Absolutely Continuous Compensators

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Abstract

We give sufficient conditions on the underlying filtration such that all totally inaccessible stopping times have compensators which are absolutely continuous. If a semimartingale, strong Markov process X has a representation as a solution of a stochastic differential equation driven by a Wiener process, Lebesgue measure, and a Poisson random measure, then all compensators of totally inaccessible stopping times are absolutely continuous with respect to the minimal filtration generated by X. However Çinlar and Jacod have shown that all semimartingale strong Markov processes, up to a change of time and slightly of space, have such a representation.

1 Introduction

The celebrated Doob-Meyer Decomposition Theorem states that if X is a submartingale, then it can be written in the form $X = M + A$ where X is a local martingale and A is a unique, càdlàg increasing predictably measurable process with $A_0 = 0$. (See, for example, [26].) In the case of a point process of $N = (N_t)_{t\geq 0}$ it is trivially a submartingale, and hence we know there exists a process A such that $N - A$ is a local martingale. A special case of interest in the theory of Credit Risk is the case

$$1_{\{t\geq R\}} - A_t = \text{ a martingale} \tag{1}$$

The process A in (1) is known as *the compensator of the stopping time R*, by an abuse of language. It is common in applications to assume, often without

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mention, that A has absolutely continuous paths. That is, one often assumes *a priori* that (1) is of the form

$$1_{\{t \geq R\}} - \int_0^t \lambda_s ds = \text{ a martingale} \tag{2}$$

for some adapted process λ. The process λ is often referred to as the *hazard rate* and has intuitive content as the instantaneous likelihood of the stopping time R occurring in the next infinitesimal time interval. Of course this is not true in general, and for example in the theory of credit risk K. Giesecke and L. Goldberg have given a natural example where it does not hold [11, C3, p. 7]. The goal of this paper is to give simple and natural conditions on the generating underlying filtration to show when the compensators of *all* of the totally inaccessible stopping times are absolutely continuous; that is, to give sufficient conditions on the filtration such that they all have hazard rates.

2 Prior Results

Previous work has been restricted to giving conditions on a given stopping time in relation to the underlying filtration that ensures the compensator is absolutely continuous. Perhaps the most well known of these conditions is that of S. Ethier and T.G. Kurtz [8], which we restate here.

Theorem 1 (Ethier–Kurtz Criterion) *Let $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ be a given filtration satisfying the usual hypotheses (see [26] for the "usual hypotheses.") Let A be an increasing (not necessarily adapted) and integrable cádlág process, with $A_0 = 0$. Let $\tilde{A}$ be the $\mathbb{G}$ compensator of A. If there is a constant K such that for all $0 \leq s \leq t$*

$$E\{A_t - A_s | \mathcal{G}_s\} \leq K(t-s) \text{ a.s.} \tag{3}$$

then the compensator of A has absolutely continuous paths, a.s. That is, it is of the form $\tilde{A}_t = \int_0^t \lambda_s ds$.

An extension of Theorem 1 to necessary and sufficient conditions for the $\mathbb{G}$ compensator to have such an intensity process is given in the Cornell PhD thesis of Yan Zeng [28]. A trivial extension is to replace the constant K with an increasing predictable process $(K_t)_{t \geq 0}$, and then the inequality (3) becomes:

$$E\{A_t - A_s | \mathcal{G}_s\} \leq K_s(t-s) \text{ a.s.} \tag{4}$$

and of course the conclusion in Theorem 1 still holds. Zeng [28, p. 14] did a little better:

Theorem 2 (Yan Zeng) *Let A be an increasing (not necessarily adapted) and integrable measurable process, with $A_0 = 0$. Let $\tilde{A}$ be the compensator of A. Then $d\tilde{A}_t \ll dt$ if and only if there exists an increasing and integrable measurable process D with $D_0 = 0$, such that $d\tilde{D}_t \ll dt$ and for all $t \geq 0, h \geq 0$,*

$$E\{A_{t+h} - A_t | \mathcal{G}_t\} \leq E\{D_{t+h} - D_t | \mathcal{G}_t\} \tag{5}$$

and if equality holds, then we have $\tilde{A} = \tilde{D}$.

Another observation is perhaps useful to make. Once a stopping time has an absolutely continuous compensator in a given filtration, say $\mathbb{G}$, then if it is also a stopping time for a smaller filtration it also has an absolutely continuous compensator in the smaller filtration. Actually one can obtain a more precise result, which is established in the book of Martin Jacobsen [13]. We provide here an original and elementary proof of this result, and *inter alia* we extend the result a little.

Theorem 3 *Let R be a $\mathbb{G}$ stopping time with compensator given by $\int_0^t \lambda_s dc(s)$ for some $\mathbb{G}$ adapted process λ, where $\mathbb{G}$ satisfies the usual hypotheses. Here $s \mapsto c(s)$ is non random, continuous, and non-decreasing. Let $\mathbb{F}$ be a subfiltration of $\mathbb{G}$ also satisfying the usual hypotheses, and suppose R is also an $\mathbb{F}$ stopping time. Then the $\mathbb{F}$ compensator of R is given by $\int_0^t {}^o\lambda_s dc(s)$.[1] That is we have*

$$\begin{aligned} &\textit{If} && 1_{\{t\geq R\}} - \int_0^t \lambda_s dc(s) &&= \textit{a martingale in } \mathbb{G} \\ &\textit{then} && 1_{\{t\geq R\}} - \int_0^t {}^o\lambda_s dc(s) &&= \textit{a martingale in } \mathbb{F}. \end{aligned} \tag{6}$$

