_\mu^\dagger\}$ be the set of probability distributions invariant under the involution $\dagger$. Since $\dagger$ is continuous, $\mathcal{P}_\mathsf{s}$ is a closed and thus compact subset of $\mathcal{P}$.

Remark 5.2. If $\mu \in \mathcal{P}_\mathsf{s}$, let $x_0 := 1 - \inf\{x : (x,x) \in S_\mu\}$. Then (5.2) and (5.4) imply that $F_\mu(x_0-) \leq 1 - x_0 \leq F_\mu(x_0)$, and the restriction of F_μ to $[0, x_0)$ is an increasing right-continuous function with values in $[0, 1-x_0]$ and this restriction determines $F_\mu(t)$ for $x \geq x_0$ too by (5.11).

Conversely, given any $x_0 \in [0,1]$ and increasing right-continuous $F : [0, x_0) \to [0, 1-x_0]$, there is a unique $\mu \in \mathcal{P}_\mathsf{s}$ with $F_\mu(x) = F(x)$ for $x < x_0$ and $F_\mu(x_0) \geq 1 - x_0$.

Non-bipartite case. We can now state our main theorem for (non-bipartite) threshold graph limits.

Theorem 5.3. *There are bijections between the set $\mathcal{T}_\infty$ of graph limits of threshold graphs and each of the following sets.*

- (i) *The set $\mathcal{P}_\mathsf{s}$ of probability distributions on $[0,1]$ symmetric with respect to $\dagger$.*
- (ii) *The set $\mathcal{C}_\mathcal{T}$ of symmetric increasing closed sets $S \subseteq [0,1]^2$ that contain the upper and right edges $[0,1] \times \{1\} \cup \{1\} \times [0,1]$.*
- (iii) *The set $\mathcal{O}_\mathcal{T}$ of symmetric increasing open sets $S \subseteq (0,1)^2$.*
- (iv) *The set $\mathcal{W}_\mathcal{T}$ of symmetric increasing 0–1 valued functions $W : [0,1]^2 \to \{0,1\}$ modulo a.e. equality.*

More precisely, there are commuting bijections between these sets given by the following mappings and their compositions:

$$
\begin{aligned}
\iota_{\mathcal{TP}} &: \mathcal{T}_\infty \to \mathcal{P}_{\mathsf{s}}, & \iota_{\mathcal{TP}}(\Gamma) &:= \nu(\Gamma);\\
\iota_{\mathcal{PC}} &: \mathcal{P}_{\mathsf{s}} \to \mathcal{C}_{\mathcal{T}}, & \iota_{\mathcal{PC}}(\mu) &:= S_\mu;\\
\iota_{\mathcal{CO}} &: \mathcal{C}_{\mathcal{T}} \to \mathcal{O}_{\mathcal{T}}, & \iota_{\mathcal{CO}}(S) &:= S^\circ;\\
\iota_{\mathcal{CW}} &: \mathcal{C}_{\mathcal{T}} \to \mathcal{W}_{\mathcal{T}}, & \iota_{\mathcal{CW}}(S) &:= \mathbf{1}_S;\\
\iota_{\mathcal{OW}} &: \mathcal{O}_{\mathcal{T}} \to \mathcal{W}_{\mathcal{T}}, & \iota_{\mathcal{OW}}(S) &:= \mathbf{1}_S;\\
\iota_{\mathcal{WT}} &: \mathcal{W}_{\mathcal{T}} \to \mathcal{T}_\infty, & \iota_{\mathcal{WT}}(W) &:= \Gamma_W.
\end{aligned}
\tag{5.12}
$$

In particular, a probability distribution $\mu \in \mathcal{P}_{\mathsf{s}}$ *corresponds to* $\Gamma_\mu \in \mathcal{T}_\infty$ *and to* $S_\mu \in \mathcal{C}_{\mathcal{T}}$, $S^\circ_\mu \in \mathcal{O}_{\mathcal{T}}$, *and* $W_\mu \in \mathcal{W}_{\mathcal{T}}$. *Conversely,* $\Gamma \in \mathcal{T}_\infty$ *corresponds to* $\nu(\Gamma) \in \mathcal{P}_{\mathsf{s}}$. *Thus, the mappings* $\Gamma \mapsto \nu(\Gamma)$ *and* $\mu \mapsto \Gamma_\mu$ *are the inverses of each other.*

$$
\begin{tikzcd}
\mathcal{T}_\infty \arrow[r, "\iota_{\mathcal{TP}}"] & \mathcal{P}_{\mathsf{s}} \arrow[r, "\iota_{\mathcal{PC}}"] & \mathcal{C}_{\mathcal{T}} \arrow[dl, "\iota_{\mathcal{CW}}"'] \arrow[d, "\iota_{\mathcal{CO}}"] \\
 & \mathcal{W}_{\mathcal{T}} \arrow[ul, "\iota_{\mathcal{WT}}"'] & \mathcal{O}_{\mathcal{T}} \arrow[l, "\iota_{\mathcal{OW}}"]
\end{tikzcd}
$$

Moreover, these bijections are homeomorphisms, with any of the following topologies or metrics: the standard (weak) topology on $\mathcal{P}_{\mathsf{s}} \subset \mathcal{P}$; *the Hausdorff metric, or the Fell topology, or the measure distance on* $\mathcal{C}_{\mathcal{T}}$; *the measure distance on* $\mathcal{O}_{\mathcal{T}}$; *the* L^1*-distance or the cut-distance on the set* $\mathcal{W}_{\mathcal{T}}$. *These homeomorphic topological spaces are compact metric spaces.*

Proof. The mappings $\iota_{\mathcal{PC}}$, $\iota_{\mathcal{CO}}$, $\iota_{\mathcal{CW}}$, $\iota_{\mathcal{OW}}$ are restrictions of the corresponding mappings in Theorem 5.1, and it follows from Theorem 5.1 and the definitions that these mappings are bijections and homeomorphisms for the given topologies. The spaces are closed subspaces of the corresponding spaces in Theorem 5.1, since † is continuous on these spaces, and thus compact metric spaces.

The rest is as in the proof of Theorem 5.1, and we omit some details. It follows from Theorem 4.3 that the composition $\iota_{\mathcal{WB}}\iota_{\mathcal{CW}}\iota_{\mathcal{PC}} : \mu \mapsto \Gamma(W_\mu) = \Gamma_\mu$ is a bijection of $\mathcal{P}_{\mathsf{s}}$ onto a subset $\mathcal{T}'$ of $\mathcal{T}''_{\infty,\infty}$, with $\iota_{\mathcal{TP}}$ as its inverse. It follows that these mappings too are homeomorphisms, and that the L^1-distance and cut-distance are equivalent on $\mathcal{W}_{\mathcal{T}}$.

To see that $\mathcal{T}' = \mathcal{T}_\infty$, we also follow the proof of Theorem 5.1. A minor complication is that if $G \in \mathcal{T}$ is a threshold graph, and we order the vertices with increasing degrees, then W_G is not increasing, because $W_G(x,x) = 0$ for all x since we consider loopless graphs only. However, we can define $W^*(G)$ by changing W_G to be 1 on some squares on the diagonal so that $W^*(G)$ is increasing and thus $W^*(G) \in \mathcal{W}_{\mathcal{T}}$, and the error $\|W_G - W^*(G)\|_{L^1} \le 1/v(G)$. If we define $\pi^*(G) := \Gamma(W^*(G)) \in \mathcal{T}'$, we see that if (G_n) is a sequence of threshold graphs with $v(G_n) \to \infty$, then for every graph F, by a simple estimate, see e.g. [22, Lemma 4.1],

$$|t(F, \pi^*(G_n)) - t(F, G_n)| \le e(F)\|W(G_n) - W^*(G_n)\|_{L^1} \le e(F)/v(G_n) \to 0. \tag{5.13}$$

It follows that $G_n \to \Gamma$ in $\overline{\mathcal{U}}$ if and only if $\pi^*(G_n) \to \Gamma$ in $\mathcal{U}_\infty$. If $\Gamma \in \mathcal{T}_\infty$, then there exists such a sequence $G_n \to \Gamma$, and thus $\pi^*(G_n) \to \Gamma$ in $\mathcal{U}_\infty$, and since $\pi^*(G_n) \in \mathcal{T}'$ and $\mathcal{T}'$ is compact, we find $\Gamma \in \mathcal{T}'$.

The converse follows in the same way. If $\Gamma \in \mathcal{T}'$, then $\Gamma = \iota_{\mathcal{W}\mathcal{T}}(W)$ for some function $W \in \mathcal{W}_{\mathcal{T}}$. The approximating step functions W_n constructed in the proof of Theorem 5.1 are symmetric, and if we let W_n^* by the modification that vanishes on all diagonal squares, $W_n^* = W_{G_n}$ for some threshold graph G_n, and for every graph F,

$$t(F, G_n) = t(F, W_n^*) = t(F, W_n) + o(1) = t(F, W) + o(1).$$

Hence, $G_n \to \Gamma_W = \Gamma$ in $\overline{\mathcal{U}}$, and thus $\Gamma \in \mathcal{T}_\infty$. Consequently, $\mathcal{T}' = \mathcal{T}_\infty$. □

Corollary 5.4. *The degree distribution yields a homeomorphism $\Gamma \mapsto \nu(\Gamma)$ of $\mathcal{T}_\infty$ onto the closed subspace $\mathcal{P}_{\mathsf{s}}$ of $\mathcal{P}$.*

Remark 5.3. The fact that a graph limit Γ can be represented by a function $W \in \mathcal{W}_{\mathcal{T}}$ if and only if $t_{\mathrm{ind}}(P_4, \Gamma) = t_{\mathrm{ind}}(C_4, \Gamma) = t_{\mathrm{ind}}(2K_2, \Gamma) = 0$, which by Theorem 3.1 is equivalent to the bijection $\mathcal{T}_\infty \leftrightarrow \mathcal{W}_{\mathcal{T}}$ in Theorem 5.3, is also proved by Lovász and Szegedy [23].

We have described the possible limits of sequences of threshold graphs; this makes it easy to see when such sequences converge.

