FURTHER EXAMPLES WITH MOMENTS OF GAMMA TYPE

SVANTE JANSON

This is an appendix to [20] containing further examples. See [20] for notation and for examples and equations referred to below by numbers. See also the further references in [20, Addendum].

This appendix will probably be extended with more examples in the future.

Appendix B. Further examples

Example B.1 (Rayleigh distribution). The Rayleigh distribution R is the chi distribution $\chi(2)$, with density $xe^{-x^2/2}$. This is a special case of Example 3.6, and we have

$$\mathbb{E}\, R^s = 2^{s/2}\Gamma(s/2+1), \qquad -2 < \operatorname{Re} s < \infty. \tag{B.1}$$

We have $\rho_+ = \infty$, $\rho_- = -2$, $\gamma = \gamma' = 1/2$, $\delta = 1/2$, $\varkappa = 0$, $C_1 = \pi^{1/2}$.

Example B.2 (Maxwell distribution). The Maxwell distribution M is the chi distribution $\chi(3)$, with density $(2/\pi)^{1/2}x^2e^{-x^2/2}$. This is a another special case of Example 3.6, and we have

$$\mathbb{E}\, M^s = \frac{2^{s/2}}{\Gamma(3/2)}\Gamma\Bigl(\frac{s}{2}+\frac{3}{2}\Bigr) = \frac{2^{s/2+1}}{\sqrt{\pi}}\Gamma\Bigl(\frac{s}{2}+\frac{3}{2}\Bigr), \qquad -3 < \operatorname{Re} s < \infty. \tag{B.2}$$

We have $\rho_+ = \infty$, $\rho_- = -3$, $\gamma = \gamma' = 1/2$, $\delta = 1$, $\varkappa = 0$, $C_1 = \sqrt{2}$.

Example B.3 (Type-2 Beta distribution). The type-2 Beta distribution [34, Chapter 4] has density

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1+x)^{-(\alpha+\beta)}, \qquad x > 0, \tag{B.3}$$

for two parameters $\alpha, \beta > 0$. A variable $X_{\alpha,\beta}$ with this distribution has moments given by

$$\mathbb{E}\, X_{\alpha,\beta}^s = \frac{\Gamma(\alpha+s)\Gamma(\beta-s)}{\Gamma(\alpha)\Gamma(\beta)}, \qquad -\alpha < \operatorname{Re} s < \beta. \tag{B.4}$$

A comparison with (3.1) shows that $X_{\alpha,\beta} \overset{\mathrm{d}}{=} \Gamma_\alpha/\Gamma'_\beta$, with Γ_α and Γ'_β independent. In particular, see Example 3.7, the F distribution is of this type (up to a constant factor): $F_{m,n} \overset{\mathrm{d}}{=} (n/m)X_{m/2,n/2}$.

Date: 3 December, 2010; Revised 6 February 2013.

We have $\rho_+ = \beta$, $\rho_- = -\alpha$, $\gamma = 2$, $\gamma' = 0$, $\delta = \alpha + \beta - 1$, $\varkappa = 0$, $C_1 = 2\pi/(\Gamma(\alpha)\Gamma(\beta))$.

Note that $1 + X_{\alpha,\beta} \overset{\mathrm{d}}{=} B_{\beta,\alpha}^{-1}$, the inverse of a (usual) Beta distributed variable, see Example 3.4. Thus $1+X_{\alpha,\beta}$ also has moments of Gamma type, with

$$\mathbb{E}(X_{\alpha,\beta} + 1)^s = \mathbb{E}\, B_{\beta,\alpha}^{-s} = \frac{\Gamma(\alpha+\beta)\Gamma(\beta-s)}{\Gamma(\beta)\Gamma(\alpha+\beta-s)}, \qquad \operatorname{Re} s < \beta. \tag{B.5}$$

Example B.4 (**Cauchy distribution**)**.** The Cauchy distribution with density $1/(\pi(1+x^2))$, $-\infty < x < \infty$, equals the t-distribution in Example 3.8 with $n = 1$. Hence, if X is a random variable with a Cauchy distribution, then $|X| \overset{\mathrm{d}}{=} |\mathcal{T}_1| \overset{\mathrm{d}}{=} F_{1,1}^{1/2}$ and $|X|$ has moments of Gamma type

$$\mathbb{E}\,|X|^s = \frac{1}{\pi}\Gamma\Big(\frac{1}{2}+\frac{s}{2}\Big)\Gamma\Big(\frac{1}{2}-\frac{s}{2}\Big) = \frac{1}{\cos(\pi s/2)}, \qquad -1 < \operatorname{Re} s < 1. \tag{B.6}$$

Cf. Example 3.19, where $A \overset{\mathrm{d}}{=} \frac{2}{\pi}\log|X|$.

We have $\rho_+ = 1$, $\rho_- = -1$, $\gamma = 1$, $\gamma' = 0$, $\delta = 0$, $\varkappa = 0$, $C_1 = 2$.

Example B.5 (**Beta product distribution**)**.** Dufresne [11] has shown that if a, b, c, d are real, then there exists a probability distribution $G(a, c; a+b, c+d)$ on $(0, 1)$ with moments

$$\mathbb{E}\, X^s = C\frac{\Gamma(a+s)\Gamma(c+s)}{\Gamma(a+b+s)\Gamma(c+d+s)} \tag{B.7}$$

(where necessarily $C = \Gamma(a+b)\Gamma(c+d)/(\Gamma(a)\Gamma(c))$), if and only if either

(i) $a > 0$, $c > 0$, $b+d > 0$ and $\min(a+b, c+d) > \min(a, c)$, or
(ii) (B.7) degenerates to $\mathbb{E}\, X^s = C\Gamma(\alpha+s)/\Gamma(\alpha+\beta+s)$ with $\alpha > 0$ and $\beta \geq 0$, so X has a Beta distribution or $X \equiv 1$. (This degenerate case occurs if $b = 0$, $d = 0$, $a + b = c$ or $c + d = a$.)

We have $\rho_+ = \infty$, $\rho_- = -\min\{a, c\}$, $\gamma = \gamma' = 0$, $\delta = -b - d$, $\varkappa = 0$, $C_1 = C$.

The case when all $a, b, c, d > 0$ is just a product of two independent Beta variables $B_{a,b}B_{c,d}$, see Example 3.4, but there are also other possible parameter values, for example $(2, 5, 8, -1)$ given in [11].

Moreover, if we allow complex parameters a, b, c, d, there is exactly one more case [11], viz. $a > 0$, $c > 0$, $b+d > 0$ (entailing $\operatorname{Im}(b) = -\operatorname{Im}(d)$), and $\operatorname{Re}(a+b) = \operatorname{Re}(c+d)$. In particular, we may take $a = c > 0$ and $d = \overline{b}$ for any complex b with $\operatorname{Re} b > 0$. However, complex parameters are not included in the class of distributions studied in [20], see Remark 11.3.

Example B.6 (**Density of ISE**)**.** The ISE (integrated superbrownian excursion) is a random probability measure introduced by Aldous [2]. It was shown in [6] that the ISE a.s. is absolutely continuous, and thus has a (random) density $f_{\mathrm{ISE}}(x)$, $x \in (-\infty, \infty)$.

The ISE can be described as the occupation measure of the head of the Brownian snake, see Le Gall [29, Chapter IV] or Le Gall and Weill [30] for details; see also [19, Section 4.1]. Thus $f_{\mathrm{ISE}}(x)$ is the local time of the head of the Brownian snake. Moreover, $f_{\mathrm{ISE}}(x)$ arises for example as a limit of the vertical profile of random trees, see [33], [5], [6], [8] and [10].

The distribution of $f_{\mathrm{ISE}}(x)$ for a fixed x is given by a rather complicated formula, see [5] and [6]; in the case $x = 0$ it simplifies and $f_{\mathrm{ISE}}(0) \overset{\mathrm{d}}{=} 2^{1/4}3^{-1}S_{2/3}^{-1/2}$, where $S_{2/3}$ is a positive 2/3-stable variable with Laplace transform $\mathbb{E}\, e^{-tS_{2/3}} = e^{-t^{2/3}}$, see Example 3.10. Thus $f_{\mathrm{ISE}}(0)$ has moments of Gamma type with

$$\mathbb{E}\, f_{\mathrm{ISE}}(0)^s = 2^{s/4}3^{-s}\frac{\Gamma(3s/4+1)}{\Gamma(s/2+1)}, \qquad -4/3 < \operatorname{Re} s < \infty, \tag{B.8}$$

see [5] and [6]. We have $\rho_+ = \infty$, $\rho_- = -4/3$, $\gamma = \gamma' = 1/4$, $\delta = 0$, $\varkappa = -\frac{3}{4}\log 2 - \frac{1}{4}\log 3$, $C_1 = \sqrt{3/2}$.

Example B.7 (**Average ISE**). The ISE in Example B.6 is a random probability measure μ_{ISE}; taking the expectation we obtain a deterministic probability measure $\mathbb{E}\,\mu_{\mathrm{ISE}}$, which is the distribution of a random variable X that can be seen as a random point given by a random ISE. (This is, for example, the limit distribution of the label of a random node in a random tree under suitable assumptions and normalizations.) X has a symmetric distribution, and $|X|$ has moments of Gamma type with

$$\mathbb{E}\,|X|^s = \frac{2^{3s/4}}{\sqrt{\pi}}\Gamma\Big(\frac{s}{2}+\frac{1}{2}\Big)\Gamma\Big(\frac{s}{4}+1\Big), \qquad -1 < \operatorname{Re} s < \infty, \tag{B.9}$$

see [2] and [6]. We have $\rho_+ = \infty$, $\rho_- = -1$, $\gamma = \gamma' = 3/4$, $\delta = 1/2$, $\varkappa = -\frac{1}{4}\log 2$, $C_1 = \sqrt{\pi}$.

