PHASE TRANSITIONS FOR MODIFIED ERDŐS–RÉNYI PROCESSES

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Abstract. A fundamental and very well studied region of the Erdős-Rényi process is the phase transition at $m \sim \frac{n}{2}$ edges in which a giant component suddenly appears. We examine the process beginning with an initial graph. We further examine the Bohman–Frieze process in which edges between isolated vertices are more likely. While the positions of the phase transitions vary, the three processes belong, roughly speaking, to the same universality class. In particular, the growth of the giant component in the barely supercritical region is linear in all cases.

1. Introduction

The standard Erdős–Rényi process $(G(n,m))_{m=0}^{\binom{n}{2}}$ starts with an empty graph $G(n,0) = E_n$ with n vertices and adds edges one by one in random order, uniformly over all possibilities, i.e., drawing the edges uniformly without replacement. (Hence, $G(n,m)$ has n vertices and m edges.) This random graph model has been studied a great deal, starting with Erdős and Rényi [6, 7], see for example the monographs by Bollobás [4] and Janson, Łuczak and Ruciński [12].

The purpose of this paper is to study two modifications of this process. We are interested in the sizes (orders) of the components of the random graphs; in particular whether there exists a giant component of size comparable to the entire graph and, if so, how large it is. (We ignore the internal structure of the components.) We denote the components of a graph G by $\mathcal{C}_i(G)$, $i = 1, \dots, \upsilon(G)$, where thus $\upsilon(G)$ is the number of components of G, and their sizes by $C_i(G) := |\mathcal{C}_i(G)|$, $1 \le i \le \upsilon(G)$; we will always assume that the components are ordered such that $C_1 \ge C_2 \ge \dots$. (For convenience we also define $C_i(G) = 0$ when $i > \upsilon(G)$.) We will often, as just done, omit the argument G when the graph is clear from the context. We further denote the edge set of G by $E(G)$, the number of edges by $e(G) := |E(G)|$, and the number of vertices by $|G|$ (the *order* or *size* of G).

We recall the fundamental result for $G(n,m)$ [7] that if $n \to \infty$ and $m \sim cn/2$ for some constant c, then $C_1 = \rho(c)n + o_{\mathrm{p}}(n)$, where $\rho(c) = 0$ if $c \le 1$,

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and $\rho(c) > 0$ if $c > 1$. (Furthermore, $C_2 = o_p(n)$ for every c.) This is usually expressed by saying that there is a *threshold* or *phase transition* at $m = n/2$. See further [7; 4; 12]. Moreover, as $\delta \searrow 0$, $\rho(1+\delta) \sim 2\delta$ (see [5, Theorem 3.17] for a generalization to certain other random graphs). (For the notation $o_p(n)$, and other standard notations used below such as w.h.p., see e.g. [12] and [9].)

In the first modification of the Erdős–Rényi process, we assume that some (non-random) edges are present initially; additional edges then are added randomly as above. We actually consider three slightly different versions of this process; see Section 2 for details. Our main result for these processes (Theorem 2.1) characterizes the existence and size of a giant component in terms of the initial edges (more precisely, the sizes of the components defined by them) and the number of added random edges. We define the *susceptibility* s_2 as the average size of the component containing a random vertex in the initial graph, see (2.1)–(2.3), and show the existence of a threshold when $t_{\mathsf{c}}n/2$ edges are added, where $t_{\mathsf{c}} := s_2^{-1}$. (This was also done, under a technical assumption, in Spencer and Wormald [14].) Moreover, we give upper and lower bounds for the size of the giant component after the threshold in terms of s_2 and two related quantities (higher moments of the component size) s_3 and s_4 for the initial graph, also defined in (2.1)–(2.3).