Theorem 5.5. *Let G_n be a sequence of threshold graphs such that $v(G_n) \to \infty$. Then G_n converges in $\overline{\mathcal{U}}$ as $n \to \infty$, if and only if the degree distributions $\nu(G_n)$ converge to some distribution μ. In this case, $\mu \in \mathcal{P}_{\mathsf{s}}$ and $G_n \to \Gamma_\mu$.*

Proof. As in the proof of Theorem 5.3, $G_n \to \Gamma$ if and only if $\pi^*(G_n) \to \Gamma$ in $\mathcal{T}' = \mathcal{T}_\infty$, which by Theorem 5.3 holds if and only if $\nu(\pi^*(G_n)) \to \nu(\Gamma)$. By Theorem 4.3, $\nu(\pi^*(G_n))$ equals the distribution of $\int_0^1 W^*(G_n)(U, y)\,\mathrm{d}y$, but this random variable differs by at most $1/v(G_n) = o(1)$ from the random variable $\int_0^1 W_{G_n}(U, y)\,\mathrm{d}y$, which has degree distribution $\nu(G_n)$. The result follows. □

Theorem 5.6. *Let G_n be a sequence of bipartite threshold graphs such that $v_1(G_n), v_2(G_n) \to \infty$. Then G_n converges in $\overline{\mathcal{B}}$ as $n \to \infty$, if and only if the degree distributions $\nu_1(G_n)$ converge to some distribution μ. In this case, $\nu_2(G_n) \to \mu^\dagger$ and $G_n \to \Gamma''_\mu$.*

Proof. $G_n \to \Gamma$ if and only if $\pi(G_n) \to \Gamma$ in $\widetilde{\mathcal{B}} = \mathcal{B}_{\infty\infty}$, which by Theorem 5.1 holds if and only if $\nu_1(\pi(G_n)) \to \nu_1(\Gamma)$. It follows from Theorem 4.3 that $\nu_1(\pi(G_n)) = \nu_1(G_n)$, and the result follows from Theorem 5.1. □

Remark 5.4. A threshold graph limit Γ is, by Theorem 5.3, determined by its degree distribution and the fact that it is a threshold graph limit. By Theorem 3.2 and Lemma 4.1, Γ is thus determined by $t(F,\Gamma)$ for F in the set $\{P_4, C_4, 2K_2, K_{1,1}, K_{1,2}, \dots\}$. Lovász and Szegedy [23] have shown that in some special cases, a finite set of F is enough; for example, the limit defined by the function $W(x,y) = \mathbf{1}[x+y \geq 1]$ (see Example 1.3 and Figure 4) is the unique graph limit with $t(P_4,\Gamma) = t(C_4,\Gamma) = t(2K_2,\Gamma) = 0$, $t(K_2,\Gamma) = 1/2$, $t(P_3,\Gamma) = 1/3$.

6. Random threshold graphs

We consider several ways to define random threshold graphs. We will only consider constructions with a fixed number n of vertices; in fact, we take the vertex set to be $[n] = \{1,\dots,n\}$, where $n \geq 1$ is a given parameter. By a *random threshold graph* we thus mean a random element of $\mathcal{T}_n := \{G \in \mathcal{T} : V(G) = [n]\}$ for some n; we do not imply any particular construction or distribution unless otherwise stated. (We can regard these graphs as either labeled or unlabeled.)

This section treats four classes of examples: a canonical example based on increasing sets, random weights examples, random attachment examples and uniform random threshold graphs.

6.1. Increasing set.

For any symmetric increasing $S \subseteq [0,1]^2$, we let $W = \mathbf{1}_S$ and define $T_{n;S} := G(n,W)$ as in Section 3. In other words, we take i.i.d. random variables $U_1,\dots,U_n \sim U(0,1)$ and draw an edge ij if $(U_i,U_j) \in S$.

As said in Section 3, $G(n,W) \xrightarrow{\text{a.s.}} \Gamma_W$, which in this case means that $T_{n;S} \xrightarrow{\text{a.s.}} \Gamma(\mathbf{1}_S) \in \mathcal{T}_\infty$. We denote $\Gamma(\mathbf{1}_S)$ by Γ_S and have thus the following result, using also Theorem 4.3.

Theorem 6.1. *As $n \to \infty$, $T_{n;S} \xrightarrow{\text{a.s.}} \Gamma_S$. In particular, the degree distribution $\nu(T_{n;S}) \xrightarrow{\text{a.s.}} \nu(\Gamma_S)$, which equals the distribution of*

$$\varphi_S(U) := |\{y : (U,y) \in S\}| = \mathbb{P}\big((U,U') \in S \mid U\big), \tag{6.1}$$

with $U, U' \sim U(0,1)$ independent. □

By Theorem 5.3, this construction gives a canonical representation of the limit objects in $\mathcal{T}_\infty$, and we may restrict ourselves to closed or open sets as in Theorem 5.3(ii)(iii) to get a unique representation. We can obtain any desired degree distribution $\mu \in \mathcal{P}_\mathsf{s}$ for the limit by choosing $S = S_\mu$. This construction further gives a canonical representation of random threshold graphs for finite n, provided we make two natural additional assumptions.

Theorem 6.2. *Suppose that $(G_n)_1^\infty$ is a sequence of random threshold graphs with $V(G_n) = [n]$ such that the distribution of each G_n is invariant under permutations of $[n]$ and that the restriction (induced subgraph) of G_{n+1} to $[n]$ has the same distribution as G_n, for every $n \geq 1$. If further $\nu(G_n) \xrightarrow{\text{p}} \mu$ as $n \to \infty$, for some $\mu \in \mathcal{P}$, then $\mu \in \mathcal{P}_\mathsf{s}$ and, for every n, $G_n \overset{\text{d}}{=} T_{n;S_\mu}$.*

Proof. It follows from Theorem 5.5 that $G_n \xrightarrow{\text{p}} \Gamma_\mu$. (To apply Theorem 5.5 to convergence in probability, we can use the standard trick of considering subsequences that converge a.e., since every subsequence has such a subsubsequence [19, Lemma 4.2].)

If we represent a graph by its edge indicators, the random graph G_n can be regarded as a family of 0–1-valued random variables indexed by pairs (i, j), $1 \le i < j \le n$. By assumption, these families for different n are consistent, so by the Kolmogorov extension theorem [19, Theorem 6.16], they can be defined for all n together, which means that there exists a random infinite graph G_∞ with vertex set $\mathbb{N}$ whose restriction to $[n]$ coincides (in distribution) with G_n. Moreover, since each G_n is invariant under permutations of the vertices, so is G_∞, i.e., G_∞ is exchangeable. By Aldous and Hoover [1], see also [20] and [10], every exchangeable random infinite graph can be obtained as a mixture of $G(\infty, W)$; in other words, as $G(\infty, W)$ for some random function $W \in \mathcal{W}_{\mathsf{s}}$. In this case, the subgraphs G_n converge in probability to the corresponding random Γ_W, see Diaconis and Janson [10]. Since we have shown that G_n converge to a deterministic graph limit Γ_μ, we can take W deterministic so it follows that $G_\infty \overset{\text{d}}{=} G(\infty, W)$ for some $W \in \mathcal{W}_{\mathsf{s}}$; moreover, $\Gamma_\mu = \Gamma_W$, and thus we can by Theorem 5.3 choose $W = W_\mu$. (Recall that in general, W is not unique.) Consequently,

$$G_n \overset{\text{d}}{=} G(n, W_\mu) = T_{n;S_\mu}. \qquad \square$$

6.2. **Random weights.** Definition (1.1) suggests immediately the construction (1.6):

Let $X_1, X_2, \dots,$ be i.i.d. copies of a random variable X, let $t \in \mathbb{R}$, and let $T_{n;X,t}$ be the threshold graph with vertex set $[n]$ and edges ij for all pairs ij such that $X_i + X_j > t$. (We can without loss of generality let $t = 0$, by replacing X by $X - t/2$.)

Examples 1.2 and 1.3 are in this mode.

Let $F(x) := \mathbb{P}(X \le x)$ be the distribution function of X, and let F^{-1} be its right-continuous inverse defined by

$$F^{-1}(u) := \sup\{x \in \mathbb{R} : F(x) \le u\}. \tag{6.2}$$

(Cf. (5.1), where we consider distributions on $[0, 1]$ only.) Thus $-\infty < F^{-1}(u) < \infty$ if $0 < u < 1$, while $F^{-1}(1) = \infty$. It is well-known that the random variables X_i can be constructed as $F^{-1}(U_i)$ with U_i independent uniformly distributed random variables on $(0, 1)$, which leads to the following theorem, showing that this construction is equivalent to the one in Subsection 6.1 for a suitable set S. Parts of this theorem were found earlier by Masuda, Konno and co-authors [21, 26].

Theorem 6.3. *Let S be the symmetric increasing set*

$$S := \{(x, y) \in (0, 1]^2 : F^{-1}(x) + F^{-1}(y) > t\}. \tag{6.3}$$

Then $T_{n;X,t} \overset{\text{d}}{=} T_{n;S}$ for every n.

Furthermore, as $n \to \infty$, the degree distribution $\nu(T_{n;X,t}) \xrightarrow{\text{a.s.}} \mu$ and thus $T_{n;X,t} \xrightarrow{\text{a.s.}} \Gamma_\mu$, where $\mu \in \mathcal{P}_{\mathsf{s}}$ is the distribution of the random variable $1 - F(t - X)$, i.e.

$$\mu[0, s] = \mathbb{P}\big(1 - F(t - X) \le s\big), \qquad s \in [0, 1]. \tag{6.4}$$

Proof. Taking $X_i = F^{-1}(U_i)$, we see that

$$\text{there is an edge } ij \iff F^{-1}(U_i) + F^{-1}(U_j) > t \iff (U_i, U_j) \in S,$$

which shows that $T_{n;X,t} = T_{n;S}$.

The remaining assertions now follow from Theorem 6.1 together with the calculation, with $U, U' \sim U(0, 1)$ independent and $X = F^{-1}(U)$, $X' = F^{-1}(U')$,

$$\begin{aligned}\varphi_S(U) &= \mathbb{P}\big((U, U') \in S \mid U\big) = \mathbb{P}\big(F^{-1}(U) + F^{-1}(U') > t \mid U\big) \\ &= \mathbb{P}\big(X + X' > t \mid X\big) = \mathbb{P}\big(X' > t - X \mid X\big) = 1 - F(t - X).\end{aligned}$$

□

The set S defined in (6.3) is in general neither open nor closed; the corresponding open set is

$$S^\circ = \big\{(x, y) \in (0, 1)^2 : F^{-1}(x-) + F^{-1}(y-) > t\big\},$$

and the corresponding closed set S_μ in Theorem 5.3 can be found as $\tilde{S^\circ}$ from (5.8). If we assume for simplicity that the distribution of X is continuous, then, as is easily verified,

$$S_\mu = \big\{(x, y) \in [0, 1]^2 : F^{-1}(x) + F^{-1}(y) \ge t\big\},$$

where we define $F^{-1}(1) = \infty$ (and interpret $\infty + (-\infty) = \infty$ in case $F^{-1}(0) = -\infty$). We can use these sets instead of S in (6.3) since they differ by null sets only.