Example B.8 (**Blocks in a Stirling permutation**). Let $k \geq 2$ be a fixed integer. It is shown in [22] that the number of blocks in a random k-Stirling permutation of order n (see [22] for definitions) after suitable normalization converges in distribution as $n \to \infty$ to a random variable ζ with moments of Gamma type given by

$$\mathbb{E}\,\zeta^s = (s+1)!\frac{\Gamma(1+\frac{1}{k})}{\Gamma(1+\frac{s+1}{k})} = \Gamma\Big(1+\frac{1}{k}\Big)\frac{\Gamma(s+2)}{\Gamma\big(\frac{s}{k}+\frac{k+1}{k}\big)}, \qquad -2 < \operatorname{Re} s < \infty. \tag{B.10}$$

As explained in [22], this is actually a special case of (9.1).

We have $\rho_+ = \infty$, $\rho_- = -2$, $\gamma = \gamma' = \delta = (k-1)/k$, $\varkappa = \frac{1}{k}\log k$, $C_1 = k^{(k+2)/2k}\Gamma((k+1)/k)$.

Example B.9 (**Distances in a sphere**). Let X_1 and X_2 be two independent random points, uniformly distributed in an n-dimensional ball of radius a, and let $D := |X_1 - X_2|$ be the distance between them. (Here $n \geq 1$.) Note that $0 \leq D \leq 2a$, so $D/2a \in [0,1]$. Hammersley [15] showed that the density function of $D/2a$ is

$$f_n(\lambda) = \frac{2n\Gamma(n+1)\lambda^{n-1}}{\Gamma\bigl(\frac12 n+\frac12\bigr)^2}\int_\lambda^1 (1-z^2)^{(n-1)/2}\,\mathrm{d}z \tag{B.11}$$

and as a consequence, for $\operatorname{Re} s > -n$,

$$\mathbb{E}(D/2a)^s = \frac{n\Gamma(n+1)}{\Gamma\bigl(\frac12 n+\frac12\bigr)}\cdot\frac{\Gamma\bigl(\frac12 n+\frac12 s+\frac12\bigr)}{(n+s)\Gamma\bigl(n+\frac12 s+1\bigr)} \tag{B.12}$$

and, equivalently,

$$\begin{aligned}\mathbb{E}\, D^s &= C(2a)^s\frac{\Gamma(s+n)\Gamma\bigl(\frac12 s+\frac12 n+\frac12\bigr)}{\Gamma(s+n+1)\Gamma\bigl(\frac12 s+n+1\bigr)}\\ &= C'a^s\frac{\Gamma(s+n)}{\Gamma\bigl(\frac12 s+\frac12 n+1\bigr)\Gamma\bigl(\frac12 s+n+1\bigr)}\end{aligned} \tag{B.13}$$

with $C = n\Gamma(n+1)/\Gamma\bigl(\frac12 n+\frac12\bigr)$ and $C' = \pi^{1/2}2^{-n}C$. (Hammersley [15] did not specify the range of s, and presumably intended only positive and perhaps integer values, but the formula follows by (B.11) for any s with $\operatorname{Re} s > n$. Alternatively, the result extends from positive s by Theorem 2.1.)

D thus has moments of Gamma type, with $\rho_+ = +\infty$, $\rho_- = -n$, $\gamma = \gamma' = 0$, $\delta = -(n+3)/2$, $\varkappa = \log(2a)$, $C_1 = 2^{(n+1)/2}C$.

It follows from (B.13) that if $B_{n,1}$ and $B_{(n+1)/2,(n+1)/2}$ are independent Beta distributed variables, then $\mathbb{E}(D/2a)^s = \mathbb{E}\, B_{n,1}^s B_{(n+1)/2,(n+1)/2}^{s/2}$, see Example 3.4, and thus

$$D \overset{\mathrm{d}}{=} 2aB_{n,1}B_{(n+1)/2,(n+1)/2}^{1/2}. \tag{B.14}$$

Cf. Remark 1.5. Note further that $B_{n,1} \overset{\mathrm{d}}{=} U^{1/n}$ where $U \sim \mathrm{U}(0,1)$ is uniform, see Example 3.3, so we also have $D \overset{\mathrm{d}}{=} 2aU^{1/n}B_{(n+1)/2,(n+1)/2}^{1/2}$.

Taking $n = 2$ and $s = 1$ in (B.13) we see that the average distance between two random points in a circular disc of radius a is $\frac{128}{45\pi}a$; this is an old problem.

Further examples for a ball of diameter 1 (so $a = 1/2$): if $n = 1$, then $\mathbb{E}\, D = 1/3$ and $\mathbb{E}\, D^2 = 1/6$; if $n = 2$, then $\mathbb{E}\, D = 64/45\pi$ and $\mathbb{E}\, D^2 = 1/4$; if $n = 3$, then $\mathbb{E}\, D = 18/35$ and $\mathbb{E}\, D^2 = 3/10$.

For $s = 2$, $\mathbb{E}\, D^2 = 2a^2n/(n+2)$, as is easily seen directly.

Example B.10 (**Preferential attachment random graph**). Peköz, Röllin and Ross [37] have, motivated by the study of vertex degrees in a preferential attachment random graph, studied a special case of the triangular urn in Section 9 and obtained further results. (They obtain the random variable K_α below, for $\alpha \in \{\frac12, 1, \frac32, 2, \dots\}$, as the limit in distribution, after normalization, of the degree of a fixed vertex in one of two slightly different random graphs.)

In our notation, let W_α, for $\alpha \geq 1/2$, be the (limit) variable in Section 9.1 with $a = 2$, $c = d = 1$, $w_0 = 1$ and $b = 2\alpha - 1$. Then (9.1) yields

$$\mathbb{E}\, W_\alpha^s = \Gamma(\alpha)\frac{\Gamma(s+1)}{\Gamma(s/2+\alpha)}, \qquad \operatorname{Re} s > -1. \tag{B.15}$$

(Theorem 2.1 implies that the condition $\alpha \geq 1/2$ also is necessary for the existence of such a random variable: $\alpha \in \{0, -1, -2, \dots\}$ is clearly impossible, since then $\Gamma(\alpha) = \infty$, and otherwise the function in (B.15) har $\rho_- \leq -1$ and $\rho_+ = \infty$, and if $\alpha < 1/2$ it is 0 at $s = -2\alpha \in (\rho_-, \rho_+)$, which contradicts Theorem 2.1.)

Peköz, Röllin and Ross [37] choose a different normalisation, so we define $K_\alpha := (\alpha/2)^{1/2} W_\alpha$ and obtain

$$\mathbb{E}\, K_\alpha^s = \Bigl(\frac{\alpha}{2}\Bigr)^{s/2} \frac{\Gamma(\alpha)\Gamma(s+1)}{\Gamma(s/2+\alpha)}, \qquad \operatorname{Re} s > -1. \tag{B.16}$$

In particular, K_α satifies the normalisation $EK_\alpha^2 = 1$.

K_α has $\rho_+ = \infty$, $\rho_- = -1$ (except when $\alpha = 1/2$; then $\rho_- = -2$), $\gamma = \gamma' = 1/2$, $\delta = 1 - \alpha$, $\varkappa = \frac{1}{2}\log\alpha$, $C_1 = 2^{\alpha - 1/2}\Gamma(\alpha)$.

Peköz, Röllin and Ross [37] show, among other things, that the random variable K_α has the density function

$$\varkappa_\alpha(s) = \Gamma(\alpha)\sqrt{\frac{2}{\alpha\pi}}\, e^{-x^2/2\alpha} U\Bigl(\alpha - 1, \frac{1}{2}; \frac{x^2}{2\alpha}\Bigr), \qquad x > 0, \tag{B.17}$$

where $U(a, b; z)$ denotes the confluent hypergeometric function of thesecond kind; see e.g. [1, Chapter 13] or [28] (where it is denoted Ψ). This is a considerably simpler formula than the power series expansion given in Theorem 9.1. It would be interesting to know whether the density in Theorem 9.1 can be expressed using hypergeometric functions also for other triangular urns.

Note the special case $\alpha = 1/2$; then (B.16) simplifies by the duplication formula for the Gamma function to

$$\mathbb{E}\, K_{1/2}^s = \Gamma(s/2+1), \qquad \operatorname{Re} s > -2. \tag{B.18}$$

showing that

$$K_{1/2} \overset{\mathrm{d}}{=} T^{1/2} \overset{\mathrm{d}}{=} 2^{-1/2} R, \tag{B.19}$$

with $T \sim \operatorname{Exp}(1)$, see Example 3.2, and $R \sim \chi(2)$ (the Rayleigh distribution), see Examples 3.6 and B.1. The density function of $K_{1/2}$ is thus

$$\varkappa_{1/2}(x) = 2xe^{-x^2}, \qquad x > 0. \tag{B.20}$$

Example B.11 (**The maximum of i.i.d. exponentials**). Let $(T_i)_{i=1}^\infty$ be i.i.d. exponential random variables with $T_i \sim \operatorname{Exp}(1)$, and let $M_n := \max_{1 \leq i \leq n} T_i$. Then $e^{-T_i} \sim \mathrm{U}(0,1)$, and thus, since $e^{-M_n} := \min_{1 \leq i \leq n} e^{-T_i}$,

$$\mathbb{P}\bigl(e^{-M_n} > x\bigr) = \mathbb{P}\bigl(e^{-T_1} > x\bigr)^n = (1-x)^n, \qquad 0 < x < 1, \tag{B.21}$$

so e^{-M_n} has the Beta distribution $\mathrm{B}(1, n)$.