Our second modification is known as the Bohman–Frieze process, after Bohman and Frieze [2]. The initial graph on n vertices is empty. At each round two edges $e_1 = \{v_1, w_1\}$ and $e_2 = \{v_2, w_2\}$ are selected independently and uniformly. *If* both v_1 and w_1 are isolated vertices the edge e_1 is added to the graph; *otherwise* the edge e_2 is added to the graph. We let BF_m denote this process when m edges are added. This is a natural example of an Achlioptas process, in which a choice may be made from two randomly chosen potential edges. In Bohman and Frieze [2] and Bohman, Frieze and Wormald [3] it was shown that the phase transition is deferred beyond $m \sim n/2$. More precisely, it is proved in Spencer and Wormald [14] that the Bohman–Frieze process has a phase transition at some $t_{\mathsf{c}} \approx 1.1763$. In the present paper we study further what happens just after the phase transition, using the result just described for the Erdős–Rényi process with initial edges. The idea is, as in [14], that to study the process at a time $t_1 > t_{\mathsf{c}}$, we stop the process at a suitable time t_0 just *before* the phase transition, and then approximate the evolution between t_0 and t_1 by an Erdős–Rényi process, using the graph obtained at time t_0 as our initial graph. In order to apply Theorem 2.1, we then need information on s_2, s_3 and s_4 in the subcritical phase. The analysis in Spencer and Wormald [14] of the Bohman–Frieze process (and a class of generalizations of it) is based on studying the susceptibility s_2 in the subcritical region. We will use some results from [14], reviewed in Section 3, and extend them to s_3 and s_4 in order to obtain the required results needed to apply Theorem 2.1.

In particular, we show that after the phase transition, the giant component grows at a linear rate, just as for the Erdős–Rényi process. The precise statement is given by Theorem 3.5. The original Erdős–Rényi process, the process from an appropriate starting point, and the Bohman–Frieze process appear to be in what mathematical physicists loosely call the same universality class. While the placement of the phase transitions differ the nature of the phase transitions appears to be basically the same. A very different picture was given for a related process in [1]. There, as in the Bohman–Frieze process, two random potential edges $e_1 = \{v_1, w_1\}$ and $e_2 = \{v_2, w_2\}$ are given. However the edge is selected by the *Product Rule*: we select that edge for which the product of the component sizes of the two vertices is largest. Strong computational evidence is presented indicating clearly that this process is not in the same univerality class as the three processes we compare. We feel, nonetheless, that there is likely to be a wide variety of processes in the same universality class as the bedrock Erdős–Rényi process.

The main results are stated in Sections 2 and 3, and proved in Sections 4 and 5.

Our results are asymptotic, as the size grows. All unspecified limits are as $n \to \infty$.

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2. Erdős–Rényi process with an initial graph

The purpose of this section is to study the Erdős–Rényi process when some edges are present initially. We define three different but closely related versions of the process.

Let F be a subgraph of K_n with vertex set $V(F) = V(K_n) = \{1, \dots, n\}$. Define $(G(m, n; F))_{m=0}^{\binom{n}{2}-e(F)}$ by starting with $G(n, 0; F) := F$ and adding the $\binom{n}{2} - e(F)$ edges in $E(K_n) \setminus E(F)$ one by one in random order, i.e., by drawing without replacement.

For our purposes it will be convenient to consider two modifications of this random graph process. (Both modifications are well-known for $G(n, m)$.) We define $(G^*(n, m; F))_{m=0}^{\infty}$ by starting with $G^*(n, 0; F) := F$ and then adding at each time step an edge randomly drawn (with replacement) from $E(K_n)$, provided this edge is not already present (in which case nothing happens). In particular, $G^*(n, m) := G^*(n, m; E_n)$ is defined as $G(n, m)$ but drawing the edges with replacement. In general, we have $E(G^*(n, m; F)) = E(G^*(n, m)) \cup E(F)$.

Note that the number of edges in $G^*(n, m)$ may be less than m. Alternatively, we may regard $G^*(n, m; F)$ as a multigraph and add the edges whether they already are present or not; then the number of edges is always exactly $m + e(F)$. Since we will study the component sizes only, this makes no difference for the present paper.

The second modification is to use continuous time. We may think of the $\binom{n}{2}$ edges as arriving according to independent Poisson processes with rates $1/n$; thus edges appear at a total rate $\binom{n}{2}/n = \frac{n-1}{2}$ and each edge is chosen uniformly at random and independently of all previous choices. We define $\tilde{G}(n,t;F)$ to be F together with all edges that have arrived in $[0,t]$. (As above, we can consider either a multigraph version or the corresponding process of simple graphs, obtained by ignoring all edges that already appear in the graph.) Hence, if i and j are two vertices that are not already joined by an edge in F, then the probability that they are joined in $\tilde{G}(n,t;F)$ is $1-e^{-t/n} = t/n + O(t^2/n^2)$, and these events are independent for different pairs i,j. (Starting with the empty graph we thus obtain $G(n,p)$ with $p = 1-e^{-t/n}$. We could change the time scale slightly to obtain exactly $G(n,t/n)$, and asymptotically we obtain the same results for the two versions.)