6.3. **Random addition of vertices.** Preferential attachment graphs are a rich topic of research in modern graph theory. See the monograph [24], along with the survey [30]. The versions in this section are natural because of (1.2) and the construction (1.7).

Let $T_{n,p}$ be the random threshold graph with n vertices obtained by adding vertices one by one with the new vertices chosen as isolated or dominating at random, independently of each other and with a given probability $p \in [0, 1]$ of being dominating. (Starting with a single vertex, there are thus $n - 1$ vertex additions.)

The vertices are not equivalent (for example, note that the edges $1i$, $i \ne 1$, appear independently, but not the edges ni, $i \ne n$), so we also define the random threshold graph $\widehat{T}_{n,p}$ obtained by a random permutation of the vertices in $T_{n,p}$. (When considering unlabeled graphs, there is no difference between $T_{n,p}$ and $\widehat{T}_{n,p}$.)

Remark 6.1. We may, as stated in (1.7), use different probabilities p_i for different vertices. We leave it to the reader to explore this case, for example with $p_i = f(i/n)$ for some given continuous function $f : [0,1] \to [0,1]$.

Theorem 6.4. *The degree distribution $\nu(T_{n,p})$ converges a.s. as $n \to \infty$ to a distribution μ_p that, for $0 < p < 1$, has constant density $(1-p)/p$ on $(0,p)$ and $p/(1-p)$ on $(p,1)$; μ_0 is a point mass at 0 and μ_1 is a point mass at 1. In particular, $\mu_{1/2}$ is the uniform distribution on $[0,1]$.*

Consequently, $T_{n,p} \xrightarrow{\text{a.s.}} \Gamma_{\mu_p} \in \mathcal{T}_\infty$.

Proof. Let $Z_n(t)$ be the number of vertices in $\{1,\dots,\lfloor nt\rfloor\}$ that are added as dominating. It follows from the law of large numbers that $n^{-1}Z_n(t) \xrightarrow{\text{a.s.}} pt$, uniformly on $[0,1]$, and we assume this in the sequel of the proof.

If vertex k was added as isolated, it has degree $Z_n(1) - Z_n(k/n)$, since its neighbours are the vertices that later are added as dominating. Similarly, if vertex k was added as dominating, it has degree $k-1+Z_n(1)-Z_n(k/n)$. Consequently, if μ_n is the (normalized) degree distribution of $T_{n,p}$, and ϕ is any continuous function on $[0,1]$, then

$$
\begin{aligned}
\int_0^1 \phi(t)\,\mathrm{d}\mu_n(t) &= \frac{1}{n}\sum_{k=1}^n \phi(d(k)/n) \\
&= \frac{1}{n}\sum_{k=1}^n \Bigl(\phi\bigl(n^{-1}Z_n(1) - n^{-1}Z_n(k/n)\bigr)\mathbf{1}[\Delta Z_n(k/n)=0] \\
&\qquad\qquad + \phi\bigl(n^{-1}Z_n(1) - n^{-1}Z_n(k/n) + (k-1)/n\bigr)\mathbf{1}[\Delta Z_n(k/n)=1]\Bigr).
\end{aligned}
$$

Since $n^{-1}Z_n(t) \to pt$ uniformly, and ϕ is uniformly continuous, it follows that, as $n \to \infty$,

$$
\begin{aligned}
\int_0^1 \phi(t)\,\mathrm{d}\mu_n(t) &= \frac{1}{n}\sum_{k=1}^n \Bigl(\phi\bigl(p(1-k/n)\bigr)\mathbf{1}[\Delta Z_n(k/n)=0] \\
&\qquad\qquad + \phi\bigl(p(1-k/n)+k/n\bigr)\mathbf{1}[\Delta Z_n(k/n)=1]\Bigr) + o(1) \\
&= \int_0^1 \phi\bigl(p(1-t)\bigr)\,\mathrm{d}\bigl(n^{-1}\lfloor nt\rfloor - n^{-1}Z_n(t)\bigr) \\
&\qquad + \int_0^1 \phi\bigl(p(1-t)+t\bigr)\,\mathrm{d}\bigl(n^{-1}Z_n(t)\bigr) + o(1).
\end{aligned}
$$

Since the convergence $n^{-1}Z_n(t) \to pt$ implies (weak) convergence of the corresponding measures, we finally obtain, as $n \to \infty$,

$$\begin{aligned}\int_0^1 \phi(t)\,\mathrm{d}\mu_n(t) \to \int_0^1 \phi\bigl(p(1-t)\bigr)(1-p)\,\mathrm{d}t + \int_0^1 \phi\bigl(p(1-t)+t\bigr)p\,\mathrm{d}t\\ = \frac{1-p}{p}\int_0^p \phi(x)\,\mathrm{d}x + \frac{p}{1-p}\int_p^1 \phi(x)\,\mathrm{d}x\\ = \int_0^1 \phi(x)\,\mathrm{d}\mu_p(x),\end{aligned}$$

with obvious modifications if $p = 0$ or $p = 1$. □

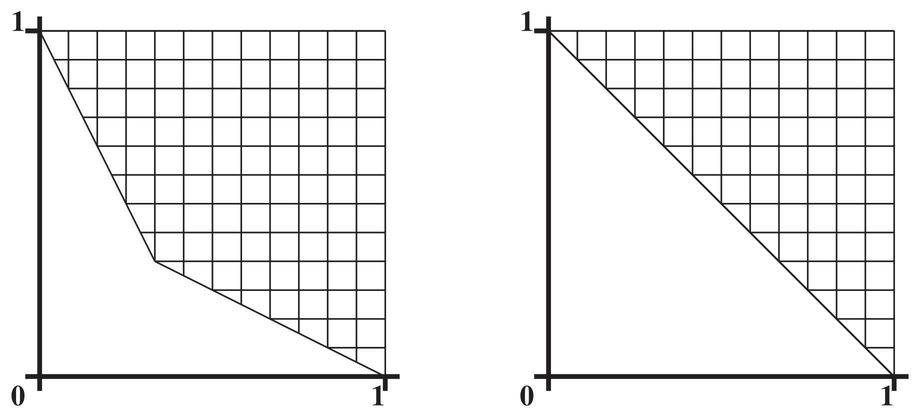


FIGURE 8. Two examples of the sets S_p; the one on the right shows the special case where $p = 0.5$.

Let $S_p := S_{\mu_p}$ be the corresponding subset of $[0,1]^2$. If $0 < p < 1$, μ_p has the distribution function

$$F_{\mu_p}(x) = \begin{cases} \frac{1-p}{p}x, & 0 \le x \le p,\\ 1 - \frac{p}{1-p}(1-x), & p \le x \le 1, \end{cases} \tag{6.5}$$

and it follows from (5.2) that S_p is the quadrilateral with vertices $(0,1)$, $(1-p, 1-p)$, $(1,0)$ and $(1,1)$, see Figure 8. In the special case $p = 1/2$, μ_p is the uniform distribution on $[0,1]$, and S_p is the triangle $\{(x,y) \in [0,1]^2 : x+y \ge 1\}$ pictured in Figure 4 with vertices $(0,1)$, $(1,0)$ and $(1,1)$. Finally, S_0 consists of the upper and right edges only, and $S_1 = [0,1]^2$.

Removing any vertex from $T_{n,p}$ (and relabeling the remaining ones) yields $T_{n-1,p}$. It follows that the same property holds for $\widehat{T}_{n,p}$, so $\widehat{T}_{n,p}$ satisfies the assumptions of Theorem 6.2. Since $\widehat{T}_{n,p}$ has the same degree distribution as $T_{n,p}$, Theorems 6.2 and 6.4 show the following equality.

Corollary 6.5. *If* $0 \le p \le 1$ *and* $n \ge 1$, *then* $\widehat{T}_{n,p} \overset{\mathrm{d}}{=} T_{n;S_p}$.

Hence the random threshold graphs in this subsection are special cases of the general construction in Subsection 6.1. We can also construct them using random weights as in Subsection 6.2.

Corollary 6.6. *If* $0 \le p \le 1$ *and* $n \ge 1$, *then* $\widehat{T}_{n,p} \overset{\mathrm{d}}{=} T_{n;X,0}$, *where* X *has the density* $1-p$ *on* $(-1,0)$ *and* p *on* $(0,1)$.

Proof. A simple calculation shows that the set S given by (6.3) is the quadrilateral S_p. $\square$

We may transform X by a linear map; for example, we may equivalently take X with density $2(1-p)$ on $(0,1/2)$ and $2p$ on $(1/2,1)$, with the threshold $t=1$. In particular, $\widehat{T}_{n,1/2} \overset{\mathrm{d}}{=} T_{n;U,1}$, where $U \sim U(0,1)$ as in Example 1.3.

6.4. **Uniform random threshold graphs.** Let T_n be a random unlabeled threshold graph of order n with the uniform distribution studied in Section 2. Similarly, let T_n^L be a random labeled threshold graph of order n with the uniform distribution. Although T_n and T_n^L have different distributions, see Section 2, the next theorem shows that they have the same limit as $n \to \infty$.

Theorem 6.7. *The degree distributions* $\nu(T_n)$ *and* $\nu(T_n^L)$ *both converge in probability to the uniform distribution* λ *on* $[0,1]$. *Hence,* $T_n \xrightarrow{\mathrm{p}} \Gamma_\lambda$ *and* $T_n^L \xrightarrow{\mathrm{p}} \Gamma_\lambda$.

By Subsection 2.1, $T_n \overset{\mathrm{d}}{=} T_{n,1/2}$; hence the result for unlabeled graphs follows from Theorem 6.4.

Proof. We use Theorem 2.4; in fact, the proof works for random threshold graphs generated by Algorithm 2.5 for any i.i.d. random variables $B_2, B_3, \dots$ with finite mean, and any B_1. (In the case when B_2 is always a multiple of some $d > 1$, there is a trivial modification.) Let $\beta := \mathbb{E}\, B_2$.

The algorithm starts by choosing (random) block lengths $B_1, B_2, \dots$ until their sum is at least n, and then rejects them and restarts (Step 3) unless the sums is exactly n. It is simpler to ignore this check, so we consider the following modified algorithm: Take $B_1, B_2, \dots$ as above. Let $S_k := \sum_{j=1}^k B_j$ be their partial sums and let $\tau(n) := \min\{k : S_k \ge n\}$. Toss a coin to determine whether the first block is isolated or dominating, and construct a random threshold graph by adding $\tau(n)$ blocks of vertices with $B_1, \dots, B_{\tau(n)}$ elements, alternatingly isolated and dominant.

This gives a random graph $\widetilde{G}_n$ with $S_{\tau(n)}$ vertices, but conditioned on $S_{\tau(n)} = n$, we obtain the desired random threshold graph. (Cf. Theorem 2.4.) Since $\mathbb{P}(S_{\tau(n)} = n)$ converges to $1/\beta > 0$ by renewal theory, it suffices to prove that $\nu(\widetilde{G}_n) \xrightarrow{\mathrm{p}} \lambda$ as $n \to \infty$. In fact, we will show that $\nu(\widetilde{G}_n) \xrightarrow{\mathrm{a.s.}} \lambda$ if we first choose an infinite sequence $B_1, B_2, \dots$ and then let $n \to \infty$.

Let $S^O_m := \sum_{2k+1\le m} B_{2k+1}$ and $S^E_m := \sum_{2k\le m} B_{2k}$ be the partial sums of the odd and even terms. By the law of large numbers, a.s. $S_n/n \to \beta$ and $S^O_n/n \to \frac12\beta$, $S^E_n/n \to \frac12\beta$. We now consider a fixed sequence $(B_j)_1^\infty$ such that these limits hold. Since $S_{\tau(n)-1} < n \le S_{\tau(n)}$, it follows, as is well-known, that $n/\tau(n) \to \beta$, so $\tau(n) = n/\beta + o(n)$.

Suppose for definiteness that the first block is chosen to be isolated; then every odd block is isolated and every even block is dominating. (In the opposite case, interchange even and odd below.) If $i \in (S_{2k}, S_{2k+1}]$, then i belongs to block $2k+1$, so i is added as isolated, and the neighbors of i will be only the vertices added after i as dominating, i.e. $\bigcup_{k<\ell\le\tau(n)/2}(S_{2\ell-1}, S_{2\ell}]$, and

$$d(i) = \sum_{2k<2\ell\le\tau(n)} B_{2\ell} = S^E_{\tau(n)} - S^E_{\tau(i)}.$$

If instead $i \in (S_{2k-1}, S_{2k}]$, then i is also joined to all vertices up to S_{2k}, and thus

$$d(i) = \sum_{2\ell\le\tau(n)} B_{2\ell} + \sum_{2\ell+1\le\tau(i)} B_{2\ell+1} = S^E_{\tau(n)} + S^O_{\tau(i)}.$$

Hence, if i is in an odd block,

$$\frac{d(i)}{n} = \frac1n\Bigl(\tau(n)\frac{\beta}{2} - \tau(i)\frac{\beta}{2} + o(n)\Bigr) = \frac{n-i+o(n)}{2n} = \frac12 - \frac{i}{2n} + o(1),$$

and if i is in an even block, similarly,

$$\frac{d(i)}{n} = \frac12 + \frac{i}{2n} + o(1).$$

Now fix $t \in (0, 1/2)$ and let $\varepsilon > 0$. Then the following holds if n is large enough: If i is in an even block, then $d(i)/n \ge 1/2 + o(1) > t$. If i is in an odd block and $i \le i_1 := (1-2t-2\varepsilon)n$, then $d(i)/n = \frac12(n-i)/n + o(1) \ge t + \varepsilon + o(1) > t$. If i is in an odd block and $i \ge i_2 := (1-2t+2\varepsilon)n$, then $d(i)/n = \frac12(n-i)/n + o(1) \le t - \varepsilon + o(1) < t$. Consequently, for large n, $d(i)/n \le t$ only if i is in an odd block $(S_{2k}, S_{2k+1}]$, and in this case $2k+1 > \tau(i_1)$ is necessary and $2k+1 > \tau(i_2)$ is sufficient. Hence,

$$S^O_{\tau(n)} - S^O_{\tau(i_2)} \le |\{i : d(i)/n \le t\}| \le S^O_{\tau(n)} - S^O_{\tau(i_1)}.$$

Since $\nu(\widetilde G_n)[0,t] = \frac1n|\{i : d(i)/n \le t\}|$ and

$$\frac1n\bigl(S^O_{\tau(n)} - S^O_{\tau(i_j)}\bigr) = \frac{\beta(\tau(n)-\tau(i_j)) + o(n)}{2n} = \frac{n-i_j+o(n)}{2n} = t \pm \varepsilon + o(1),$$

it follows that

$$t - \varepsilon + o(1) \le \nu(\widetilde G_n)[0,t] \le t + \varepsilon + o(1).$$

Since ε is arbitrary, this shows that $\nu(\widetilde G_n)[0,t] \to t$, for every $t \in (0, \frac12)$. We clearly obtain the same result if the first block is dominating.

For $t \in (\frac12, 1)$ we can argue similarly, now analysing the dominant blocks. Alternatively, we may apply the result just obtained to the complement of

$\widetilde{G}_n$, which is obtained from the same B_j by switching the types of the blocks. This shows that $\nu(\widetilde{G}_n)[0,t] \to t$ for $t \in (\frac{1}{2}, 1)$ too.

Hence, $\nu(\widetilde{G}_n)[0,t] \to t$ for every $t \in (0,1)$ except possibly $\frac{1}{2}$, which shows that $\nu(\widetilde{G}_n) \to \lambda$. □

7. Vertex degrees in uniform random threshold graphs

We have seen in Theorem 6.7 that the normalized degree distributions $\nu(T_n)$ and $\nu(T_n^L)$ for uniform unlabeled and labeled random threshold graphs both converge to the uniform distribution on $[0,1]$. This is for weak convergence of distributions in $\mathcal{P}$, which is equivalent to averaging over degrees in intervals (an, bn); we here refine this by studying individual degrees.

Let $N_d(G)$ be the number of vertices of degree d in the graph G. Thus, D_G, the degree of a random vertex in G has distribution $\mathbb{P}(D_G = d) = N_d/v(G)$. (Recall that $\nu(G)$ is the distribution of $D_G/v(G)$, see Section 4.)

We will study the random variables $N_d(T_n)$ and $N_d(T_n^L)$ describing the numbers of vertices of a given degree d in a uniform random unlabeled or labeled threshold graph, and in particular their expectations $\mathbb{E}\, N_d(T_n)$ and $\mathbb{E}\, N_d(T_n^L)$; note that $\mathbb{E}\, N_d(T_n)/n$ and $\mathbb{E}\, N_d(T_n^L)/n$ are the probabilities that a given (or random) vertex in the random graph T_n or T_n^L has degree d. By symmetry under complementation,

$$N_d(T_n) \overset{\mathrm{d}}{=} N_{n-1-d}(T_n) \qquad \text{and} \qquad N_d(T_n^L) \overset{\mathrm{d}}{=} N_{n-1-d}(T_n^L).$$

Let us first look at N_0, the number of isolated vertices. (By symmetry, we have the same results for N_{n-1}, the number of dominating vertices). Note that, for every $n \geq 2$, $\mathbb{P}(N_0(T_n) = 0) = \mathbb{P}(N_0(T_n^L) = 0) = 1/2$ by symmetry.

Theorem 7.1. (i) *For any $n \geq 1$,*

$$\mathbb{P}\big(N_0(T_n) = j\big) = \begin{cases} 2^{-j-1}, & 0 \leq j \leq n-2, \\ 0, & j = n-1, \\ 2^{-n+1}, & j = n. \end{cases} \tag{7.1}$$

In other words, if $X \sim \mathrm{Ge}(1/2)$, then $N_0(T_n) \overset{\mathrm{d}}{=} X_n'$, where $X_n' := X_n$ if $x < n-1$ and $X_n' := n$ if $X_n \geq n-1$. Furthermore, $\mathbb{E}\, N_0(T_n) = 1$, and $N_0(T_n) \xrightarrow{\mathrm{d}} \mathrm{Ge}(1/2)$ as $n \to \infty$, with convergence of all moments.

(ii) $\mathbb{P}\big(N_0(T_n^L) = j\big) = t(n,j)/t(n)$, *where $t(n,j)$ is given by (2.7); in particular, if $0 \leq j \leq n-2$, then*

$$\mathbb{P}\big(N_0(T_n^L) = j\big) = \frac{1}{2j!}\frac{t(n-j)/(n-j)!}{t(n)/n!} = \frac{1}{2j!}(\log 2)^j\big(1 + O(\rho^{n-j})\big)$$

with $\rho = \log 2/(2\pi) \approx 0.11$. Hence, $N_0(T_n^L) \xrightarrow{\mathrm{d}} \mathrm{Po}(\log 2)$ as $n \to \infty$ with convergence of all moments; in particular, $\mathbb{E}\, N_0(T_n^L) \to \log 2$.

Proof. (i): A threshold graph has j isolated vertices if and only if the extended binary code $\alpha_1 \cdots \alpha_n$ in Section 2 ends with exactly j 0's. For a random unlabeled threshold graph T_n, the binary code $\alpha_2 \cdots \alpha_n$ is uniformly distributed, and thus (7.1) follows. The remaining assertions follow directly.

(ii): In the labeled case, the exact distribution is given by (2.7), and the asymptotics follow by (2.4). Uniform integrabilit of any power $N_0(T_n^L)^m$ follows by the same estimates, and thus moment convergence holds. □

For higher degrees, we begin with an exact result for the unlabeled case.

Theorem 7.2. *$\mathbb{E}\, N_d(T_n) = 1$ for every $d = 0, \dots, n-1$.*

Actually, this is the special case $p = 1/2$ of a more general theorem for the random threshold graph $T_{n,p}$ defined in Subsection 6.3: (Cf. Theorem 6.4, which is for weak convergence, but on the other hand yields an a.s. limit while we here study the expectations.)