Hence, by Example 3.4,

$$\mathbb{E}\, e^{sM_n} = \mathbb{E}\, B_{1,n}^{-s} = \frac{\Gamma(n+1)\Gamma(1-s)}{\Gamma(n+1-s)}, \qquad \operatorname{Re} s < 1. \tag{B.22}$$

Hence M_n has moment generating function of Gamma type. The special case $n = 1$ gives $M_1 = T_1 \sim \mathrm{Exp}(1)$ treated in Example 3.16.

M_n has $\rho_+ = 1$, $\rho_- = -\infty$, $\gamma = \gamma' = 0$, $\delta = -n$, $\varkappa = 0$, $C_1 = n!$, cf. Example 3.4 and Remark 2.8.

For an alternative proof of (B.22), note that if $T_{(1)} < \dots < T_{(n)} = M_n$ are $T_1, \dots, T_n$ arranged in increasing order, then it is a standard observation (e.g. by regarding $T_1, \dots, T_n$ as the first points in independent Poisson processes) that $T_{(1)}, T_{(2)} - T_{(1)}, \dots T_{(n)} - T_{(n-1)}$ are independent exponential variables with $T_{(k)} - T_{(k-1)} \sim \mathrm{Exp}(1/(n-k+1))$ (wih $T_{(0)} := 0$), and hence, for $\operatorname{Re} s < 1$

$$\begin{aligned}\mathbb{E}\, e^{sM_n} &= \prod_{k=1}^{n} \frac{1}{1 - s/(n-k+1)} = \prod_{j=1}^{n} \frac{1}{1-s/j} = \prod_{j=1}^{n} \frac{j}{j-s} \\ &= \Gamma(n+1) \frac{\Gamma(1-s)}{\Gamma(n+1-s)}. \end{aligned} \tag{B.23}$$

Note further that, as $n \to \infty$,

$$\mathbb{E}\, e^{s(M_n - \log n)} = n^{-s} \frac{\Gamma(n+1)}{\Gamma(n+1-s)} \Gamma(1-s) \to \Gamma(1-s), \qquad \operatorname{Re} s < 1, \tag{B.24}$$

and thus

$$M_n - \log n \xrightarrow{\mathrm{d}} W, \tag{B.25}$$

where W has the Gumbel distribution $\mathbb{P}(W \le x) = e^{-e^{-x}}$ which has moment generating function $\mathbb{E}\, e^{sW} = \Gamma(1-s)$, $\operatorname{Re} s < 1$, see Example 3.19. Note that it is also easy to prove (B.25) directly, since, for $x \in \mathbb{R}$ and n large enough,

$$\mathbb{P}(M_n - \log n \le x) = \mathbb{P}(T_1 \le \log n + x)^n = \left(1 - \frac{e^{-x}}{n}\right)^n \to e^{-e^{-x}}. \tag{B.26}$$

The decomposition

$$M_n = \sum_{k=1}^{n} \bigl(T_{(k)} - T_{(k-1)}\bigr) \stackrel{\mathrm{d}}{=} \sum_{k=1}^{n} \frac{1}{n-k+1} T_k \stackrel{\mathrm{d}}{=} \sum_{j=1}^{n} \frac{1}{j} T_j \tag{B.27}$$

shows also that

$$\mathbb{E}\, M_n = \sum_{j=1}^{n} \frac{1}{j}, \tag{B.28}$$

the n:th harmonic number H_n. We have $\mathbb{E}\, W = -\Gamma'(1) = \gamma$, Euler's gamma, and thus, since (B.24) implies convergence of all moments,

$$H_n - \log n = \mathbb{E}(M_n - \log n) \to \mathbb{E}\, W = \gamma, \tag{B.29}$$

a well-known result by Euler [12].

Moreover, it follows from (B.24), or from (B.25) and (B.29), that

$$M_n - \mathbb{E}\,M_n \xrightarrow{\mathrm{d}} W - \mathbb{E}\,W = W - \gamma, \tag{B.30}$$

and thus by (B.27)

$$\sum_{j=1}^{\infty} \frac{1}{j}(T_j - 1) \overset{\mathrm{d}}{=} W - \gamma, \tag{B.31}$$

where the infinite sum converges in L^2 and thus a.s. [23, Lemma 4.16].

Example B.12 (**The largest values of i.i.d. exponentials**)**.** Generalizing Example B.11, let $M_n^{(m)}$ be the m:th largest of the n i.i.d. exponential random variables $T_1, \dots, T_n \sim \mathrm{Exp}(1)$; here $1 \le m \le n$. (The special case $m = 1$ gives M_n treated in Example B.11.)

Let $U_i := e^{-T_i} \sim \mathrm{U}(0,1)$. Then $e^{-M_n^{(m)}}$ is the m:th smallest of the i.i.d. uniform $U_1, \dots, U_n$, and thus $e^{-M_n^{(m)}}$ has the Beta distribution $\mathrm{B}(m, n - m + 1)$.

Hence, by Example 3.4,

$$\mathbb{E}\,e^{sM_n^{(m)}} = \mathbb{E}\,B_{m,n-m+1}^{-s} = \frac{\Gamma(n+1)\Gamma(m-s)}{\Gamma(m)\Gamma(n+1-s)}, \qquad \operatorname{Re} s < m. \tag{B.32}$$

Thus $M_n^{(m)}$ has moment generating function of Gamma type.

$M_n^{(m)}$ has $\rho_+ = m$, $\rho_- = -\infty$, $\gamma = \gamma' = 0$, $\delta = -(n - m + 1)$, $\varkappa = 0$, $C_1 = n!/(m-1)!$, cf. Example 3.4 and Remark 2.8.

Alternatively, (B.32) can be obtained by the argument in (B.23). Moreover, by the lack of memory for the exponential distribution, $M_n - M_n^{(m)}$ is independent of $M_n^{(m)}$ and has the same distribution as M_{m-1}; thus $M_n = M_n^{(m)} + M'_{m-1}$, where M'_{m-1} is a copy of M_{m-1} that is independent of $M_n^{(m)}$; this yields $\mathbb{E}\,e^{sM_n} = \mathbb{E}\,e^{sM_n^{(m)}}\,\mathbb{E}\,e^{sM_{m-1}}$, and (B.32) follows from (B.22).

As $n \to \infty$,

$$\mathbb{E}\,e^{s(M_n^{(m)} - \log n)} = n^{-s}\frac{\Gamma(n+1)}{\Gamma(n+1-s)} \cdot \frac{\Gamma(m-s)}{\Gamma(m)} \to \frac{\Gamma(m-s)}{\Gamma(m)}, \qquad \operatorname{Re} s < m, \tag{B.33}$$

and thus

$$M_n^{(m)} - \log n \xrightarrow{\mathrm{d}} W^{(m)}, \tag{B.34}$$

where $W^{(m)}$ has the moment generating function of Gamma type

$$\mathbb{E}\,e^{sW^{(m)}} = \frac{\Gamma(m-s)}{\Gamma(m)}, \qquad \operatorname{Re} s < m. \tag{B.35}$$

Comparing with Example 3.1, we see that $\mathbb{E}\,e^{sW^{(m)}} = \mathbb{E}\,\Gamma_m^{-s} = \mathbb{E}\,e^{-s\log\Gamma_m}$ and thus $W^{(m)} \overset{\mathrm{d}}{=} -\log\Gamma_m$, where Γ_m has a Gamma distribution $\Gamma(m)$.

$W^{(m)}$ has $\rho_+ = m$, $\rho_- = -\infty$, $\gamma = 1$, $\gamma' = -1$, $\delta = m - 1/2$, $\varkappa = 0$, $C_1 = \sqrt{2\pi}/(m-1)!$, cf. Example 3.1 and Remark 2.8.

As in (B.27), there is a decomposition

$$M_n^{(m)} = \sum_{k=1}^{n-m+1} \bigl(T_{(k)} - T_{(k-1)}\bigr) \overset{\mathrm{d}}{=} \sum_{k=1}^{n-m+1} \frac{1}{n-k+1} T_k \overset{\mathrm{d}}{=} \sum_{j=m}^{n} \frac{1}{j} T_j, \tag{B.36}$$

which shows that

$$\mathbb{E}\, M_n = \sum_{j=m}^{n} \frac{1}{j} = H_n - H_{m-1}. \tag{B.37}$$

Since (B.33) implies convergence of all moments, this yields

$$\mathbb{E}\, W^{(m)} = \lim_{n\to\infty} \mathbb{E}(M_n^{(m)} - \log n) = \lim_{n\to\infty} \bigl(H_n - H_{m-1} - \log n\bigr) = \gamma - H_{m-1}. \tag{B.38}$$

Moreover, (B.36), (B.34) and (B.38) imply

$$\sum_{j=m}^{\infty} \frac{1}{j}(T_j - 1) \overset{\mathrm{d}}{=} W^{(m)} - \mathbb{E}\, W^{(m)} \overset{\mathrm{d}}{=} W^{(m)} - \gamma + H_{m-1}, \tag{B.39}$$

where the infinite sum converges in L^2 and thus a.s. [23, Lemma 4.16].

We can also study the joint distribution for several m. In particular, (B.36) holds jointly for all $m \le n$, and thus (B.34) and (B.39) hold jointly for all m.

Moreover, the conditional distribution of $M_n^{(m+1)}$ given $M_n^{(1)}, \dots, M_n^{(m)}$ equals the distribution of the maximum of $n-m$ i.i.d. Exp(1) random variables, conditioned on this maximum being at most $M_n^{(m)}$. In particular, $M_n^{(1)}, M_n^{(2)}, \dots, M_n^{(n)}$ form a Markov chain. It is easy to see that this holds also in the limit as $n \to \infty$. Thus $W^{(1)}, W^{(2)}, \dots$ is a Markov chain, and the conditional distribution of $W^{(m+1)}$ given $W^{(1)}, \dots, W^{(m)}$ equals the distribution of $(W \mid W \le W^{(m)})$, where W is a Gumbel variabel independent of $W^{(m)}$. Explicitly, for $x \le y$,

$$\mathbb{P}\bigl(W^{(m+1)} \le x \mid W^{(m)} = y\bigr) = \mathbb{P}(W \le x \mid W \le y) = \exp\bigl(e^{-y} - e^{-x}\bigr). \tag{B.40}$$

Note that this does not depend on m, so the Markov chain is homogeneous. (The chain $M_n^{(1)}, \dots, M_n^{(n)}$ is not.)

This Markov chain was used by Fristedt [14] to describe the asymptotic distribution of sizes the largest parts in a random partition (after suitable normalization); the largest parts have the same asymptotic distribution as the largest in a sequence of i.i.d. exponential random variables.

The Markov chain becomes simpler if we transform to $e^{-W^{(m)}}$. The conditional distribution of $e^{-M_n^{(m+1)}}$ given $e^{-M_n^{(1)}}, \dots, e^{-M_n^{(m)}}$ equals the distribution of the minimum $e^{-M'_{n-m}}$ of $n-m$ independent uniform random variables conditioned on this minimum being at least $e^{-M_n^{(m)}}$; in the limit it follows that the conditional distribution of $e^{-W^{(m+1)}}$ given $e^{-W^{(1)}}, \dots, e^{-W^{(m)}}$ equals the distribution of $\bigl(e^{-W'} \mid e^{-W'} \ge e^{-W^{(m)}}\bigr)$, where W' is a copy of W independent of $W^{(m)}$. Since $e^{-W'} \overset{\mathrm{d}}{=} e^{-W} \sim \mathrm{Exp}(1)$, it follows that, conditionally given $W^{(1)}, \dots, W^{(m)}$,

$$e^{-W^{(m+1)}} \overset{\mathrm{d}}{=} \bigl(T \mid T \geq e^{-W^{(m)}}\bigr) \overset{\mathrm{d}}{=} T + e^{-W^{(m)}}, \tag{B.41}$$

where $T \sim \mathrm{Exp}(1)$ is independent of $W^{(m)}$. Consequently, the sequence $e^{-W^{(1)}}, e^{-W^{(2)}}, \dots$ has the same distribution as the sequence of partial sums of the i.i.d. $\mathrm{Exp}(1)$ sequence $T_1, T_2, \dots$:

$$\bigl(e^{-W^{(1)}}, e^{-W^{(2)}}, \dots\bigr) \overset{\mathrm{d}}{=} \bigl(T_1, T_1 + T_2, \dots\bigr). \tag{B.42}$$

In particular, this shows again that $e^{-W^{(m)}} \sim \Gamma(m)$ and thus $W^{(m)} \overset{\mathrm{d}}{=} -\log \Gamma_m$.

The same asymptotic distributions $W^{(m)}$ appear for the largest variables in many other situations, see e.g. [27, Sections 2.2–2.3].

Example B.13 (**Logistic distribution**). Let $\widetilde{W}$ have the *logistic distribution* with distribution function $e^x/(e^x+1)$, or equivalently

$$\mathbb{P}(\widetilde{W} > x) = \frac{1}{e^x+1}, \qquad -\infty < x < \infty. \tag{B.43}$$

By differentiation, the density function is

$$\frac{e^x}{(e^x+1)^2} = \frac{1}{(e^{x/2}+e^{-x/2})^2} = \frac{1}{4\cosh^2(x/2)}. \tag{B.44}$$

If $\widetilde{P}_1$ has the shifted Pareto distribution with density $(x+1)^{-2}$, $x>0$, see Example 3.14, then $\mathbb{P}(\widetilde{P}_1 > e^x) = (e^x+1)^{-1} = \mathbb{P}(\widetilde{W} > x)$ and thus

$$\widetilde{W} \overset{\mathrm{d}}{=} \log \widetilde{P}_1. \tag{B.45}$$

As a consequence, by Example 3.14, $\widetilde{W}$ has moment generating function of Gamma type with

$$\mathbb{E}\, e^{s\widetilde{W}} = \mathbb{E}\, \widetilde{P}_1^s = \Gamma(1-s)\Gamma(1+s) = \frac{\pi s}{\sin \pi s}, \qquad -1 < \operatorname{Re} s < 1. \tag{B.46}$$

Equivalently, $\widetilde{W}$ has the characteristic function

$$\mathbb{E}\, e^{\mathrm{i}t\widetilde{W}} = \Gamma(1-\mathrm{i}t)\Gamma(1+\mathrm{i}t) = \frac{\pi t}{\sinh \pi t}. \tag{B.47}$$

We have $\rho_+ = 1$, $\rho_- = -1$, $\gamma = 2$, $\gamma' = 0$, $\delta = 1$, $\varkappa = 0$, $C_1 = 2\pi$.

One way the logistic distribution appears is as the symmetrization of the Gumbel distribution. Let W and W' be i.i.d. with the Gumbel distribution (3.26), see Examples 3.19, and consider $W - W'$, which by (3.35) has the moment generating function, for $-1 < \operatorname{Re} s < 1$,

$$\mathbb{E}\, e^{s(W-W')} = \mathbb{E}\, e^{sW}\, \mathbb{E}\, e^{-sW} = \Gamma(1-s)\Gamma(1+s) = \mathbb{E}\, e^{s\widetilde{W}}. \tag{B.48}$$

Hence,

$$\widetilde{W} \overset{\mathrm{d}}{=} W - W'. \tag{B.49}$$

By (B.49) and (B.31) we further have the representation

$$\widetilde{W} \overset{\mathrm{d}}{=} \sum_{j\neq 0} \frac{1}{j}(T_j - 1) = \sum_{j=1}^{\infty} \frac{1}{j}(T_j - T_{-j}) \tag{B.50}$$

where T_j, $j \in \mathbb{Z}$, are i.i.d. with the distribution Exp(1). Since $T_j - T_{-j}$ has the moment generating function

$$\mathbb{E}\, e^{s(T_j - T_{-j})} = \mathbb{E}\, e^{sT_1}\, \mathbb{E}\, e^{-sT_1} = \frac{1}{1-s}\frac{1}{1+s} = \frac{1}{1-s^2}, \qquad -1 < \operatorname{Re} s < 1, \tag{B.51}$$

(B.50) is equivalent to

$$\frac{\pi s}{\sin \pi s} = \prod_{j=1}^{\infty} \frac{1}{1 - s^2/j^2}, \tag{B.52}$$

which is a version of the product formula for sin [1, 4.3.89]

$$\sin z = z \prod_{j=1}^{\infty} \Bigl(1 - \frac{z^2}{j^2\pi^2}\Bigr). \tag{B.53}$$

The random variable with the characteristic function $t/\sinh t$, and thus the distribution of $\widetilde{W}/\pi$, is studied by Pitman and Yor [38] (there denoted $\hat{S}_1$); among other things, they give the following construction: Let $B(t)$ be a standard Brownian motion and let T be the stopping time when an independent standard 3-dimensional Brownian motion hits the unit sphere in $\mathbb{R}^3$. Then

$$B(T) \overset{\mathrm{d}}{=} \widetilde{W}/\pi. \tag{B.54}$$

The random variable $\widetilde{W}/\pi$ (or $\widetilde{W}$, depending on the choice of normalization) appears also as the asymptotic distribution of the rank of a random partition, see [9].

Example B.14 (Discriminants and Selberg's integral formula). For a vector $(x_1, \dots, x_n)$ of real (or complex) numbers, define

$$\Delta(x_1, \dots, x_n) := \prod_{1\le i<j\le n} (x_j - x_i). \tag{B.55}$$

Thus $\Delta(x_1, \dots, x_n)^2$ is the discriminant of the monic polynomial with roots $x_1, \dots, x_n$. Furthermore, $\Delta(x_1, \dots, x_n)$ is the well-known value of the Vandermonde determinant $\det\bigl(x_i^{j-1}\bigr)_{i,j=1}^n$ (which apparently was never considered by Vandermonde, see [35]).

Selberg [39] proved the following integral formula, for $n \ge 2$ and $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\beta > 0$, $\operatorname{Re} s > \max\{-1/n, -\operatorname{Re}\alpha/(n-1), -\operatorname{Re}\beta/(n-1)\}$,

$$\int_0^1 \cdots \int_0^1 |\Delta(x_1, \dots, x_n)|^{2s} \prod_{i=1}^n x_i^{\alpha-1}(1-x_i)^{\beta-1}\, \mathrm{d}x_1 \cdots \mathrm{d}x_n$$

$$= \prod_{j=1}^{n} \frac{\Gamma\bigl(\alpha + (j-1)s\bigr)\Gamma\bigl(\beta + (j-1)s\bigr)\Gamma(1+js)}{\Gamma\bigl(\alpha+\beta+(n+j-2)s\bigr)\Gamma(1+s)}. \tag{B.56}$$

(For applications of this formula, see e.g. [13] and [3].) This leads to the following probabilistic interpretaions, see Lu and Richards [32].