Note that if $N(t)$ is the total number of edges arriving in $[0,t]$, then $N(t) \sim \operatorname{Po}\bigl(\binom{n}{2}t/n\bigr) = \operatorname{Po}\bigl(\frac{n-1}{2}t\bigr)$, and, with an obvious coupling of the processes, $\tilde{G}(n,t;F) = G^*(n,N(t);F)$. For constant t, $N(t)/(n/2) \xrightarrow{\text{p}} t$ as $n\to\infty$ by the law of large numbers. Moreover, the expected number of repeated edges in $G^*(n,m;F)$ is at most $\binom{m}{2}/\binom{n}{2} + m|E(F)|/\binom{n}{2}$; if for example, as in Theorem 2.1 $m = O(n)$ and $|E(F)| = O(n)$, then this is $O(1)$, which will be negligible. Standard arguments, comparing the processes at times t and $(1\pm\varepsilon)t$, show that for the properties considered here, and asymptotically as $n\to\infty$, we then obtain the same results for $G(n,\lfloor nt/2\rfloor;F)$, $G^*(n,\lfloor nt/2\rfloor;F)$, and $\tilde{G}(n,t;F)$.

We define, for a graph G with components of sizes $C_1,\dots,C_v$, and $k\ge 1$,

$$S_k = S_k(G) := \sum_i C_i^k, \tag{2.1}$$

summing over all components of G. Thus $S_1(G) = |G|$, the number of vertices. We normalize these sums by dividing by $|G|$ and define

$$s_k = s_k(G) := \frac{S_k(G)}{|G|} = \frac{S_k(G)}{S_1(G)}. \tag{2.2}$$

Hence, $s_1(G) = 1$ for every G. Note that

$$s_k(G) = \sum_i \frac{C_i}{|G|} C_i^{k-1}, \tag{2.3}$$

which is the $(k-1)$:th moment of the size of the component containing a randomly chosen vertex. In particular, $s_2(G)$ is the average size of the component containing a random vertex. The number $s_2(G)$ is called the *susceptibility*; see e.g. [11; 13; 10] for results on the susceptibility in $G(n,m)$ and some other random graphs.

It follows from the definitions (2.1) and (2.2) that S_k and s_k are (weakly) increasing in k; in particular, $s_k(G) \geq s_1(G) = 1$ for every k and G. Moreover, Hölder's inequality and (2.3) imply that the stronger result that $s_k^{1/k}$ (and even $s_k^{1/(k-1)}$, $k \geq 2$) is (weakly) increasing in k.

Note further that the number of edges in a component of size C_i is at most $\binom{C_i}{2} \leq C_i^2$; hence, for any graph G,

$$|E(G)| \leq S_2(G). \tag{2.4}$$

We will use these functionals for the initial graph F to characterize the existence and size of a giant component in the random graph processes starting with F. An informal summary of the following theorem (our main result in this section) is that there is a phase transition at $t_{\mathsf{c}} := 1/s_2(F)$, and that for $t = t_{\mathsf{c}} + \delta$ with δ small, there is a giant component of size $\approx 2(s_2(F)^3/s_3(F))\delta n$. For the special case when $F = E_n$ is empty, $s_2 = s_3 = 1$ and we recover the well-known result for the Erdős–Rényi process mentioned above that there is a phase transition at $t_{\mathsf{c}} = 1$ (i.e., at $n/2$ edges) and further for $t = 1 + \delta$, there is a giant component of size $\approx 2\delta n$. The formal statement is asymptotic, and we thus consider a sequence F_n.