Theorem 7.3. *Let $0 < p < 1$. If $q = 1 - p$ and $X \sim \mathrm{Bin}(n, p)$, then, for $0 \le d \le n-1$,*

$$\mathbb{E}\, N_d(T_{n,p}) = \frac{q}{p} + \Bigl(\frac{p}{q} - \frac{q}{p}\Bigr)\mathbb{P}(X \le d).$$

Proof. We use the definition in Subsection 6.3. (For the uniform case $p = 1/2$, this is Algorithm 2.1.) Let d_i be the degree of vertex i. Then, if $\alpha_1 \cdots \alpha_n$ is the extended binary code of the graph, we have

$$d_i = (i-1)\alpha_i + \sum_{j=i+1}^{n} \alpha_j.$$

Since the α_i are i.i.d. $\mathrm{Be}(p)$ for $i = 2, \dots, n$, the probability generating function of d_i is

$$\mathbb{E}\, x^{d_i} = \mathbb{E}\, x^{(i-1)\alpha_i} \prod_{j=i+1}^{n} \mathbb{E}\, x^{\alpha_j} = (px^{i-1} + q)(px+q)^{n-i}.$$

Consequently,

$$\begin{aligned}
\sum_d \mathbb{E}\, N_d(T_{n,p}) x^d = \sum_{i=1}^{n} \mathbb{E}\, x^{d_i} &= \sum_{i=1}^{n} px^{i-1}(px+q)^{n-i} + \sum_{i=1}^{n} q(px+q)^{n-i} \\
&= p\frac{x^n - (px+q)^n}{x - (px+q)} + q\frac{1-(px+q)^n}{1-(px+q)} \\
&= \frac{(q/p) + (p/q - q/p)(px+q)^n - (p/q)x^n}{1-x}.
\end{aligned}$$

In the special case $p = 1/2$, this is $(1-x^n)/(1-x) = \sum_{d=0}^{n-1} x^d$, which shows Theorem 7.2 by identifying coefficients. For general p, Theorem 7.3 follows in the same way. □

Recall that R_d denotes the number of preferential arrangements, or surjection numbers, given in (2.5).

Theorem 7.4. (i) *In the unlabeled case, for any sequence* $d = d(n)$ *with* $0 \le d \le n-1$, $N_d(T_n) \xrightarrow{\mathrm{d}} \mathrm{Ge}(1/2)$ *with convergence of all moments.*

(ii) *In the labeled case, let* X_d, $0 \le d \le \infty$, *have the modified Poisson distribution given by*

$$\mathbb{P}(X_d = \ell) = \begin{cases} \frac{\gamma_d}{\log 2} \mathbb{P}\bigl(\mathrm{Po}(\log 2) = \ell\bigr) = \gamma_d \frac{(\log 2)^{\ell-1}}{2\cdot \ell!}, & \ell \ge 1, \\ 1 - \frac{\gamma_d}{2 \log 2}, & \ell = 0, \end{cases}$$

where $\gamma_0 := \log 2$, $\gamma_d := 2R_d(\log 2)^{d+1}/d!$ *for* $d \ge 1$, *and* $\gamma_\infty := 1$. *Then, for every fixed* $d \ge 0$, $N_d(T_n^L) \overset{\mathrm{d}}{=} N_{n-1-d}(T_n^L) \xrightarrow{\mathrm{d}} X_d$, *and for every sequence* $d = d(n) \to \infty$ *with* $n-d \to \infty$, $N_d(T_n^L) \overset{\mathrm{d}}{=} N_{n-1-d}(T_n^L) \xrightarrow{\mathrm{d}} X_\infty$ *as* $n \to \infty$, *in both cases with convergence of all moments.*

In particular, $\mathbb{E}\, N_d(T_n^L) = \mathbb{E}\, N_{n-1-d}(T_n^L)$ *converges to* γ_d *for every fixed* d, *and to* $\gamma_\infty = 1$ *if* $d \to \infty$ *and* $n - d \to \infty$.

In the labeled case we thus have, in particular, $\mathbb{E}\, N_0(T_n^L) \to \log 2 \approx 0.69315$, $\mathbb{E}\, N_1(T_n^L) \to 2(\log 2)^2 \approx 0.96091$, $\mathbb{E}\, N_2(T_n^L) \to 3(\log 2)^3 \approx 0.99907$, $\mathbb{E}\, N_3(T_n^L) \to \frac{13}{3}(\log 2)^4 \approx 1.00028$. The values for degrees 0 and 1 (and symmetrically $n-1$ and $n-2$) are thus substantially smaller than 1, which is clearly seen in Figure 7. (We can regard this as an edge effect; the vertices with degrees close to 0 or $n-1$ are the ones added last in Algorithm 2.5. Figure 7 also shows an edge effect at the other side; there is a small bump for degrees arond $n/2$, which correspond to the vertices added very early in the algorithm; this bump vanishes asymptotically, as shown by Theorem 7.4; we believe that it has height of order $n^{-1/2}$ and width of order $n^{1/2}$, but we have not analyzed it in detail.)

Proof. The cases $d = 0$ and $d = n-1$ follow from Theorem 7.1. We may thus suppose $1 \le d \le n-2$. We use Algorithm 2.5. We know that vertices in each block have the same degree, while different blocks have different degrees; thus there is at most one block with degrees d.

Let $p_d(\ell)$ be the probability that there is such a block of length $\ell \ge 1$, and that this block is added as isolated. By symmetry, the probability that there is a dominating block of length ℓ with degrees d is p_{n-1-d} and thus

$$\mathbb{P}(N_d = \ell) = p_d(\ell) + p_{n-1-d}(\ell), \qquad \ell \ge 1. \tag{7.2}$$

If block j is an isolated block, then the degree of the vertices in it equals the number of vertices added as dominating after it, i.e., $B_{j+1} + B_{j+3} + \cdots + B_{j+2k-1}$, if the total number τ of blocks is $j+2k-1$ or $j+2k$. Consequently, there is an isolated block of length ℓ with vertices of degree d if and only if there exist $j \ge 1$ and $k \ge 1$ with

- $B_j = \ell$,
- block j is isolated,
- $\sum_{i=1}^{k} B_{j+2i-1} = d$,
- $\sum_{i=1}^{j+2k-1} B_i = n$ or $\sum_{i=1}^{j+2k} B_i = n$.

Recall that $B_1, B_2, \dots$ are independent and that $B_2, B_3, \dots$ have the same distribution while B_1 has a different one. (The distributions differ between the unlabeled and labeled cases.) Let

$$\hat{S}_n := \sum_{i=1}^{m} B_i \qquad \text{and} \qquad S_n := \sum_{i=1}^{m} B_{i+1}.$$

Further, let

$$\begin{aligned} \hat{u}(n) &= \sum_{m=0}^{\infty} \mathbb{P}(\hat{S}_m = n) = \mathbb{P}(B_\tau = n) = \mathbb{P}\Bigl(\sum_{i=1}^{\tau} B_i = n\Bigr), \\ u(n) &= \sum_{m=0}^{\infty} \mathbb{P}(S_m = n), \end{aligned}$$

and recall that $u(n), \hat{u}(n) \to 1/\mu := 1/\mathbb{E}\, B_2$ (exponentially fast) by standard renewal theory (for example by considering generating functions). For any $m \geq j + 2k - 1$,

$$\sum_{i=1}^{m} B_i - B_j - \sum_{i=1}^{k} B_{j+2i-1} \overset{\mathrm{d}}{=} \begin{cases} S_{m-1-k}, & j = 1, \\ \hat{S}_{m-1-k}, & j \geq 2, \end{cases}$$

and it follows that, since $B_1, B_2, \dots$ are independent and we condition on $\hat{S}_\tau = n$,

$$\begin{aligned} p_d(\ell) = \frac{1}{2\hat{u}(n)} \Bigl\{ &\sum_{k=1}^{\infty} \mathbb{P}(B_1 = \ell)\, \mathbb{P}(S_k = d) \\ &\qquad \cdot \bigl(\mathbb{P}(S_{k-1} = n - \ell - d) + \mathbb{P}(S_k = n - \ell - d)\bigr) \\ &+ \sum_{j=2}^{\infty} \sum_{k=1}^{\infty} \mathbb{P}(B_j = \ell)\, \mathbb{P}(S_k = d) \\ &\qquad \cdot \bigl(\mathbb{P}(\hat{S}_{j+k-2} = n - \ell - d) + \mathbb{P}(\hat{S}_{j+k-1} = n - \ell - d)\bigr) \Bigr\} \end{aligned} \tag{7.3}$$

In the double sum, $\mathbb{P}(B_j = \ell) = \mathbb{P}(B_2 = \ell)$ does not depend on j, so the sum is at most

$$\begin{aligned} \mathbb{P}(B_2 = \ell) \sum_k \mathbb{P}(S_k = d) 2\hat{u}(n - \ell - d) &= 2\, \mathbb{P}(B_2 = \ell) u(d) \hat{u}(n - \ell - d) \\ &= O(\mathbb{P}(B_2 = \ell)). \end{aligned}$$

Similarly, the first sum is $O(\mathbb{P}(B_1 = \ell)) = O(\mathbb{P}(B_2 = \ell))$, and it follows that $p_d(\ell) = O(\mathbb{P}(B_2 = \ell))$ and thus, by (7.2),

$$\mathbb{P}(N_d = \ell) = O(\mathbb{P}(B_2 = \ell)), \tag{7.4}$$

uniformly in n, d and ℓ. This shows tightness, so convergence $\mathbb{P}(N_d = \ell) \to \mathbb{P}(X = \ell)$ for some non-negative integer valued random variable X and each fixed $\ell \geq 1$ implies convergence in distribution (i.e., for $\ell = 0$ too). Further, since all moments of B_2 are finite, (7.4) implies that all moments $\mathbb{E}\, N_d^m$ are bounded, uniformly in d and n; hence convergence in distribution implies that all moments converge too. In the rest of the proof we thus let $\ell \geq 1$ be fixed.

If $d \leq n/2$ it is easy to see that $\mathbb{P}(S_{k-1} = n-\ell-d) + \mathbb{P}(S_k = n-\ell-d) = O\bigl((n-\ell-d)^{-1/2}\bigr) = O(n^{-1/2})$, uniformly in k, so the first sum in (7.3) is $O\bigl(n^{-1/2}u(d)\bigr) = O\bigl(n^{-1/2}\bigr)$. If $d > n/2$, we similarly have $\mathbb{P}(S_k = d) = O\bigl(d^{-1/2}\bigr) = O\bigl(n^{-1/2}\bigr)$ and thus the sum is $O\bigl(n^{-1/2}u(n-\ell-d)\bigr) = O\bigl(n^{-1/2}\bigr)$. Hence (7.3) yields

$$p_d(\ell) = O\bigl(n^{-1/2}\bigr) + \frac{\mathbb{P}(B_2 = \ell)}{2\hat{u}(n)} \sum_{k=1}^{\infty} \mathbb{P}(S_k = d) \cdot \Bigl(\sum_{i=k} \mathbb{P}(\hat{S}_i = n-\ell-d) + \sum_{i=k+1} \mathbb{P}(\hat{S}_i = n-\ell-d)\Bigr). \tag{7.5}$$

The term with $i = k$ can be taken twice, just as the ones with $i > k$, since $\sum_k \mathbb{P}(S_k = d)\,\mathbb{P}(\hat{S}_k = n-\ell-d) = O\bigl(n^{-1/2}\bigr)$ by the same argument as for the first sum in (7.3). Further, for $i \geq k$, $\hat{S}_i - \hat{S}_k \overset{\mathrm{d}}{=} S_{i-k}$ and is independent of $\hat{S}_k$; thus

$$\mathbb{P}(\hat{S}_i = n-\ell-d) = \mathbb{P}(S_{i-k} = n-\ell-d-\hat{S}_k) = \mathbb{E}\,\mathbb{P}\bigl(S_{i-k} = n-\ell-d-\hat{S}_k \mid \hat{S}_k\bigr)$$

and $\sum_{i=k}^{\infty} \mathbb{P}(\hat{S}_i = n-\ell-d) = \mathbb{E}\,u(n-\ell-d-\hat{S}_k)$. Hence, (7.5) yields

$$p_d(\ell) = \frac{\mathbb{P}(B_2 = \ell)}{\hat{u}(n)} \sum_{k=1}^{\infty} \mathbb{P}(S_k = d)\,\mathbb{E}\,u(n-\ell-d-\hat{S}_k) + O\bigl(n^{-1/2}\bigr). \tag{7.6}$$

If d is fixed, then $\mathbb{E}\,u(n-\ell-d-\hat{S}_k) \to \mu^{-1}$ by dominated convergence as $n \to \infty$ for each k, and thus (7.6) yields, by dominated convergence again,

$$p_d(\ell) \to \mathbb{P}(B_2 = \ell) \sum_{k=1}^{\infty} \mathbb{P}(S_k = d) = u(d)\,\mathbb{P}(B_2 = \ell). \tag{7.7}$$

If $d \to \infty$, we use the fact that $u(m) - \mathbf{1}[m \geq 0]\mu^{-1}$ is summable over $\mathbb{Z}$ to see that

$$\mathbb{E}\,u(n-\ell-d-\hat{S}_k) - \mu^{-1}\,\mathbb{P}(n-\ell-d-\hat{S}_k \geq 0) = O\bigl(\max_m \mathbb{P}(\hat{S}_k = m)\bigr),$$

which tends to 0 as $k \to \infty$; on the other hand, $\mathbb{P}(S_k = d) \to 0$ for every fixed k. It follows that (7.6) yields

$$p_d(\ell) = \mathbb{P}(B_2 = \ell) \sum_{k=1}^{\infty} \mathbb{P}(S_k = d)\,\mathbb{P}(\hat{S}_k \leq n-\ell-d) + o(1).$$

If $\tau_d := \min\{k : S_k \geq d\}$, and $\hat{S}'_k$ denotes a copy of $\hat{S}_k$ independent of $\{S_j\}_1^{\infty}$, then

$$\sum_{k=1}^{\infty} \mathbb{P}(S_k = d)\,\mathbb{P}(\hat{S}_k \leq n-\ell-d) = u(d)\,\mathbb{P}\bigl(\hat{S}'_{\tau_d} \leq n-\ell-d \mid S_{\tau_d} = d\bigr).$$

It is easy to see that, with $\sigma^2 := \operatorname{Var}(B_2)$, as $d \to \infty$,

$$\bigl((\hat{S}'_{\tau_d} - d)/\sqrt{d} \mid S_{\tau_d} = d\bigr) = \bigl((\hat{S}'_{\tau_d} - S_{\tau_d})/\sqrt{d} \mid S_{\tau_d} = d\bigr) \xrightarrow{\mathrm{d}} N(0, 2\sigma^2/\mu),$$

cf. [14] (the extra conditioning on $S_{\tau_d} = d$ makes no difference). Hence, when $d \to \infty$,

$$p_d(\ell) = \mathbb{P}(B_2 = \ell)u(d)\Phi\bigl((n - \ell - 2d)/\sqrt{d}\bigr) + o(1).$$

(By (7.7), this holds for fixed d too.) We next observe that $\Phi\bigl((n - \ell - 2d)/\sqrt{d}\bigr) = \Phi\bigl((n - 2d)/\sqrt{n/2}\bigr) + o(1)$; this is easily seen by considering separately the three cases $d/n \to a \in [0, 1/2)$, $d/n \to a \in (1/2, 1]$, and $d/n \to 1/2$ and $(n - 2d)/\sqrt{n/2} \to b \in [-\infty, \infty]$ (the general case follows by considering suitable subsequences). Hence, we have when $d \to \infty$, recalling that then $u(d) \to \mu^{-1}$,

$$p_d(\ell) = \mu^{-1}\,\mathbb{P}(B_2 = \ell)\Phi\bigl((n - 2d)/\sqrt{n/2}\bigr) + o(1).$$

For fixed d, this implies that $p_{n-d-1}(\ell) \to 0$, and thus (7.2) and (7.7) yield

$$\mathbb{P}(N_d = \ell) = p_d(\ell) + p_{n-1-d}(\ell) = u(d)\,\mathbb{P}(B_2 = \ell) + o(1).$$

Similarly, if $d \to \infty$ and $n - d \to \infty$,

$$\begin{aligned}
\mathbb{P}(N_d = \ell) &= p_d(\ell) + p_{n-1-d}(\ell) \\
&= \mu^{-1}\,\mathbb{P}(B_2 = \ell)\bigl(\Phi\bigl((n - 2d)/\sqrt{n/2}\bigr) + \Phi\bigl((2d + 2 - n)/\sqrt{n/2}\bigr)\bigr) + o(1) \\
&= \mu^{-1}\,\mathbb{P}(B_2 = \ell) + o(1).
\end{aligned}$$

We have thus proven convergence as $n \to \infty$, with all moments, $N_d \xrightarrow{\mathrm{d}} X_d$ for fixed d and $N_d \xrightarrow{\mathrm{d}} X_\infty$ for $d = d(n) \to \infty$ with $n - d \to \infty$, where

$$\mathbb{P}(X_d = \ell) = u(d)\,\mathbb{P}(B_2 = \ell) = 2u(d)\,\mathbb{P}(B^* = \ell), \qquad \ell \geq 1, \tag{7.8}$$

$$\mathbb{P}(X_d = 0) = 1 - \mathbb{P}(X_d \geq 1) = 1 - u(d), \tag{7.9}$$

for $1 \leq d \leq \infty$, with $u(\infty) := \mu^{-1}$.

In the unlabeled case, $B_2 = (B^* \mid B^* \geq 1) \overset{\mathrm{d}}{=} B^* + 1$ with $B^* \sim \operatorname{Ge}(1/2)$. Consider a random infinite string $\alpha_1\alpha_2\cdots$ of i.i.d. $\operatorname{Be}(1/2)$ binary digits, and define a block as a string of $m \geq 0$ 0's followed by a single 1. Then B_{j+1}, $j \geq 1$, can be interpreted as the successive block lengths in $\alpha_1\alpha_2\cdots$, and thus $u(d)$ is the probability that some block ends at d, i.e., $u(d) = \mathbb{P}(\alpha_d = 1) = 1/2$, for every $d \geq 1$. It follows from (7.8)–(7.9) that $X_d \overset{\mathrm{d}}{=} B^* \sim \operatorname{Ge}(1/2)$ for every $d \geq 1$, and (i) follows.

In the labeled case, when $B^* \sim \mathrm{Po}(\log 2)$, we use generating functions:

$$\sum_{d=0}^{\infty} u(d)x^d = \sum_{k=0}^{\infty} \mathbb{E}\, x^{S_k} = \sum_{k=0}^{\infty} \bigl(\mathbb{E}\, x^{B_2}\bigr)^k = \frac{1}{1-\mathbb{E}\, x^{B_2}} = \frac{\mathbb{P}(B^* \geq 1)}{1-\mathbb{E}\, x^{B^*}}$$

$$= \frac{1/2}{1-e^{(x-1)\log 2}} = \frac{1}{2-e^{x\log 2}} = \sum_{d=0}^{\infty} \frac{R_d}{d!}(x\log 2)^d,$$

where we recognize the gererating function (2.6). Thus, $u(d) = R_d(\log 2)^d/d!$. (A direct combinatorial proof of this is also easy.)

We let, using $\mu := \mathbb{E}\, B_2 = \mathbb{E}\, B^*/\,\mathbb{P}(B^* \geq 1) = 2\log 2$,

$$\gamma_d := \mathbb{E}\, X_d = u(d)\,\mathbb{E}\, B_2 = \mu u(d) = 2\log 2 u(d) = 2R_d(\log 2)^{d+1}/d!$$

and note that $\gamma_d \to \gamma_\infty = 1$ as $d \to \infty$ since $u(d) \to \mu^{-1}$, or by the known asymptotics of R_d [12, (II.16)]. The description of X_d in the statement now follows from (7.8)–(7.9). $\square$

8. Random bipartite threshold graphs

The constructions and results in Section 6 have analogues for bipartite threshold graphs. The proofs are simple modifications of the ones above and are omitted.

8.1. **Increasing set.** For any increasing $S \subseteq [0,1]^2$, define $T_{n_1,n_2;S} := G(n_1,n_2,\mathbf{1}_S)$. In other words, take i.i.d. random variables $U_1',\dots,U_{n_1}'$, $U_1'',\dots,U_{n_2}'' \sim U(0,1)$ and draw an edge ij if $(U_i',U_j'') \in S$.

Theorem 8.1. *As $n_1,n_2 \to \infty$, $T_{n_1,n_2;S} \xrightarrow{\text{a.s.}} \Gamma_S''$. In particular, the degree distribution $\nu_1(T_{n;S}) \xrightarrow{\text{a.s.}} \nu_1(\Gamma_S'')$, which equals the distribution of $\varphi_S(U)$ defined by* (6.1). $\square$

As in Section 6, this gives a canonical representation of random bipartite threshold graphs under natural assumptions.

Theorem 8.2. *Suppose that $(G_{n_1,n_2})_{n_1,n_2\geq 1}$ are random bipartite threshold graphs with $V_1(G_{n_1,n_2}) = [n_1]$ and $V_2(G_{n_1,n_2}) = [n_2]$ such that the distribution of each G_{n_1,n_2} is invariant under permutations of V_1 and V_2 and that the restrictions (induced subgraphs) of G_{n_1+1,n_2} and G_{n_1,n_2+1} to $V(G)$ both have the same distribution as G_{n_1,n_2}, for every $n_1,n_2 \geq 1$. If further $\nu_1(G_{n_1,n_2}) \xrightarrow{\text{p}} \mu$ as $n_1,n_2 \to \infty$, for some $\mu \in \mathcal{P}$, then, for every n_1,n_2, $G_{n_1,n_2} \overset{\text{d}}{=} T_{n_1,n_2;S_\mu}$.* $\square$

8.2. **Random weights.** Definition (1.9) suggests the following construction:

(8.1) Let X and Y be two random variables and let $t \in \mathbb{R}$. Let $X_1,X_2,\dots,$ be copies of X and $Y_1,Y_2,\dots,$ copies of Y, all independent, and let $T_{n_1,n_2;X,Y,t}$ be the bipartite threshold graph with vertex sets $[n_1]$ and $[n_2]$ and edges ij for all pairs ij such that $X_i + Y_j > t$.

Theorem 8.3. *Let S be the increasing set*

$$S := \{(x, y) \in (0, 1]^2 : F_X^{-1}(x) + F_Y^{-1}(y) > t\}. \tag{8.2}$$

Then $T_{n_1,n_2;X,Y,t} \overset{\mathrm{d}}{=} T_{n_1,n_2;S}$ for every $n_1, n_2 \geq 1$.

Furthermore, as $n_1, n_2 \to \infty$, the degree distribution $\nu_1(T_{n_1,n_2;X,Y,t}) \xrightarrow{\text{a.s.}} \mu$ and thus $T_{n_1,n_2;X,Y,t} \xrightarrow{\text{a.s.}} \Gamma''_\mu$, where $\mu \in \mathcal{P}$ is the distribution of the random variable $1 - F_Y(t - X)$, i.e.

$$\mu[0, s] = \mathbb{P}\bigl(1 - F_Y(t - X) \leq s\bigr), \qquad s \in [0, 1]. \tag{8.3}$$

$\square$

In the special case when $\mathbb{P}(X \in [0, 1]) = 1$, $Y \sim U(0, 1)$ and $t = 1$, (8.3) yields $\mu[0, s] = \mathbb{P}(X \leq s)$, so μ is the distribution of X; further, the set S in (8.2) is a.e. equal to S_μ in (5.3).

Corollary 8.4. *If $\mu \in \mathcal{P}_{\mathsf{s}}$, let X have distribution μ and let $Y \sim U(0, 1)$. Then $T_{n_1,n_2;X,Y,t} \overset{\mathrm{d}}{=} T_{n_1,n_2;S_\mu}$ for every $n_1, n_2 \geq 1$. Furthermore, as $n_1, n_2 \to \infty$, $\nu_1(T_{n_1,n_2;X,Y,t}) \xrightarrow{\text{p}} \mu$ and $T_{n_1,n_2;X,Y,t} \xrightarrow{\text{p}} \Gamma''_\mu$.*

This yields another canonical construction for every $\mu \in \mathcal{P}$. (We claim only convergence in probability in Corollary 8.4; convergence a.s. holds at least along every increasing subsequence $(n_1(m), n_2(m))$, see [10, Remark 8.2].)

8.3. **Random addition of vertices.** Definition (1.10) suggests the following construction:

(8.4) Let $T_{n_1,n_2;p_1,p_2}$ be the random bipartite threshold graph with n_1+n_2 vertices obtained as follows: Take n_1 'white' vertices and n_2 'black' vertices, and arrange them in random order. Then, join each white vertex with probability p_1 to all earlier black vertices, and join each black vertex with probability p_2 to all earlier white vertices (otherwise, the vertex is joined to no earlier vertex), the decisions being made independently by tossing a biased coin once for each white vertex, and another biased coin once for each black vertex.

Let, for $p_1, p_2 \in [0, 1]$, μ_{p_1,p_2} be the probability measure in $\mathcal{P}$ with distribution function

$$F_{\mu_{p_1,p_2}}(x) = \begin{cases} \frac{1-p_1}{p_2}x, & 0 \leq x < p_2, \\ 1 - \frac{p_1}{1-p_2}(1 - x), & p_2 \leq x < 1. \end{cases} \tag{8.5}$$

Hence, μ_{p_1,p_2} has density $(1 - p_1)/p_2$ on $(0, p_2)$ and $p_1/(1 - p_2)$ on $(p_2, 1)$; if $p_2 = 0$ there is also a point mass $1 - p_1$ at 0, and if $p_2 = 1$ there is also a point mass p_1 at 1. It follows from (5.2) that the corresponding subset $S_{p_1,p_2} := S_{\mu_{p_1,p_2}}$ of $[0, 1]^2$ is the quadrilateral with vertices $(0, 1)$, $(1 - p_1, 1 - p_2)$, $(1, 0)$ and $(1, 1)$ (including degenerate cases when p_1 or p_2 is 0 or 1).

This is an extension of the definitions in Subsection 6.3; we have $\mu_{p,p} = \mu_p$ and $S_{p,p} = S_p$. Note also that $\mu^\dagger_{p_1,p_2} = \mu_{p_2,p_1}$. In particular, $\mu_{p_1,p_2} \in \mathcal{P}_\mathsf{s}$ only if $p_1 = p_2$.

Theorem 8.5. *As $n_1, n_2 \to \infty$, the degree distributions $\nu_1(T_{n_1,n_2;p_1,p_2}) \xrightarrow{\text{p}} \mu_{p_1,p_2}$ and $\nu_2(T_{n_1,n_2;p_1,p_2}) \xrightarrow{\text{p}} \mu_{p_2,p_1}$; consequently, $T_{n_1,n_2;p_1,p_2} \xrightarrow{\text{p}} \Gamma''_{p_1,p_2} := \Gamma''_{\mu_{p_1,p_2}} \in \mathcal{T}''_{\infty,\infty}$.*

Corollary 8.6. *If $p_1, p_2 \in [0,1]$ and $n_1, n_2 \geq 1$, then*

$$T_{n_1,n_2;p_1,p_2} \overset{\text{d}}{=} T_{n_1,n_2;S_{p_1,p_2}} \overset{\text{d}}{=} T_{n_1,n_2;X_1,X_2,0},$$

where X_j has the density $1 - p_j$ on $(-1,0)$ and p_j on $(0,1)$, $j = 1,2$.

Note that if $p_1 + p_2 = 1$, then S_{p_1,p_2} is the upper triangle $S_{1/2} := \{(x,y) : x + y \geq 1\}$. Hence the distribution of $T_{n_1,n_2;p_1,p_2}$ does not depend on p_1 as long as $p_2 = 1 - p_1$. In particular, we may then choose $p_1 = 1$ and $p_2 = 0$. In this case, Definition (8.4) simplifies as follows.

(8.6) Let T_{n_1,n_2} be the random bipartite threshold graph with $n_1 + n_2$ vertices obtained as follows: Take n_1 'white' vertices and n_2 'black' vertices, and arrange them in random order. Join every white vertex to every earlier black vertex.

If $p_1 = 1$ and $p_2 = 0$, then further $X_1 \overset{\text{d}}{=} U \sim U(0,1)$ and $X_2 \overset{\text{d}}{=} U - 1$ in Corollary 8.6. Hence, we have found a number of natural constructions that yield the same random bipartite threshold graph.

Corollary 8.7. *If $p_1 \in [0,1]$ and $n_1, n_2 \geq 1$, then*

$$T_{n_1,n_2;p_1,1-p_1} \overset{\text{d}}{=} T_{n_1,n_2;1,0} = T_{n_1,n_2} \overset{\text{d}}{=} T_{n_1,n_2;S_{1/2}} \overset{\text{d}}{=} T_{n_1,n_2;U,U,1},$$

with $U \sim U(0,1)$. □

We will see in the next subsection that this random bipartite threshold graph is uniformly distributed as an unlabeled bipartite threshold graph.

8.4. **Uniform random bipartite threshold graphs.** It is easy to see that for every bipartite threshold graph, if we color the vertices in V_1 white and the vertices in V_2 black, then there is an ordering of the vertices such that a white vertex is joined to every earlier black vertex but not to any later. (For example, if there are weights as in (1.9), order the vertices according to w'_i and w''_j, taking the white vertices first in case of a tie.) This yields a 1–1 correspondence between unlabeled bipartite threshold graphs on $n_1 + n_2$ vertices and sequences of n_1 white and n_2 black balls. Consequently, the number of unlabeled bipartite threshold graphs is

$$|\mathcal{T}_{n_1,n_2}| = \binom{n_1+n_2}{n_1}, \qquad n_1, n_2 \geq 1.$$

Moreover, it follows that T_{n_1,n_2} is uniformly distributed in $\mathcal{T}_{n_1,n_2}$; hence Corollary 8.7 yields the following:

Theorem 8.8. *The random bipartite threshold graphs T_{n_1,n_2}, $T_{n_1,n_2;p_1,1-p_1}$ ($0 \le p_1 \le 1$), $T_{n_1,n_2;S_{1/2}}$, $T_{n_1,n_2;U,U,1}$ are all uniformly distributed, regarded as unlabeled bipartite threshold graphs.* □

We have not studied uniform random labeled bipartite threshold graphs.

9. Spectrum of Threshold Graphs

There is a healthy literature on the eigenvalue distribution of the adjacency matrix for various classes of random graphs. Much of this is focused on the spectral gap (e.g., most k-regular graphs are Ramanujan [9]). See Jakobson, Miller, Rivin, Rudnick [18] for evidence showing that random k-regular graphs have the same limiting eigenvalue distribution as the Gaussian orthogonal ensemble. The following results show that random threshold graphs give a family of examples with highly controlled limiting spectrum.

There is a tight connection between the degree distribution of a threshold graph and the spectrum of its Laplacian, see [28, 16, 29]. Recall that the *Laplacian* of a graph G, with $V(G) = [n]$, say, is the $n \times n$ matrix $\mathcal{L} = D - A$, where A is the adjacency matrix of G and D is the diagonal matrix with entries $d_{ii} = d_G(i)$. (Thus $\mathcal{L}$ is symmetric and has row sums 0.) It is easily seen that $\langle \mathcal{L}x, y\rangle = \sum_{ij\in E(G)}(x_i - x_j)(y_i - y_j)$ for $x, y \in \mathbb{R}^n$. The eigenvalues λ_i of $\mathcal{L}$ satisfy $0 \le \lambda_i \le n$, $i = 1, \dots, n$, and we define the normalized spectral distribution $\nu_L \in \mathcal{P}$ as the empirical distribution of $\{\lambda_i/n\}_{i=1}^n$.

For a threshold graph, it is easily seen that if we order the vertices as in (1.2) and Subsection 2.1, then for each $i = 2, \dots, n$ the function

$$\varphi_i(j) := \begin{cases} -1, & j < i, \\ i-1, & j = i, \\ 0, & j > i. \end{cases}$$

is an eigenfunction of $\mathcal{L}$ with eigenvalue $d(i)$ or $d(i)+1$, depending on whether i is added as isolated or dominating, i.e., whether $\alpha_i = 0$ or 1 in the binary code of the graph. Together with $\varphi_1 := 1$ (which is an eigenfunction with eigenvalue 0 for any graph), these form an orthogonal basis of eigenfunctions. The Laplacian spectrum thus can be written

$$\{0\} \cup \{d(i) + \alpha_i : i = 2, \dots, n\}. \tag{9.1}$$

In particular, the eigenvalues are all integers.

Moreover, (9.1) shows that the spectrum $\{\lambda_i\}_1^n$ is closely related to the degree sequence; in particular, asymptotically they are the same in the sense that if G_n is a sequence of threshold graphs with $v(G_n) \to \infty$ and $\mu \in \mathcal{P}$, then

$$\nu_L(G_n) \to \mu \iff \nu(G_n) \to \mu. \tag{9.2}$$

(See [16] for a detailed comparison of the Laplacian spectrum and the degree sequence for threshold graphs.) In particular, Theorem 5.5 can be restated using the spectral distribution:

Theorem 9.1. *Let G_n be a sequence of threshold graphs such that $v(G_n) \to \infty$. Then G_n converges in $\overline{\mathcal{U}}$ as $n \to \infty$, if and only if the spectral distributions $\nu_L(G_n)$ converge to some distribution μ. In this case, $\mu \in \mathcal{P}_{\mathsf{s}}$ and $G_n \to \Gamma_\mu$.* □

Remark 9.1. It can be shown that the spectrum and the degree sequence are asymptotically close in the sense that (9.2) holds for any graphs G_n with $v(G_n) \to \infty$, even though in general there is no simple relation like (9.1).

Another relation between the spectrum and the degree sequence for a threshold graph is that their Ferrers diagrams are transposes of each other, see [28, 29]; this is easily verified from (9.1) by induction. If we scale the Ferrers diagrams by n, so that they fit in the unit square $[0,1]^2$ with a corner at $(0,1)$, then the lower boundary is the graph of the empirical distribution function of the corresponding normalized values, i.e., the distribution function of $\nu(G)$ or $\nu_L(G)$. Hence, these distribution functions are related by reflection in the diagonal between $(0,1)$ and $(1,0)$, so by (5.11) (and the comment after it), for any threshold graph G,

$$\nu_L(G) = \nu(G)^\dagger.$$

Acknowledgements. Large parts of this research was done during visits of SJ to Université de Nice and of PD and SH to Uppsala University in January and March 2007, partly funded by the ANR Chaire d'excellence to PD. Work was continued during a visit of SJ to the Institut Mittag-Leffler, Djursholm, Sweden, 2009. SH was supported by grants NSF DMS-02-41246 and NIGMS R01GM086884-2. We thank Adam Guetz and Sukhada Fadnavis for careful reading of a preliminary draft.

References

[1] D. Aldous. Exchangeability and related topics. *Lecture Notes in Math*, Jan 1985.
[2] J. S. Beissinger and U. N. Peled. Enumeration of labelled threshold graphs and a theorem of Frobenius involving Eulerian polynomials. *Graphs Combin.*, 3(3):213–219, 1987.
[3] P. Billingsley. *Convergence of Probability Measures.* Wiley, New York, 1968.
[4] C. Borgs, J. T. Chayes, and L. Lovász. Unique limits of dense graph sequences. 2007. Preprint, `http://arxiv.org/math/0803.1244v1`.
[5] C. Borgs, J. T. Chayes, L. Lovász, V. T. Sós, and K. Vesztergombi. Convergent sequences of dense graphs I: subgraph frequencies, metric properties and testing., 2007. Preprint, January 2007. `http://arxiv.org/math.CO/0702004`.
[6] A. Brandstädt, V. B. Le, and J. P. Spinrad. *Graph Classes: a Survey.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1999.
[7] K. Brown and P. Diaconis. Random walks and hyperplane arrangements. *Ann. Probab.*, pages 1813–1854, 1998.
[8] V. Chvátal and P. L. Hammer. *Studies in Integer Programming (Proc. Workshop, Bonn, 1975)*, chapter Aggregation of inequalities in integer programming, pages 145–162. Number 1 in Annals of Discrete Math. North-Holland, Amsterdam, 1977.
[9] G. Davidoff, P. Sarnak, and A. Valette. *Elementary number theory, group theory, and Ramanujan graphs*, volume 55 of *London Mathematical Society Student Texts.* Cambridge University Press, Cambridge, 2003.

[10] P. Diaconis and S. Janson. Graph limits and exchangeable random graphs. *Rendiconti di Matematica*, 28(VII):33–61, 2008.
[11] P. Erdös, A. Gyárfás, E. T. Ordman, and Y. Zalcstein. The size of chordal, interval and threshold subgraphs. *Combinatorica*, 9:245–253, 1989.
[12] P. Flajolet and R. Sedgewick. *Analytic Combinatorics*. Cambridge Univ. Press, Cambridge, 2009.
[13] A. Gut. *Probability: A Graduate Course*. Springer-Verlag, New York, 2005.
[14] A. Gut and S. Janson. The limiting behaviour of certain stopped sums and some applications. *Scand. J. Statist.*, 10(4):281–292, 1983.
[15] A. Hagberg, P. J. Swart, and D. A. Schult. Designing threshold networks with given structural and dynamical properties. *Phys. Rev. E*, 74:056116, 2006. See also `https://networkx.lanl.gov/wiki`.
[16] P. L. Hammer and A. K. Kelmans. Laplacian spectra and spanning trees of threshold graphs. *Discrete Appl. Math.*, 65(1-3):255–273, 1996. First International Colloquium on Graphs and Optimization (GOI), 1992 (Grimentz).
[17] P. L. Hammer, U. N. Peled, and X. Sun. Difference graphs. *Discrete Appl. Math.*, 28(1):35–44, 1990.
[18] D. Jakobson, S. D. Miller, I. Rivin, and Z. Rudnick. Eigenvalue spacings for regular graphs. In *Emerging applications of number theory (Minneapolis, MN, 1996)*, volume 109 of *IMA Vol. Math. Appl.*, pages 317–327. Springer, New York, 1999.
[19] O. Kallenberg. *Foundations of Modern Probability*. Springer-Verlag, New York, 2nd edition, 2002.
[20] O. Kallenberg. *Probabilistic Symmetries and Invariance Principles*. Springer-Verlag, New York, 2005.
[21] N. Konno, N. Masuda, R. Roy, and A. Sarkar. Rigorous results on the threshold network model. *Journal of Physics A*, 38:6277–6291, 2005.
[22] L. Lovász and B. Szegedy. Limits of dense graph sequences. *J. Comb. Theory B*, 96:933–957, 2006.
[23] L. Lovász and B. Szegedy. Finitely forcible graphons. 2009. Preprint, `http://arxiv.org/math/0901.0929v1`.
[24] L. Lu and F. R. K. Chung. *Complex graphs and networks*. American Mathematical Society, Providence, RI, 2006.
[25] N. Mahadev and U. Peled. *Threshold graphs and related topics*. Number 56 in Annals of Discrete Math. North-Holland, Elsevier, 1995.
[26] N. Masuda, H. Miwa, and N. Konno. Geographical threshold graphs with small-world and scale-free properties. *Physical Review E*, 71(036108), 2005.
[27] T. A. McKee and F. R. McMorris. *Topics in intersection graph theory*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1999.
[28] R. Merris. Laplacian matrices of graphs: a survey. *Linear Algebra Appl.*, 197/198:143–176, 1994. Second Conference of the International Linear Algebra Society (ILAS) (Lisbon, 1992).
[29] R. Merris and T. Roby. The lattice of threshold graphs. *JIPAM. J. Inequal. Pure Appl. Math.*, 6(1):Article 2, 21 pp. (electronic), 2005.
[30] M. Mitzenmacher. A brief history of generative models for power law and lognormal distributions. *Internet Mathematics*, 1:226–251, 2004.
[31] M. Penrose. *Random Geometric Graphs*. Oxford University Press, Oxford, 2003.
[32] N. J. A. Sloane. *The On-Line Encyclopedia of Integer Sequences*. 2009. `www.research.att.com/∼njas/sequences/`.
[33] M. Yannakakis. The complexity of the partial order dimension problem. *SIAM J. Algebraic Discrete Methods*, 3(3):351–358, 1982.

Stanford and CNRS, Université de Nice Sophia Antipolis, Department of Mathematics, Stanford, CA 94305, USA
E-mail address: diaconis@math.stanford.edu
URL: http://www-stat.stanford.edu/∼CGATES/persi

Stanford, Department of Statistics, Stanford, CA 94305, USA
E-mail address: susan@stat.stanford.edu
URL: http://www-stat.stanford.edu/∼susan/

Department of Mathematics, Uppsala University, PO Box 480, SE-751 06 Uppsala, Sweden
E-mail address: svante.janson@math.uu.se
URL: http://www.math.uu.se/∼svante/