For real $\alpha, \beta > 0$, let $X_1, \dots, X_n$ be i.i.d. random variables with the Beta distribution $\mathrm{B}(\alpha, \beta)$; then (B.56) can equivalently be written as the expectation

$$\mathbb{E}\,|\Delta(X_1, \dots, X_n)|^{2s} = \prod_{j=1}^{n} \frac{\Gamma(\alpha+\beta)\Gamma\bigl(\alpha + (j-1)s\bigr)\Gamma\bigl(\beta + (j-1)s\bigr)\Gamma(1+js)}{\Gamma(\alpha)\Gamma(\beta)\Gamma\bigl(\alpha+\beta+(n+j-2)s\bigr)\Gamma(1+s)}. \tag{B.57}$$

for $\operatorname{Re} s > \max\{-1/n, -\operatorname{Re}\alpha/(n-1), -\operatorname{Re}\beta/(n-1)\}$. This shows that $\Delta(X_1, \dots, X_n)^2$ has moments of Gamma type. We have $\gamma = \gamma' = 0$, $\delta = 1 - \alpha - \beta - n/2$, $\rho_+ = \infty$ and $\rho_- = \max\{-1/n, -\alpha/(n-1), -\beta/(n-1)\}$ (for $n \geq 2$).

Equivalently, (B.57) shows that the absolute value $|\Delta(X_1, \dots, X_n)|$ has moments of Gamma type. In this case, see Remark 2.8, $\gamma = \gamma' = 0$, $\delta = 1 - \alpha - \beta - n/2$, $\rho_+ = \infty$ and $\rho_- = 2\max\{-1/n, -\alpha/(n-1), -\beta/(n-1)\}$ (for $n \geq 2$).

Note that for $n = 1$, $\Delta(X_1) = 1$ is trivial, so the simplest non-trivial case is $n = 2$, when (B.57) says that if $X_1, X_2 \sim \mathrm{B}(\alpha, \beta)$ are independent, then

$$\mathbb{E}\,|X_1 - X_2|^{2s} = \frac{\Gamma(\alpha+\beta)^2}{\Gamma(\alpha)\Gamma(\beta)} \cdot \frac{\Gamma(\alpha+s)\Gamma(\beta+s)\Gamma(1+2s)}{\Gamma(\alpha+\beta+s)\Gamma(\alpha+\beta+2s)\Gamma(1+s)}. \tag{B.58}$$

We obtain further results by taking suitable limits above, cf. [3]. First, note that if $X_j \sim \mathrm{B}(\alpha, \beta)$, then $\beta X_j \xrightarrow{\mathrm{d}} Y_j \sim \Gamma(\alpha)$ as $\beta \to \infty$. (For example by the method of moments, see (3.6) and (3.1).) By taking limits in (B.57), using the facts that

$$\Delta(\beta X_1, \dots, \beta X_n) = \beta^{n(n-1)/2} \Delta(X_1, \dots, X_n) \tag{B.59}$$

and

$$\beta^{-a} \frac{\Gamma(\beta + a)}{\Gamma(\beta)} \to 1 \qquad \text{as } \beta \to \infty, \text{ for every fixed } a, \tag{B.60}$$

it follows that if $Y_1, \dots, Y_n \sim \Gamma(\alpha)$ are i.i.d., then

$$\mathbb{E}\,|\Delta(Y_1, \dots, Y_n)|^{2s} = \prod_{j=2}^{n} \frac{\Gamma\bigl(\alpha + (j-1)s\bigr)\Gamma(1+js)}{\Gamma(\alpha)\Gamma(1+s)} \tag{B.61}$$

for $\operatorname{Re} s > \max\{-1/n, -\alpha/(n-1)\}$. Thus $\Delta(Y_1, \dots, Y_n)^2$ has moments of Gamma type, with $\gamma = \gamma' = n^2 - n$, $\delta = (n-1)(\alpha - 1/2)$, $\rho_+ = \infty$ and $\rho_- = \max\{-1/n, -\alpha/(n-1)\}$. In particular, $n = 2$ yields

$$\mathbb{E}\,|Y_1 - Y_2|^{2s} = \frac{\Gamma\bigl(\alpha + s\bigr)\Gamma(1+2s)}{\Gamma(\alpha)\Gamma(1+s)}. \tag{B.62}$$

(For $\alpha = 1$, when $Y_1, Y_2 \sim \mathrm{Exp}(1)$, this is an immediate consequence of the fact that $|Y_1 - Y_2| \sim \mathrm{Exp}(1)$ by the lack of memory in the exponential distribution.)

Secondly, taking $\beta = \alpha$, if $X_j \sim \mathrm{B}(\alpha, \alpha)$, then $\sqrt{8\alpha}(X_j - 1/2) \xrightarrow{\mathrm{d}} Z_j \sim \mathrm{N}(0,1)$ as $\alpha \to \infty$. By taking limits in (B.57), using (B.59)–(B.60) and the translation invariance

$$\Delta(X_1 + a, \dots, X_n + a) = \Delta(X_1, \dots, X_n), \tag{B.63}$$

it follows that if $Z_1, \dots, Z_n \sim \mathrm{N}(0,1)$ are i.i.d., then

$$\mathbb{E}\,|\Delta(Z_1, \dots, Z_n)|^{2s} = \prod_{j=2}^{n} \frac{\Gamma(1+js)}{\Gamma(1+s)}, \qquad \operatorname{Re} s > -1/n. \tag{B.64}$$

(This also follows by letting $\alpha \to \infty$ in (B.61), using $(Y_\alpha - \alpha)/\sqrt{\alpha} \xrightarrow{\mathrm{d}} \mathrm{N}(0,1)$, which for integer α is just the central limit theorem for $\Gamma(1) = \mathrm{Exp}(1)$.) Thus $\Delta(Z_1, \dots, Z_n)^2$ has moments of Gamma type, with $\gamma = \gamma' = n(n-1)/2$, $\delta = 0$, $\rho_+ = \infty$ and $\rho_- = -1$. Using the multiplication formula (A.5) for the Gamma function, (B.64) can be rewritten as

$$\begin{aligned}
\mathbb{E}\,|\Delta(Z_1, \dots, Z_n)|^{2s} &= (2\pi)^{-n(n-1)/4}(n!)^{1/2} \prod_{j=1}^{n} j^{js} \prod_{j=2}^{n} \prod_{i=1}^{j-1} \Gamma(s + i/j) \\
&= \Bigl(\prod_{j=1}^{n} j^j\Bigr)^s \prod_{1 \le i < j \le n} \frac{\Gamma(s+i/j)}{\Gamma(i/j)}, \qquad \operatorname{Re} s > -1/n.
\end{aligned} \tag{B.65}$$

The special case $n = 2$ now just yields

$$\mathbb{E}\,|Z_1 - Z_2|^{2s} = \frac{\Gamma(1+2s)}{\Gamma(1+s)} = \frac{2^{2s}}{\sqrt{\pi}}\Gamma(s + 1/2), \qquad \operatorname{Re} s > -1/2, \tag{B.66}$$

which is immediate because $Z_1 - Z_2 \sim \mathrm{N}(0,2)$, see (3.9).

The formulas for the moment imply some factorization formulas. Thus, a comparison between (B.64) and Example 3.10 shows the equality in distribution

$$\Delta(Z_1, \dots, Z_n)^2 \overset{\mathrm{d}}{=} \prod_{j=2}^{n} S_{1/j}^{-1}, \tag{B.67}$$

where $S_{1/j}$ is stable with index $1/j$, and the variables are independent. Similarly, (B.65) and (3.1) show the alternative factorization [32]

$$\Delta(Z_1, \dots, Z_n)^2 \overset{\mathrm{d}}{=} \prod_{j=1}^{n} j^j \prod_{1 \le i < j \le n} G_{ij} \tag{B.68}$$

with $G_{ij} \sim \Gamma(i/j)$ independent.

Similarly, (B.61), (B.64) and (3.1) yield

$$\Delta(Y_1, \dots, Y_n)^2 \overset{\mathrm{d}}{=} \Delta(Z_1, \dots, Z_n)^2 \prod_{j=2}^{n} V_j^{j-1}, \tag{B.69}$$

where $V_j \sim \Gamma(\alpha)$ are independent of each other and $Z_1, \dots, Z_n$; by (B.68) this leads to a factorization of $\Delta(Y_1, \dots, Y_n)^2$ into independent Gamma variables. Another such factorization is given by [32], where also similar factorizations of $\Delta(X_1, \dots, X_n)^2$ with $X \sim \mathrm{B}(\alpha, \beta)$ are given for $n \le 4$.

Example B.15 (Symmetric stable variables). Consider a symmetric stable random variable $\bar{S}_\alpha$ with characteristic function $\varphi(t) := \mathbb{E}\, e^{\mathrm{i}t\bar{S}_\alpha} = e^{-|t|^\alpha}$, where $0 < \alpha \le 2$. (With this normalization, the Lévy measure has density $c|x|^{-\alpha-1}$, where $c = \bigl(-2\Gamma(-\alpha)\cos\frac{\pi\alpha}{2}\bigr)^{-1}$, see e.g. [21, Theorem 3.3].)

The (locally integrable function) $|x|^{s-1}$, where $0 < \operatorname{Re} s < 1$, has the Fourier transform, in distribution sense, $c_s|x|^{-s}$ for a constant c_s given by

$$c_s = 2^s\sqrt{\pi}\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}, \tag{B.70}$$

see e.g. [40, Theorem IV.4.1]; this means that if ψ is in the Schwartz class $\mathcal{S}$, then

$$\int_{-\infty}^{\infty} |t|^{s-1}\widehat{\psi}(t)\,\mathrm{d}t = c_s\int_{-\infty}^{\infty} |x|^{-s}\psi(x)\,\mathrm{d}x, \tag{B.71}$$

where we define the Fourier transform on $\mathcal{S}$ by $\widehat{\psi}(x) := \int_{-\infty}^{\infty} e^{\mathrm{i}xt}\psi(t)\,\mathrm{d}t$. (The fact that the Fourier transform is of this type follows by a simple homogeneity argument, and the value of c_s then can be found by considering the special case $\alpha = 2$ below.)

Since $\varphi(t) \to 0$ rapidly as $|t| \to \infty$, $\bar{S}_\alpha$ has a bounded and infinitely differentiable density function $f(x)$; however, f is not in $\mathcal{S}$. Thus we regularize. Let $\eta(x)$ be symmetric and infinitely differentiable with compact support and $\widehat{\eta}(0) = \int_{-\infty}^{\infty} \eta(x)\,\mathrm{d}x = 1$, and define, for $\varepsilon > 0$, $\eta_\varepsilon(x) := \varepsilon^{-1}\eta(x/\varepsilon)$, which has the Fourier transform $\widehat{\eta}_\varepsilon(x) = \widehat{\eta}(\varepsilon x)$. We then consider the product $f_\varepsilon(x) := f(x)\widehat{\eta}_\varepsilon(x) = f(x)\widehat{\eta}(\varepsilon x)$, whose Fourier transform is $\widehat{f} * \eta_\varepsilon = \varphi * \eta_\varepsilon$; the function f_ε belongs to $\mathcal{S}$, and by applying (B.71) with $\psi = f_\varepsilon$ and then letting $\varepsilon \to 0$, it follows that (B.71) holds with $\psi(x) = f(x)$ too, and thus, using (B.70)

$$\begin{aligned}
\mathbb{E}\,|\bar{S}_\alpha|^{-s} &= \int_{-\infty}^{\infty} |x|^{-s} f(x)\,\mathrm{d}x = c_s^{-1}\int_{-\infty}^{\infty} |t|^{s-1}\varphi(t)\,\mathrm{d}t \\
&= 2c_s^{-1}\int_0^{\infty} t^{s-1}e^{-t^\alpha}\,\mathrm{d}t = 2c_s^{-1}\alpha^{-1}\int_0^{\infty} u^{s/\alpha-1}e^{-u}\,\mathrm{d}u \\
&= 2c_s^{-1}\alpha^{-1}\Gamma(s/\alpha) = 2^{1-s}\pi^{-1/2}\frac{\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\frac{s}{\alpha}\right)}{\alpha\Gamma\left(\frac{s}{2}\right)}, \\
&= 2^{-s}\pi^{-1/2}\frac{\Gamma\left(\frac{1-s}{2}\right)\Gamma\bigl(1+\frac{s}{\alpha}\bigr)}{\Gamma\bigl(1+\frac{s}{2}\bigr)}.
\end{aligned} \tag{B.72}$$

We have proved this for $0 < \operatorname{Re} s < 1$, but by analytic continuation, it extends to $-\alpha < \operatorname{Re} s < 1$, and thus

$$\mathbb{E}\,|\bar{S}_\alpha|^s = 2^s \frac{\Gamma\left(\frac{1+s}{2}\right)\Gamma\left(1-\frac{s}{\alpha}\right)}{\sqrt{\pi}\,\Gamma\left(1-\frac{s}{2}\right)}, \qquad -1 < \operatorname{Re} s < \alpha. \tag{B.73}$$

Hence, $\bar{S}_\alpha$ has moments of Gamma type. We have $\rho_+ = \alpha$ (except when $\alpha = 2$; then $\rho_+ = \infty$) and $\rho_- = -1$; furthermore, $\gamma = 1/\alpha$, $\gamma' = 1 - 1/\alpha$, $\delta = 0$, $\varkappa = \alpha^{-1}\log\alpha$, $C_1 = \sqrt{4/\alpha}$.

In the special case $\alpha = 2$, we have (with our choice of normalization) $\bar{S}_\alpha \overset{\mathrm{d}}{=} \sqrt{2}\,N$ with $N \sim \mathrm{N}(0,1)$, and thus (B.73) is equivalent to (3.9). (As said above, this yields a method to calculate c_s.)

In the special case $\alpha = 1$, $\bar{S}_1$ has a Cauchy distribution with density $1/(\pi(1+x^2))$, see Example B.4. In this case (B.73) yields, using (A.3) and (A.6),

$$\mathbb{E}\,|\bar{S}_1|^s = 2^s \frac{\Gamma\left(\frac{1+s}{2}\right)\Gamma\left(1-s\right)}{\sqrt{\pi}\,\Gamma\left(1-\frac{s}{2}\right)} = \frac{\Gamma\left(\frac{1+s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)}{\pi} = \frac{1}{\cos\frac{\pi s}{2}}, \quad -1 < \operatorname{Re} s < 1. \tag{B.74}$$

This is the same as (B.6), and also as (3.13) with $n = 1$. Indeed, it is well-known that $\mathcal{T}_1 \overset{\mathrm{d}}{=} \bar{S}_1$, i.e., $\mathcal{T}_1$ has a Cauchy distribution.

Example B.16 (**Products of Cauchy variables**)**.** Let $X_1, X_2, \dots$ be i.i.d. random variables with the Cauchy distribution in Example B.4, and let $\Pi_k := \prod_1^k X_i$ be the product of k such variables. (Note that $X_i \overset{\mathrm{d}}{=} X_i^{-1}$; thus e.g. also $\Pi_2 \overset{\mathrm{d}}{=} X_1/X_2$.) It follows from Example B.4 that $|\Pi_k|$ has moments of Gamma type

$$\mathbb{E}\,|\Pi_k|^s = \frac{1}{\pi^k}\Gamma\Big(\frac{1}{2}+\frac{s}{2}\Big)^k\Gamma\Big(\frac{1}{2}-\frac{s}{2}\Big)^k = \frac{1}{\cos^k(\pi s/2)}, \qquad -1 < \operatorname{Re} s < 1. \tag{B.75}$$

We have $\rho_+ = 1$, $\rho_- = -1$, $\gamma = k$, $\gamma' = 0$, $\delta = 0$, $\varkappa = 0$, $C_1 = 2^k$.

The density of $\Pi_1 = X_1$ is $1/(\pi(1+x^2))$, and the density of $\Pi_2 = X_1X_2$ is

$$\frac{2\log|x|}{\pi^2(x^2-1)}, \qquad -\infty < x < \infty, \tag{B.76}$$

see e.g. Pace [36]. Formulas for the density of Π_k for any integer $k \ge 1$ are given by Bourgade, Fujita and Yor [4].

Example B.17 (**Generalized hyperbolic secant distribution**)**.** The distribution of the Lévy stochastic area A in Example 3.20 is also known as the *hyperbolic secant distribution*, since both the density function and the characteristic function are given by the hyperbolic secant $1/\cosh$ (up to normalization constants). This distribution is infinitely divisible, and thus, there exists a Lévy process $\hat{C}_t$, $t \ge 0$, such that $\hat{C}_1 = A$; consequently $\hat{C}_t$

has the characteristic function, cf. (3.36),

$$\mathbb{E}\, e^{\mathrm{i}s\hat{C}_t} = \frac{1}{\cosh^t s}, \qquad s \in \mathbb{R}. \tag{B.77}$$

The density is

$$\frac{2^{t-2}}{\pi\Gamma(t)} \left|\Gamma\left(\frac{t+\mathrm{i}x}{2}\right)\right|^2, \qquad x \in \mathbb{R}, \tag{B.78}$$

see e.g. Pitman and Yor [38] where many further results are given.

When $t = k$ is an integer, $\hat{C}_k$ is the sum of k independent copies of A, so by Example 3.20, $\hat{C}_k$ has moment generating function of Gamma type, with

$$\mathbb{E}\, e^{s\hat{C}_k} = \pi^{-k}\Gamma\Big(\frac{1}{2}+\frac{s}{\pi}\Big)^k\Gamma\Big(\frac{1}{2}-\frac{s}{\pi}\Big)^k = \frac{1}{\cos^k s}, \qquad |\operatorname{Re} s| < \frac{\pi}{2}; \tag{B.79}$$

we have $\rho_\pm = \pm\pi/2$, $\gamma = 2k/\pi$, $\gamma' = 0$, $\delta = 0$, $\varkappa = 0$, $C_1 = 2^k$. On the other hand, if t is not an integer, then $\hat{C}_t$ does not have moment generating function of Gamma type, since the characteristic function (B.77) then cannot be extended to a meromorphic function in $\mathbb{C}$.