Theorem 2.1. *Suppose that for each n (at least in some subsequence), F_n is a given graph with n vertices, and suppose that $\sup_n s_3(F_n) < \infty$. Let the random variable Z_n be the size of the component containing a random vertex in F_n.*

Consider the random graph processes $\tilde{G}(n,t;F_n)$. Then, for any fixed $t > 0$, the following hold as $n \to \infty$, with $s_k := s_k(F_n)$,

(i) *If $t \leq 1/s_2$, then $C_1(\tilde{G}(n,t;F_n)) = o_{\mathrm{p}}(n)$.*

(ii) *If $t > 1/s_2$, then there is a unique $\rho_n > 0$ such that*

$$\rho_n = 1 - \mathbb{E}\, e^{-\rho_n t Z_n},$$

and we have

$$C_1(\tilde{G}(n,t;F_n)) = \rho_n n + o_{\mathrm{p}}(n).$$

(iii) *If $t > 1/s_2$, let $\delta_n := t - 1/s_2 > 0$. Then*

$$\frac{C_1(\tilde{G}(n,t;F_n))}{n} \geq 2\delta_n \frac{s_2^3}{s_3}(1 - 2\delta_n s_2) + o_{\mathrm{p}}(1).$$

If further $\delta_n s_2^2 s_4/s_3^2 \leq \frac{3}{8}$, then also

$$\frac{C_1(\tilde{G}(n,t;F_n))}{n} \leq 2\delta_n \frac{s_2^3}{s_3}\left(1 + \frac{8}{3}\delta_n \frac{s_2^2 s_4}{s_3^2}\right) + o_{\mathrm{p}}(1).$$

(iv) *In (iii), if in addition $\liminf_{n\to\infty} \delta_n > 0$, then moreover w.h.p.*

$$\frac{C_1(\tilde{G}(n,t;F_n))}{n} \geq 2\delta_n \frac{s_2^3}{s_3}(1 - 2\delta_n s_2)$$

and, if $\delta_n s_2^2 s_4/s_3^2 \le \frac{3}{8}$,

$$\frac{C_1(\tilde{G}(n,t;F_n))}{n} \le 2\delta_n \frac{s_2^3}{s_3}\left(1+\frac{8}{3}\delta_n\frac{s_2^2 s_4}{s_3^2}\right).$$

The same results hold for the random graph processes $G(n,\lfloor nt/2\rfloor;F)$ *and* $G^*(n,\lfloor nt/2\rfloor;F)$.

The proof is given in Section 4. Note that by (2.3),

$$\mathbb{E}\, Z_n^k = s_{k+1}(F_n), \qquad k \ge 1. \tag{2.5}$$

3. The Bohman–Frieze process

Recall the definition of the Bohman–Frieze process from Section 1, see [2; 3; 14]: we are at each round presented with two random edges $e_1 = \{v_1, w_1\}$ and $e_2 = \{v_2, w_2\}$ in the complete graph K_n and choose one of them; we choose e_1 if both its endpoints v_1 and w_1 are isolated, and otherwise we choose e_2. We let BF_m denote the random graph created by this process when m edges are added. (The size n is not shown explicitly.) We further define, using the natural time scale, $\mathsf{BF}(t) := \mathsf{BF}_{\lfloor nt/2\rfloor}$. (For convenience, we sometimes omit rounding to integers in expressions below.)

Note that if we add e_1, then it always joins two previously isolated vertices, while if we add e_2, it is uniformly distributed and independent of the existing graph. We call the added edges e_2 *Erdős–Rényi edges*, since all edges in the Erdős–Rényi process are of this type.

Remark 3.1. We have talked about edges e_1 and e_2, but it is technically convenient in the proofs to allow also loops (as in [14]); we thus assume in the proofs below that in each round, the vertices v_1, w_1, v_2, w_2 are independent, uniformly distributed, random vertices. It is easily seen that the results proved for this version hold also if we assume that there are no loops, for example by conditioning on the event that no loops are presented during the first $nt/2$ rounds; we omit the details.

For a graph G, let $n_i = n_i(G)$ be the number of vertices in components of order i, and let $x_i = x_i(G) := n_i(G)/|G|$ be the proportion of the total number of vertices that are in such components. (Thus, $s_k(G) = \sum_i i^{k-1} x_i(G)$.) For the Bohman–Frieze process, we need only n_1, the number of isolated vertices, and the corresponding proportion $x_1 := n_1/n$.