Proof. Let

$$M_t = 1_{\{t\geq R\}} - \int_0^t \lambda_s dc(s). \tag{7}$$

Then M is a $\mathbb{G}$ martingale. Since $\lambda_s \geq 0$, the optional projection ${}^o\lambda_s$ exists with $0 \leq {}^o\lambda_s \leq \infty$. For every s, $E^o\lambda_s = E\lambda_s$, and thus, by Fubini's theorem, $E\int_0^\infty {}^o\lambda_s\, dc(s) = E\int_0^\infty \lambda_s\, dc(s) = E1_{\{R<\infty\}} < \infty$. Thus, $A_t = \int_0^t {}^o\lambda_s\, dc(s)$ is an integrable increasing continuous adapted process; in particular, ${}^o\lambda_s < \infty$ for a.e. s a.s. We define

$$L_t = 1_{\{t\geq R\}} - A_t = 1_{\{t\geq R\}} - \int_0^t {}^o\lambda_s\, dc(s). \tag{8}$$

If $0 \leq s \leq t$ and H is bounded and $\mathcal{F}_s$ measurable, then, by Fubini's theorem and the fact that for fixed r, ${}^o\lambda_r = E\{\lambda_r|\mathcal{F}_r\}$ a.s.,

$$\begin{aligned} E\big(H(L_t - L_s)\big) &= E\big(H1_{\{t\geq R>s\}}\big) - \int_s^t E\big(HE(\lambda_r|\mathcal{F}_r)\big)\, dc(r) \\ &= E\big(H1_{\{t\geq R>s\}}\big) - \int_s^t E(H\lambda_r)\, dc(r) \\ &= E\big(H(M_t - M_s)\big) = 0. \end{aligned} \tag{9}$$

Hence the uniformly integrable process L_t is an $\mathbb{F}$ martingale. This gives the $\mathbb{F}$ canonical decomposition of the $\mathbb{F}$ submartingale $1_{\{t\geq R\}}$ as $1_{\{t\geq R\}} = L_t + A_t$, and thus A is the $\mathbb{F}$ compensator of $1_{\{t\geq R\}}$. ∎

We include for emphasis the following obvious but important (and well known) corollary:

[1] Note that for fixed s, ${}^o\lambda_s = E\{\lambda_s|\mathcal{F}_s\}$ a.s.; the optional projection gives a method to define the projection via conditional expectation for all $s \geq 0$ simultaneously. Since λ is positive, the optional projection exists. See [26] for more details.

Corollary 4 *Let R be a $\mathbb{G}$ stopping time with compensator given by $\int_0^t \lambda_s ds$ for some $\mathbb{G}$ adapted process λ, where $\mathbb{G}$ satisfies the usual hypotheses. Let $\mathbb{F}$ be a subfiltration of $\mathbb{G}$ also satisfying the usual hypotheses, and suppose R is also an $\mathbb{F}$ stopping time. Then the $\mathbb{F}$ compensator of R is given by $\int_0^t {}^o\lambda_s ds$. That is we have*

$$\begin{aligned} \textit{If} \qquad & 1_{\{t \geq R\}} - \int_0^t \lambda_s ds &=& \text{ a martingale in } \mathbb{G} \\ \textit{then} \qquad & 1_{\{t \geq R\}} - \int_0^t {}^o\lambda_s ds &=& \text{ a martingale in } \mathbb{F}. \end{aligned} \tag{10}$$

Theorem 3 and its Corollary 4 show that once there is a filtration $\mathbb{H}$ such that a stopping time R is totally inaccessible, if R has an AC compensator in $\mathbb{H}$, then it has an AC compensator in any smaller filtration $\mathbb{G}$ as well. In particular Dellacherie's result (Theorem 5 below) implies that *the law of R is absolutely continuous (ie, has a density) as well.* We recall Dellacherie's result here for the reader's convenience. A proof can be found in [26, p. 120].

Theorem 5 (Dellacherie's Theorem) *Let R be a nonnegative random variable with $P(R = 0) = 0, P(R > t) > 0$ for each $t > 0$. Let $\mathcal{F}_t = \sigma(t \wedge R)$, the minimal filtration which renders R a stopping time. Let F denote the law of R. That is, $F(x) = P(R \leq x)$ for $x \geq 0$. Then the compensator $A = (A_t)_{t\geq 0}$ of the process $1_{\{R \geq t\}}$ is given by*

$$A_t = \int_0^t \frac{1}{1 - F(u-)} dF(u).$$

If F is continuous, then A is continuous, R is totally inaccessible, and $A_t = -\ln(1 - F(R \wedge t))$.

One may ask if that, once a compensator of a stopping time R is a.s. singular with respect to Lebesgue measure, does that propagate down to smaller filtrations, and in particular does it imply that the law of the stopping time is singular as well? The next example shows that this is not true in general.

Example 6 Let B be a standard one dimensional Brownian motion with natural filtration $\mathbb{F}$ and with a local time at zero $L = (L_t)_{t\geq 0}$. Define the change of time

$$\tau_t = \inf\{s > 0 : L_s > t\}.$$

Then $(\tau_t)_{t\geq 0}$ is a family of $\mathbb{F}$ stopping times. Also, $M_t = B_{\tau_t}$ is a local martingale for the filtration $\mathbb{G}$ given by $\mathcal{G}_t = \mathcal{F}_{\tau_t}$ for $t \geq 0$. Let N be an independent Poisson process, and consider the vector of processes on the appropriate product space $(N_t - t, M_t)_{t\geq 0}$, with filtration $\mathbb{H}$ such that $(N_t - t, M_t)$ is a vector of two martingales. Then the family $(L_t)_{t\geq 0}$ are stopping times for $\mathbb{H}$, and

$$N_{L_t} - L_t = \text{ a local martingale for the filtration } \tilde{\mathbb{H}}$$

where $(\tilde{\mathcal{H}}_t) = (\mathcal{H}_{L_t})_{t\geq 0}$. Since L is Brownian local time at zero, it has paths which are singular with respect to Lebesgue measure, a.s. However by Tanaka's formula we have

$$|B_t| = \int_0^t \operatorname{sign}(B_s)dB_s + L_t$$

and therefore $E(L_t) = E(|B_t|) = \sqrt{\frac{2}{\pi}}\sqrt{t}$.

Next let

$$R = \inf\{s > 0 : N_{L_s} \geq 1\}.$$

Then $1_{\{t\geq R\}} - L_{t\wedge R} =$ a martingale for the filtration $\tilde{\mathbb{H}}$, and the compensator of R is $L_{t\wedge R}$ which inherits the singular nature of the paths of L. That is, the compensator of R has paths which are a.s. singular with respect to Lebesgue measure. However the law F of R satisfies

$$F(t) = P(R \leq t) = E(1_{\{R\leq t\}}) = E(L_{t\wedge R}),$$

which is absolutely continuous, since $t \mapsto E(L_t) = \sqrt{\frac{2}{\pi}}\sqrt{t}$ is absolutely continuous. Therefore by Dellacherie's theorem the compensator of R in the minimal filtration that makes it a stopping time is absolutely continuous.

Corollary 7 *Let R be a stopping time for a filtration $\mathbb{H}$, and suppose there is a subfiltration $\mathbb{G}$ such that R is totally inaccessible, and that the compensator of $1_{\{t\geq R\}}$ is given by $\int_0^t \lambda_s dc(s)$ where $s \mapsto c(s)$ is non random, continuous, and non-decreasing. Let F denote the law of R. Then $dF(s) \ll dc(s)$. In particular if $c(s) = s$, then F is absolutely continuous; if $s \mapsto dc(s)$ is singular with respect to Lebesgue measure, then F is also singular.*

Corollary 8 *Let R be a stopping time for a filtration $\mathbb{H}$. Suppose there is a subfiltration $\mathbb{G}$ such that R is totally inaccessible, and that the compensator of $1_{\{t\geq R\}}$ is given by $\int_0^t \lambda_s dc(s)$ where $s \mapsto c(s)$ is non random, continuous, and non-decreasing. Then for any sub-subfiltration $\mathbb{F}$ where R is still a stopping time, R will still be totally inaccessible, and its compensator will be absolutely continuous with respect to dc. In particular, a compensator cannot be singular of this form for a filtration $\mathbb{G}$ and then become absolutely continuous for a subfiltration $\mathbb{F}$.*

3 Filtration Level Results

In their seminal paper of 1981, E. Çinlar and J. Jacod [5, Theorem 3.33 on page 206] showed that any $\mathbb{R}^d$ valued strong Markov process which is a Hunt process, and which is also a semimartingale, up to a change of time via an additive functional "clock," can be represented as the solution of a stochastic differential equation driven by dt, dW_t, and $n(ds, dz)$; where W is a standard multidimensional Brownian motion, and n is a standard Poisson random measure with mean measure given by $ds\nu(dz)$.

Therefore we assume as given a strong Markov Hunt process semimartingale which can be represented on a space $(\Omega, \mathcal{F}, \mathbb{F}, P^x)$ where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, as follows:

$$\begin{aligned} X_t = X_0 \quad &+ \quad \int_0^t b(X_s)ds + \int_0^t c(X_s)dW_s \\ &+ \quad \int_0^t \int_{\mathbb{R}} k(X_{s-}, z)1_{\{|k(X_{s-},z)|\leq 1\}}[n(ds, dz) - ds\nu(dz)] \\ &+ \quad \int_0^t \int_{\mathbb{R}} k(X_{s-}, z)1_{\{|k(X_{s-},z)|> 1\}}n(ds, dz) \end{aligned} \tag{11}$$

We let P^μ denote the probability measure governing X where the law of X_0 is μ, and $\mathbb{F}^\mu$ denote the filtration containing $\mathbb{F}$ but such that $\mathcal{F}_0^\mu$ contains all of the P^μ null sets. Our goal is to prove that the predictable compensator process A, of the process $1_{\{t\geq R\}}$ where R is a given but arbitrary totally inaccessible stopping time, has absolutely continuous paths a.s. The next theorem makes this precise.

Theorem 9 *For any totally inaccessible stopping time R on the space $(\Omega, \mathcal{F}, \mathbb{F}^\mu, P^\mu)$ the predictable increasing process A, with $A_0 = 0$, such that $1_{\{t\geq R\}} - A_t = M_t$ is a martingale, has the form $A_t = \int_0^t \lambda_s ds$ for some adapted process λ.*

Proof. It is now perhaps mostly forgotten, but most of the ingredients for the proof of this theorem are contained in an old paper of P.A. Meyer, published in 1973 [23]; see also [22]. The idea is to recall that for each law P^μ, each square integrable martingale M with $M_0 = 0$ is null, if it is orthogonal to all martingales of the form

$$C_t^n = g_n(X_t) - g_n(X_0) - \int_0^t Lg_n(X_s)ds \tag{12}$$

where L is the infinitesimal generator of the underlying strong Markov process X, and the functions g_n are a suitable sequence of functions that are "nearly Borel" measurable, belong to the domain of L, and each Lg_n is bounded. Next let D^n be the sequence of (still square integrable) martingales obtained by an orthogonalization procedure under P^μ, for the sequence C^n. Let M denote an arbitrary and chosen square integrable martingale under P^μ. Then M can be represented as a sum of stochastic integrals with respect to the collection D^n, and hence $\langle M, M\rangle$ is absolutely continuous with respect to the collection $\langle D^n, D^n\rangle$, hence also absolutely continuous with respect to the collection $\langle C^n, C^n\rangle$. We next choose constants λ_n such that $E(\sum_n \lambda_n \langle C^n, C^n\rangle_t) < \infty$ for all $t > 0$, and we let

$$K_t = \sum_n \lambda_n \langle C^n, C^n\rangle_t. \tag{13}$$

Thus, $\langle M, M\rangle$ is absolutely continuous with respect to K.

In the case of the martingale M which is the compensated indicator function of the stopping time R: $M_t = 1_{\{R\geq t\}} - A_t$, with A continuous (which is equivalent to R being totally inaccessible), then we have that

$$[M,M]_t = \sum_{s\leq t}(\Delta M_s)^2 = 1_{\{t\geq R\}}.$$

We conclude that $\langle M,M\rangle_t = A_t$. Thus $\langle M,M\rangle = A$ is also continuous, and we conclude that $\langle M,M\rangle$ is absolutely continuous with respect to the continuous part of K. The continuous part of K, however, is a version of the continuous additive functional H of the Lévy system of X, as given in (14) which follows this proof.

Therefore $A = \langle M,M\rangle$ must be absolutely continuous with respect to H. Finally for a Markov process X of the type given in (11), we know that dH_t is absolutely continuous with respect to dt. This completes the proof. ∎

Remark: Theorem 9 gives a sufficient condition for compensators of all totally inaccessible stopping times to be absolutely continuous, within a semimartingale Hunt process framework. One might ask for necessary and sufficient conditions. The same proof plus a use of Lévy systems can provide this result, given in Theorem 10. The connection to Lévy systems was recently recalled in the work of X. Guo and Y. Zeng [12], and examples of intensities arising in the field of Credit Risk can be found there and in their references, as well as in [11] and [18], for example. See also [17]. Examples of intensities arising in the field of Survival Analysis can be found in the book of Fleming and Harrington [9].

Corollary 10 *Let $X = (\Omega, X, P^\mu)$ be a semimartingale Hunt process with a Lévy system (K,H), where K is a kernel on $\mathbb{R}$ and H is a continuous additive functional, given by the following relationship:*

$$E^\mu\left(\sum_{0<s\leq t} f(X_{s-},X_s)1_{\{X_{s-}\neq X_s\}}\right) = E^\mu\left(\int_0^t dH_s\int_{\mathbb{R}} K(X_s,dy)f(X_s,y)\right) \tag{14}$$

Then all totally inaccessible stopping times have absolutely continuous compensators if and only if the continuous additive functional of equation (14) *is absolutely continuous with respect to Lebesgue measure, a.s. That is, if and only if $dH_s \ll ds$ a.s.*

Proof. The representation of a semimartingale Markov process given in equation (11) assumes there has already been a time change, if necessary, to arrive at a Poisson random measure with compensator $ds\,\nu(dx)$. Here we are not making that assumption. The results contained in (for example) [23] and [3] show that for the additive functional H of the Lévy system, any representation such as (11) must have that the compensator of the corresponding "Poisson random measure" will be absolutely continuous in the t variable with respect to H. The additive functional H is not necessarily unique within the framework of Lévy systems, but any other version will be mutually absolutely continuous with respect to it. Therefore by the proof of Theorem 9 we have that all totally inaccessible times are absolutely continuous with respect to dt if $dH_t \ll dt$.

For the necessity, suppose that every totally inaccessible stopping time has absolutely continuous compensator. Since C^n in (12) jumps only when X jumps, and the jumps of the Hunt process X can be covered by a countable collection of totally inaccessible stopping times, it follows that $d\langle C^n, C^n\rangle_t \ll dt$. Hence, by (13), $dK_t \ll dt$. In particular K_t is continuous and is thus a version of the additive functional H, so $dH_t \ll dK_t \ll dt$. ■

A useful result related to Theorem 9 is the following. Jacod and Skorohod [16] define a *jumping filtration* $\mathbb{F}$ to be a filtration such that there exists a sequence of stopping times $(T_n)_{n=0,1,\dots}$ increasing to ∞ a.s. with $T_0 = 0$ and such that for all $n \in \mathbb{N}, t > 0$, the σ-fields $\mathcal{F}_t$ and $\mathcal{F}_{T_n}$ coincide on $\{T_n \leq t < T_{n+1}\}$. We then have:

Theorem 11 *Let $N = (N_t)_{t\geq 0}$ be a point process without explosions that generates a quasi-left continuous jumping filtration, and suppose there exists a process $(\lambda_s)_{s\geq 0}$ such that*

$$N_t - \int_0^t \lambda_s ds = \text{ a martingale.} \tag{15}$$

Let $\mathbb{D} = (\mathcal{D}_t)_{t\geq 0}$ be the (automatically right continuous) filtration generated by N and completed in the usual way. Then for any $\mathbb{D}$ totally inaccessible stopping time R we have that the compensator of $1_{\{t\geq R\}}$ has absolutely continuous paths, a.s.

Proof. By Theorem 2 of [16] we know that $\{R < \infty\} \subset \bigcup_{n\geq 1}\{R = T_n\}$, a.s. This implies that $1_{\{t\geq R\}} \leq N_t$. We write

$$N_t = 1_{\{t\geq R\}} + (N_t - 1_{\{t\geq R\}}) = 1_{\{t\geq R\}} + C_t \tag{16}$$

Let us now take compensators of both sides of (16), and let A denote the compensator of $1_{\{t\geq R\}}$, and $\tilde{C}$ denote the compensator of C. We want to show $dA_t \ll dt$. Then the compensators version of equation (16) becomes

$$\int_0^t \lambda_s ds = A_t + \tilde{C}_t \tag{17}$$

since taking compensators is a linear operation. Since both dA_t and $d\tilde{C}_t$ are positive measures on $\mathbb{R}_+$, it follows that $dA_t \ll \lambda_t dt$ and $d\tilde{C}_t \ll \lambda_t dt$. ■

Corollary 12 *Let N be a Poisson process with parameter λ, and R be a totally inaccessible stopping time on the minimal space generated by N. Then the compensator[2] of R has paths which are absolutely continuous.*

A result which is related to Theorem 11, but does not involve a hypothesis on the filtration, is the following. For convenience we define $\Delta Z_\infty = 0$; hence $\{\Delta Z_R > 0\} \subseteq \{R < \infty\}$.

[2] The compensator of a stopping time R refers to the compensator of the increasing process $1_{\{t\geq R\}}$.

Theorem 13 *Suppose Z is an increasing process which has an absolutely continuous compensator; that is, suppose there exists a nonnegative adapted process λ such that $Z_t - \int_0^t \lambda_s ds =$ a martingale. Let R be a stopping time such that $P(\Delta Z_R > 0 \cap \{R < \infty\}) = P(R < \infty)$, i.e., $\Delta Z_R > 0$ a.s. on $\{R < \infty\}$. Then R has an absolutely continuous compensator. That is, there exists a nonnegative adapted process μ such that $1_{\{t \geq R\}} - \int_0^t \mu_s ds =$ a martingale.*

Proof. Let

$$\begin{aligned} \Lambda_1 &= \{\Delta Z_R \geq 1\}, \\ \Lambda_n &= \{\frac{1}{n} \leq \Delta Z_R < \frac{1}{n-1}\}, \qquad n \geq 2, \\ Y_t^n &= 1_{\{t \geq R_{\Lambda_n}\}}. \end{aligned} \tag{18}$$

We have that $nZ_t - Y_t^n$ is again an increasing process, and if we observe that

$$nZ_t = (nZ_t - Y_t^n) + Y_t^n, \tag{19}$$

then by taking compensators of both sides in (19) we have, by the same argument as in the proof of Theorem 10, that the compensator of Y^n is absolutely continuous. Therefore we can write the compensator as $\int_0^t \mu_s^n ds$, so

$$Y_t^n - \int_0^t \mu_s^n ds = \text{ a martingale.} \tag{20}$$

Furthermore, $1_{\{t \geq R\}} = \sum_{n=1}^\infty Y_t^n$ and thus the compensator A of $1_{\{t \geq R\}}$ is

$$A_t = \sum_{n=1}^\infty \int_0^t \mu_s^n ds = \int_0^t \left(\sum_{n=1}^\infty \mu_s^n \right) ds \tag{21}$$

by the Fubini–Tonelli theorem, and the theorem is proved. ■

4 Related Results

In this section we relate the results of the preceding part of the paper to some situations that arise in Mathematical Finance Theory. Indeed, it is often the case in Finance that one begins with a system $(\Omega, \mathcal{F}, P, \mathbb{F}, X)$ and then one changes to an equivalent probability measure Q such that X is a sigma martingale (or less generally, a local martingale) under Q. We will say that a probability measure Q has **Property AC** if all totally inaccessible stopping times have absolutely continuous compensators under Q.

Theorem 14 *Suppose that $(\Omega, \mathcal{F}, P, \mathbb{F}, X)$ is a given system, and that there exists an equivalent probability measure $Q^\star$ (which can be P itself) such that $Q^\star$ has Property AC. If $\mathcal{Q}$ is the set of all probability measures equivalent to P, we have that Property AC holds under any $Q \in \mathcal{Q}$.*

Proof. Suppose $Q^\star \in \mathcal{Q}$, and let $R \in \mathcal{Q}$, so that R is equivalent to $Q^\star$. Let τ be a totally inaccessible stopping time, so that we can write

$$1_{\{t\geq\tau\}} - \int_0^t \lambda_s ds = \text{ a martingale, under } Q^\star.$$

By the predictable version of the Meyer–Girsanov theorem (see, eg, [26, p. 135])

$$\begin{aligned}
Z &= \frac{dR}{dQ^\star} \\
Z_t &= E_{Q^\star}\{\frac{dR}{dQ^\star}|\mathcal{F}_t\} \\
M_t &= 1_{\{t\geq\tau\}} - \int_0^t \lambda_s ds \\
M_t &= \left(1_{\{t\geq\tau\}} - \int_0^t \lambda_s ds - \int_0^t \frac{1}{Z_{s-}} d\langle Z, M\rangle_s\right) + \int_0^t \frac{1}{Z_{s-}} d\langle Z, M\rangle_s \quad (22)
\end{aligned}$$

and we have the term in parentheses on the right side of (22) is a martingale under R. Therefore the compensator of τ under R is $\int_0^t \lambda_s ds + \int_0^t \frac{1}{Z_{s-}} d\langle Z, M\rangle_s$, and is absolutely continuous as soon as $d\langle Z, M\rangle_t \ll dt$. We next note that

$$[M, M]_t = \sum_{s\leq t} (\Delta M_s)^2 = 1_{\{t\geq\tau\}}$$

since M has only one jump, and it is of size one. Since compensators are unique, this means that $\langle M, M\rangle_t = \int_0^t \lambda_s ds$, since $\langle M, M\rangle_t$ is the compensator of $[M, M]$, and it exists since $[M, M]_t \in L^1$. Moreover we also know that $\langle Z, M\rangle_t$ exists, and by the Kunita–Watanabe inequality (see, eg, [26, p. 150]) we have that $d\langle Z, M\rangle_t \ll d\langle M, M\rangle_t \ll dt$, a.s. The result follows. ■

A topic that has achieved importance recently in the theory of Credit Risk is that of the Expansion of Filtrations. See, for example, [2], [6], or [19]. In the case of initial expansions, we can expand using Jacod's criterion (see [14] or [26, p. 371]) by adding a random variable L to the filtration $\mathbb{F}$ at time 0, provided that for each $t \geq 0$ the (regular) conditional distribution of L given $\mathcal{F}_t$, denoted $\Upsilon_t(\omega, dx)$, is such that $\Upsilon_t(\omega, dx) \ll \eta_t(dx)$, where $\eta_t(dx)$ is a σ-finite measure. (The key part is that $\eta_t(dx)$ does not depend on ω.) It is shown ([14], [26]) that we can in general replace $\eta_t(dx)$ with $\eta(dx)$ which does not depend on t. We define q_t^x to be a càdlàg martingale such that $\Upsilon_t(\omega, dx) = q_t^x \eta(dx)$. Finally, we let $\mathcal{G}_t^0 = \mathcal{F}_t \vee \sigma(t \wedge L)$, and $(\mathcal{G}_t)_{t\geq 0}$ be the completed, right continuous version of the filtration $\mathbb{G}^0$.

We return to considering the probability measure P only; the results generalize immediately to all measures $Q^\star \in \mathcal{Q}$ by Theorem 14.

Theorem 15 *Suppose we expand the filtration $\mathbb{F}$ by adding a random variable L initially, where its conditional distribution $\Upsilon_t(\omega, dx) \ll \eta_t(dx)$ for some σ-finite measure $\eta_t(dx)$. Suppose also that P has Property AC. Then P has Property AC under the expanded filtration $\mathbb{G}$.*

Proof. Let τ be a totally inaccessible stopping time and recall that $M_t = 1_{\{t\geq\tau\}} - \int_0^t \lambda_s ds$ is our decomposition for the filtration $\mathbb{F}$. Jacod's theorem [14, Théorème (2.5)] gives that, for some predictable process k_s^x and with $\langle\cdot,\cdot\rangle$ computed always in $\mathbb{F}$,

$$\begin{aligned} \langle q^x, M\rangle_t &= \int_0^t k_s^x q_{s-}^x d\langle M,M\rangle_s, \quad \text{when } q_{t-}^x > 0, \text{ for } \eta\text{-a.e. } x, \\ M_t - \int_0^t k_s^L d\langle M,M\rangle_s &= \text{a martingale in } \mathbb{G}. \end{aligned} \tag{23}$$

Since $M_t = 1_{\{t\geq\tau\}} - \int_0^t \lambda_s ds$ in $\mathbb{F}$, we have that the compensator of $1_{\{t\geq\tau\}}$ in $\mathbb{G}$ is again absolutely continuous, since as we saw in the proof of Theorem 14, $d\langle M,M\rangle_t$ is absolutely continuous. ∎

We have an analogous result for progressive expansions. Using the notation and results presented in [26, Chapter VI, Section 3], we let L be a positive random variable that avoids all $\mathbb{F}$ stopping times. That is, $P(L=\nu)=0$ for all $\mathbb{F}$ stopping times ν. Since constants are stopping times, we note that this implies that L has a continuous distribution function. By Dellacherie's theorem (Theorem 5) we have that this implies L is totally inaccessible, at least in the minimal filtration that turns L into a stopping time; see further Lemma 18 below.

We let $Z_t = {}^o 1_{\{L>t\}}$, where the o superscript denotes optional projection onto the $\mathbb{F}$ filtration. We let $A^L = (A_t^L)_{t\geq 0}$ denote the (predictable) compensator of $1_{\{t\geq L\}}$ for the filtration $\mathbb{F}$. (The process $1_{\{t\geq L\}}$ need not be adapted in order to have a compensator.) We then have that the Doob–Meyer decomposition of Z in $\mathbb{F}$ is $Z_t = M_t^L - A_t^L$, where M_t^L is defined by $M_t^L \equiv Z_t + A_t^L$.

Recall that a random time L is called an **honest time** if it is the end of an optional set. More precisely a random variable L is called **honest** if for every $t\leq\infty$ there exists an $\mathcal{F}_t$ measurable random variable L_t such that $L = L_t$ on $\{L\leq t\}$. (See, e.g., [26, p. 381–382] for more on honest times.) If L is honest, we let $\mathbb{F}^L$ be the filtration defined by

$$\mathcal{F}_t^L = \{\Gamma : \Gamma = (A\cap\{L>t\})\cup(B\cap\{L\leq t\}) \text{ for some } A,B\in\mathcal{F}_t\}. \tag{24}$$

(It is easy to see that $\mathbb{F}^L$ is a filtration satisfying the usual hypotheses when L is honest, see [26, Theorem VI.17].) We note that this definition of $\mathbb{F}^L$ is not the standard one. In [26] it is called the filtration $\mathbb{G}$. The standard definition of $\mathbb{F}^L$ (which does not require L to be honest) is as follows, and to avoid confusion, we denote it $\mathbb{K}^L$:

$$\mathcal{K}_t^L = \{\Gamma\in\mathcal{F} : \text{there exists } \Gamma_t\in\mathcal{F}_t : \Gamma\cap\{L>t\} = \Gamma_t\cap\{L>t\}\}$$

(Thus $\mathcal{F}_t^L \subseteq \mathcal{K}_t^L$.) The filtration $\mathbb{F}^L$ is called the **progressive expansion** of $\mathbb{F}$ under L. We assume that L is honest, and consider the filtration $\mathbb{F}^L$. It is easy to see that L becomes a stopping time for $\mathbb{F}^L$. ($\mathbb{F}^L$ is the smallest expansion of $\mathbb{F}$ that makes L a stopping time.)

Theorem 16 *Let L be a positive honest random variable which avoids all $\mathbb{F}$ stopping times. If P has Property AC for $\mathbb{F}$, then for any totally inaccessible $\mathbb{F}$ stopping time τ, the compensator of τ in $\mathbb{F}^L$ is again absolutely continuous.*

Proof. We begin by assuming that τ is a totally inaccessible $\mathbb{F}$ stopping time. It is shown in [26, Theorem VI.18] that for a square integrable martingale X, its decomposition under $\mathbb{F}^L$, is given by

$$\begin{aligned} X_t &= \left(X_t - \int_0^{t\wedge L} \frac{1}{Z_{s-}} d\langle X, M^L\rangle_s + 1_{\{t\geq L\}} \int_L^t \frac{1}{1-Z_{s-}} d\langle X, M^L\rangle_s \right) \\ &\quad + \left(\int_0^{t\wedge L} \frac{1}{Z_{s-}} d\langle X, M^L\rangle_s - 1_{\{t\geq L\}} \int_L^t \frac{1}{1-Z_{s-}} d\langle X, M^L\rangle_s \right). \end{aligned}$$

In our case, the $\mathbb{F}$ martingale X is equal to M, where $M_t = 1_{\{t\geq\tau\}} - \int_0^t \lambda_s ds$. But we already know that $d\langle M, M\rangle_t \ll dt$, whence again by the Kunita–Watanabe inequality we have that $d\langle X, M^L\rangle_t \ll d\langle X, X\rangle_t = d\langle M, M\rangle_t \ll dt$, and the result follows for all totally inaccessible stopping times τ. ■

Theorem 16 shows the AC property only for $\mathbb{F}$ stopping times τ. In order to extend it to $\mathbb{F}^L$ stopping times, we need two lemmas (and an extra condition). Note that the first part of Lemma 17(i) is a special case of the Lemma in [26, p. 378] (with essentially the same proof).

Lemma 17 *Let T be a stopping time for $\mathbb{F}^L$. Then:*

(i) *There exists an $\mathbb{F}$ stopping time S such that $T \wedge L = S \wedge L$ a.s. If T is $\mathbb{F}^L$ totally inaccessible, then S can be chosen to be $\mathbb{F}$ totally inaccessible.*

(ii) *There exists a sequence $(S_n)_{n\geq 1}$ of $\mathbb{F}$ stopping times such that $[T] \subseteq [L] \cup \bigcup_{n=1}^{\infty} [S_n]$ a.s. If T is $\mathbb{F}^L$ totally inaccessible, then all S_n can be chosen to be $\mathbb{F}$ totally inaccessible.*

Proof. The process $U_t = 1_{\{t>T\}}$ is $\mathbb{F}^L$ predictable, so by [26, Theorem VI.17], there exist two $\mathbb{F}$ predictable processes H and K such that

$$U = H 1_{[0,L]} + K 1_{(L,\infty)}.$$

Define

$$R_0 = \inf\{t \geq 0 : H_t \neq 0\}$$

and, for rational $r > 0$,

$$R_r = \inf\{t > r : K_t \neq 0\}.$$

These are $\mathbb{F}$ stopping times. If $T < L$, then $H = 0$ on $[0, T]$ and $H = 1$ on $(T, L]$, so $R_0 = T$. If $T > L$, then $K = 0$ on $(L, T]$ and $K = 1$ on (T, ∞), so $R_r = T$ for every $r \in (L, T)$. Consequently, $[T] \subseteq [L] \cup \bigcup_{r\geq 0} [R_r]$, and the first assertion in (ii) follows by rearranging the countable set of stopping times $(R_r)_{r\geq 0}$ into a sequence $(S_n)_{n\geq 1}$.

Furthermore, if $T \geq L$, then $R_0 \geq L$, and thus $T \wedge L = R_0 \wedge L$, so the first part of (i) follows with $S = R_0$.

Now suppose that T is totally inaccessible. Each $\mathbb{F}$ stopping time R_r can be decomposed into two $\mathbb{F}$ stopping times as $[R_r] = [R_r^a] \cup [R_r^i]$ with R_r^a accessible and R_r^i totally inaccessible [26, p. 104]. Then R_r^a is also for $\mathbb{F}^L$ an accessible stopping time, and since T is totally inaccessible, $P(T = R_r^a) = 0$. Hence $[T] \subseteq [L] \cup \bigcup_{r\geq 0}[R_r^i]$ a.s. and we can replace R_r by R_r^i above. ■

Lemma 18 *L is a totally inaccessible stopping time for $\mathbb{F}^L$.*

Proof. L is a stopping time by the definition of $\mathbb{F}^L$.

Suppose that T is a $\mathbb{F}^L$ predictable stopping time, and let T_n be a sequence of $\mathbb{F}^L$ stopping times that announces T, i.e., $T_n \nearrow T$ and $T_n < T$ for all n when $T > 0$ [26, Section III.2]. By Lemma 17, there exist $\mathbb{F}$ stopping times S_n such that $T_n \wedge L = S_n \wedge L$. Let $S = \liminf S_n$; this is an $\mathbb{F}$ stopping time. On the set $\{0 < T \leq L\}$, each $T_n < T \leq L$, so $T_n \wedge L = T_n$ and $S_n = T_n$; thus $S = T$. Further, on $\{T = 0\}$, trivially each $T_n = 0 < L$ a.s., so $S_n = 0$ and $S = 0 = T$ a.s. Hence, $S = T$ a.s. on $\{T \leq L\}$. Consequently,

$$P(T = L) = P(S = T = L) \leq P(S = L) = 0,$$

because S is an $\mathbb{F}$ stopping time. Since T is an arbitrary $\mathbb{F}^L$ predictable stopping time, this shows that L is totally inaccessible. ■

We can now extend Theorem 16 to $\mathbb{F}^L$ totally inaccessible stopping times, but we need an extra condition.

Theorem 19 *Let L be a positive honest random variable which avoids all $\mathbb{F}$ stopping times, and suppose that P has Property AC for $\mathbb{F}$. Then P has Property AC for $\mathbb{F}^L$ if and only if the compensator A^L of L in $\mathbb{F}$ is absolutely continuous on $[0, L]$.*

Proof. By the Jeulin–Yor theorem [21, 12], the $\mathbb{F}^L$ compensator of L is $\int_0^{t\wedge L} \frac{1}{Z_{s-}} dA_s^L$. Hence this compensator is absolutely continuous if and only if A_t^L is absolutely continuous on $[0, L]$.

The necessity of this condition is now clear, since L is totally inaccessible by Lemma 18.

Conversely, suppose that this condition holds so that L has an absolutely continuous compensator. Let T be a totally inaccessible stopping time for $\mathbb{F}^L$. By Lemma 17, there exists a sequence $(S_n)_{n\geq 1}$ of totally inaccessible $\mathbb{F}$ stopping times such that $[T] \subseteq [L] \cup \bigcup_{n=1}^{\infty}[S_n]$. For notational convenience, let $S_0 = L$, so $[T] \subseteq \bigcup_{n=0}^{\infty}[S_n]$.

By assumption (for $n = 0$) and Theorem 16 (for $n \geq 1$), each S_n has an absolutely continuous compensator A_L^n for $\mathbb{F}^L$; we write $A_t^n = \int_0^t \lambda_s^n \, ds$. Let $T_n = T$ if $T = S_k$ for some $k \leq n$, and $T_n = \infty$ otherwise. Then T_n is a stopping time with $[T_n] \subseteq \bigcup_{k=0}^n [S_k]$, and it follows that T_n has a compensator B^n for $\mathbb{F}^L$ such that $\sum_{k=0}^n A_t^k - B_t^n$ is an increasing process; thus $B_t^n = \int_0^t f_s^n \, ds$ with $0 \leq f_s^n \leq \sum_{k=0}^n \lambda_s^k$; in particular the compensator B^n of T^n is absolutely continuous. It now follows by monotone convergence that B^n converges to the compensator B of T for $\mathbb{F}^L$, and thus this compensator too is absolutely continuous. ■

The condition that A^L be absolutely continuous on $[0, L]$ seems, unfortunately, to be rather restrictive. As we see in Example 22 below, in several natural examples, A^L is, on the contrary, singular, which by the proof above implies that the $\mathbb{F}^L$ compensator of L is singular, and thus that P does *not* have Property AC for $\mathbb{F}^L$. Nevertheless, Theorem 16 still applies, and more generally, the $\mathbb{F}^L$ compensator of every totally inaccessible $\mathbb{F}^L$ stopping time T such that $P(T = L) = 0$ is absolutely continuous.

Problem 20 (i) *Does there exist an honest time L (for some $(\Omega, \mathcal{F}, P, \mathbb{F})$) such that A^L is absolutely continuous a.s.?*

(ii) *In particular, does there exist such an honest time for the natural filtration of a standard Brownian motion?*

We note that the relatively recent work of A. Nikeghbali [25] makes the positive resolution of Problem 20 seem unlikely.

Remark 21 It is easily seen that A^L always is continuous, since otherwise the set of jump times would be an $\mathbb{F}$ predictable set and thus there would exist a predictable $\mathbb{F}$ stopping time R such that $P(\Delta A_R^L > 0) > 0$. But then $E(\Delta 1_{\{t \geq L\}})_R = E\Delta A_R^L > 0$ so $P(L = R) > 0$, a contradiction.

Example 22 Typical examples of honest times are exit times. For a simple example, consider a standard Brownian motion B_t with its standard filtration $\mathbb{F}$, and let $L = \sup\{t \leq 1 : B_t = 0\}$. Then L is an honest time, and $P(L = T) = 0$ for every stopping time T (by the strong Markov property of B, which implies that a.s. either $B_T \neq 0$ or $B_{T+t} = 0$ for some sequence of $t \searrow 0$). Since L belongs to the predictable set $\{t : B_t = 0\}$, the compensator dA^L is a.s. supported by this set, but this set has Lebesgue measure 0, so dA^L is a.s. singular.

In fact [1, 27], a simple calculation shows that for $t < 1$, $Z_t = P(L > t|\mathcal{F}_t) = 2\Phi(-|B_t|/\sqrt{1-t})$, where Φ is the standard normal distribution function, and $dA_t^L = \sqrt{\frac{2}{\pi(1-t)}} dL_t^0$, where L^0 is the local time at 0.

Several similar examples with singular A^L are treated in [1] and [27], for example $\sup\{t < 1 : B_t \in F\}$ for a given finite set F, $\sup\{t < \tau_1 : B_t = 0\}$ with $\tau_1 = \inf\{t > 0 : B_t = 1\}$, and $\sup\{t : |B_t^{(3)}| \geq 1\}$ where $B^{(3)}$ is a three-dimensional Brownian motion and thus $|B^{(3)}|$ is a BES(3) process (see also [10] for this exit time).

The results above are easily extended by induction to the case of the filtration $\mathbb{F}^{\{L^1, L^2, \ldots\}}$ extended by a finite or infinite, strictly increasing sequence $(L_n)_{n=1}^N$ of positive honest variables that avoid all $\mathbb{F}$ stopping times; cf. [7, p. 190–191].

We omit the details. One could also consider a more general setting, where for example L^{i+1} is honest only for $\mathbb{F}^{L^i}$, instead of requiring it and all subsequent random times to be honest for $\mathbb{F}$, or the case where the L^i need not be required to be strictly increasing. We do not treat these cases here. The necessary theory to do so is available, however, within the book of Th. Jeulin [20].

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