The density (B.78) is for $t = 1$

$$\frac{1}{2\cosh\frac{\pi x}{2}} \tag{B.80}$$

as stated in Example 3.20, and for $t = 2$

$$\frac{x}{2\sinh\frac{\pi x}{2}}. \tag{B.81}$$

Similarly, for every integer $t = k \geq 1$, the density (B.78) is a polynomial in x divided by $\cosh(\pi x/2)$ (k odd) or $\sinh(\pi x/2)$ (k even); see Harkness and Harkness [16] for explicit formulas. See also [17] for an application.

Note that $\hat{C}_k \overset{\mathrm{d}}{=} \frac{2}{\pi}\log|\Pi_k|$, where Π_k is the product of Cauchy variables in Example B.16; this is an immediate consequence of the case $k = 1$ mentioned in Example B.4.

As a curiosity, we remark also that the distribution of $\hat{C}_2$ is related to the logistic distribution in Example B.13 in the sense that the density function of one distribution equals, up to constant factors and a rescaling, the characteristic function of the other, see (B.44), (B.47), (B.77), (B.81). In other words, the two density functions are essentially the Fourier transforms of each other.

Example B.18 (**Lamperti variables**)**.** Let $0 < \alpha < 1$ and consider $L_\alpha := S_\alpha/S'_\alpha$ where S_α, S'_α are two independent copies of the positive stable variable in Example 3.10; thus $\mathbb{E}\, e^{-tS_\alpha} = \mathbb{E}\, e^{-tS'_\alpha} = e^{-t^\alpha}$, $t > 0$. By (3.16), L_α has moments of Gamma type, using (A.6),

$$\begin{aligned}\mathbb{E}\, L_\alpha^s = \mathbb{E}\, S_\alpha^s\, \mathbb{E}\, S_\alpha^{-s} &= \frac{\Gamma(1-s/\alpha)\Gamma(1+s/\alpha)}{\Gamma(1-s)\Gamma(1+s)} \\ &= \frac{\Gamma(s/\alpha)\Gamma(1-s/\alpha)}{\alpha\Gamma(s)\Gamma(1-s)} = \frac{\sin(\pi s)}{\alpha\sin(\pi s/\alpha)}, \qquad -\alpha < \operatorname{Re} s < \alpha. \end{aligned}\tag{B.82}$$

We have $\rho_\pm = \pm\alpha$, $\gamma = 2\alpha^{-1} - 2$, $\gamma' = \delta = \varkappa = 0$, $C_1 = 1/\alpha$, cf. Remarks 2.8 and 2.10.

It is somewhat simpler to consider the power $L_\alpha^\alpha = M'_\alpha/M_\alpha$ where M_α, M'_α are i.i.d. with the Mittag-Leffler distribution in Example 3.11. By (B.82), cf. (3.17),

$$\mathbb{E}(L_\alpha^\alpha)^s = \frac{\Gamma(1-s)\Gamma(1+s)}{\Gamma(1-\alpha s)\Gamma(1+\alpha s)} = \frac{\sin(\pi\alpha s)}{\alpha\sin(\pi s)}, \qquad -1 < \operatorname{Re} s < 1. \tag{B.83}$$

We now have $\rho_\pm = \pm 1$, $\gamma = 2 - 2\alpha$, $\gamma' = \delta = \varkappa = 0$, $C_1 = 1/\alpha$, cf. Remark 2.9.

The density of L_α^α can be found by Fourier inversion, see e.g. [41, p. 445], and can be written as

$$\frac{\sin(\pi\alpha)}{\pi\alpha}\frac{1}{x^2 + 2\cos(\pi\alpha)x + 1}, \qquad x > 0. \tag{B.84}$$

Consequently, the density of L_α is

$$\frac{\sin(\pi\alpha)}{\pi}\frac{x^{\alpha-1}}{x^{2\alpha} + 2\cos(\pi\alpha)x^\alpha + 1}, \qquad x > 0. \tag{B.85}$$

The random variable L_α was studied (at least implicitly) by Lamperti [26], and is therefore called a *Lamperti* variable by James [18], where also further references are given.

In the special case $\alpha = 1/2$, (B.83) simplifies to $1/\cos(\pi s/2)$, so $L_{1/2}^{1/2}$ is the absolute value of a Cauchy variable, see Example B.4, which also follows directly from (B.84).

Kotz and Ostrovskii [24] defined, for $0 < \alpha < \beta \le 2$, a random variable $Y_{\alpha,\beta} = \bigl(L_{\alpha/\beta}^{\alpha/\beta}\bigr)^{1/\alpha} = L_{\alpha/\beta}^{1/\beta}$. (The defined $Y_{\alpha,\beta}$ by giving its density function; that the definitions are equivalent follows from (B.84).) By (B.82) or (B.83), $Y_{\alpha,\beta}$ has moments of Gamma type

$$\mathbb{E}\, Y_{\alpha,\beta}^s = \mathbb{E}\, L_{\alpha/\beta}^{s/\beta} = \frac{\Gamma(1+s/\alpha)\Gamma(1-s/\alpha)}{\Gamma(1+s/\beta)\Gamma(1-s/\beta)} = \frac{\beta\sin(\pi s/\beta)}{\alpha\sin(\pi s/\alpha)}, \qquad -\alpha < \operatorname{Re} s < \alpha. \tag{B.86}$$

We have $\rho_\pm = \pm\alpha$, $\gamma = 2\alpha^{-1} - 2$, $\gamma' = \delta = \varkappa = 0$, $C_1 = 1/\alpha$.

James [18] also considers the more general $X_{\alpha,\theta} := S_\alpha/S_{\alpha,\theta}$, for $\alpha > 0$ and $\theta > -\alpha$, where S_α and $S_{\alpha,\theta}$ are independent, S_α is a stable variable as above, and $S_{\alpha,\theta}$ has a distribution that is the same stable law tilted by $x^{-\theta}$, see Remark 2.11. Thus $S_{\alpha,\theta}$ has moments of Gamma type given by

$$\mathbb{E}\, S_{\alpha,\theta}^s = \frac{\mathbb{E}\, S_\alpha^{s-\theta}}{\mathbb{E}\, S_\alpha^{-\theta}} = \frac{\Gamma(1+\theta)}{\Gamma(1+\theta/\alpha)}\frac{\Gamma(1-s/\alpha+\theta/\alpha)}{\Gamma(1-s+\theta)}, \qquad \operatorname{Re} s < \alpha+\theta, \tag{B.87}$$

and $X_{\alpha,\theta}$ has moments of Gamma type given by, for $-\alpha-\theta < \operatorname{Re} s < \alpha$,

$$\mathbb{E}\, X_{\alpha,\theta}^s = \mathbb{E}\, S_\alpha^s\, \mathbb{E}\, S_{\alpha,\theta}^{-s} = \frac{\Gamma(1+\theta)}{\Gamma(1+\theta/\alpha)}\frac{\Gamma(1-s/\alpha)\Gamma(1+s/\alpha+\theta/\alpha)}{\Gamma(1-s)\Gamma(1+s+\theta)}. \tag{B.88}$$

We have $\rho_+ = \alpha$, $\rho_- = -\alpha - \theta$, $\gamma = 2(1/\alpha - 1)$, $\gamma' = 0$, $\delta = \theta(1/\alpha - 1)$, $\varkappa = 0$.

Example B.19 (**A generalized exponential distribution**). Let $\beta > 0$ and let V_β be a positive random variable with the density function

$$\frac{1}{\Gamma(1+1/\beta)} e^{-x^\beta}, \qquad x > 0. \tag{B.89}$$

A simple change of variables verifies that this is a probability density function, and more generally that, for $\operatorname{Re} s > -1$,

$$\mathbb{E}\, V_\beta^s = \frac{1}{\Gamma(1+1/\beta)} \int_0^\infty x^s e^{-x^\beta}\, \mathrm{d}x = \frac{\Gamma(s/\beta + 1/\beta)}{\beta\Gamma(1+1/\beta)} = \frac{\Gamma(s/\beta + 1/\beta)}{\Gamma(1/\beta)}. \tag{B.90}$$

V_β thus has moments of Gamma type, with $\rho_+ = \infty$, $\rho_- = -1$, $\gamma = \gamma' = 1/\beta$, $\delta = \frac{1}{\beta} - \frac{1}{2}$, $\varkappa = \frac{1}{\beta}\log\frac{1}{\beta}$.

Note that $\beta = 1$ gives the exponential distribution in Example 3.2. In general, the distribution of V_β can be seen as a tilted version of the Weibull distribution in Example 3.7.

Example B.20 (**Linnik distribution**). The Linnik distribution [31] has characteristic function

$$\frac{1}{1+|t|^\alpha}, \tag{B.91}$$

where $0 < \alpha \le 2$. As shown by Devroye [7], a random variable X_α with this distribution is easily constructed as

$$X_\alpha := \bar{S}_\alpha V_1^{1/\alpha}, \tag{B.92}$$

where $\bar{S}_\alpha$ is the symmetric stable random variable in Example B.15, V_1 has the exponential distribution $\mathrm{Exp}(1)$, and these are independent.

By (B.73) and (3.2), for $-\min(\alpha,1) < \operatorname{Re} s < \alpha$,

$$\mathbb{E}\,|X_\alpha|^s = \mathbb{E}\,|\bar{S}_\alpha|^s\, \mathbb{E}\, V_1^{s/\alpha} = 2^s \frac{\Gamma\left(\frac{1+s}{2}\right)\Gamma\left(1+\frac{s}{\alpha}\right)\Gamma\left(1-\frac{s}{\alpha}\right)}{\sqrt{\pi}\,\Gamma\left(1-\frac{s}{2}\right)}. \tag{B.93}$$

Hence X_α has moments of Gamma type, with $\rho_+ = \alpha$, $\rho_- = -\min(\alpha,1)$, $\gamma = 2/\alpha$, $\gamma' = 1$, $\delta = 1/2$, $\varkappa = 0$, $C_1 = \sqrt{2\pi}$.

More generally, Devroye [7] showed that if $0 < \alpha \le 2$ and $\beta > 0$, and $\bar{S}_\alpha$ is as above and Y_β as in Example B.19 and independent of $\bar{S}_\alpha$, then

$$X_{\alpha,\beta} := \bar{S}_\alpha V_\beta^{\beta/\alpha} \tag{B.94}$$

has characteristic function

$$\frac{1}{(1+|t|^\alpha)^{1/\beta}}. \tag{B.95}$$

(This implies that $X_\alpha = X_{\alpha,1}$, and more generally every $X_{\alpha,\beta}$, is infinitely divisible, and that there is a Lévy process $\hat{X}_{\alpha,t}$, $t \ge 0$, such that $\hat{X}_{\alpha,t} \overset{\mathrm{d}}{=} X_{\alpha,1/t}$ for all $t > 0$.)

By (B.73) and (B.90), for $-\alpha/\beta < \operatorname{Re} s < \alpha$,

$$\mathbb{E}\,|X_{\alpha,\beta}|^s = \mathbb{E}\,|\bar{S}_\alpha|^s\,\mathbb{E}\,V_\beta^{s\beta/\alpha} = 2^s \frac{\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(1-\frac{s}{\alpha}\right)\Gamma\left(\frac{s}{\alpha}+\frac{1}{\beta}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{1}{\beta}\right)\Gamma\left(1-\frac{s}{2}\right)}. \tag{B.96}$$

Hence $X_{\alpha,\beta}$ has moments of Gamma type, with $\rho_+ = \alpha$, $\rho_- = -\alpha/\beta$, $\gamma = 2/\alpha$, $\gamma' = 1$, $\delta = 1/\beta - 1/2$, $\varkappa = 0$.

Kotz and Ostrovskii [24] showed that $X_\alpha \overset{\mathrm{d}}{=} X_\beta Y_{\alpha,\beta}$ where $Y_{\alpha,\beta}$ is as in Example B.18 and independent of X_β; this follows also directly from (B.86) and (B.93). For the Linnik distribution see further [25].

References

[1] M. Abramowitz & I. A. Stegun, eds., *Handbook of Mathematical Functions.* Dover, New York, 1972.

[2] D. Aldous, Tree-based models for random distribution of mass. *J. Statist. Phys.* **73** (1993), 625–641.

[3] G. W. Anderson, A. Guionnet, & O. Zeitouni, *An Introduction to Random Matrices.* Cambridge Univ. Press, Cambridge, 2010.

[4] P. Bourgade, T. Fujita & M. Yor, Euler's formulae for $\zeta(2n)$ and products of Cauchy variables. *Electron. Comm. Probab.* **12** (2007), 73–80.

[5] M. Bousquet-Mélou, Limit laws for embedded trees. Applications to the integrated superBrownian excursion. *Random Struct. Alg.* **29**, no. 4, 475–523 (2006).

[6] M. Bousquet-Mélou & S. Janson, The density of the ISE and local limit laws for embedded trees. *Ann. Appl. Probab.* **16**, no. 3, 1597–1632 (2006).

[7] L. Devroye, A note on Linnik's distribution. *Statist. Probab. Lett.* **9** (1990), no. 4, 305–306.

[8] L. Devroye & S. Janson, Distances between pairs of vertices and vertical profile in conditioned Galton–Watson trees. *Random Struct. Alg.* **38** (2011), no. 4, 381–395.

[9] P. Diaconis, S. Janson and R. C. Rhoades, Note on a partition limit of Bringman–Mahlburg–Rhoades. *Bull. London Math. Soc.*, to appear. `arXiv:1205.1252.`

[10] M. Drmota, *Random Trees.* SpringerWienNewYork, Vienna, 2009.

[11] D. Dufresne, The beta product distribution with complex parameters. *Comm. Statistics – Theory and Methods* **39** (2010), no. 5, 837–854.

[12] L. Euler, De progressionibus harmonicis observationes. *Commentarii academiae scientiarum imperialis Petropolitanae* **VII** (1734–1735; printed 1740), 150–161. Reprinted in his *Opera Omnia*, Series 1, Volume 14, 87–100.

[13] P. J. Forrester & S. O. Warnaar, The importance of the Selberg integral. *Bull. Amer. Math. Soc.* **45** (2008), 489–534.

[14] B. Fristedt, The structure of random partitions of large integers. *Trans. Amer. Math. Soc.* **337** (1993), no. 2, 703–735.

[15] J. M. Hammersley, The distribution of distance in a hypersphere. *Ann. Math. Statist.* **21** (1950), no. 3, 447–452.

[16] W. L. Harkness & M. L. Harkness, Generalized hyperbolic secant distributions. *J. Amer. Statist. Assoc.* **63** (1968) 329–337.

[17] L. Holst, Euler's $\zeta(2) = \pi^2/6$ and the hyperbolic secant distribution. Preprint, 2011.

[18] L. F. James, Lamperti-type laws. *Ann. Appl. Probab.* **20** (2010), no. 4, 1303–1340.

[19] S. Janson, Left and right pathlengths in random binary trees, *Algorithmica*, **46** (2006), no. 3/4, 419–429.

[20] S. Janson, Moments of Gamma type and the Brownian supremum process area. *Probability Surveys* **7** (2010), 1–52. Addendum. *Probability Surveys* **7** (2010), 207–208.

[21] S. Janson, Stable distributions. Preprint, 2012. `arXiv:1112.0220v2.`

[22] S. Janson, M. Kuba & A. Panholzer, Generalized Stirling permutations, families of increasing trees and urn models. *J. Combin. Theory Ser. A*, **118** (2010), 94–114.

[23] O. Kallenberg, *Foundations of Modern Probability.* 2nd ed., Springer, New York, 2002.

[24] S. Kotz & I. V. Ostrovskii, A mixture representation of the Linnik distribution. *Statist. Probab. Lett.* **26** (1996), no. 1, 61–64.

[25] S. Kotz, I. V. Ostrovskii & A. Hayfavi, Analytic and asymptotic properties of Linnik's probability densities. I, II. *J. Math. Anal. Appl.* **193** (1995), no. 1, 353–371; no. 2, 497–521.

[26] J. Lamperti, An occupation time theorem for a class of stochastic processes. *Trans. Amer. Math. Soc.* **88** (1958), 380–387.

[27] M. R. Leadbetter, G. Lindgren & H. Rootzén, *Extremes and Related Properties of Random Sequences and Processes.* Springer-Verlag, New York, 1983.

[28] N. N. Lebedev, *Special Functions and Their Applications.* (Translated from Russian.) Dover, New York, 1972.

[29] J.-F. Le Gall, *Spatial Branching Processes, Random Snakes and Partial Differential Equations.* Lectures in Mathematics ETH Zürich, Birkhäuser, Basel, 1999.

[30] J.-F. Le Gall & M. Weill, Conditioned Brownian trees. *Ann. Inst. H. Poincaré Probab. Statist.* **42** (2006), no. 4, 455–489.

[31] Yu. V. Linnik, Linear forms and statistical criteria. I, II. (Russian) *Ukrain. Mat. Žurnal* **5**, (1953), 207–243; 247–290. English transl. in *Selected Transl. Math. Statist. and Prob.* **3** (1963), 1–40; 41–90.

[32] I. Lu & D. Richards, Random discriminants. *Ann. Statist.* **21** (1993), no. 4, 1982–2000.

[33] J.-F. Marckert, The rotation correspondence is asymptotically a dilatation. *Random Struct. Alg.* **24** (2004), no. 2, 118–132.

[34] A. M. Mathai, R. K. Saxena & H. J. Haubold. *The H-Function. Theory and Applications.* Springer, New York, 2010. xiv+268 pp. ISBN: 978-1-4419-0915-2

[35] J. J. O'Connor & E. F. Robertson, *Alexandre-Théophile Vandermonde.* The MacTutor History of Mathematics archive. `http://www-history.mcs.st-andrews.ac.uk/Biographies/Vandermonde.html`

[36] L. Pace, Probabilistically proving that $\zeta(2) = \pi^2/6$. *Amer. Math. Monthly* **118** (2011), no. 7, 641–643.

[37] E. A. Peköz, A. Röllin & N. Ross, Degree asymptotics with rates for preferential attachment random graphs. Preprint, 2011. `arXiv:1108.5236`.

[38] J. Pitman & M. Yor, Infinitely divisible laws associated with hyperbolic functions. *Canad. J. Math.* **55** (2003), no. 2, 292–330.

[39] A. Selberg, Bemerkninger om et multipelt integral. *Norsk. Mat. Tidsskr.* **24** (1944), 71–78.

[40] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces.* Princeton Univ. Press, Princeton, NJ, 1971.

[41] V. M. Zolotarev, Mellin–Stieltjes transformations in probability theory. *Teor. Veroyatnost. i Primenen.* **2** (1957), 444–469.

Department of Mathematics, Uppsala University, PO Box 480, SE-751 06 Uppsala, Sweden

E-mail address: `svante.janson@math.uu.se`

URL: `http://www.math.uu.se/~svante/`