For the Bohman–Frieze process (and some generalizations of it), it is shown in Spencer and Wormald [14] that the random variables $x_1(\mathsf{BF}(t))$ (for any fixed $t < \infty$) and $s_2(\mathsf{BF}(t))$ (for any fixed $t < t_{\mathsf{c}}$) converge in probability, as $n \to \infty$, to some deterministic values $\bar{x}_1(t)$ and $\bar{s}_2(t)$; these limit values are given as solutions to differential equations. We extend this to s_3 and s_4 as follows.

We first define, as in [14], the deterministic function $\bar{x}_1(t)$ as the solution to the differential equation

$$\bar{x}_1'(t) = -\bar{x}_1^2(t) - \bigl(1-\bar{x}_1^2(t)\bigr)\bar{x}_1(t), \qquad t \ge 0, \tag{3.1}$$

with initial condition $\bar{x}_1(0) = 1$; by [14, Theorem 2.1], $\bar{x}_1(t)$ is defined and positive for all $t \geq 0$, and by [14, Theorem 1.1], $x_1(\mathsf{BF}(t)) \xrightarrow{\mathrm{p}} \bar{x}_1(t)$ for every fixed $t \geq 0$.

We further define functions $\bar{s}_2(t)$, $\bar{s}_3(t)$, $\bar{s}_4(t)$ as the solutions to the differential equations

$$\bar{s}_2'(t) = \bar{x}_1^2(t) + \bigl(1 - \bar{x}_1^2(t)\bigr)\bar{s}_2^2(t), \tag{3.2}$$

$$\bar{s}_3'(t) = 3\bar{x}_1^2(t) + 3\bigl(1 - \bar{x}_1^2(t)\bigr)\bar{s}_2(t)\bar{s}_3(t), \tag{3.3}$$

$$\bar{s}_4'(t) = 7\bar{x}_1^2(t) + \bigl(1 - \bar{x}_1^2(t)\bigr)\bigl(4\bar{s}_2(t)\bar{s}_4(t) + 3\bar{s}_3^2(t)\bigr), \tag{3.4}$$

with initial conditions

$$\bar{s}_2(0) = \bar{s}_3(0) = \bar{s}_4(0) = 1. \tag{3.5}$$

The function $\bar{s}_2(t)$ is studied in Spencer and Wormald [14], Theorem 2.2, and it is shown there that it explodes at some finite t_{c}, i.e., the solution $\bar{s}_2(t)$ is (uniquely) defined for $t \in [0, t_{\mathsf{c}})$, but $\bar{s}_2(t) \nearrow +\infty$ as $t \nearrow t_{\mathsf{c}}$; it is further shown [14, Theorem 1.1] that this t_{c} is the time of the phase transition for the Bohman–Frieze process, when a giant component first appears, and that for any fixed $t < t_{\mathsf{c}}$, $\bar{s}_2(\mathsf{BF}(t)) \xrightarrow{\mathrm{p}} \bar{s}_2(t)$. We extend these results to $\bar{s}_3$ and $\bar{s}_4$ as follows.

Theorem 3.2. *The functions $\bar{s}_2(t)$, $\bar{s}_3(t)$, $\bar{s}_4(t)$ are uniquely defined by (3.2)–(3.5) for all $t \in [0, t_{\mathsf{c}})$. As $t \nearrow t_{\mathsf{c}}$, there exist positive constants α and β such that*

$$\begin{aligned}
\bar{s}_2(t) &\sim \frac{\alpha}{t_{\mathsf{c}} - t},\\
\bar{s}_3(t) &\sim \beta \bar{s}_2(t)^3 \sim \frac{\beta\alpha^3}{(t_{\mathsf{c}} - t)^3},\\
\bar{s}_4(t) &\sim 3\beta^2 \bar{s}_2(t)^5 \sim \frac{3\beta^2\alpha^5}{(t_{\mathsf{c}} - t)^5}.
\end{aligned}$$

More precisely, $\bar{s}_k(t) = a_k(t_{\mathsf{c}} - t)^{-(2k-3)}(1 + O(t_{\mathsf{c}} - t))$ for $k = 2, 3, 4$ with $a_2 = \alpha$, $a_3 = \beta\alpha^3$, $a_4 = 3\beta^2\alpha^5$.

We have $\alpha = \bigl(1 - \bar{x}_1^2(t_{\mathsf{c}})\bigr)^{-1}$, while $\beta = g(t_{\mathsf{c}})$ is given by (5.2) and (5.5).

Theorem 3.3. *For any fixed $t \in [0, t_{\mathsf{c}})$, and $k = 2, 3, 4$, $s_k(\mathsf{BF}(t)) \xrightarrow{\mathrm{p}} \bar{s}_k(t)$.*

Remark 3.4. It is straightforward to extend Theorem 3.3 to any $k \geq 2$, with $\bar{s}_k(t)$ given by a differential equation similar to (3.2)–(3.4) (involving $\bar{s}_j$ for $j < k$, so the functions are defined recursively). We leave the details to the reader since we only use $k \leq 4$ in the present paper.

Proofs are given in Section 5. Using these results for the subcritical phase, we obtain the following for the supercritical phase; again the proof is given in Section 5.

Theorem 3.5. *There exists constants $\gamma = 2(1-\bar{x}_1^2(t_{\mathsf{c}}))/\beta > 0$ and $K < \infty$ such that for any fixed $\delta > 0$, w.h.p.*

$$\gamma\delta - K\delta^{4/3} \le \frac{C_1(\mathsf{BF}(t_{\mathsf{c}}+\delta))}{n} \le \gamma\delta + K\delta^{4/3}.$$

Remark 3.6. Numerical calculations of Will Perkins give $t_{\mathsf{c}} \approx 1.1763$, $\bar{x}_1(t_{\mathsf{c}}) \approx 0.2438$, $\alpha \approx 1.063$, $\beta \approx 0.764$, $a_2 = \alpha$, $a_3 \approx 0.917$, $a_4 \approx 2.375$ and $\gamma \approx 2.463$.

There is an obvious conjecture (made explicit in [14]) that $C_1(\mathsf{BF}(t))/n \xrightarrow{\mathrm{p}} \rho_{\mathsf{BF}}(t)$ for some function $\rho_{\mathsf{BF}} : [0,\infty) \to [0,1]$; equivalently, $C_1(\mathsf{BF}(t)) = \rho_{\mathsf{BF}}(t)n + o_{\mathrm{p}}(n)$. (For $t < t_{\mathsf{c}}$, clearly this holds with $\rho_{\mathsf{BF}}(t) = 0$.) In Spencer and Wormald [14] it was further conjectured that $\lim_{\delta\to t_{\mathsf{c}}^+} \rho_{\mathsf{BF}}(t) = 0$; in the language of Mathematical Physics, this says that the phase transition is not first order. If such an ρ_{BF} exists, Theorem 3.5 resolves the latter conjecture positively and further gives the asymptotic behavior $\rho_{\mathsf{BF}}(t_{\mathsf{c}}+\delta) \sim \gamma\delta$ as $\delta \to 0^+$.

Remark 3.7. We further conjecture that the function ρ_{BF} is smooth on $[t_{\mathsf{c}},\infty)$; if this is the case, then Theorem 3.5 shows that $\rho'_{\mathsf{BF}}(t_{\mathsf{c}}^+) = \gamma$. This conjecture would imply that $\delta^{4/3}$ in Theorem 3.5 could be replaced by δ^2; unfortunately, our approximations are not sharp enough to show this.

4. Proof of Theorem 2.1

We begin with a simple lemma (related to results in [5, Section 5]).

Lemma 4.1. *Let $Y \ge 0$ be a random variable with $1 < \mathbb{E}\,Y \le \infty$.*

(i) *There is a unique $\rho > 0$ such that*

$$\rho = 1 - \mathbb{E}\,e^{-\rho Y}. \tag{4.1}$$

(ii) *If $\mathbb{E}\,Y^2 < \infty$, then*

$$\rho > \frac{2(\mathbb{E}\,Y-1)}{\mathbb{E}\,Y^2}.$$

(iii) *If $\mathbb{E}\,Y^3 < \infty$ and $8(\mathbb{E}\,Y-1)\,\mathbb{E}\,Y^3 \le 3(\mathbb{E}\,Y^2)^2$, then*

$$\begin{aligned}\rho &< \frac{3\,\mathbb{E}\,Y^2 - \sqrt{9(\mathbb{E}\,Y^2)^2 - 24(\mathbb{E}\,Y-1)\,\mathbb{E}\,Y^3}}{2\,\mathbb{E}\,Y^3}\\ &= \frac{4(\mathbb{E}\,Y-1)}{\mathbb{E}\,Y^2 + \sqrt{(\mathbb{E}\,Y^2)^2 - \frac{8}{3}(\mathbb{E}\,Y-1)\,\mathbb{E}\,Y^3}}\\ &\le \frac{2(\mathbb{E}\,Y-1)}{\mathbb{E}\,Y^2}\left(1 + \frac{8(\mathbb{E}\,Y-1)\,\mathbb{E}\,Y^3}{3(\mathbb{E}\,Y^2)^2}\right).\end{aligned}$$

(iv) *Let Y_n, $n \ge 1$, be random variables with $Y_n \ge 0$ and $\mathbb{E}\,Y_n > 1$ and let $\rho_n > 0$ be the corresponding numbers such that $\rho_n = 1 - \mathbb{E}\,e^{-\rho_n Y_n}$. If $Y_n \xrightarrow{\mathrm{d}} Y$ for some Y with $\mathbb{E}\,Y > 1$, then $\rho_n \to \rho > 0$ satisfying* (4.1). *On the other hand, if $Y_n \xrightarrow{\mathrm{d}} Y$ with $\mathbb{E}\,Y \le 1$, then $\rho_n \to 0$.*

Remark 4.2. In fact, (4.1) is the standard equation for the survival probability of a Galton–Watson process with a mixed Poisson $\mathrm{Po}(Y)$ offspring distribution. Parts (i) and (iv) follow easily from standard results on branching processes. We prefer, however, to give direct proofs (also easy). Note further that $\rho = 0$ always is another solution to (4.1). If $\mathbb{E}\, Y \leq 1$, then $\rho = 0$ is the only non-negative solution, either by branching process theory, or because

$$1 - \mathbb{E}\, e^{-sY} = \mathbb{E}(1 - e^{-sY}) \leq \mathbb{E}(sY) \leq s \tag{4.2}$$

for every $s \geq 0$, with strict inequality unless $sY = 0$ a.e. and $\mathbb{E}(sY) = s$, which together imply $s = 0$.

Proof. The function $\varphi(s) := 1 - \mathbb{E}\, e^{-sY}$, $s \in [0, \infty)$, is increasing and concave with $0 \leq \varphi(s) < 1$, $\varphi(0) = 0$ and $\varphi'(0) = \mathbb{E}\, Y > 1$. Consequently, $\varphi(s) > s$ for small $s > 0$, but $\varphi(s) < s$ for $s > 1$, say, and there is a unique $\rho > 0$ such that $\varphi(\rho) = \rho$. This proves (i). Note that $\varphi(s) > s$ for $0 < s < \rho$ and $\varphi(s) < s$ for $s > \rho$.

We next prove (iv). If $\mathbb{E}\, Y > 1$, let $0 < \varepsilon < \rho$. Then $\varphi(\rho - \varepsilon) > \rho - \varepsilon$ and thus, because $Y_n \xrightarrow{\mathrm{d}} Y$,

$$1 - \mathbb{E}\, e^{-(\rho-\varepsilon)Y_n} \to 1 - \mathbb{E}\, e^{-(\rho-\varepsilon)Y} > \rho - \varepsilon,$$

so for large n, $1 - \mathbb{E}\, e^{-(\rho-\varepsilon)Y_n} > \rho - \varepsilon$ and thus $\rho - \varepsilon < \rho_n$. Similarly, for large n, $1 - \mathbb{E}\, e^{-(\rho+\varepsilon)Y_n} < \rho + \varepsilon$ and thus $\rho + \varepsilon > \rho_n$. Since ε is arbitrarily small, it follows that $\rho_n \to \rho$.

If instead $\mathbb{E}\, Y \leq 1$, then $\varphi(s) < s$ for every $s > 0$ by (4.2) and the comment after it. Hence the same argument shows that for every $\varepsilon > 0$, $\rho_n < \varepsilon$ for large n; thus $\rho_n \to 0$.

To see (ii), observe that $e^{-x} \leq 1 - x + x^2/2$ for $x \geq 0$, with strict inequality unless $x = 0$, and thus, when $\mathbb{E}\, Y^2 < \infty$,

$$\rho = \mathbb{E}\big(1 - e^{-\rho Y}\big) > \mathbb{E}\left(\rho Y - \frac{\rho^2 Y^2}{2}\right) = \rho\, \mathbb{E}\, Y - \frac{\rho^2}{2}\, \mathbb{E}\, Y^2.$$

Hence, $1 > \mathbb{E}\, Y - \rho\, \mathbb{E}\, Y^2/2$, which yields (ii).

For (iii), we first note that, similarly, $e^{-x} \geq 1 - x + x^2/2 - x^3/6$ for $x \geq 0$, again with strict inequality unless $x = 0$, and thus, provided $\mathbb{E}\, Y^3 < \infty$,

$$\rho = \mathbb{E}\big(1 - e^{-\rho Y}\big) < \mathbb{E}\left(\rho Y - \frac{\rho^2 Y^2}{2} + \frac{\rho^3 Y^3}{6}\right) = \rho\, \mathbb{E}\, Y - \frac{\rho^2}{2}\, \mathbb{E}\, Y^2 + \frac{\rho^3}{6}\, \mathbb{E}\, Y^3.$$

This can be written

$$\mathbb{E}\, Y^3\, \rho^2 - 3\mathbb{E}\, Y^2\, \rho + 6(\mathbb{E}\, Y - 1) > 0. \tag{4.3}$$

As long as the discriminant $9(\mathbb{E}\, Y^2)^2 - 24(\mathbb{E}\, Y - 1)\, \mathbb{E}\, Y^3 \geq 0$, the corresponding quadratic equation (with equality instead of $>$) has two roots

$$\rho_\pm = \frac{3\, \mathbb{E}\, Y^2 \pm \sqrt{9(\mathbb{E}\, Y^2)^2 - 24(\mathbb{E}\, Y - 1)\, \mathbb{E}\, Y^3}}{2\, \mathbb{E}\, Y^3}$$

and we have either $\rho < \rho_-$ or $\rho > \rho_+$. In order to rule out the latter possibility, we consider the random variable $Y_t := tY$ for $t_0 < t \le 1$, where $t_0 = 1/\mathbb{E}\,Y$. Note that for $t_0 < t \le 1$, $\mathbb{E}\,Y_t > 1$ and thus there is an $\rho(t) > 0$ such that $\rho(t) = 1 - \mathbb{E}\,e^{-\rho(t)Y_t}$; by (iv), $\rho(t)$ is a continuous function of t. Further, for $t_0 < t \le 1$,

$$\begin{aligned}9(\mathbb{E}\,Y_t^2)^2 &= 9t^4(\mathbb{E}\,Y^2)^2 \ge 24t^4(\mathbb{E}\,Y-1)\,\mathbb{E}\,Y^3 = 24(\mathbb{E}\,Y_t - t)\,\mathbb{E}\,Y_t^3 \\ &\ge 24(\mathbb{E}\,Y_t - 1)\,\mathbb{E}\,Y_t^3;\end{aligned}$$

hence the discriminant is non-negative for each Y_t, and there are corresponding roots $\rho_\pm(t)$. These are continuous functions of t and for each $t \in (t_0, 1)$, $\rho(t) < \rho_-(t)$ or $\rho(t) > \rho_+(t)$. As $t \searrow t_0$, $\mathbb{E}\,Y_t \to 1$ and $\rho_+(t) \to 3\,\mathbb{E}\,Y_{t_0}^2/(2\,\mathbb{E}\,Y_{t_0}^3) > 0$ while, by (iv) again, $\rho(t) \to 0$. Hence, $\rho(t) < \rho_+(t)$ for t close to t_0, and by continuity, $\rho(t) < \rho_+(t)$ for all $t \in (t_0, 1]$ (since equality is impossible by (4.3)). