THE MAXIMUM OF BROWNIAN MOTION WITH PARABOLIC DRIFT

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ABSTRACT. We study the maximum of a Brownian motion with a parabolic drift; this is a random variable that often occurs as a limit of the maximum of discrete processes whose expectations have a maximum at an interior point. We give new series expansions and integral formulas for the distribution and the first two moments, together with numerical values to high precision.

1. INTRODUCTION

Let $W(t)$ be a two-sided Brownian motion with $W(0) = 0$; i.e., $(W(t))_{t\geq 0}$ and $(W(-t))_{t\geq 0}$ are two independent standard Brownian motions. We are interested in the process

$$W_\gamma(t) := W(t) - \gamma t^2 \tag{1.1}$$

for a given $\gamma > 0$, and in particular in its maximum

$$M_\gamma := \max_{-\infty<t<\infty} W_\gamma(t) = \max_{-\infty<t<\infty} \bigl(W(t) - \gamma t^2\bigr). \tag{1.2}$$

We also consider the corresponding one-sided maximum

$$N_\gamma := \max_{0\leq t<\infty} W_\gamma(t) = \max_{0\leq t<\infty} \bigl(W(t) - \gamma t^2\bigr). \tag{1.3}$$

Since the restrictions of W to the positive and negative half-axes are independent, we have the relation

$$M_\gamma \overset{\mathrm{d}}{=} \max(N_\gamma, N'_\gamma) \tag{1.4}$$

where N'_γ is an independent copy of N_γ.

Note that (a.s.) $W_\gamma(t) \to -\infty$ as $t \to \pm\infty$, so the maxima in (1.2) and (1.3) exist and are finite; moreover, they are attained at unique points and $M_\gamma, N_\gamma > 0$. It is easily seen (e.g., by Cameron–Martin) that M_γ and N_γ have absolutely continuous distributions.

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1.1. **Scaling.** For any $a > 0$, $W(at) \overset{\mathrm{d}}{=} a^{1/2}W(t)$ (as processes on $(-\infty,\infty)$), and thus

$$\begin{aligned} M_\gamma &= \max_{-\infty<t<\infty} \bigl(W(at) - \gamma(at)^2\bigr) \\ &\overset{\mathrm{d}}{=} \max_{-\infty<t<\infty} \bigl(a^{1/2}W(t) - a^2\gamma t^2\bigr) = a^{1/2}M_{a^{3/2}\gamma}, \end{aligned} \tag{1.5}$$

and similarly

$$N_\gamma \overset{\mathrm{d}}{=} a^{1/2}N_{a^{3/2}\gamma}. \tag{1.6}$$

The parameter γ is thus just a scale parameter, and it suffices to consider a single choice of γ. We choose the normalization $\gamma = 1/2$ as the standard case, and write $M := M_{1/2}$, $N := N_{1/2}$. In general, (1.5)–(1.6) with $a = (2\gamma)^{-2/3}$ yield

$$M_\gamma \overset{\mathrm{d}}{=} (2\gamma)^{-1/3}M, \qquad N_\gamma \overset{\mathrm{d}}{=} (2\gamma)^{-1/3}N. \tag{1.7}$$

Remark 1.1. More generally, if $a, b > 0$, then

$$\max_{-\infty<t<\infty} \bigl(aW(t) - bt^2\bigr) = aM_{b/a} \overset{\mathrm{d}}{=} \Bigl(\frac{a^4}{2b}\Bigr)^{1/3} M. \tag{1.8}$$

The distributions of N and M were characterized by Groeneboom [10, Theorem 3.1 – Corollary 3.2]; see also Daniels and Skyrme [8]. ([8] gives also formulas for the mean $\mathbb{E}\, M$.) The descriptions there are, however, rather complicated. The purpose of this paper is to provide further, more explicit, formulas for the distribution function of M and, in particular, for its moments; our formulas are either integrals or series, and invariably involve the Airy function Ai. We also provide companion result for N. (Our results are based on Groeneboom [10] and the related results by Salminen [21].) Other simple integral formulas for the distribution and density functions of N and M have been given by Groeneboom [11]; these are not obviously equivalent to our formulas. It seems likely that there may be further similar formulas that have not yet been discovered, possibly including some even simpler ones. In particular, it would be interesting to find simple integral formulas for higher moments of M.

The main results are given in Section 2, with proofs and further details in Sections 3–5. We illustrate the use of the formulas by using them to compute the density and the first two moments of M and N numerically; we discuss the numerical computations in Section 6.

We use many more or less well-known results for Airy functions; for convenience we have collected them in Appendices A–B. Finally, Appendix C discusses an interesting integral equation, while Appendix D contains an alternative proof of an important formula used in our proofs.

1.2. **Background.** The random variable M is studied by Barbour [4], Daniels and Skyrme [8] and Groeneboom [10]. (Further results on N are given by

Lachal [16].) M arises as a natural limit distribution in many different problems, and in many related problems its expectation $\mathbb{E}\, M$ enters in a second order term for the asymptotics of means or in improved normal approximations. For various examples and general results, see for example Daniels [6; 7], Daniels and Skyrme [8], Barbour [4; 5], Smith [22], Louchard, Kenyon and Schott [19], Steinsaltz [23], Janson [15]. As discussed in several of these papers, the appearance of M in these limit results can be explained as follows, ignoring technical conditions: Consider the maximum over time t of a random process $X_n(t)$, defined on a compact interval I, for example $[0,1]$, such that as $n \to \infty$, the mean $\mathbb{E}\, X_n(t)$, after scaling, converges to deterministic function $f(t)$, and that the fluctuations $X_n(t) - \mathbb{E}\, X_n(t)$ are of smaller order and, after a different scaling, converge to a gaussian process $G(t)$. If we assume that f is continuous on I and has a unique maximum at a point $t_0 \in I$, then the maximum of the process $X_n(t)$ is attained close to t_0. Assuming that t_0 is an interior point of I and that f is twice differentiable at t_0 with $f''(t_0) \neq 0$, we can locally at t_0 approximate f by a parabola and $G(t) - G(t_0)$ by a two-sided Brownian motion (with some scaling), and thus $\max_t X_n(t) - X_n(t_0)$ is approximated by a scaling constant times the variable M, see Barbour [4]. In the typical case where the mean of $X_n(t)$ is of order n and the Gaussian fluctuations are of order $n^{1/2}$, it is easily seen that the correct scaling is that $n^{-1/3}(\max_t X_n(t) - X_n(t_0)) \xrightarrow{\mathrm{d}} cM$, for some $c > 0$, which for the mean gives $\mathbb{E} \max_t X_n(t) = nf(t_0) + n^{1/3} c\, \mathbb{E}\, M + o(n^{1/3})$, see [4; 6; 7]. As examples of applications in algorithmic and data structures analysis, this type of asymptotics appears in the analysis of linear lists, priority queues and dictionaries [18; 19] and in a sorting algorithm [15].

Remark 1.2. The *location* of the maximum in (1.2) is also of interest in various applications, see Groeneboom and Wellner [12]. It has been studied by several authors; in particular, Groeneboom [9, 10] gave a description of the distribution and Groeneboom and Wellner [12] give more explicit analytical and numerical formulas. (Groeneboom [10] describes even the joint distribution of the maximum M and its location, see also Daniels and Skyrme [8].) The location of the maximum is not considered in the present paper.

2. Main results

The mean of M can be expressed as integrals involving the Airy functions Ai and Bi, for example as follows. (For general definitions and properties of Airy functions, see [1, Section 10.4]. We remind the reader that $\mathrm{Ai}(t)$ and $\mathrm{Bi}(t)$ are linearly independent solutions of $f''(t) = tf(t)$, and that $\mathrm{Ai}(t) \to 0$ as $t \to +\infty$.)

Theorem 2.1 (Daniels and Skyrme [8])**.**

$$\mathbb{E}\, M = -\frac{2^{-1/3}}{2\pi i} \int_{-i\infty}^{i\infty} \frac{z\,\mathrm{d}z}{\mathrm{Ai}(z)^2} = \frac{2^{-1/3}}{2\pi i} \int_{-\infty}^{\infty} \frac{y\,\mathrm{d}y}{\mathrm{Ai}(iy)^2} \tag{2.1}$$

$$= 2^{2/3} \int_0^\infty \frac{\mathrm{Ai}(t)^2 + \sqrt{3}\mathrm{Ai}(t)\mathrm{Bi}(t)}{\mathrm{Ai}(t)^2 + \mathrm{Bi}(t)^2} \, \mathrm{d}t \tag{2.2}$$

$$= 2^{2/3} \operatorname{Re} \left(\bigl(1 + i\sqrt{3}\bigr) \int_0^\infty \frac{\mathrm{Ai}(t)}{\mathrm{Ai}(t) + i\mathrm{Bi}(t)} \, \mathrm{d}t \right) \tag{2.3}$$

$$= \frac{2^{2/3}}{\pi} \int_0^\infty \frac{\sqrt{3}\mathrm{Bi}(t)^2 - \sqrt{3}\mathrm{Ai}(t)^2 + 2\mathrm{Ai}(t)\mathrm{Bi}(t)}{\bigl(\mathrm{Ai}(t)^2 + \mathrm{Bi}(t)^2\bigr)^2} \, t \, \mathrm{d}t. \tag{2.4}$$

The expressions (2.1) and (2.2) (unfortunately with typos in the latter) are given by Daniels and Skyrme [8]. Since detailed proofs of the formulas are not given there, we for completeness give a complete proof in Section 5. (The proof includes a direct analytical verification of the equivalence of (2.1) and (2.2), which was left open in [8].)

By (A.1) and (A.8), $|\mathrm{Ai}(iy)|$ increases superexponentially as $y \to \pm\infty$, while $\mathrm{Ai}(t)$ decreases superexponentially and $\mathrm{Bi}(t)$ increases superexponentially as $t \to \infty$; hence, the integrands in the integrals in Theorem 2.1 all decrease superexponentially and the integrals converge rapidly, so they are suited for numerical calculations. We obtain by numerical integration (using `Maple`), improving the numerical values in [4; 5; 8; 7],

$$\mathbb{E}\, M = 0.99619\,30199\,28363\,11660\,37766 \ldots \tag{2.5}$$

We do not know any similar integral formulas for the second moment of M (or higher moments). Instead we give expressions using infinite series, summing over the zeros a_k, $k \geq 1$, of the Airy function, see Appendix A. Recall that these zeros all are real and negative, so we have $0 > a_1 > a_2 > \dots$, see [1, (10.4.94)] and Appendix A; note that $|a_k| \asymp k^{2/3}$, see (A.30). (We use $x_n \asymp y_n$, for two sequences of positive numbers x_n and y_n, to denote that $0 < \liminf_{n\to\infty} x_n/y_n \leq \limsup_{n\to\infty} x_n/y_n < \infty$; this is also denoted $x_n = \Theta(y_n)$.)

We first introduce more notation. Let $F_N(x)$ be the distribution function of N, i.e., $F_N(x) := \mathbb{P}(N \leq x)$, and let $F_M(x)$ be the distribution function of M; further, let $G_N(x) = 1 - F_N(x) = \mathbb{P}(N > x)$ and $G_M(x) = 1 - F_M(x) = \mathbb{P}(M > x)$ be the corresponding tail probabilities. Then, by (1.4),

$$F_M(x) := \mathbb{P}(M \leq x) = \mathbb{P}(N \leq x)^2 = F_N(x)^2. \tag{2.6}$$

and, equivalently,

$$G_M(x) := 1 - (1 - G_N(x))^2 = 2G_N(x) - G_N(x)^2. \tag{2.7}$$

If we know $G_N(x)$, we thus know the distribution of both N and M, and we can compute moments by

$$\mathbb{E}\, N^p = p \int_0^\infty x^{p-1} G_N(x) \, \mathrm{d}x, \tag{2.8}$$

$$\mathbb{E}\, M^p = p \int_0^\infty x^{p-1} G_M(x) \, \mathrm{d}x = p \int_0^\infty x^{p-1} \bigl(2G_N(x) - G_N(x)^2\bigr) \, \mathrm{d}x. \tag{2.9}$$

Two formulas for the distribution function are given in the following theorem. Others are given in (3.3) and Lemma 3.5. The proof is given in Section 3. Hi is the function defined in (A.16).

Theorem 2.2. *The distribution functions of M and N are $F_M(x) = (1 - G_N(x))^2$ and $F_N(x) = 1 - G_N(x)$, where*

$$G_N(x) = \pi \sum_{k=1}^{\infty} \frac{\mathrm{Hi}(a_k)}{\mathrm{Ai}'(a_k)} \mathrm{Ai}(a_k + 2^{1/3}x), \qquad x > 0. \tag{2.10}$$

The sum converges conditionally but not absolutely for every $x > 0$. Alternatively, with an absolutely convergent sum, for $x \geq 0$,

$$G_N(x) = \frac{\mathrm{Ai}(2^{1/3}x)}{\mathrm{Ai}(0)} + \sum_{k=1}^{\infty} \frac{\pi \mathrm{Hi}(a_k) + a_k^{-1}}{\mathrm{Ai}'(a_k)} \mathrm{Ai}(a_k + 2^{1/3}x). \tag{2.11}$$

The function $G_N(x)$ is plotted in Figure 1.

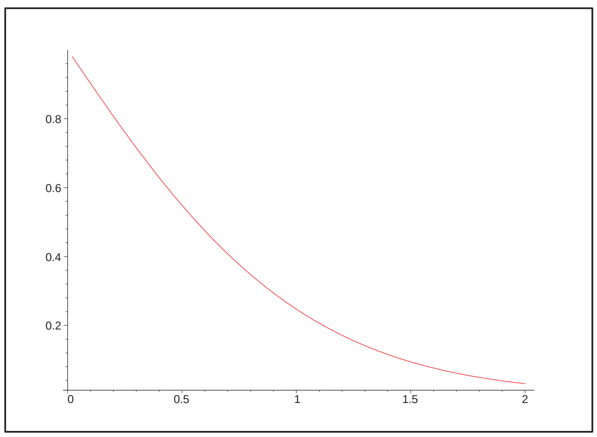

FIGURE 1. $G_N(x)$

Remark 2.3. By (A.4) and (A.30), for any fixed $x > 0$, $|\mathrm{Ai}(a_k + 2^{1/3}x)|$ is usually of the order $|a_k|^{-1/4} \asymp k^{-1/6}$, and using also (A.31) and (A.33), the summands in (2.10) are (typically) of the order $k^{-1/6-1/6-2/3} = k^{-1}$, so this sum is *not* absolutely convergent. (For some values of k, the term may be smaller than k^{-1} because $a_k + 2^{1/3}x$ may be close to another zero, but such cases are infrequent and do not prevent the series from being absolutely divergent.)

On the other hand, by (A.22), $\pi\mathrm{Hi}(z) + z^{-1} = O(|z|^{-4})$ on the negative real axis, and it follows that the terms in (2.11) are $O(k^{-3})$, so the series is absolutely convergent. Moreover, since Ai is bounded on the real axis (see (A.1) and (A.4)), the sum in (2.11) converges uniformly for $x \geq 0$, and is thus a continuous function of x; this is no surprise since we already have remarked that N has an absolutely continuous distribution, so G is continuous. Note also that (2.11) for $x = 0$ is the trivial $G_N(0) = 1$, since each $\mathrm{Ai}(a_k) = 0$, while (2.10) does not hold for $x = 0$.

The sum in (2.11) can be differentiated termwise and we have the following result, proved in Section 3.

Theorem 2.4. *N and M have absolutely continuous distributions with infinitely differentiable density functions, for $x > 0$,*

$$f_N(x) = -2^{1/3}\frac{\mathrm{Ai}'(2^{1/3}x)}{\mathrm{Ai}(0)} - 2^{1/3}\sum_{k=1}^{\infty}\frac{\pi\mathrm{Hi}(a_k) + a_k^{-1}}{\mathrm{Ai}'(a_k)}\mathrm{Ai}'(a_k + 2^{1/3}x), \tag{2.12}$$

$$f_M(x) = 2(1 - G_N(x))f_N(x). \tag{2.13}$$

Integral formulas for $f_N(x)$ will be given in (3.10) and (5.10). The density functions $f_N(x)$ and $f_M(x)$ are plotted in Figures 2 and 3.

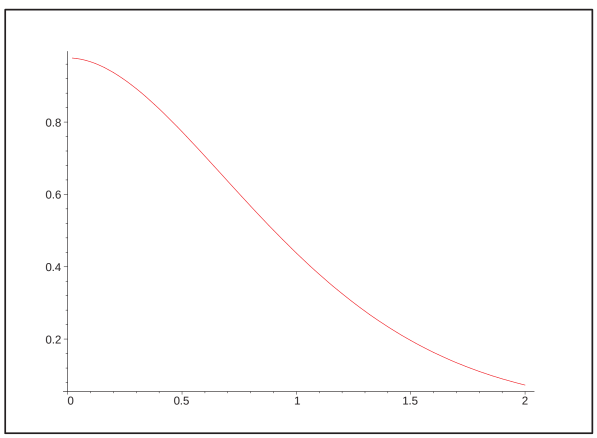

FIGURE 2. $f_N(x)$

Remark 2.5. In contrast, the sum

$$2^{1/3}\pi\sum_{k=1}^{\infty}\frac{\mathrm{Hi}(a_k)}{\mathrm{Ai}'(a_k)}\mathrm{Ai}'(a_k + 2^{1/3}x) \tag{2.14}$$

obtained by termwise differentiation of (2.10) is *not* convergent for any $x \geq 0$, as will be seen in Section 3.

Moments of M and N now can be obtained from (2.8) and (2.9) by integrating (2.11) termwise. This yields the following result; see Section 4

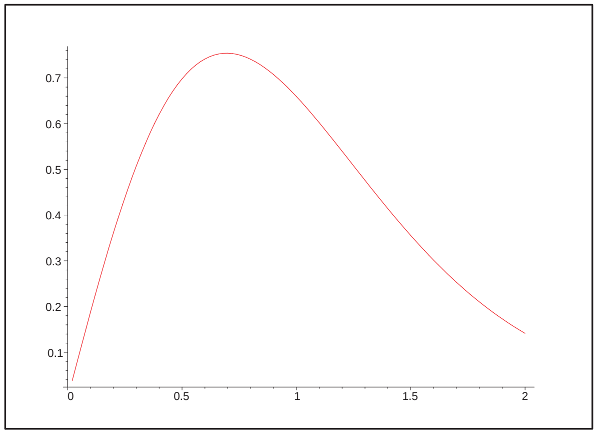

FIGURE 3. $f_M(x)$

for proofs as well as related integral formulas. For higher moments, see Remark 4.4.

We define for convenience

$$\varphi(k) := \pi \mathrm{Hi}(a_k) + a_k^{-1}. \tag{2.15}$$

By (A.22), $\varphi(k) = O(|a_k|^{-4}) = O(k^{-8/3})$. Recall also the function $\mathrm{Gi}(z) = \mathrm{Bi}(z) - \mathrm{Hi}(z)$, see (A.17).

Theorem 2.6. *The means and second moments of M and N are given by the absolutely convergent sums*

$$\mathbb{E}\, N = \frac{1}{2^{1/3} 3 \mathrm{Ai}(0)} - \frac{\pi}{2^{1/3}} \sum_{k=1}^{\infty} \varphi(k) \mathrm{Gi}(a_k) \tag{2.16}$$

$$= \frac{1}{2^{1/3} 3 \mathrm{Ai}(0)} - \frac{\pi}{2^{1/3}} \sum_{k=1}^{\infty} [\varphi(k) \mathrm{Bi}(a_k) - \varphi(k) \mathrm{Hi}(a_k)], \tag{2.17}$$

$$\mathbb{E}\, M = \frac{2^{2/3}}{3\mathrm{Ai}(0)} - \frac{\mathrm{Ai}'(0)^2}{2^{1/3}\mathrm{Ai}(0)^2} - \frac{1}{2^{1/3}} \sum_{k=1}^{\infty} \left[2\pi\varphi(k)\mathrm{Bi}(a_k) - \varphi(k)^2 \right], \tag{2.18}$$

$$\mathbb{E}\, N^2 = -\frac{2^{1/3}\mathrm{Ai}'(0)}{\mathrm{Ai}(0)} + 2^{1/3} \sum_{k=1}^{\infty} \varphi(k) \left[\pi a_k \mathrm{Gi}(a_k) - 1 \right] \tag{2.19}$$

$$= -\frac{2^{1/3}\mathrm{Ai}'(0)}{\mathrm{Ai}(0)} + 2^{1/3} \sum_{k=1}^{\infty} \left[\pi a_k \varphi(k) \mathrm{Bi}(a_k) - a_k \varphi(k)^2 \right]; \tag{2.20}$$

$$\mathbb{E}\, M^2 = -\frac{2^{1/3} 5\, \mathrm{Ai}'(0)}{3\mathrm{Ai}(0)} + 2^{4/3} \sum_{k=1}^{\infty} \varphi(k) \left[\pi a_k \mathrm{Bi}(a_k) - \frac{2}{3} a_k \varphi(k) + \frac{2}{a_k^3} + \frac{2\mathrm{Ai}'(0)}{\mathrm{Ai}(0) a_k^2} \right]$$

$$+ 2^{7/3} \sum_{k=1}^{\infty} \sum_{j=1}^{k-1} \frac{\varphi(k)\varphi(j)}{(a_k - a_j)^2}. \tag{2.21}$$

Numerically we have

$$\begin{aligned}
\mathbb{E}(N) &= 0.6955289995\dots, \\
\mathbb{E}(M) &= 0.9961930199\dots, \\
\mathbb{E}(N^2) &= 1.1027982645\dots, \\
\mathbb{E}(M^2) &= 1.8032957042\dots,
\end{aligned}$$

and thus

$$\begin{aligned}
\mathrm{Var}(N) &= \mathbb{E}(N^2) - \mathbb{E}(N)^2 = 0.6190376754\dots, \\
\mathrm{Var}(M) &= \mathbb{E}(M^2) - \mathbb{E}(M)^2 = 0.8108951713\dots.
\end{aligned}$$

The numerical value for $\mathbb{E}\, M$ agrees with the one in (2.5).

Further formulas for moments are given in Theorems 4.5 and 4.6.

3. Distributions

Salminen [21, Example 3.2] studied the hitting time

$$\tau := \inf\{t \ge 0 : x + W(t) = -\beta t^2\}, \tag{3.1}$$

and gave the formula [21, (3.10)], for $x, \beta > 0$, (with $\alpha = -\beta$ in his notation),

$$f_\tau(t) = 2^{1/3}\beta^{2/3} \sum_{k=1}^{\infty} \exp\bigl(2^{1/3}\beta^{2/3} a_k t - \tfrac{2}{3}\beta^2 t^3\bigr) \frac{\mathrm{Ai}(a_k + 2^{2/3}\beta^{1/3}x)}{\mathrm{Ai}'(a_k)} \tag{3.2}$$

for the density function of τ. Note that τ is a defective random variable, and that $\tau = \infty$ if and only if $\min_{t\ge 0}(x + W(t) + \beta t^2) > 0$. By symmetry, $W \overset{\mathrm{d}}{=} -W$, and thus

$$\mathbb{P}(\tau = \infty) = \mathbb{P}\bigl(\max_{t\ge 0}(W(t) - \beta t^2 - x) < 0\bigr) = \mathbb{P}(N_\beta < x).$$

Hence, choosing $\beta = 1/2$,

$$\begin{aligned}
G_N(x) = 1 - F_N(x) = \mathbb{P}(N \ge x) = \mathbb{P}(\tau < \infty) &= \int_0^\infty f_\tau(t)\,\mathrm{d}t \\
&= \int_0^\infty 2^{-1/3} \sum_{k=1}^{\infty} \frac{\mathrm{Ai}(a_k + 2^{1/3}x)}{\mathrm{Ai}'(a_k)} \exp\bigl(2^{-1/3} a_k t - \tfrac{1}{6}t^3\bigr)\,\mathrm{d}t \\
&= \int_0^\infty \sum_{k=1}^{\infty} \frac{\mathrm{Ai}(a_k + 2^{1/3}x)}{\mathrm{Ai}'(a_k)} \exp\bigl(a_k t - \tfrac{1}{3}t^3\bigr)\,\mathrm{d}t.
\end{aligned} \tag{3.3}$$

If we formally integrate termwise we obtain (2.10), from (A.16). However, as seen in Remark 2.3, the sum is not absolutely convergent, so we cannot use e.g. Fubini's theorem, and we have to justify the termwise integration by a more complicated argument.

Remark 3.1. We have $|a_k| = \Theta(k^{2/3})$ by (A.30), and $|\mathrm{Ai}'(a_k)| = \Theta(k^{1/6})$ by (A.31); further, for fixed $x > 0$, $\mathrm{Ai}(a_k + 2^{1/3}x) = O(|a_k|^{-1/4}) = O(k^{-1/6})$ by (A.4). Hence the sums in (3.2) and (3.3) converge rapidly for each fixed t, because of the negative term $a_k t$ in the exponent. But the convergence rate is small for small t, and when integrating we have the problem just described.

We begin by converting the sum in (3.2) into a residue integral. Fix $\theta_0 \in (0, \pi/2)$ and $x_0 \in (a_1, 0)$, and let $\Gamma = \Gamma(\theta_0, x_0)$ be the contour consisting of the ray $\{re^{i(\pi+\theta_0)}\}$ for r from ∞ to $r_0 := |x_0|/\cos\theta_0$, the line segment $\{x_0 + iy\}$ for $y \in [-r_0 \sin\theta_0, r_0 \sin\theta_0]$ and the ray $\{re^{i(\pi-\theta_0)}\}$ for $r \in (r_0, \infty)$.

For (large) integers $N \in \mathbb{N}$, let $R_N := (\frac{3}{2}\pi N)^{2/3}$, and let $\Gamma_N := \Gamma_N(\theta_0, x_0)$ be the closed contour obtained from Γ by cutting the infinite rays at $r = R_N$ and connecting them by the arc $\Gamma'_N := \{R_N e^{i\theta} : \theta \in [\pi - \theta_0, \pi + \theta_0]\}$.

Note that by (A.30), $|a_N| < R_N < |a_{N+1}|$ (at least for large N; in fact for all $N \geq 1$). Thus, Γ_N goes around the N first zeros of Ai; moreover, Γ_N does not come too close to any of the zeros; this is made more precise by the estimates in Lemma A.2 and Lemma A.3.

Lemma 3.2. *Let $\tau = \tau_x$ be the hitting time* (3.1) *for $\beta = 1/2$ and some $x > 0$. Then the defective random variable τ has the density function, for $t > 0$,*

$$f_\tau(t) = \frac{1}{2\pi i} \int_\Gamma 2^{-1/3} e^{2^{-1/3}zt - t^3/6} \frac{\mathrm{Ai}(z + 2^{1/3}x)}{\mathrm{Ai}(z)} \,\mathrm{d}z. \tag{3.4}$$

Proof. For $x > 0$ and $z \in \Gamma$ or $z \in \Gamma_N$, Lemmas A.1 and A.2 yield $\mathrm{Ai}(z + 2^{1/3}x)/\mathrm{Ai}(z) = O(1)$, and thus the integrand in (3.4), $\Phi(z)$ say, is bounded by

$$\Phi(z) = O\big(\exp\big(-2^{-1/3}t|\operatorname{Re} z| - t^3/6\big)\big) = O\big(\exp\big(-2^{-1/3}t|\operatorname{Re} z|\big)\big).$$

This shows both that $\int_\Gamma \Phi(z)\,\mathrm{d}z$ is absolutely convergent, and that

$$\int_{\Gamma_N} \Phi(z)\,\mathrm{d}z - \int_\Gamma \Phi(z)\,\mathrm{d}z \to 0 \quad \text{as } N \to \infty. \tag{3.5}$$

$\Phi(z)$ has simple poles at the zeros a_k of Ai, and evaluating $\int_{\Gamma_N} \Phi(z)\,\mathrm{d}z$ by residues, we see that it equals the partial sum of the N first terms of (3.2). Consequently, (3.2) yields $\int_{\Gamma_N} \Phi(z)\,\mathrm{d}z \to f_\tau(t)$ as $N \to \infty$, and the result follows by (3.5). ∎

Remark 3.3. The contour Γ in (3.4) can be deformed to the imaginary axis, using the estimate in Lemma A.1. Hence, setting $z = 2^{1/3}si$,

$$e^{t^3/6} f_\tau(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ist} \frac{\mathrm{Ai}(2^{1/3}(si + x))}{\mathrm{Ai}(2^{1/3}si)} \,\mathrm{d}s. \tag{3.6}$$

Moreover, this holds also for $t \leq 0$, with $f_\tau(t) = 0$, since the right-hand side of (3.6) then easily is shown to vanish: writing it again as a line integral along the imaginary axis we can move the line of integration to $\operatorname{Re} z = \sigma$, for any $\sigma > a_1$, and for $t \ge 0$ we may let $\sigma \to +\infty$, again using Lemma A.1.

This exhibits $e^{t^3/6} f_\tau(t)$ as the inverse Fourier transform of $s \mapsto \operatorname{Ai}(2^{1/3}(si + x))/\operatorname{Ai}(2^{1/3}si)$. By Lemma A.1, this function is integrable and in L^2, and using (A.1) and (A.2), it is seen that so is its derivative, which implies that the Fourier transform $e^{t^3/6} f_\tau(t)$ is integrable. The Fourier inversion formula yields

$$\int_{-\infty}^{\infty} e^{t^3/6} f_\tau(t) e^{-ist}\,\mathrm{d}t = \frac{\operatorname{Ai}(2^{1/3}(si+x))}{\operatorname{Ai}(2^{1/3}si)}, \qquad -\infty < s < \infty, \tag{3.7}$$

and, more generally, by analytic continuation,

$$\int_{-\infty}^{\infty} e^{t^3/6} f_\tau(t) e^{-zt}\,\mathrm{d}t = \frac{\operatorname{Ai}(2^{1/3}(z+x))}{\operatorname{Ai}(2^{1/3}z)}, \qquad \operatorname{Re} z \ge 0 \tag{3.8}$$

(and, in fact, for $\operatorname{Re} z > a_1$). This formula for the Laplace transform is (in a more general version) given by Groeneboom [10, Theorem 2.1], where $e^{t^3/6} f_\tau(t)$ is denoted $h_{1/2,x}(t)$; see also (5.5) below. Conversely, this formula from [10] yields by Fourier inversion (3.6) and (3.4), so we could have used it instead of (3.2) from [21] as our starting point. We give an alternative proof of (3.8) in Appendix D, which thus gives us a self-contained proof of Lemma 3.2. (Groeneboom [10], Salminen [21] and our Appendix D use similar methods. See also Appendix C for another approach.)

Remark 3.4. For our purposes we consider only $x > 0$ in (3.1). For $x < 0$, the hitting time is a.s. finite; its distribution is found in Martin-Löf [20].

Lemma 3.5. *For $x > 0$,*

$$G_N(x) = \frac{1}{2i} \int_\Gamma \frac{\operatorname{Hi}(z)\operatorname{Ai}(z + 2^{1/3}x)}{\operatorname{Ai}(z)}\,\mathrm{d}z. \tag{3.9}$$

Proof. We have $G_N(x) = \mathbb{P}(\tau < \infty) = \int_0^\infty f_\tau(t)\,\mathrm{d}t$. Now integrate (3.4) with respect to t and interchange the order of integration, which is allowed by Fubini's theorem and the estimate Lemma A.1, which implies that for $z \in \Gamma$, the integrand in (3.4) is bounded by $O\bigl(\exp(-cx|z|^{1/2} - t^3/6)\bigr)$. The result follows by (A.16) (and a change of variables $t = 2^{1/3}t_1$). □

The integral in (3.9) is absolutely convergent and converges rapidly for any fixed $x > 0$ by Lemma A.1 and (A.21). We denote the integrand in (3.9) by $\Psi(z) = \Psi(z; x)$.

Proof of Theorem 2.2. Fix $x > 0$. By (A.21) and Lemmas A.1 and A.2, for $z = R_N e^{i(\varphi+\pi)}$ with $|\varphi| \le \frac{\pi}{2}$, $\Psi(z) = O\bigl(|z|^{-1}\exp\bigl(-x|z|^{1/2}|\varphi|/\pi\bigr)\bigr)$. It follows that $\int_{\Gamma_N} \Psi(z)\,\mathrm{d}z - \int_\Gamma \Psi(z)\,\mathrm{d}z \to 0$ as $N \to \infty$, and thus $G_N(z) = \frac{1}{2i}\lim_{N\to\infty} \int_{\Gamma_N} \Psi(z)\,\mathrm{d}z$. Evaluating $\int_{\Gamma_N}$ by residues, we obtain (2.10).

To obtain (2.11), we take out the first term in the expansion (A.20) of $\mathrm{Hi}(z)$, and write (3.9) as

$$G_N(x) = \frac{1}{2i}\int_\Gamma \frac{(\mathrm{Hi}(z)+\pi^{-1}z^{-1})\mathrm{Ai}(z+2^{1/3}x)}{\mathrm{Ai}(z)}\,\mathrm{d}z - \frac{1}{2i\pi}\int_\Gamma \frac{\mathrm{Ai}(z+2^{1/3}x)}{z\mathrm{Ai}(z)}\,\mathrm{d}z.$$

The first integral can be converted to a sum of residues by the argument just given for (3.9), which yields the sum in (2.11). Indeed, we have better estimates now, and the resulting sum is absolutely convergent as seen in Remark 2.3.

For the second integral we instead close the contour on the right, by a large circular arc $\{Re^{it}\}$ for t from $\pi-\theta_0$ to $-(\pi-\theta_0)$; it follows by Lemma A.1 that the error tends to 0 as $R\to\infty$. Inside this closed contour, Ai has no zeros, so the only pole is at $z=0$ where the residue is $\mathrm{Ai}(2^{1/3}x)/\mathrm{Ai}(0)$. The result follows, noting that we go around this contour in the negative direction. □

Remark 3.6. We may also use the expansion (A.20) of Hi with more terms. In general, subtracting the sum with L terms in (A.20) from Hi in (3.9) yields an integral that can be converted to a sum of residues as above; this sum is similar to the ones in (2.10) and (2.11), and the terms are now of order k^{-1-2L}. We also have the integral with the subtracted terms; this is a linear combination of terms of the type $\int_\Gamma z^{-n-1}\mathrm{Ai}(z+2^{1/3}x)/\mathrm{Ai}(z)$, which as above equals $-2\pi i$ times the residue at 0, so this integral can be written as a combination of derivatives of Ai at $2^{1/3}x$ and 0; by the equation $\mathrm{Ai}''(z)=z\mathrm{Ai}(z)$, and successive derivations of this equation, the result can be written as $p_1(x)\mathrm{Ai}(2^{1/3}x)+p_2(x)\mathrm{Ai}'(2^{1/3}x)$ for some polynomials p_1 and p_2 (depending on L), whose coefficients are rational functions in $\mathrm{Ai}(0)$ and $\mathrm{Ai}'(0)$. We leave the details to the reader.

Proof of Theorem 2.4. If $|\arg z|<\pi-\delta$ and $|z|\ge 1$, say, then by Lemma A.3, $|\mathrm{Ai}'(z)/\mathrm{Ai}(z)|\asymp|z|^{1/2}$. More generally, using also $\mathrm{Ai}''(z)=z\mathrm{Ai}(z)$, and differentiating this equation further, by induction, $\mathrm{Ai}^{(m)}(z)/\mathrm{Ai}(z)=O(|z|^{m/2})$ for every fixed $m\ge 0$.

It follows by Lemma A.1 and (A.21) that for every fixed $m\ge 0$, x in a fixed interval (x_0,x_1) with $0<x_0<x_1$, and $z\in\Gamma$,

$$\frac{\partial^m}{\partial x^m}\Psi(z;x) = O\left(|z|^{-1}|z+x|^{m/2}e^{-cx|z|^{1/2}}\right) = O\left(|z|^{m/2-1}e^{-cx_0|z|^{1/2}}\right).$$

Consequently, we can differentiate (3.9) under the integral sign an arbitrary number of times; this shows that G is infinitely differentiable on $(0,\infty)$ and that N has an infinitely differentiable density $f_N=-G'$ given by

$$f_N(x) = -G'(x) = \frac{2^{1/3}i}{2}\int_\Gamma \frac{\mathrm{Hi}(z)\mathrm{Ai}'(z+2^{1/3}x)}{\mathrm{Ai}(z)}\,\mathrm{d}z. \tag{3.10}$$

This integral can be evaluated as a sum of residues as for (2.11) in the proof of Theorem 2.2 above, by first adding $\pi^{-1}z^{-1}$ to $\mathrm{Hi}(z)$, which yields (2.12). Alternatively, and perhaps simpler, (A.5) and (A.31) imply that, uniformly for $0 \le x \le x_0$ for any fixed $x_0 > 0$, $\mathrm{Ai}'(a_k+2^{1/3}x) = O(|a_k|^{1/4}) = O(\mathrm{Ai}'(a_k))$, and thus the terms in the sum in (2.12) are $O(|\pi\mathrm{Hi}(a_k)+a_k^{-1}|) = O(|a_k|^{-4}) = O(k^{-8/3})$. Hence we can integrate the sum in (2.12) termwise; equivalently, we can differentiate (2.11) termwise, which yields (2.12).

The result for M and (2.13) follow from $F_M(x) = F_N(x)^2 = (1-G_N(x))^2$. ∎

To see that the sum (2.14) does not converge for any $x > 0$, let $y := 2^{1/3}x$. Take $x = |a_k| - y$ in (A.5). Since then, by Taylor's formula and (A.30),

$$\tfrac{2}{3}(|a_k| - y)^{3/2} = \tfrac{2}{3}|a_k|^{3/2} - y|a_k|^{1/2} + o(1) = \frac{\pi(4k-1)}{4} - y(3\pi k/2)^{1/3} + o(1),$$

we obtain

$$\mathrm{Ai}'(a_k + y) = -\pi^{-1/2}|a_k|^{1/4}\big(\cos(\pi k - y(3\pi k/2)^{1/3} + o(1)) + o(1)\big)$$

and thus

$$\frac{\mathrm{Ai}'(a_k+y)}{\mathrm{Ai}'(a_k)} = \frac{\cos(\pi k - y(3\pi k/2)^{1/3}) + o(1)}{\cos(\pi k) + o(1)} = \cos(y(3\pi k/2)^{1/3}) + o(1).$$

Let I_n be the interval $[c(2\pi n)^3, c(2\pi n + 1)^3]$, with $c := 2y^{-3}/(3\pi)$. For $k \in I_n$, we have $y(3\pi k/2)^{1/3} \in [2\pi n, 2\pi n+1]$ and thus $\mathrm{Ai}'(a_k+y)/\mathrm{Ai}'(a_k) \ge \cos 1 + o(1) > 0.5$, if n is large. Further, by (A.19) and (A.30), $\mathrm{Hi}(a_k) \sim \pi^{-1}|a_k|^{-1} \sim \pi^{-1}(3\pi/2)^{-2/3}k^{-2/3}$. Hence the term in (2.14), t_k say, satifies, for some constants $c_1, c_2 > 0$ and $k \in I_n$,

$$t_k \ge c_1 k^{-2/3} \ge c_2 n^{-2}.$$

Since there are $\Theta(n^2)$ integers in I_n, the sum over them is $\Theta(1)$, and thus the sum in (2.14) diverges. (The case $x = 0$ is simple.)

4. Moments

Lemma 4.1. *For every fixed $p \ge 1$, uniformly in $a \ge 0$,*

$$\int_0^\infty x^{p-1}|\mathrm{Ai}(x-a)|\,\mathrm{d}x = O(a^{p-1/4} + 1), \tag{4.1}$$

$$\int_0^\infty x^{p-1}|\mathrm{Ai}(x-a)|^2\,\mathrm{d}x = O(a^{p-1/2} + 1). \tag{4.2}$$

Proof. For $0 \le x < a$ we have $x^{p-1} \le a^{p-1}$ and $|\mathrm{Ai}(x-a)| = O((a-x)^{-1/4})$ by (A.4), so

$$\int_0^a x^{p-1}|\mathrm{Ai}(x-a)|\,\mathrm{d}x = O\left(a^{p-1}\int_0^a (a-x)^{-1/4}\,\mathrm{d}x\right) = O\big(a^{p-1}a^{3/4}\big),$$

and similarly $\int_0^a x^{p-1}|\mathrm{Ai}(x-a)|^2\,\mathrm{d}x = O\big(a^{p-1}a^{1/2}\big)$. For larger x we use the rapid decrease in (A.1), which implies

$$\int_a^\infty x^{p-1}|\mathrm{Ai}(x-a)|\,\mathrm{d}x = \int_0^\infty (y+a)^{p-1}|\mathrm{Ai}(y)|\,\mathrm{d}y = O(1 + a^{p-1}),$$

and similarly for $\int_a^\infty x^{p-1}|\mathrm{Ai}(x-a)|^2\,\mathrm{d}x$. The result follows. ∎

Proof of Theorem 2.6. Write (2.11) as

$$G_N(2^{-1/3}x) = \sum_{k=0}^{\infty} c(k)\mathrm{Ai}(x+a_k), \tag{4.3}$$

where we for convenience define $a_0 = 0$ and

$$\begin{aligned} c(0) &:= \frac{1}{\mathrm{Ai}(0)}, \\ c(k) &:= \frac{\pi\mathrm{Hi}(a_k)+a_k^{-1}}{\mathrm{Ai}'(a_k)} = \frac{\varphi(k)}{\mathrm{Ai}'(a_k)}, \qquad k \geq 1. \end{aligned}$$

By (A.22), (A.31) and (A.30),

$$|c(k)| = O\bigl(|a_k|^{-4-1/4}\bigr) = O\bigl(k^{-17/6}\bigr), \qquad k \geq 1. \tag{4.4}$$

By (4.1), $\|\mathrm{Ai}(x+a_k)\|_{L^1((0,\infty),\,\mathrm{d}x)} = O(|a_k|^{3/4}) = O(k^{1/2})$, $k \geq 1$, and thus the sum in (4.3) converges absolutely in $L^1((0,\infty),\,\mathrm{d}x)$, so it may be integrated termwise. Consequently, using (2.8) and (A.23),

$$\mathbb{E}\,N = \int_0^\infty G_N(x)\,\mathrm{d}x = 2^{-1/3}\int_0^\infty G_N(2^{-1/3}x)\,\mathrm{d}x = 2^{-1/3}\sum_{k=0}^{\infty} c(k)\mathrm{AI}(a_k).$$

We have $\mathrm{AI}(0) = \int_0^\infty \mathrm{Ai}(x)\,\mathrm{d}x = 1/3$ [1, 10.4.82], see (A.23), and, for $k \geq 1$, $\mathrm{AI}(a_k) = -\pi\mathrm{Ai}'(a_k)\mathrm{Gi}(a_k)$ by (A.26) since $\mathrm{Ai}(a_k) = 0$. Thus (2.16) follows, and so does (2.17) by (A.17).

Lemma 4.1 and (4.4) similarly imply that (4.3) converges absolutely also in $L^2((0,\infty),\,\mathrm{d}x)$. Hence, the integral $\int_0^\infty G_N(x)^2\,\mathrm{d}x$ can be obtained by termwise integration in (4.3), using (B.5), (B.16), (B.20), (B.24):

$$\begin{aligned} \int_0^\infty G_N(x)^2\,\mathrm{d}x &= 2^{-1/3}\int_0^\infty G_N(2^{-1/3}x)^2\,\mathrm{d}x \\ &= 2^{-1/3}\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} c(k)c(\ell)\int_0^\infty \mathrm{Ai}(x+a_k)\mathrm{Ai}(x+a_\ell)\,\mathrm{d}x \\ &= 2^{-1/3}c(0)^2(\mathrm{Ai}'(0))^2 - 2^{-1/3}2\sum_{k=1}^{\infty} c(0)c(k)\frac{\mathrm{Ai}(0)\mathrm{Ai}'(a_k)}{a_k} \\ &\qquad + 2^{-1/3}\sum_{k=1}^{\infty} c(k)^2(\mathrm{Ai}'(a_k))^2 \\ &= 2^{-1/3}\left(\frac{\mathrm{Ai}'(0)}{\mathrm{Ai}(0)}\right)^2 - 2^{2/3}\sum_{k=1}^{\infty}\frac{\varphi(k)}{a_k} + 2^{-1/3}\sum_{k=1}^{\infty}\varphi(k)^2. \end{aligned}$$

Thus, by (2.9) and (2.17),

$$\mathbb{E}\,M = 2\int_0^\infty G_N(x)\,\mathrm{d}x - \int_0^\infty G_N(x)^2\,\mathrm{d}x = 2\,\mathbb{E}\,N - \int_0^\infty G_N(x)^2\,\mathrm{d}x$$

$$= \frac{2^{2/3}}{3\mathrm{Ai}(0)} - 2^{-1/3}\left(\frac{\mathrm{Ai}'(0)}{\mathrm{Ai}(0)}\right)^2 - \frac{1}{2^{1/3}}\sum_{k=1}^{\infty}\varphi(k)\left[2\pi\mathrm{Bi}(a_k) - 2\pi\mathrm{Hi}(a_k) - \frac{2}{a_k} + \varphi(k)\right],$$

and (2.18) follows by the definition of $\varphi(k)$.

Similarly, (4.3) converges absolutely in $L^1((0,\infty), x\,\mathrm{d}x)$ and $L^2((0,\infty), x\,\mathrm{d}x)$ too, and termwise integration in (4.3) yields, using (B.15), (A.26), (B.6), (B.25), (B.19), (B.21),

$$\begin{aligned}
\int_0^\infty G_N(x)x\,\mathrm{d}x &= 2^{-2/3}\int_0^\infty G_N(2^{-1/3}x)x\,\mathrm{d}x \\
&= 2^{-2/3}\sum_{k=0}^{\infty}\int_0^\infty c(k)\mathrm{Ai}(x+a_k)x\,\mathrm{d}x \\
&= 2^{-2/3}\sum_{k=0}^{\infty}c(k)\big(-\mathrm{Ai}'(a_k) - a_k\mathrm{AI}(a_k)\big) \\
&= -\frac{\mathrm{Ai}'(0)}{2^{2/3}\mathrm{Ai}(0)} + 2^{-2/3}\sum_{k=1}^{\infty}\varphi(k)\big(-1+\pi a_k\mathrm{Gi}(a_k)\big); \\
\int_0^\infty G_N(x)^2x\,\mathrm{d}x &= 2^{-2/3}\int_0^\infty G_N(2^{-1/3}x)^2x\,\mathrm{d}x \\
&= 2^{-2/3}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}c(k)c(j)\int_0^\infty x\mathrm{Ai}(x+a_k)\mathrm{Ai}(x+a_j)\,\mathrm{d}x \\
&= -\frac{\mathrm{Ai}'(0)}{2^{2/3}3\mathrm{Ai}(0)} + \frac{2}{2^{2/3}}\sum_{k=1}^{\infty}\varphi(k)\left[-\frac{2}{a_k^3} - \frac{2\mathrm{Ai}'(0)}{\mathrm{Ai}(0)a_k^2}\right] \\
&\qquad + \frac{1}{2^{2/3}}\left[-\sum_{k=1}^{\infty}\varphi(k)^2\frac{2}{3}a_k - 2\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}[\![k\neq j]\!]\frac{\varphi(k)\varphi(j)}{(a_k-a_j)^2}\right].
\end{aligned}$$

By (2.8), $\mathbb{E}\,N^2 = 2\int_0^\infty G_N(x)x\,\mathrm{d}x$, and (2.19)–(2.20) follow. Similarly, $\mathbb{E}\,M^2 = 4\int_0^\infty G_N(x)x\,\mathrm{d}x - 2\int_0^\infty G_N(x)^2x\,\mathrm{d}x = 2\,\mathbb{E}\,N^2 - 2\int_0^\infty G_N(x)^2x\,\mathrm{d}x$, and (2.21) follows. The numerical evaluation is done by `Maple`, using the method discussed in Section 6. □

Remark 4.2. The formula above for $\int_0^\infty G_N(x)^2\,\mathrm{d}x$ may be simplified. In fact, $(\mathrm{Ai}'(0)/\mathrm{Ai}(0))^2 = \sum_{k=1}^\infty a_k^{-2}$ by (B.27), and thus the formula can be written

$$\int_0^\infty G_N(x)^2\,\mathrm{d}x = 2^{-1/3}\sum_{k=1}^{\infty}(\varphi(k)-1/a_k)^2 = 2^{-1/3}\pi^2\sum_{k=1}^{\infty}\mathrm{Hi}(a_k)^2. \qquad (4.5)$$

This can be seen as an instance of Parseval's formula, see Remark B.1. However, although simpler than our expression above, this sum converges more slowly and is less suitable for our purposes.

Remark 4.3. As is well-known, see [1, 10.4.4–5], $\mathrm{Ai}(0) = 3^{-2/3}/\Gamma(2/3) = 3^{-1/6}\Gamma(1/3)/(2\pi)$ and $\mathrm{Ai}'(0) = -3^{-1/3}/\Gamma(1/3) = -3^{1/6}\Gamma(2/3)/(2\pi)$. We prefer to keep $\mathrm{Ai}(0)$ and $\mathrm{Ai}'(0)$ in our formulas.

Remark 4.4. Higher moments can be computed by the same method, with Airy integrals evaluated as shown in Appendix B, but in order to get convergence, one may have to use a version of (2.11) with more terms taken out of the expansion (A.20) of Hi, as discussed in Remark 3.6. We do not pursue the details.

We can also give integral formulas based on Lemma 3.5.

Theorem 4.5. *The moments of M and N are given by, for any real $p > 0$,*

$$\begin{aligned}
\mathbb{E}\, N^p &= -p2^{-p/3-1} i \int_\Gamma \int_0^\infty x^{p-1}\mathrm{Ai}(z+x)\,\mathrm{d}x\, \frac{\mathrm{Hi}(z)}{\mathrm{Ai}(z)}\,\mathrm{d}z,\\
\mathbb{E}\, M^p &= -p2^{-p/3} i \int_\Gamma \int_0^\infty x^{p-1}\mathrm{Ai}(z+x)\,\mathrm{d}x\, \frac{\mathrm{Hi}(z)}{\mathrm{Ai}(z)}\,\mathrm{d}z\\
&\quad + p2^{-p/3-2} \int_\Gamma \int_\Gamma \int_0^\infty x^{p-1}\mathrm{Ai}(z+x)\mathrm{Ai}(w+x)\,\mathrm{d}x\, \frac{\mathrm{Hi}(z)}{\mathrm{Ai}(z)}\frac{\mathrm{Hi}(w)}{\mathrm{Ai}(w)}\,\mathrm{d}z\,\mathrm{d}w.
\end{aligned}$$

Proof. Immediate from (2.8)–(2.9) and Lemma 3.5 (with a change of variables $x \to 2^{-1/3}x$). The double and triple integrals converge absolutely by Lemma A.1 and (A.21). $\square$

For integer p, the integrals over x in Theorem 4.5 can be evaluated by the formulas in Appendix B. In particular, by (B.1) and (B.22),

$$\begin{aligned}
\mathbb{E}\, M &= -2^{-1/3} i \int_\Gamma \frac{\mathrm{AI}(z)\mathrm{Hi}(z)}{\mathrm{Ai}(z)}\,\mathrm{d}z\\
&\quad + 2^{-7/3}\int_\Gamma\int_\Gamma \frac{\mathrm{Ai}(z)\mathrm{Ai}'(w)-\mathrm{Ai}'(z)\mathrm{Ai}(w)}{z-w}\,\frac{\mathrm{Hi}(z)\mathrm{Hi}(w)}{\mathrm{Ai}(z)\mathrm{Ai}(w)}\,\mathrm{d}z\,\mathrm{d}w. \qquad (4.6)
\end{aligned}$$

Although there is no singularity when $z = w$ in the double integral in (4.6), it may be advantageous to use different, disjoint, contours for z and w. Remember that $\Gamma = \Gamma(\theta_0, x_0)$. We choose $\theta_1 \in (\theta_0, \pi/2)$ and $x_1 \in (x_0, 0)$, and let $\Gamma' := \Gamma(\theta_1, x_1)$. Then Γ and Γ' are disjoint; moreover, if $z \in \Gamma$ and $w \in \Gamma'$, then

$$|z-w| \geq c\max(|z|,|w|) \qquad (4.7)$$

for some $c > 0$. Furthermore, (4.7) holds also if $z \in \Gamma_n$ and $w \in \Gamma'_M$ with $M \geq 2N$. We can replace the double integrals $\int_\Gamma\int_\Gamma$ in Theorem 4.5 and (4.6) by $\int_\Gamma\int_{\Gamma'}$. Taking residues, this leads to another formula with sums over Airy zeros.

Theorem 4.6.

$$\mathbb{E}\, N = 2^{-1/3}\pi^2 \sum_{k=1}^\infty \mathrm{Hi}(a_k)(\mathrm{Hi}(a_k)-\mathrm{Bi}(a_k)), \qquad (4.8)$$

$$\mathbb{E}\, M = 2^{-1/3}\pi^2 \sum_{k=1}^{\infty} \mathrm{Hi}(a_k)(\mathrm{Hi}(a_k) - 2\mathrm{Bi}(a_k)). \tag{4.9}$$

These formulas are closely related to (2.17)–(2.18). They are simpler, but less suitable for numerical calculations since they do not even converge absolutely; the terms in the sums decrease as $k^{-5/6}$ by (A.32) and (A.33). (However, they alternate in sign, and the sums converge.) The formulas (4.8) and (4.9) are what we obtain if we substitute (2.10) in (2.8) and (2.9) (with $p = 1$) and integrate termwise; however, since the resulting sums are not absolutely convergent, termwise integration has to be justified carefully, and we use a detour via complex integration.

Proof. Let

$$Q(z, w) := \int_0^\infty \mathrm{Ai}(z + x)\mathrm{Ai}(w + x)\,\mathrm{d}x.$$

The integral converges absolutely by (A.1) for all complex z and w, uniformly in compact sets, and thus Q is an entire function of two variables; moreover, (B.5) and (B.9) yield the explicit formulas

$$Q(z, z) = \mathrm{Ai}'(z)^2 - z\mathrm{Ai}(z)^2, \tag{4.10}$$

$$Q(z, w) = \frac{\mathrm{Ai}(z)\mathrm{Ai}'(w) - \mathrm{Ai}'(z)\mathrm{Ai}(w)}{z - w}, \qquad z \neq w. \tag{4.11}$$

By Theorem 4.5 and (4.6), using Γ' as discussed above,

$$\mathbb{E}\, N = -2^{-4/3} i \int_\Gamma \frac{\mathrm{AI}(z)\mathrm{Hi}(z)}{\mathrm{Ai}(z)}\,\mathrm{d}z,$$

$$\mathbb{E}\, M = -2^{-1/3} i \int_\Gamma \frac{\mathrm{AI}(z)\mathrm{Hi}(z)}{\mathrm{Ai}(z)}\,\mathrm{d}z + 2^{-7/3} \int_\Gamma \int_{\Gamma'} \frac{Q(z, w)\mathrm{Hi}(z)\mathrm{Hi}(w)}{\mathrm{Ai}(z)\mathrm{Ai}(w)}\,\mathrm{d}z\,\mathrm{d}w.$$

First consider the simple integral, $\int_\Gamma \Phi(z)\,\mathrm{d}z$ say. It follows from Lemma A.3 and (A.21) that $\int_\Gamma \Phi(z)\,\mathrm{d}z - \int_{\Gamma_n} \Phi(z) \to 0$ as $n \to \infty$, and we find by the residue theorem applied to Γ_n, letting $n \to \infty$, together with (A.26) and (A.17),

$$\begin{aligned}
\int_\Gamma \frac{\mathrm{AI}(z)\mathrm{Hi}(z)}{\mathrm{Ai}(z)}\,\mathrm{d}z &= 2\pi i \sum_{k=1}^{\infty} \frac{\mathrm{AI}(a_k)\mathrm{Hi}(a_k)}{\mathrm{Ai}'(a_k)} \\
&= -2\pi^2 i \sum_{k=1}^{\infty} \mathrm{Gi}(a_k)\mathrm{Hi}(a_k) \\
&= -2\pi^2 i \sum_{k=1}^{\infty} \mathrm{Bi}(a_k)\mathrm{Hi}(a_k) + 2\pi^2 i \sum_{k=1}^{\infty} \mathrm{Hi}(a_k)^2.
\end{aligned}$$

The sums converge, e.g. by the argument just given, but only the final sum $\sum \mathrm{Hi}(a_k)^2$ converges absolutely, by (A.32)–(A.34). This yields the result (4.8) for N.

Next consider the double integral. If $z \in \Gamma$ and $w \in \Gamma'$, or $z \in \Gamma_n$ and $w \in \Gamma'_m$ with $m \geq 2n$, then by (4.7), (4.11) and Lemma A.3,

$$\begin{aligned}\left|\frac{Q(z,w)}{\mathrm{Ai}(z)\mathrm{Ai}(w)}\right| &\leq \frac{C}{|z|+|w|}\left(\left|\frac{\mathrm{Ai}'(z)}{\mathrm{Ai}(z)}\right| + \left|\frac{\mathrm{Ai}'(w)}{\mathrm{Ai}(w)}\right|\right)\\ &\leq \frac{C}{|z|+|w|}\bigl(|z|^{1/2}+|w|^{1/2}\bigr) \leq C|z|^{-1/4}|w|^{-1/4}.\end{aligned}$$

It follows that $\int_\Gamma\int_{\Gamma'} - \int_{\Gamma_n}\int_{\Gamma'_m} \to 0$ as $m \geq 2n \to \infty$. Using the residue theorem for first $\int_{\Gamma_n}$ and then $\int_{\Gamma'_m}$, with $m = 2n$, we find that the double integral above equals

$$\lim_{n\to\infty}(2\pi i)^2\sum_{j=1}^{n}\sum_{k=1}^{2n}\frac{Q(a_j,a_k)\mathrm{Hi}(a_j)\mathrm{Hi}(a_k)}{\mathrm{Ai}'(a_j)\mathrm{Ai}'(a_k)}.$$

By (4.11), $Q(a_j,a_k) = 0$ when $j \neq k$, and $Q(a_k,a_k) = \mathrm{Ai}'(a_k)^2$ by (4.10). Hence the double integral equals $-4\pi^2\sum_{k=1}^\infty \mathrm{Hi}(a_k)^2$.

The result (4.9) follows by combining the terms. □

5. Proof of Theorem 2.1

Proof of (2.1). We shall use Groeneboom [10]. Fix $\gamma > 0$. (We may choose e.g. $\gamma = 1/2$ as in other parts of this paper by (1.7), but for ease of comparison with [10], and because we find it instructive to see how the homogeneity works, we write the proof for a general γ.) Fix also $s \geq 0$ and define

$$V_s := \max_{t\geq -s}(W(t)-\gamma t^2). \tag{5.1}$$

Thus, $V_0 = N_\gamma$, and $V_s \nearrow V_\infty = M_\gamma$ as $s \to \infty$.

For $t \geq -s$, the process $W_s(t) := W(t) - W(-s)$ is a Brownian motion, starting at 0 at time $-s$. Define

$$\widetilde{M} = \widetilde{M}_s := \max_{t\geq -s}\bigl(W_s(t)-\gamma t^2\bigr) = V_s - W(-s), \tag{5.2}$$

and let $\tau = \tau_s$ be the (a.s. unique) time with $W_s(\tau) - \gamma\tau^2 = \widetilde{M}$. Note that $\widetilde{M} \geq W_s(-s) - \gamma s^2 = -\gamma s^2$ and $\tau \geq -s$ (strict inequalities hold a.s.) Groeneboom [10, Corollary 3.1] applies to $W_s(t) - \gamma t^2$ (with s replaced by $-s$ and $x = -\gamma s^2$), and shows that τ and $\widetilde{M}$ have a joint density, for $t > -s$ and $y > x = -\gamma s^2$,

$$\begin{aligned}f_{\tau,\widetilde{M}}(t,y) &= \exp\bigl(-\tfrac{2}{3}\gamma^2(t^3+s^3) + 2\gamma s(y+\gamma s^2)\bigr)h_{\gamma,\,y+\gamma s^2}(t+s)k_\gamma(t)\\ &= \exp\bigl(\tfrac{4}{3}\gamma^2 s^3 + 2\gamma s y\bigr)h_{\gamma,\,y+\gamma s^2}(t+s)g_\gamma(t).\end{aligned} \tag{5.3}$$

where the functions $h_{\gamma,\,y+\gamma s^2}$, k_γ and g_γ are given in [10]. Integrating over $t \geq -s$ we find, with $h_{\gamma,a}(t) = 0$ for $t < 0$ and $\check{g}_\gamma(t) := g_\gamma(-t)$, the density $f_{\widetilde{M}}$ of $\widetilde{M}$ as, for $y > -\gamma s^2$,

$$f_{\widetilde{M}}(y) = \int_{-s}^\infty f(t,y)\,\mathrm{d}t = \exp\bigl(\tfrac{4}{3}\gamma^2 s^3 + 2\gamma s y\bigr)\int_{-s}^\infty h_{\gamma,\,y+\gamma s^2}(t+s)g_\gamma(t)\,\mathrm{d}t$$

$$= \exp\bigl(\tfrac{4}{3}\gamma^2 s^3 + 2\gamma s y\bigr) h_{\gamma,\, y+\gamma s^2} * \check{g}_\gamma(s). \tag{5.4}$$

By [10, Theorem 2.1], for $a > 0$, $h_{\gamma,\,a} \geq 0$ and $h_{\gamma,\,a}$ has the Laplace transform, for $\lambda > 0$, (see also Remark 3.3)

$$\int_0^\infty e^{-\lambda u} h_{\gamma,\,a}(u)\,\mathrm{d}u = \frac{\mathrm{Ai}\bigl((4\gamma)^{1/3}a + \xi\bigr)}{\mathrm{Ai}(\xi)}, \quad \xi := (2\gamma^2)^{-1/3}\lambda. \tag{5.5}$$

Letting $\lambda \searrow 0$ we see that

$$\int_0^\infty h_{\gamma,\,a}(u)\,\mathrm{d}u = \mathrm{Ai}\bigl((4\gamma)^{1/3}a\bigr)/\mathrm{Ai}(0) < \infty,$$

so $h_{\gamma,\,a} \in L^1(\mathbb{R})$ and (5.5) holds for all complex λ with $\operatorname{Re}\lambda \geq 0$ by analytic continuation. In particular, $h_{\gamma,\,a}$ has the Fourier transform, see (3.7),

$$\widehat{h_{\gamma,\,a}}(\omega) = \int_{-\infty}^\infty e^{-i\omega u} h_{\gamma,\,a}(u)\,\mathrm{d}u = \frac{\mathrm{Ai}\bigl((4\gamma)^{1/3}a + i(2\gamma^2)^{-1/3}\omega\bigr)}{\mathrm{Ai}\bigl(i(2\gamma^2)^{-1/3}\omega\bigr)}, \qquad \omega \in \mathbb{R}.$$

Furthermore, by (5.3), $g_\gamma \geq 0$, and by [10, Corollary 3.1], $g_\gamma \in L^1(\mathbb{R})$ with the Fourier transform

$$\widehat{g_\gamma}(\omega) = \int_{-\infty}^\infty e^{-i\omega u} g_\gamma(u)\,\mathrm{d}u = \frac{2^{1/3}\gamma^{-1/3}}{\mathrm{Ai}\bigl(-i(2\gamma^2)^{-1/3}\omega\bigr)}, \qquad \omega \in \mathbb{R}. \tag{5.6}$$

Hence, for $y > -\gamma s^2$, $h_{\gamma,\, y+\gamma s^2} * \check{g}_\gamma$ has the Fourier transform

$$\widehat{h_{\gamma,\, y+\gamma s^2}}(\omega)\widehat{g_\gamma}(-\omega) = 2^{1/3}\gamma^{-1/3} \frac{\mathrm{Ai}\bigl((4\gamma)^{1/3}(y+\gamma s^2) + i(2\gamma^2)^{-1/3}\omega\bigr)}{\mathrm{Ai}\bigl(i(2\gamma^2)^{-1/3}\omega\bigr)^2}. \tag{5.7}$$

Since $|\mathrm{Ai}(iy)| \asymp |y|^{-1/4}\exp\bigl(\frac{\sqrt{2}}{3}|y|^{3/2}\bigr)$ as $y \to \pm\infty$ by (A.3), (5.6) implies $|\widehat{g_\gamma}(\omega)| \asymp |\omega|^{1/4}\exp\bigl(-\frac{1}{3\gamma}|\omega|^{3/2}\bigr)$; consequently $\widehat{g_\gamma} \in L^1(\mathbb{R})$. Furthermore, $\widehat{h_{\gamma,\,a}}(\omega)$ is bounded for each a (by $h_{\gamma,\,a} \in L^1$ or by Lemma A.1), and thus the product $\widehat{h_{\gamma,\, y+\gamma s^2}}(\omega)\widehat{g_\gamma}(-\omega) \in L^1(\mathbb{R})$. Consequently, the Fourier inversion formula applies to (5.7), and (5.4) thus yields, for $y > -\gamma s^2$,

$$\begin{aligned} f_{\widetilde{M}}(y) &= \exp\bigl(\tfrac{4}{3}\gamma^2 s^3 + 2\gamma s y\bigr)\frac{2^{1/3}\gamma^{-1/3}}{2\pi} \\ &\qquad \times \int_{-\infty}^\infty e^{is\omega} \frac{\mathrm{Ai}\bigl((4\gamma)^{1/3}(y+\gamma s^2) + i(2\gamma^2)^{-1/3}\omega\bigr)}{\mathrm{Ai}(i(2\gamma^2)^{-1/3}\omega)^2}\,\mathrm{d}\omega \\ &= \frac{(4\gamma)^{1/3}}{2\pi}\exp\bigl(\tfrac{4}{3}\gamma^2 s^3 + 2\gamma s y\bigr)\int_{-\infty}^\infty e^{is(2\gamma^2)^{1/3}v}\frac{\mathrm{Ai}\bigl((4\gamma)^{1/3}(y+\gamma s^2) + iv\bigr)}{\mathrm{Ai}(iv)^2}\,\mathrm{d}v. \end{aligned} \tag{5.8}$$

Multiplying by e^{zy} and integrating, we obtain for any $z \in \mathbb{C}$ the following, where the double integral is absolutely convergent by Lemma A.1,

$$\begin{aligned}
\mathbb{E}\, e^{z\widetilde{M}} &= \int_{-\gamma s^2}^{\infty} e^{zy} f_{\widetilde{M}}(y)\,\mathrm{d}y = \int_0^\infty e^{zx - z\gamma s^2} f_{\widetilde{M}}(x-\gamma s^2)\,\mathrm{d}x \\
&= \frac{(4\gamma)^{1/3}}{2\pi}\int_{x=0}^\infty\int_{v=-\infty}^\infty e^{\frac43\gamma^2 s^3 + 2\gamma s x - 2\gamma^2 s^3 + zx - \gamma s^2 z + is(2\gamma^2)^{1/3}v}\frac{\mathrm{Ai}\big((4\gamma)^{1/3}x + iv\big)}{\mathrm{Ai}(iv)^2}\,\mathrm{d}v\,\mathrm{d}x \\
&= \frac{1}{2\pi}e^{-\frac23\gamma^2 s^3 - \gamma s^2 z}\int_{v=-\infty}^\infty\int_{x=0}^\infty e^{(z+2\gamma s)(4\gamma)^{-1/3}x + is(2\gamma^2)^{1/3}v}\frac{\mathrm{Ai}\big(x+iv\big)}{\mathrm{Ai}(iv)^2}\,\mathrm{d}x\,\mathrm{d}v \\
&= \frac{1}{2\pi}e^{-\frac23\gamma^2 s^3 - \gamma s^2 z}\int_{v=-\infty}^\infty \frac{e^{is(2\gamma^2)^{1/3}v}}{\mathrm{Ai}(iv)^2}\int_{x=0}^\infty e^{(z+2\gamma s)(4\gamma)^{-1/3}x}\mathrm{Ai}\big(x+iv\big)\,\mathrm{d}x\,\mathrm{d}v.
\end{aligned} \tag{5.9}$$

Since Ai is bounded on the negative real axis by (A.4), Lemma B.2 implies that, for $\operatorname{Re} z > 0$,

$$\begin{aligned}
\int_0^\infty e^{zt}\mathrm{Ai}(t)\,\mathrm{d}t &= e^{t^3/3} - \int_{-\infty}^0 e^{zt}\mathrm{Ai}(t)\,\mathrm{d}t = e^{t^3/3} + O\Big(\int_{-\infty}^0 e^{\operatorname{Re} zt}\,\mathrm{d}t\Big) \\
&= e^{t^3/3} + O\Big(\frac{1}{\operatorname{Re} z}\Big).
\end{aligned}$$

Moreover, (A.3) implies that for $y \in \mathbb{R}$ and $z \in \mathbb{C}$,

$$\int_0^{iy} e^{zt}\mathrm{Ai}(w)\,\mathrm{d}w = O\Big(e^{\frac{\sqrt2}{3}|y|^{3/2} + |\operatorname{Im} z||y|}\Big).$$

Hence, if $\operatorname{Re} z \geq (4\gamma)^{-1/3}$, say, and $v \in \mathbb{R}$, then, using Cauchy's integral formula on a large rectangle with vertices $\{0, iy, R, R+iy\}$ and letting $R \to \infty$, using (A.1) to control the tails,

$$\begin{aligned}
\int_0^\infty e^{zx}\mathrm{Ai}(x+iv)\,\mathrm{d}x &= \int_{iv}^{\infty+iv} e^{zw - ivz}\mathrm{Ai}(w)\,\mathrm{d}w \\
&= e^{-ivz}\int_0^\infty e^{zw}\mathrm{Ai}(w)\,\mathrm{d}w - e^{-ivz}\int_0^{iv} e^{zw}\mathrm{Ai}(w)\,\mathrm{d}w \\
&= e^{z^3/3 - ivz} + O\Big(e^{\frac{\sqrt2}{3}|v|^{3/2} + |\operatorname{Im} z||v|}\Big).
\end{aligned}$$

Hence, if $\operatorname{Re} z + 2\gamma s > 1$, (5.9) yields, using (A.1) again for the error term,

$$\begin{aligned}
\mathbb{E}\, e^{z\widetilde{M}} &= \frac{1}{2\pi}e^{-\frac23\gamma^2 s^3 - \gamma s^2 z}\int_{-\infty}^\infty \frac{e^{is(2\gamma^2)^{1/3}v}}{\mathrm{Ai}(iv)^2} e^{(z+2\gamma s)^3/(12\gamma) - iv(z+2\gamma s)(4\gamma)^{-1/3}}\,\mathrm{d}v \\
&\qquad + O\Big(e^{-\frac23\gamma^2 s^3 - \gamma s^2 \operatorname{Re} z}\int_{-\infty}^\infty \frac{1}{|\mathrm{Ai}(iv)|^2} e^{\frac{\sqrt2}{3}|v|^{3/2} + |\operatorname{Im} z(4\gamma)^{-1/3}||v|}\,\mathrm{d}v\Big) \\
&= \frac{1}{2\pi}e^{\frac{z^3}{12\gamma} + \frac{z^2 s}{2}}\int_{-\infty}^\infty \frac{e^{-i(4\gamma)^{-1/3}zv}}{\mathrm{Ai}(iv)^2}\,\mathrm{d}v + O\big(e^{-\frac23\gamma^2 s^3 - \gamma s^2 \operatorname{Re} z + O(|\operatorname{Im} z|^2)}\big)
\end{aligned}$$

In particular, if $2\gamma s > 2$, say, this holds uniformly for $|z| < 1$, and we may differentiate the analytic functions at $z = 0$ and obtain

$$\mathbb{E}\,\widetilde{M} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{-i(4\gamma)^{-1/3}v}{\operatorname{Ai}(iv)^2}\,\mathrm{d}v + O\bigl(e^{-\frac{2}{3}\gamma^2 s^3+\gamma s^2}\bigr).$$

Since $\mathbb{E}\,V_s = \mathbb{E}(\widetilde{M} + W(-s)) = \mathbb{E}\,\widetilde{M}$ by (5.2), we find by letting $s \to \infty$ and choosing $\gamma = 1/2$,

$$\mathbb{E}\,M = \lim_{s\to\infty}\mathbb{E}\,V_s = \frac{2^{-1/3}}{2\pi i}\int_{-\infty}^{\infty}\frac{v}{\operatorname{Ai}(iv)^2}\,\mathrm{d}v,$$

which is (2.1). $\square$

Remark 5.1. Setting $s = 0$ and $\gamma = 1/2$ in (5.8), we obtain another formula for the density of N: For $y > 0$,

$$f_N(y) = \frac{2^{1/3}}{2\pi}\int_{-\infty}^{\infty}\frac{\operatorname{Ai}\bigl(2^{1/3}y + iv\bigr)}{\operatorname{Ai}(iv)^2}\,\mathrm{d}v. \tag{5.10}$$

By residue calculus, as with similar integrals in e.g. the proof of Theorem 2.2, this may be written as a sum of residues of $2^{1/3}\operatorname{Ai}(2^{1/3}y + z)/\operatorname{Ai}(z)^2$ at the poles a_k; however, now the poles are double and we omit the details.

The integral formulas (2.1)–(2.4) can be transformed into each other by properties of the Airy functions as follows.

Proof of (2.4). The integrand in (2.1) is analytic except at the zeros of Ai, which lie on the negative real axis. Furthermore, by (A.1), $|\operatorname{Ai}(z)|$ is exponentially large, so the integrand in (2.1) is exponentially small, as $|z| \to \infty$ with $\pi/3 + \delta \le |\arg(z)| \le \pi - \delta$; in particular when $\pi/2 \le |\arg(z)| \le 2\pi/3$. Hence, we can deform the integration path from the imaginary axis to any reasonable path in this domain. We choose to integrate along the two rays from the origin with $\arg z = \pm 2\pi/3$, and obtain thus

$$\mathbb{E}\,M = -\frac{2^{-1/3}}{2\pi i}\int_{-e^{-2\pi i/3}\infty}^{e^{2\pi i/3}\infty}\frac{z\,\mathrm{d}z}{\operatorname{Ai}(z)^2} = -\frac{2^{-1/3}}{2\pi i}\int_0^{\infty}\sum_{\pm}\pm\frac{e^{\pm 4\pi i/3}t\,\mathrm{d}t}{\operatorname{Ai}(e^{\pm 2\pi i/3}t)^2}, \tag{5.11}$$

which by the formula [1, 10.4.9]

$$\operatorname{Ai}(ze^{\pm 2\pi i/3}) = \tfrac{1}{2}e^{\pm\pi i/3}\bigl(\operatorname{Ai}(z) \mp i\operatorname{Bi}(z)\bigr) \tag{5.12}$$

yields

$$\begin{aligned}
\mathbb{E}\,M &= -\frac{2^{-1/3}}{2\pi i}\int_0^\infty \sum_{\pm} \pm 4\frac{e^{\pm 2\pi i/3}t\,\mathrm{d}t}{\big(\mathrm{Ai}(t)\mp i\mathrm{Bi}(t)\big)^2} \\
&= -\frac{2^{2/3}}{\pi i}\int_0^\infty \sum_{\pm} \pm\frac{e^{\pm 2\pi i/3}\big(\mathrm{Ai}(t)\pm i\mathrm{Bi}(t)\big)^2}{\big(\mathrm{Ai}(t)^2+\mathrm{Bi}(t)^2\big)^2}\,t\,\mathrm{d}t \\
&= -\frac{2^{2/3}}{\pi i}\int_0^\infty \sum_{\pm} \pm\frac{(-1\pm\sqrt{3}i)\big(\mathrm{Ai}(t)^2-\mathrm{Bi}(t)^2\pm 2i\mathrm{Ai}(t)\mathrm{Bi}(t)\big)}{2\big(\mathrm{Ai}(t)^2+\mathrm{Bi}(t)^2\big)^2}\,t\,\mathrm{d}t \\
&= -\frac{2^{2/3}}{\pi}\int_0^\infty \frac{\sqrt{3}\mathrm{Ai}(t)^2-\sqrt{3}\mathrm{Bi}(t)^2-2\mathrm{Ai}(t)\mathrm{Bi}(t)}{\big(\mathrm{Ai}(t)^2+\mathrm{Bi}(t)^2\big)^2}\,t\,\mathrm{d}t,
\end{aligned}$$

which proves (2.4). □

Proof of (2.2). To prove (2.2), we as above deform the integration path in (2.1) and integrate along the two rays from the origin with $\arg z = \pm 2\pi/3$ and obtain (5.11). We now use the indefinite integral

$$\int \frac{\mathrm{d}z}{\mathrm{Ai}(z)^2} = \pi\frac{\mathrm{Bi}(z)}{\mathrm{Ai}(z)} \tag{5.13}$$

given by [3] (and easily verified by derivation, using the Wronskian $\mathrm{Ai}(z)\mathrm{Bi}'(z)-\mathrm{Ai}'(z)\mathrm{Bi}(z)=1/\pi$ [1, 10.4.10]). We have by (5.12) and [1, 10.4.6]

$$\mathrm{Bi}(ze^{2\pi i/3}) = e^{\pi i/6}\mathrm{Ai}(ze^{-2\pi i/3}) + e^{-\pi i/6}\mathrm{Ai}(z) = \tfrac{1}{2}e^{-\pi i/6}\big(3\mathrm{Ai}(z)+i\mathrm{Bi}(z)\big)$$

and thus, by (5.12) again,

$$\frac{\mathrm{Bi}(ze^{2\pi i/3})}{\mathrm{Ai}(ze^{2\pi i/3})} = e^{-\pi i/2}\frac{3\mathrm{Ai}(z)+i\mathrm{Bi}(z)}{\mathrm{Ai}(z)-i\mathrm{Bi}(z)} = i - 4i\frac{\mathrm{Ai}(z)}{\mathrm{Ai}(z)-i\mathrm{Bi}(z)}.$$

By (A.1) and (A.8), this converges rapidly to i as $z\to\infty$ along the positive real axis. Hence an integration by parts yields

$$\begin{aligned}
\int_0^{e^{2\pi i/3}\infty} \frac{z}{\mathrm{Ai}(z)^2}\,\mathrm{d}z &= \Big[\pi z\Big(\frac{\mathrm{Bi}(z)}{\mathrm{Ai}(z)}-i\Big)\,\mathrm{d}z\Big]_0^{e^{2\pi i/3}\infty} - \int_0^{e^{2\pi i/3}\infty}\pi\Big(\frac{\mathrm{Bi}(z)}{\mathrm{Ai}(z)}-i\Big)\,\mathrm{d}z \\
&= 0 + e^{2\pi i/3}\pi\int_0^\infty 4i\frac{\mathrm{Ai}(t)}{\mathrm{Ai}(t)-i\mathrm{Bi}(t)}\,\mathrm{d}t. && (5.14) \\
&= 2\pi\int_0^\infty (-i-\sqrt{3})\frac{\mathrm{Ai}(t)^2+i\mathrm{Ai}(t)\mathrm{Bi}(t)}{\mathrm{Ai}(t)^2+\mathrm{Bi}(t)^2}\,\mathrm{d}t. && (5.15)
\end{aligned}$$

The integral along the line from $e^{-2\pi i/3}\infty$ to 0 equals -1 times the complex conjugate of the integral in (5.15), so we obtain from (5.11) and (5.15),

$$\mathbb{E}\,M = -\frac{2^{-1/3}}{\pi}\,\mathrm{Im}\int_0^{e^{2\pi i/3}\infty}\frac{z\,\mathrm{d}z}{\mathrm{Ai}(z)^2} = 2^{2/3}\int_0^\infty \frac{\mathrm{Ai}(t)^2+\sqrt{3}\mathrm{Ai}(t)\mathrm{Bi}(t)}{\mathrm{Ai}(t)^2+\mathrm{Bi}(t)^2}\,\mathrm{d}t,$$

which is (2.2). □

Proof of (2.3). Follows similarly by (5.11) and (5.14), we omit the details. $\square$

6. Numerical computation

The sums in Theorem 2.6 converge, but rather slowly. For example, in (2.16), $|\varphi(k)\mathrm{Gi}(k)| = |\varphi(k)|\,|\mathrm{Bi}(k) - \mathrm{Hi}(k)| \sim ck^{-17/6}$ for some $c > 0$, see the asymptotic expansions below.

To obtain numerical values with high accuracy of the sums in (2.16)–(2.21), we therefore use asymptotic expansions of the summands. More precisely, for each sum, we first compute $\sum_{k=1}^{199}$ numerically. (All computations are done by `Maple`.) We then evaluate $\sum_{200}^{\infty}$ by replacing the summands by the first terms in the following asymptotic expansions. (We provide here only one or two terms in each asymptotic expansions, but we use at least 5 terms in our numerical computations.) We note that $\mathrm{Bi}(a_k)$ alternates in sign, so for terms containing it, we group the terms with $k = 2j$ and $k = 2j + 1$ together, for $j \geq 100$. The resulting infinite sums are readily computed numerically. (They can be expressed in the Riemann zeta function at some points.)

We use the expansions, see [1, (10.4.94), (10.4.105)], (A.20), (2.15), (A.9),

$$
\begin{aligned}
|a_k| &\sim \frac{3^{2/3}\pi^{2/3}}{2^{2/3}}\Big(k^{2/3} - \frac{1}{6}k^{-1/3} + \dots\Big)\\
\mathrm{Hi}(a_k) &\sim -\frac{1}{\pi}a_k^{-1} - \frac{2}{\pi}a_k^{-4} + \dots = \frac{2^{2/3}}{3^{2/3}\pi^{5/3}}\Big(k^{-2/3} + \frac{1}{6}k^{-5/3} + \dots\Big)\\
\varphi(k) &\sim -2a_k^{-4} + \dots = -\frac{2^{11/3}}{3^{8/3}\pi^{8/3}}k^{-8/3} + \dots,\\
\mathrm{Bi}(a_k) &\sim (-1)^k\frac{2^{1/6}}{3^{1/6}\pi^{2/3}}k^{-1/6} + \dots,\\
g_1(j) &:= \varphi(2j)\mathrm{Bi}(a_{2j}) + \varphi(2j+1)\mathrm{Bi}(a_{2j+1}) \sim -\frac{17\cdot 3^{1/6}}{162\,\pi^{10/3}}j^{-23/6} + \dots,\\
g_2(j) &:= a_{2j}\varphi(2j)\mathrm{Bi}(a_{2j}) + a_{2j+1}\varphi(2j+1)\mathrm{Bi}(a_{2j+1}) \sim \frac{13\cdot 3^{5/6}}{162\,\pi^{8/3}}j^{-19/6} + \dots.
\end{aligned}
$$

Thus, for example, $\mathbb{E}\,N = S_0^N + S_1^N + S_2^N$, where

$$
\begin{aligned}
S_0^N &= 0.6955290109\dots,\\
S_1^N &:= -\frac{\pi}{2^{1/3}}\sum_{k=200}^{\infty}\varphi(k)\mathrm{Bi}(a_k) = -\frac{\pi}{2^{1/3}}\sum_{j=100}^{\infty}g_1(j) = 0.5317\dots\cdot 10^{-8},\\
S_2^N &:= \frac{\pi}{2^{1/3}}\sum_{k=200}^{\infty}\varphi(k)\mathrm{Hi}(a_k) = -0.16722\dots\cdot 10^{-7},
\end{aligned}
$$

yielding $\mathbb{E}(N) = 0.6955289995\dots$. Similarly, $\mathbb{E}\,M = S_0^M + S_1^M + S_2^M$, with

$$S_0^M = 0.9961930092\dots,$$

$$S_1^M := -\frac{\pi}{2^{1/3}} \sum_{k=200}^{\infty} 2\varphi(k)\mathrm{Bi}(a_k) = 2S_1^N = 0.10635\ldots \cdot 10^{-7},$$
$$S_2^M := 2^{-1/3} \sum_{k=200}^{\infty} \varphi(k)^2 = 0.20\ldots \cdot 10^{-13},$$

yielding $\mathbb{E}\, M = 0.9961930199\ldots$, which fits with (2.5). Actually, we have 15 digits of precision with the expansions we have used.

For the second moments, we compute the sums in (2.20) and (2.21) in the same way. For the double sum in (2.21), we compute the sum with $k \le 199$ exactly, and find using asymptotics the tail sum

$$2^{7/3} \sum_{k=200}^{\infty} \sum_{j=1}^{k-1} \frac{\varphi(k)\varphi(j)}{(a_k - a_j)^2} = 0.4\ldots \cdot 10^{-10}.$$

(We may, for example, use the first term asymptotics for $\varphi(j)$ and a_j for $j \ge 200$, since the sum is small and only a low relative precision is needed.)

The (complementary) distribution function $G_N(x)$ and the density function $f_N(x)$ may, for any given $x > 0$, be computed to high precision from (2.11) and (2.12) in the same way; for the tails of the sums $\sum_{k=200}^{\infty} \varphi(k)\mathrm{Ai}(a_k + 2^{1/3}x)/\mathrm{Ai}'(a_k)$ and $\sum_{k=200}^{\infty} \varphi(k)\mathrm{Ai}'(a_k + 2^{1/3}x)/\mathrm{Ai}'(a_k)$ we use the asymptotic expansion of $\varphi(k)$ given above together with, see [1, (10.4.94), (10.4.97), (10.4.105)] and (A.4)–(A.5),

$$\mathrm{Ai}(a_k + 2^{1/3}x) \sim (-1)^{k+1} \frac{2^{1/6}}{3^{1/6}\pi^{2/3}} \sin\bigl((3\pi)^{1/3} x k^{1/3}\bigr)\, k^{-1/6} + \ldots$$
$$\mathrm{Ai}'(a_k + 2^{1/3}x) \sim (-1)^{k+1} \frac{3^{1/6}}{2^{1/6}\pi^{1/3}} \cos\bigl((3\pi)^{1/3} x k^{1/3}\bigr)\, k^{1/6} + \ldots$$

for fixed x, and, as a special case,

$$\mathrm{Ai}'(a_k) \sim (-1)^{k+1} \frac{3^{1/6}}{2^{1/6}\pi^{1/3}} k^{1/6} + \ldots$$

The distribution and density functions of M then are given by (2.6)–(2.7) and (2.13).

As an illustration, we plot the tail sum (from $k = 200$) for $f_N(x)$ in Figure 4.

Appendix A. Some Airy function estimates

The Airy function $\mathrm{Ai}(x)$ and its derivative have along the positive real axis, and more generally as $|z| \to \infty$ with $|\arg z| < \pi - \delta$ for any fixed $\delta > 0$, asymptotic expansions given in full in [1, 10.4.59 and 10.4.61]; we need only the leading terms:

$$\mathrm{Ai}(z) \sim \tfrac{1}{2\sqrt{\pi}} z^{-1/4} e^{-\frac{2}{3} z^{3/2}}, \tag{A.1}$$

$$\mathrm{Ai}'(z) \sim -\tfrac{1}{2\sqrt{\pi}} z^{1/4} e^{-\frac{2}{3} z^{3/2}}. \tag{A.2}$$

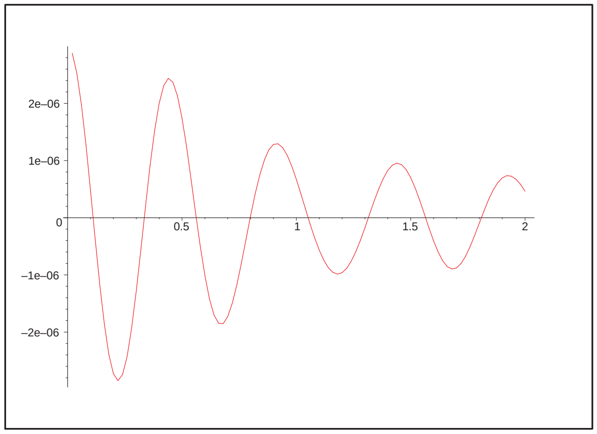

FIGURE 4. Tail sum from $k = 200$ for $f_N(x)$

In particular, along the imaginary axis, for $y \in \mathbb{R}$,

$$|\mathrm{Ai}(iy)| \sim \tfrac{1}{2\sqrt{\pi}}|y|^{-1/4}e^{\frac{\sqrt{2}}{3}|y|^{3/2}}. \tag{A.3}$$

Along the negative real axis, Ai and Ai′ oscillate and have zeros; by [1, (10.4.60), (10.4.62)] (where further terms are given), we have the asymptotic formulas

$$\mathrm{Ai}(-x) = \pi^{-1/2}x^{-1/4}\big(\sin\big(\tfrac{2}{3}x^{3/2} + \tfrac{\pi}{4}\big) + o(1)\big) = O(x^{-1/4}), \tag{A.4}$$

$$\mathrm{Ai}'(-x) = -\pi^{-1/2}x^{1/4}\big(\cos\big(\tfrac{2}{3}x^{3/2} + \tfrac{\pi}{4}\big) + o(1)\big). \tag{A.5}$$

and more generally, as $|z| \to \infty$ in any domain $|\arg z| < \frac{2}{3}\pi - \delta$,

$$\mathrm{Ai}(-z) = \pi^{-1/2}z^{-1/4}\left(\sin\Big(\frac{2}{3}z^{3/2} + \frac{\pi}{4}\Big)\big(1 + O(|z|^{-3/2})\big) + O\big(|z|^{-3/2}\big)\right), \tag{A.6}$$

$$\mathrm{Ai}'(-z) = -\pi^{-1/2}z^{1/4}\left(\cos\Big(\frac{2}{3}z^{3/2} + \frac{\pi}{4}\Big)\big(1 + O(|z|^{-3/2})\big) + O\big(|z|^{-3/2}\big)\right). \tag{A.7}$$

For the companion Airy function $\mathrm{Bi}(x)$, we use the following estimate [1, 10.4.63], valid along the positive real axis, and more generally as $|z| \to \infty$ with $|\arg z| < \pi/3 - \delta$,

$$\mathrm{Bi}(z) \sim \pi^{-1/2}z^{-1/4}e^{\frac{2}{3}z^{3/2}}. \tag{A.8}$$

On the negative axis, we have [1, 10.4.64],

$$\mathrm{Bi}(-x) = \pi^{-1/2}x^{-1/4}\big(\cos\big(\tfrac{2}{3}x^{3/2} + \tfrac{\pi}{4}\big) + o(1)\big). \tag{A.9}$$

Lemma A.1. *For every $\delta > 0$ there exists $c = c(\delta)$ such that if $|\arg z| \le \pi - \delta$ and $x \ge 0$, then*

$$\frac{\mathrm{Ai}(z+x)}{\mathrm{Ai}(z)} = O\Bigl(e^{-cx|z|^{1/2} - cx^{3/2}}\Bigr).$$

Proof. We may assume that $0 < \delta < \pi/2$. It follows from (A.1) that for $|\arg z| \le \pi - \delta$,

$$|\mathrm{Ai}(z)| \asymp (1+|z|)^{-1/4} \exp\bigl(-\tfrac{2}{3}\operatorname{Re}(z^{3/2})\bigr). \tag{A.10}$$

Since $|\arg(z+x)| \le |\arg(z)| \le \pi - \delta$, (A.10) can be used for $z + x$ too. If $\operatorname{Re} z \le 0$, then $|z+x| \ge |\operatorname{Im} z| \ge (\sin\delta)|z|$, and if $\operatorname{Re} z \ge 0$, then $|z+x| \ge |z|$. Hence, (A.10) implies

$$\left|\frac{\mathrm{Ai}(z+x)}{\mathrm{Ai}(z)}\right| = O\Bigl(e^{-\frac{2}{3}\operatorname{Re}((z+x)^{3/2}) + \frac{2}{3}\operatorname{Re}(z^{3/2})}\Bigr).$$

Further,

$$\begin{aligned}\frac{2}{3}\operatorname{Re}((z+x)^{3/2}) - \frac{2}{3}\operatorname{Re}(z^{3/2}) &= \operatorname{Re}\int_0^x (z+t)^{1/2}\,\mathrm{d}t \\ &\ge \cos\frac{\pi-\delta}{2}\int_0^x |z+t|^{1/2}\,\mathrm{d}t.\end{aligned}$$

Moreover, $|z+t| \ge c_1(|z|+t)$ by elementary geometry (use, e.g., the sine theorem on the triangle with vertices $0, z, -t$); hence $|z+t|^{1/2} \ge c_2(|z|+t)^{1/2}$, and the result follows. ∎

Recall that $R_N := \bigl(\frac{3}{2}\pi N\bigr)^{2/3}$.

Lemma A.2. *Let $x \ge 0$ be fixed. Assume that $z = R_N e^{i\theta}$ with $\theta \in [\frac{\pi}{2}, \frac{3\pi}{2}]$ and $N \ge 1$. Then (with implicit constants depending on x but not on N or θ),*

$$\frac{\mathrm{Ai}(z+x)}{\mathrm{Ai}(z)} = O\Bigl(e^{-xR_n^{1/2}|\theta-\pi|/\pi}\Bigr) \tag{A.11}$$

and

$$|\mathrm{Ai}(z)| \asymp |z|^{-1/4} e^{\left|\operatorname{Im}\frac{2}{3}(-z)^{3/2}\right|} = R_N^{-1/4} e^{\frac{2}{3}R_n^{3/2}\sin(3|\varphi|/2)}. \tag{A.12}$$

Proof. By symmetry, it suffices to consider $\theta = \pi + \varphi$ with $0 \le \varphi \le \pi/2$. Thus $z = -R_N e^{i\varphi}$. We have, by Taylor's formula,

$$\frac{2}{3}(-z-x)^{3/2} = \frac{2}{3}(-z)^{3/2} - x(-z)^{1/2} + O\bigl(R_N^{-1/2}\bigr) \tag{A.13}$$

and thus

$$\operatorname{Im}\frac{2}{3}(-z-x)^{3/2} = \frac{2}{3}R_N^{3/2}\sin\frac{3\varphi}{2} - xR_N^{1/2}\sin\frac{\varphi}{2} + O\bigl(R_N^{-1/2}\bigr). \tag{A.14}$$

If $\varphi \ge R_N^{-3/2}$, this shows that, for large N,

$$\operatorname{Im}\frac{2}{3}(-z-x)^{3/2} \ge \frac{4}{3\sqrt{2}\,\pi}R_N^{3/2}\varphi - \frac{1}{2}xR_N^{1/2}\varphi + O\bigl(R_N^{-1/2}\bigr) \ge 0.2,$$

and thus, by (A.6),

$$\begin{aligned} |\operatorname{Ai}(z+x)| &\asymp |z+x|^{-1/4}\left|\sin\!\left(\tfrac{2}{3}(-z-x)^{3/2}+\tfrac{\pi}{4}\right)\right| \\ &\asymp R_N^{-1/4}\exp\!\left(\operatorname{Im}\tfrac{2}{3}(-z-x)^{3/2}\right). \end{aligned} \tag{A.15}$$

In particular, taking $x=0$ we obtain (A.12) for these φ. Moreover, comparing (A.15) with the special case $x=0$, and using (A.14),

$$\left|\frac{\operatorname{Ai}(z+x)}{\operatorname{Ai}(z)}\right| \asymp \exp\!\Big(\operatorname{Im}\tfrac{2}{3}(-z-x)^{3/2}-\operatorname{Im}\tfrac{2}{3}(-z)^{3/2}\Big) \asymp \exp\!\Big(-xR_N^{1/2}\sin\frac{\varphi}{2}\Big).$$

This yields (A.11) for $R_N^{-3/2}\le\varphi\le\pi/2$, since $\sin(\varphi/2)\ge\varphi/\pi=|\theta-\pi|/\pi$.

For $0\le\varphi\le R_N^{-3/2}$, we find from (A.14) $\operatorname{Im}\frac{2}{3}(-z-x)^{3/2}=O(1)$, so (A.6) yields $\operatorname{Ai}(z+x)=O(|z|^{-1/4})$. Similarly, by our choice of R_N,

$$\operatorname{Re}\Big(\frac{2}{3}(-z)^{3/2}\Big)=\frac{2}{3}R_N^{3/2}\cos\frac{3\varphi}{2}=\frac{2}{3}R_N^{3/2}+O\big(R_N^{-3/2}\big)=\pi N+O\big(R_N^{-3/2}\big).$$

Hence, for large N at least, $\left|\sin\!\left(\frac{2}{3}(-z)^{3/2}+\frac{\pi}{4}\right)\right|\asymp 1$, and, by (A.6) again, $\operatorname{Ai}(z)=\Theta(|z|^{-1/4})$; consequently $\operatorname{Ai}(z+x)/\operatorname{Ai}(z)=O(1)$. These are the results we want in this case because $R_n^{1/2}\varphi=O(1)$ and $R_n^{3/2}\varphi=O(1)$. $\square$

Recall also the entire functions Gi and Hi defined by [1, (10.4.44), (10.4.46)]:

$$\operatorname{Hi}(z) := \pi^{-1}\int_0^\infty \exp\Big(-\frac{1}{3}t^3+zt\Big)\,\mathrm{d}t \tag{A.16}$$

$$\operatorname{Gi}(z) := \operatorname{Bi}(z)-\operatorname{Hi}(z). \tag{A.17}$$

The integral defining Hi evidently converges for all complex z. There is a similar integral yielding Gi [1, (10.4.42)]:

$$\operatorname{Gi}(z) := \pi^{-1}\int_0^\infty \sin\Big(\frac{1}{3}t^3+zt\Big)\,\mathrm{d}t, \qquad z\in\mathbb{R}, \tag{A.18}$$

but this integral converges only conditionally for real z, and not at all for $z\notin\mathbb{R}$.

We have the asymptotics, as $x\to\infty$, [1, (10.4.91)]

$$\operatorname{Hi}(-x)\sim\pi^{-1}x^{-1}. \tag{A.19}$$

More generally, for $\operatorname{Re} z<0$, expanding $\exp(-t^3/3)$ in (A.16) yields, for any $L\ge 0$,

$$\operatorname{Hi}(z) = -\pi^{-1}\sum_{\ell=0}^{L-1}\frac{(3\ell)!}{3^\ell}z^{-3\ell-1}+O\left(|\operatorname{Re} z|^{-3L-1}\right). \tag{A.20}$$

In particular, for any $\delta>0$ and $\arg z\in(\frac{\pi}{2}+\delta,\frac{3\pi}{2}-\delta)$,

$$\operatorname{Hi}(z)=O\left(|\operatorname{Re} z|^{-1}\right)=O\left(|z|^{-1}\right). \tag{A.21}$$

More precisely, for such z,

$$\operatorname{Hi}(z) = -\pi^{-1}z^{-1}+O\left(|z|^{-4}\right). \tag{A.22}$$

We further define, using $\int_0^\infty \mathrm{Ai}(x)\,\mathrm{d}x = 1/3$ [1, 10.4.82],

$$\mathrm{AI}(z) := \int_z^{+\infty} \mathrm{Ai}(t)\,\mathrm{d}t = \frac{1}{3} - \int_0^z \mathrm{Ai}(t)\,\mathrm{d}t; \tag{A.23}$$

this is well-defined for all complex z and yields an entire function provided the first integral is taken along, for example, a path that eventually follows the positive real axis to $+\infty$. Note that $\mathrm{AI}'(z) = -\mathrm{Ai}(z)$ and that $\mathrm{AI}(0) = 1/3$. Along the real axis we have the limits, by [1, 10.4.82–83],

$$\lim_{x\to+\infty} \mathrm{AI}(x) = 0, \tag{A.24}$$

$$\lim_{x\to-\infty} \mathrm{AI}(x) = \int_{-\infty}^{\infty} \mathrm{Ai}(x)\,\mathrm{d}x = 1. \tag{A.25}$$

In terms of the functions Gi and Hi, we have, see [1, 10.4.47–48],

$$\mathrm{AI}(z) = \pi\big(\mathrm{Ai}(z)\mathrm{Gi}'(z) - \mathrm{Ai}'(z)\mathrm{Gi}(z)\big) \tag{A.26}$$

$$= 1 + \pi\big(\mathrm{Ai}'(z)\mathrm{Hi}(z) - \mathrm{Ai}(z)\mathrm{Hi}'(z)\big). \tag{A.27}$$

We have, see [14, Appendix A] and (for real z) [1, 10.4.82–83], as $|z| \to \infty$ with $|\arg(z)| < \pi - \delta$,

$$\mathrm{AI}(z) \sim \frac{1}{2\sqrt{\pi}} z^{-3/4} e^{-\frac{2}{3}z^{3/2}}, \tag{A.28}$$

and, along the negative real axis, and more generally for $-z$ with $|\arg(z)| < 2\pi/3 - \delta$,

$$\mathrm{AI}(-z) = 1 - \pi^{-1/2} z^{-3/4} \left(\cos\Big(\frac{2}{3}z^{3/2} + \frac{\pi}{4}\Big)\big(1 + O(|z|^{-3/2})\big) + O\big(|z|^{-3/2}\big) \right). \tag{A.29}$$

We have the following estimates.

Lemma A.3. (i) *For every fixed $\delta > 0$, if $|\arg z| \le \pi - \delta$ and $|z| \ge 1$, then*

$$\left|\frac{\mathrm{Ai}'(z)}{\mathrm{Ai}(z)}\right| \asymp |z|^{1/2},$$

$$\left|\frac{\mathrm{AI}(z)}{\mathrm{Ai}(z)}\right| \asymp |z|^{-1/2}.$$

(ii) *For $z = R_N e^{i\theta}$ with $\theta \in [\frac{\pi}{2}, \frac{3\pi}{2}]$ and $N \ge 1$,*

$$\left|\frac{\mathrm{Ai}'(z)}{\mathrm{Ai}(z)}\right| = O\left(|z|^{1/2}\right),$$

$$\left|\frac{\mathrm{AI}(z)}{\mathrm{Ai}(z)}\right| = O\left(|z|^{-1/2} + |\mathrm{Ai}(z)|^{-1}\right) = O\left(|z|^{-1/2} + |z|^{1/4} e^{-\frac{2}{3}\left|\mathrm{Im}(-z)^{3/2}\right|}\right).$$

Proof. Part (i) follows from (A.1), (A.2) and (A.28).

For (ii), we use (A.12), (A.7) and (A.29). ∎

Let a_k, $k \geq 1$, denote the zeros of the Airy function; these are all real and negative, so we have $0 > a_1 > a_2 > \dots$. By [1, (10.4.94), (10.4.105)],

$$-a_k = \Bigl(\frac{3\pi(4k-1)}{8}\Bigr)^{2/3}\bigl(1 + O(k^{-2})\bigr) \sim \Bigl(\frac{3\pi}{2}\Bigr)^{2/3} k^{2/3} \asymp k^{2/3}. \tag{A.30}$$

Thus, by (A.5), see also [1, (10.4.96)], (A.9), (A.19) and (A.17),

$$|\mathrm{Ai}'(a_k)| \asymp |a_k|^{1/4} \asymp k^{1/6}, \tag{A.31}$$

$$|\mathrm{Bi}(a_k)| \asymp |a_k|^{-1/4} \asymp k^{-1/6}, \tag{A.32}$$

$$|\mathrm{Hi}(a_k)| \asymp |a_k|^{-1} \asymp k^{-2/3}. \tag{A.33}$$

$$|\mathrm{Gi}(a_k)| \asymp |a_k|^{-1/4} \asymp k^{-1/6}, \tag{A.34}$$

Appendix B. Some Airy integrals

Integrals of the Airy functions (and their derivatives) times powers of x are easily reduced using the relations $\mathrm{Ai}''(x) = x\mathrm{Ai}(x)$ and $\mathrm{Bi}''(x) = x\mathrm{Bi}(x)$ and integration by parts. We have, for example, using also the definition (A.23),

$$\int \mathrm{Ai}(x)\,\mathrm{d}x = -\mathrm{AI}(x) \tag{B.1}$$

$$\int x\mathrm{Ai}(x)\,\mathrm{d}x = \int \mathrm{Ai}''(x)\,\mathrm{d}x = \mathrm{Ai}'(x) \tag{B.2}$$

$$\int x^2\mathrm{Ai}(x)\,\mathrm{d}x = \int x\mathrm{Ai}''(x)\,\mathrm{d}x = x\mathrm{Ai}'(x) - \int \mathrm{Ai}'(x)\,\mathrm{d}x = x\mathrm{Ai}'(x) - \mathrm{Ai}(x) \tag{B.3}$$

and in general the recursion

$$\begin{aligned}\int x^n\mathrm{Ai}(x)\,\mathrm{d}x &= \int x^{n-1}\mathrm{Ai}''(x)\,\mathrm{d}x = x^{n-1}\mathrm{Ai}'(x) - (n-1)\int x^{n-2}\mathrm{Ai}'(x)\,\mathrm{d}x \\ &= x^{n-1}\mathrm{Ai}'(x) - (n-1)x^{n-2}\mathrm{Ai}(x) + (n-1)(n-2)\int x^{n-3}\mathrm{Ai}(x)\,\mathrm{d}x. \end{aligned} \tag{B.4}$$

Integrals of products of two Airy functions and powers of x can be treated similarly, see [2]; we quote the following (that are easily verified by differentiation):

$$\int \mathrm{Ai}(x)^2\,\mathrm{d}x = x\mathrm{Ai}(x)^2 - (\mathrm{Ai}'(x))^2, \tag{B.5}$$

$$\int x\mathrm{Ai}(x)^2\,\mathrm{d}x = \tfrac{1}{3}\bigl(x^2\mathrm{Ai}(x)^2 - x(\mathrm{Ai}'(x))^2 + \mathrm{Ai}'(x)\mathrm{Ai}(x)\bigr) \tag{B.6}$$

$$\int x^2\mathrm{Ai}(x)^2\,\mathrm{d}x = \tfrac{1}{5}\bigl(x^3\mathrm{Ai}(x)^2 - x^2(\mathrm{Ai}'(x))^2 + 2x\mathrm{Ai}'(x)\mathrm{Ai}(x) - \mathrm{Ai}(x)^2\bigr) \tag{B.7}$$

and in general the recursion

$$\int x^n \mathrm{Ai}(x)^2\,\mathrm{d}x = \tfrac{1}{2n+1}\Bigl(x^{n+1}\mathrm{Ai}(x)^2 - x^n(\mathrm{Ai}'(x))^2 + nx^{n-1}\mathrm{Ai}(x)\mathrm{Ai}'(x) - \frac{n(n-1)}{2}x^{n-2}(\mathrm{Ai}(x))^2 + \frac{n(n-1)(n-2)}{2}\int x^{n-3}(\mathrm{Ai}(x))^2\,\mathrm{d}x\Bigr). \tag{B.8}$$

By the same method, we can also treat products involving two different translates of Airy functions; this gives for example the following (again, these are easily verified by differentiation), if $a \neq b$ and $c = (a+b)/2$:

$$\int \mathrm{Ai}(x+a)\mathrm{Ai}(x+b)\,\mathrm{d}x = \frac{1}{a-b}\bigl(\mathrm{Ai}'(x+a)\mathrm{Ai}(x+b) - \mathrm{Ai}(x+a)\mathrm{Ai}'(x+b)\bigr), \tag{B.9}$$

$$\begin{aligned}\int (x+c)\mathrm{Ai}(x+a)\mathrm{Ai}(x+b)\,\mathrm{d}x &= \frac{1}{(a-b)^2}\Bigl((a-b)(x+c)(\mathrm{Ai}'(x+a)\mathrm{Ai}(x+b) - \mathrm{Ai}(x+a)\mathrm{Ai}'(x+b)) \\ &\qquad - 2(x+c)\mathrm{Ai}(x+a)\mathrm{Ai}(x+b) + 2\mathrm{Ai}'(x+a)\mathrm{Ai}'(x+b)\Bigr) \\ &\quad + \frac{2}{(a-b)^3}\bigl(\mathrm{Ai}'(x+a)\mathrm{Ai}(x+b) - \mathrm{Ai}(x+a)\mathrm{Ai}'(x+b)\bigr),\end{aligned} \tag{B.10}$$

and the recursion

$$\begin{aligned}&\int (x+c)^n\mathrm{Ai}(x+a)\mathrm{Ai}(x+b)\,\mathrm{d}x = \\ &\frac{1}{(a-b)^2}\Bigl((a-b)(x+c)^n\bigl(\mathrm{Ai}'(x+a)\mathrm{Ai}(x+b) - \mathrm{Ai}(x+a)\mathrm{Ai}'(x+b)\bigr) \\ &\qquad - 2n(x+c)^n\mathrm{Ai}(x+a)\mathrm{Ai}(x+b) + 2n(x+c)^{n-1}\mathrm{Ai}'(x+a)\mathrm{Ai}'(x+b) \\ &\qquad - n(n-1)(x+c)^{n-2}\bigl(\mathrm{Ai}'(x+a)\mathrm{Ai}(x+b) + \mathrm{Ai}(x+a)\mathrm{Ai}'(x+b)\bigr) \\ &\qquad + n(n-1)(n-2)(x+c)^{n-3}\mathrm{Ai}(x+a)\mathrm{Ai}(x+b) \\ &\qquad + 2n(2n-1)\int (x+c)^{n-1}\mathrm{Ai}(x+a)\mathrm{Ai}(x+b)\,\mathrm{d}x \\ &\qquad - n(n-1)(n-2)(n-3)\int (x+c)^{n-4}\mathrm{Ai}(x+a)\mathrm{Ai}(x+b)\,\mathrm{d}x\Bigr).\end{aligned} \tag{B.11}$$

In particular, if a_k and a_ℓ are zeros of Ai, with $k \neq \ell$, the formulas above yield, recalling the rapid decay (A.1) and (A.2) at ∞:

$$\int_0^\infty \mathrm{Ai}(x+a_k)\,\mathrm{d}x = \int_{a_k}^\infty \mathrm{Ai}(x)\,\mathrm{d}x = \mathrm{AI}(a_k) = -\pi\mathrm{Ai}'(a_k)\mathrm{Gi}(a_k), \tag{B.12}$$

$$\int_{a_k}^\infty x\mathrm{Ai}(x)\,\mathrm{d}x = -\mathrm{Ai}'(a_k), \tag{B.13}$$

$$\int_{a_k}^{\infty} x^2 \mathrm{Ai}(x)\,\mathrm{d}x = -a_k \mathrm{Ai}'(a_k), \tag{B.14}$$

$$\int_0^{\infty} x\mathrm{Ai}(x+a_k)\,\mathrm{d}x = -\mathrm{Ai}'(a_k) - a_k \mathrm{AI}(a_k), \tag{B.15}$$

$$\int_0^{\infty} \mathrm{Ai}(x+a_k)^2\,\mathrm{d}x = \int_{a_k}^{\infty} \mathrm{Ai}(x)^2\,\mathrm{d}x = (\mathrm{Ai}'(a_k))^2, \tag{B.16}$$

$$\int_{a_k}^{\infty} x\mathrm{Ai}(x)^2\,\mathrm{d}x = \tfrac{1}{3}a_k(\mathrm{Ai}'(a_k))^2, \tag{B.17}$$

$$\int_{a_k}^{\infty} x^2\mathrm{Ai}(x)^2\,\mathrm{d}x = \tfrac{1}{5}a_k^2(\mathrm{Ai}'(a_k))^2, \tag{B.18}$$

$$\int_0^{\infty} x\mathrm{Ai}(x+a_k)^2\,\mathrm{d}x = -\tfrac{2}{3}a_k(\mathrm{Ai}'(a_k))^2, \tag{B.19}$$

$$\int_0^{\infty} \mathrm{Ai}(x+a_k)\mathrm{Ai}(x+a_\ell)\,\mathrm{d}x = 0, \tag{B.20}$$

$$\int_0^{\infty} x\mathrm{Ai}(x+a_k)\mathrm{Ai}(x+a_\ell)\,\mathrm{d}x = -\frac{2}{(a_k-a_\ell)^2}\mathrm{Ai}'(a_k)\mathrm{Ai}'(a_\ell). \tag{B.21}$$

More generally, for arbitrary complex $a \neq b$, by (B.9) and (B.10), again using (A.1), (A.2),

$$\int_0^{\infty} \mathrm{Ai}(x+a)\mathrm{Ai}(x+b)\,\mathrm{d}x = \frac{1}{a-b}\big(\mathrm{Ai}(a)\mathrm{Ai}'(b) - \mathrm{Ai}'(a)\mathrm{Ai}(b)\big), \tag{B.22}$$

$$\begin{aligned}\int_0^{\infty} x\mathrm{Ai}(x+a)\mathrm{Ai}(x+b)\,\mathrm{d}x &= \frac{a+b}{(a-b)^2}\mathrm{Ai}(a)\mathrm{Ai}(b) - \frac{2}{(a-b)^2}\mathrm{Ai}'(a)\mathrm{Ai}'(b)\\ &\quad + \frac{2}{(a-b)^3}\big(\mathrm{Ai}(a)\mathrm{Ai}'(b) - \mathrm{Ai}'(a)\mathrm{Ai}(b)\big).\end{aligned} \tag{B.23}$$

In particular,

$$\int_0^{\infty} \mathrm{Ai}(x)\mathrm{Ai}(x+a_k)\,\mathrm{d}x = \frac{\mathrm{Ai}(0)\mathrm{Ai}'(a_k)}{-a_k} \tag{B.24}$$

$$\int_0^{\infty} x\mathrm{Ai}(x)\mathrm{Ai}(x+a_k)\,\mathrm{d}x = -\frac{2}{a_k^2}\mathrm{Ai}'(0)\mathrm{Ai}'(a_k) - \frac{2}{a_k^3}\mathrm{Ai}(0)\mathrm{Ai}'(a_k). \tag{B.25}$$

Remark B.1. The Sturm–Liouville operator $Tf(x) = -f''(x) + xf(x)$ on $[0,\infty)$ with the boundary condition $f(0) = 0$ has the eigenfunctions $\mathrm{Ai}(x + a_k)$ with eigenvalues $-a_k = |a_k|$. (B.20) thus expresses the orthogonality of the eigenfunctions which also follows by general operator theory; in fact, the operator T is self-adjoint and these eigenfunctions form an orthogonal basis in $L^2(0,\infty)$. The corresponding ON basis is by (B.16) given by the functions $\mathrm{Ai}(x+a_k)/\mathrm{Ai}'(a_k)$, $k \geq 1$.

For example, (2.10) is the expansion of $G_N(2^{-1/3}x)$ in this ON basis, with coefficients $\pi\mathrm{Hi}(a_k)$, which yields (4.5) by Parseval's formula. Similarly,

$\mathrm{Ai}(x)$ has the expansion (convergent in L^2)

$$\mathrm{Ai}(x)=\sum_{k=1}^{\infty}\frac{\mathrm{Ai}(0)}{|a_k|\mathrm{Ai}'(a_k)}\mathrm{Ai}(x+a_k), \tag{B.26}$$

where the coefficients are given by (B.24), and thus, using (B.5),

$$\frac{\mathrm{Ai}'(0)^2}{\mathrm{Ai}(0)^2}=\int_0^\infty\left(\frac{\mathrm{Ai}(x)}{\mathrm{Ai}(0)}\right)^2\,\mathrm{d}x=\sum_{k=1}^{\infty}\frac{1}{a_k^2}. \tag{B.27}$$

We will also use the Laplace transform of the Airy function, which is easily found. (Taking z imaginary, we obtain the Fourier transform $e^{i\xi^3/3}$ of Ai; this is sometimes taken as the definition of Ai, see e.g. [13, Definition 7.6.8].)

Lemma B.2. *If* $\operatorname{Re} z>0$, *then*

$$\int_{-\infty}^{\infty}e^{zt}\mathrm{Ai}(t)\,\mathrm{d}t=e^{z^3/3}.$$

Proof. By (A.1) and (A.4), the integral converges absolutely for every z with $\operatorname{Re} z>0$, and thus the integral is an analytic function of z in the right halfplane, say $F(z)$. We have, for $\operatorname{Re} z>0$, by $\mathrm{Ai}''(t)=t\mathrm{Ai}(t)$ and two integrations by parts,

$$\begin{aligned}F'(z)&=\int_{-\infty}^{\infty}te^{zt}\mathrm{Ai}(t)\,\mathrm{d}t=\int_{-\infty}^{\infty}e^{zt}\mathrm{Ai}''(t)\,\mathrm{d}t\\&=-\int_{-\infty}^{\infty}ze^{zt}\mathrm{Ai}'(t)\,\mathrm{d}t=\int_{-\infty}^{\infty}z^2e^{zt}\mathrm{Ai}(t)\,\mathrm{d}t=z^2F(z).\end{aligned}$$

Hence, $F(z)=Ce^{z^3/3}$ for some C.

For $z>0$, another integration by parts yields

$$F(z)=\int_{-\infty}^{\infty}e^{zt}\mathrm{Ai}(t)\,\mathrm{d}t=\int_{-\infty}^{\infty}ze^{zs}\int_s^\infty\mathrm{Ai}(t)\,\mathrm{d}t\,\mathrm{d}s=\int_{-\infty}^{\infty}e^u\int_{u/z}^\infty\mathrm{Ai}(t)\,\mathrm{d}t\,\mathrm{d}u. \tag{B.28}$$

Since $\mathrm{AI}(x):=\int_x^\infty\mathrm{Ai}(t)\,\mathrm{d}t\to0$ as $x\to+\infty$ and $\mathrm{AI}(x)\to1$ as $x\to-\infty$ by by (A.24)–(A.25), dominated convergence shows that, letting $z\searrow0$ in (B.28),

$$C=\lim_{z\searrow0}F(z)=\int_{-\infty}^{\infty}e^u\mathbf{1}[u<0]\,\mathrm{d}u=\int_{-\infty}^{0}e^u\,\mathrm{d}u=1.\qquad\square$$

Remark B.3. By Fourier inversion we find, for any $\sigma>0$,

$$\mathrm{Ai}(t)=\frac{1}{2\pi i}\int_{\sigma-\infty i}^{\sigma+\infty i}e^{-zt+z^3/3}\,\mathrm{d}z. \tag{B.29}$$

By analytic extension, this holds for any complex t.

Appendix C. An integral equation for $f_\tau(t)$

We give here another approach, based on Daniels [personal communication, 1993], to find the density function f_τ of the defective stopping time $\tau = \tau_x$, which was the basis of our development in Section 3. Unfortunately, we have not succeeded to make this approach rigorous, but we find it intriguing that it nevertheless yields the right result, so we present it here as an inspiration for further research.

Let as in (3.1) be the first passage time of $W(t)$ to the barrier $b(t) := -x - t^2/2$, where $x > 0$ is fixed. Let $f_\tau(t)$ be the (defective) density of τ, and $\phi(y;t) = e^{-y^2/2t}/\sqrt{2\pi t}$ the density of $W(t)$.

The first entrance decomposition of $W(t)$ to the region $w < b(t)$ gives the integral equation (using the strong Markov property)

$$\phi(w;t) = \int_0^t f_\tau(u)\phi\bigl(w - b(u); t-u\bigr)\,\mathrm{d}u$$

for $w < b(t)$. Letting $w \nearrow b(t)$ we get the equation

$$\phi(b(t);t) = \int_0^t f_\tau(u)\phi\bigl(b(t) - b(u); t-u\bigr)\,\mathrm{d}u \tag{C.1}$$

for $t > 0$. (Similar arguments using the last exit decomposition, which leads to another functional equation involving also another unknown function, are used by Daniels [6] and Daniels and Skyrme [8].) Since

$$b(t) - b(u) = (u^2 - t^2)/2 = -(t-u)(t+u)/2,$$

we have by (C.1)

$$\frac{e^{-(x+t^2/2)^2/2t}}{\sqrt{2\pi t}} = \int_0^t f_\tau(u)\frac{e^{-(t-u)(t+u)^2/8}}{\sqrt{2\pi(t-u)}}\,\mathrm{d}u. \tag{C.2}$$

The exponents can be written as $-(t-u)(t+u)^2/8 = -t^3/6 + u^3/6 + (t-u)^3/24$ and $-(x+t^2/2)^2/2t = -x^2/2t - xt/2 + t^3/24 - t^3/6$, so the integral equation (C.2) can be transformed into

$$\frac{e^{-x^2/2t - xt/2 + t^3/24}}{\sqrt{2\pi t}} = \int_0^t f_\tau(u)e^{u^3/6}\frac{e^{(t-u)^3/24}}{\sqrt{2\pi(t-u)}}\,\mathrm{d}u. \tag{C.3}$$

This is a convolution equation of the form

$$h(t) = \int_0^t g(u)k(t-u)\,\mathrm{d}u$$

with

$$\begin{aligned}
g(t) &:= f_\tau(t)e^{t^3/6},\\
h(t) &:= \frac{e^{-x^2/2t - xt/2 + t^3/24}}{\sqrt{2\pi t}} = \frac{e^{-b(t)^2/2t + t^3/6}}{\sqrt{2\pi t}},\\
k(t) &:= \frac{e^{t^3/24}}{\sqrt{2\pi t}}.
\end{aligned}$$

If the Laplace transforms $\widetilde{g}(s) := \int_0^\infty e^{-st} g(t)\,\mathrm{d}t$ etc. were finite for $\operatorname{Re} s$ large enough, we could get the solution from $\widetilde{g}(s) = \widetilde{h}(s)/\widetilde{k}(s)$. However, the factors $e^{t^3/24}$ in $h(t)$ and $k(t)$ grow too fast, so $\widetilde{h}(s)$ and $\widetilde{k}(s)$ are not finite for any $s > 0$ and this method does not work. Nevertheless, if we instead define $\widehat{h}(s)$ and $\widehat{k}(s)$ by $\widehat{h}(s) := \int_{\sigma-\infty i}^{\sigma+\infty i} e^{-st} h(t)\,\mathrm{d}t$ and $\widehat{k}(s) := \int_{\sigma-\infty i}^{\sigma+\infty i} e^{-st} k(t)\,\mathrm{d}t$, integrating along vertical lines in the complex plane with real part $\sigma > 0$, then the formula $\widetilde{g}(s) = \widehat{h}(s)/\widehat{k}(s)$ yields the correct formula for $\widetilde{g}(s)$ and thus for $g(t)$. (Note that $\widehat{h}$ and $\widehat{k}$ can be seen as Fourier transform of h and k restricted to vertical lines. The value of $\sigma > 0$ is arbitrary and does not affect $\widehat{h}$ and $\widehat{k}$.) Let us show this remarkable fact by calculating $\widehat{h}(s)$ and $\widehat{k}(s)$.

Consider first $h(t)$ and express the Gaussian factor by Fourier inversion:

$$\frac{e^{-b(t)^2/2t}}{\sqrt{2\pi t}} = \int_{\sigma-\infty i}^{\sigma+\infty i} e^{b(t)u + tu^2/2} \frac{\mathrm{d}u}{2\pi i}, \qquad \operatorname{Re} t > 0.$$

The exponent $b(t)u + tu^2/2$ can then be written as $-ux + u^3/6 + (t-u)^3/6 - t^3/6$, so that

$$h(t) = \int_{\sigma-\infty i}^{\sigma+\infty i} e^{-ux + u^3/6 + (t-u)^3/6} \frac{\mathrm{d}u}{2\pi i},$$

and, choosing $\sigma_1 > \sigma > 0$ and letting $\sigma_2 := \sigma_1 - \sigma$,

$$\begin{aligned}
\widehat{h}(s) &= \int_{\sigma_1-\infty i}^{\sigma_1+\infty i} \int_{\sigma-\infty i}^{\sigma+\infty i} e^{-st - ux + u^3/6 + (t-u)^3/6} \frac{\mathrm{d}u\,\mathrm{d}t}{2\pi i} \\
&= \int_{\sigma-\infty i}^{\sigma+\infty i} \int_{\sigma_2-\infty i}^{\sigma_2+\infty i} e^{-s(u+v) - ux + u^3/6 + v^3/6} \frac{\mathrm{d}v\,\mathrm{d}u}{2\pi i} \\
&= \int_{\sigma-\infty i}^{\sigma+\infty i} \int_{\sigma_2-\infty i}^{\sigma_2+\infty i} e^{-sv + v^3/6} e^{-(s+x)u + u^3/6} \frac{\mathrm{d}v\,\mathrm{d}u}{2\pi i} \\
&= 2\pi i \operatorname{Ai}\bigl(c(s+x)\bigr) \operatorname{Ai}(cs) c^2,
\end{aligned}$$

with $c := 2^{1/3}$, using (B.29).

Since $k(t)$ is obtained by putting $x = 0$ in $h(t)$, we get directly

$$\widehat{k}(s) = 2\pi i \operatorname{Ai}(cs)^2 c^2$$

and hence

$$\widehat{h}(s)/\widehat{k}(s) = \operatorname{Ai}\bigl(2^{1/3}(s+x)\bigr)/\operatorname{Ai}(2^{1/3}s).$$

This is indeed the Laplace transform of $g(t) = f_\tau(t) e^{t^3/6}$ given in (3.8), which by inversion yields the formulas (3.7) and (3.6) for $f_\tau(t)$.

It seems likely that it should be possible to verify the crucial formula $\widetilde{g}(s)\widehat{k}(s) = \widehat{h}(s)$ by suitable manipulations of integrals, which would give another proof of the formulas (3.6)–(3.8) for $f_\tau(t)$ and $g(t)$. For example, if we define, for $\operatorname{Re} t > 0$,

$$F(t) := h(t) - \int_0^{\operatorname{Re} t} g(u) k(t-u)\,\mathrm{d}u$$

(note that F is not analytic), then $F(t)=0$ for real $t>0$, and it is easily verified that the equation $\widetilde{g}(s)\widehat{k}(s)=\widehat{h}(s)$ is equivalent to $\int_{\sigma-\infty i}^{\sigma+\infty i} e^{-st}F(t)\,\mathrm{d}t \to 0$ as $\sigma\to\infty$. However, we do not know how to verify this directly. We therefore leave finding a direct proof of $\widetilde{g}(s)\widehat{k}(s)=\widehat{h}(s)$ as an open problem.

Note that, by Lefebvre [17], see also Groeneboom [10, Theorem 2.1],

$$\mathbb{E}_x\left(e^{-s\tau_0-\int_0^{\tau_0} W(t)dt}\right)=\mathrm{Ai}\big(2^{1/3}(s+x)\big)/\mathrm{Ai}(2^{1/3}s)=\widehat{h}(s)/\widehat{k}(s),$$

where τ_0 is the first hitting time of $W(t)$ to 0, with $W(0)=x$.

Appendix D. An alternative derivation of (3.8)

A proof of the formula (3.8) for the Laplace transform of the density of the passage time τ is given by Groeneboom [10] (in a more general form, allowing an arbitrary starting point and not just $t=0$, or, equivalently, linear term bt in (1.1) or (3.1); for simplicity we do not consider this extension). His proof uses partly quite technical methods. For the service of the reader we here present an alternative proof based on the same ideas but from a different point of view; we believe that this yields a more straightforward proof for our purposes. As discussed in Remark 3.3, this implies (3.6) and (3.4), so it gives us a self-contained proof of the central Lemma 3.2.

As in [21; 20] we consider the process with drift $t^2/2$ defined by $X(t):=x+t^2/2+W(t)$, so that τ defined by (3.1) (with $\beta=1/2$ as in (3.4)–(3.8)) is the first hitting time of $X(t)=0$. If P_x is the probability measure (on the space $C[0,\infty)$) corresponding to $X(\cdot)$, and Q_x that corresponding to $x+W(\cdot)$, then the Cameron–Martin formula tells us that, considering the restriction to a finite time interval $[0,t]$, the Radon–Nikodym derivative is, using $\mathrm{d}X(s)=s\,\mathrm{d}s+\mathrm{d}W(s)$,

$$\begin{aligned}\frac{\mathrm{d}P_x(X)}{\mathrm{d}Q_x(X)}&=\exp\left(-\frac12\int_0^t\left(\frac{(\mathrm{d}X(s)-s\,\mathrm{d}s)^2}{\mathrm{d}s}-\frac{\mathrm{d}X(s)^2}{\mathrm{d}s}\right)\right)\\&=\exp\left(\int_0^t s\,\mathrm{d}X(s)-\frac12\int_0^t s^2\,\mathrm{d}s\right)\\&=\exp\left(-\frac{t^3}{6}+tX(t)-\int_0^t X(s)\,\mathrm{d}s\right),\end{aligned}$$

using integration by parts. Hence, letting $\mathbb{E}_x$ denote expectation with respect to the Wiener measure Q_x,

$$P_x(\tau>t)=\mathbb{E}_x\Big(\frac{\mathrm{d}P_x}{\mathrm{d}Q_x};\tau>t\Big)=e^{-t^3/6}\,\mathbb{E}_x\left(e^{tX(t)-\int_0^t X(s)\,\mathrm{d}s};\tau>t\right)\tag{D.1}$$

(cf. [21, Theorem 2.1] and [10, Lemma 2.1]). We introduce the "Green function"

$$F(x,y,t):=\mathbb{E}_x\left(e^{-\int_0^t X(s)\,\mathrm{d}s}\delta\big(X(t)-y\big);\tau>t\right),$$

where δ is the Dirac delta function (formally, $F(x,\cdot,t)$ is defined as the density of the corresponding occupation measure); (D.1) then says that

$$P_x(\tau > t) = e^{-t^3/6} \int_0^\infty F(x,y,t) e^{ty}\,\mathrm{d}y. \tag{D.2}$$

The Feynman–Kac formula tells us that for fixed $y > 0$, $F(x,y,t)$ is the fundamental solution of the equation

$$\frac{\partial F}{\partial t} = \frac{1}{2}\frac{\partial^2 F}{\partial x^2} - xF, \qquad x,t > 0, \tag{D.3}$$

with the boundary conditions $F(0,y,t) = F(\infty,y,t) = 0$. By time-reversal (or by symmetry of the resolvent $R(x,y,z)$ in (D.7) below), $F(x,y,t) = F(y,x,t)$; hence also

$$\frac{\partial F}{\partial t} = \frac{1}{2}\frac{\partial^2 F}{\partial y^2} - yF, \qquad x,y,t > 0, \tag{D.4}$$

Differentiating (D.2) under the integral sign and using the Feynman–Kac equation (D.4), we obtain (cf. [10, Lemma 2.2])

$$\begin{aligned}
e^{t^3/6} f_\tau(t) &= -e^{t^3/6}\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-t^3/6}\int_0^\infty F(x,y,t)e^{ty}\,\mathrm{d}y\right) \\
&= \int_0^\infty \left(\frac{t^2}{2}F(x,y,t) - \frac{\partial F(x,y,t)}{\partial t} - yF(x,y,t)\right)e^{ty}\,\mathrm{d}y \\
&= \int_0^\infty \left(\frac{t^2}{2}F(x,y,t) - \frac{1}{2}\frac{\partial^2 F(x,y,t)}{\partial y^2}\right)e^{ty}\,\mathrm{d}y \\
&= \frac{1}{2}\left[te^{ty}F(x,y,t) - e^{ty}\frac{\partial F(x,y,t)}{\partial y}\right]_0^\infty \\
&= \frac{1}{2}\frac{\partial F}{\partial y}(x,0,t),
\end{aligned} \tag{D.5}$$

since $F(x,0,t) = 0$ and F and its derivatives decrease rapidly as $y \to \infty$.

The Laplace transform of $F(x,y,\cdot)$ is the resolvent, defined at least for $\operatorname{Re} z > 0$,

$$R(x,y,z) := \int_0^\infty e^{-zt} F(x,y,t)\,\mathrm{d}t. \tag{D.6}$$

The Feynman–Kac equation is a second order differential equation, and the theory of such equations (see also [14, Appendix C; in particular (379)] for this particular case) tells us that

$$R(x,y,z) = \begin{cases} \frac{2}{w}\varphi_0(x;z)\varphi_\infty(y;z), & 0 < x \le y, \\ \frac{2}{w}\varphi_0(y;z)\varphi_\infty(x;z), & 0 < y \le x, \end{cases} \tag{D.7}$$

where $\varphi_0(x) = \varphi_0(x;z)$ and $\varphi_\infty(x) = \varphi_\infty(x;z)$ are solutions of the differential equation

$$\tfrac{1}{2}\varphi''(x) - x\varphi(x) = z\varphi(x) \tag{D.8}$$

with boundary conditions $\varphi_0(0) = 0$, $\varphi_\infty(\infty) = 0$, and w is the Wronskian $w := \varphi_\infty(x)\varphi_0'(x) - \varphi_0(x)\varphi_\infty'(x)$ (which is constant in x). Note that since $\varphi_0(0) = 0$,

$$w = \varphi_\infty(0)\varphi_0'(0). \tag{D.9}$$

The differential equation (D.8) has two linearly independent solutions $A(x+z)$ and $B(x+z)$ with $A(x) := \mathrm{Ai}(cx)$ and $B(x) := \mathrm{Bi}(cx)$ with $c := 2^{1/3}$, and in terms of these we have (up to arbitrary constant factors)

$$\varphi_0(x;z) = A(z)B(x+z) - B(z)A(x+z), \tag{D.10}$$

$$\varphi_\infty(x;z) = A(x+z). \tag{D.11}$$

We integrate (D.5) (multiplied by e^{-zt}) and differentiate (D.6) and obtain, using also (D.7), (D.9) and (D.11)

$$\begin{aligned}\int_0^\infty e^{-zt}e^{t^3/6}f_\tau(t)\,\mathrm{d}t &= \frac{1}{2}\int_0^\infty e^{-zt}\frac{\partial F}{\partial y}(x,0,t)\,\mathrm{d}t = \frac{1}{2}\frac{\partial R}{\partial y}(x,0,z)\\ &= \frac{1}{w}\varphi_0'(0;z)\varphi_\infty(x;z) = \frac{\varphi_\infty(x;z)}{\varphi_\infty(0;z)}\\ &= \frac{A(x+z)}{A(z)},\end{aligned}$$

which is (3.8).

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