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Addendum to Moments of Gamma type and the Brownian supremum process area

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The integral (5.6) yielding the density function $f_X(x)$ is known as an *H-function* (provided each $b_j > 0$, which we can assume by Theorem 4.1), see Fox [6] and Mathai, Saxena and Haubold [10]; more precisely, $f_X(x) = CD^{-1}H(x/D)$, where H is an H-function with appropriate parameters determined by a_j, b_j, a'_k, b'_k. (The H-functions include many special functions. However, they are in general not positive, and thus usually not density functions.)

Hence, the class of distributions studied in this paper is essentially (ignoring cases such as Example 3.13, when the integral (5.6) does not converge) the same as the class of distributions with a density of the type $kH(cx)$ for an H-function H. Such distributions are called *H-function distributions* by Carter and Springer [2] and *H distributions* by Kaluszka and Krysicki [7], see also [10, Chapter 4]. Formulas (rather complicated) for the density of a sum of several independent such variables are given by Mathai and Saxena [8].

Braaksma [1] developed asymptotic expansions of H-functions in great detail and generality, including large parts of the results in our Section 6.

A special case of the H-function is the *Meijer G-function* [11], obtained when all $a_j, a'_k = \pm 1$ in our notation. Distributions with moments of Gamma type with all $a_j, a'_k = \pm 1$ (and $D = 1$) are thus essentially the same as distributions with a density that is a constant times a G function; such distributions are called *G distributions* by Dufresne [4, 5]; see also Mathai and Saxena [9]. (Dufresne [4, 5] include the case when some of our b_j, b'_k are complex and give an interesting example of this, cf. our Remark 11.3.) The special case when all $a_j, a'_k = 1$ is studied further by, e.g., Chamayou and Letac [3] (there called *Dufresne laws*).

The Meijer G-function is implemented in both Mathematica and Maple as `MeijerG`. This allows the use of these programs to plot densities of random variables identified only by their moments if these are of Gamma type with all $a_j, a'_k = \pm 1$.

See also Weisstein [12, 13] and the further references given there.

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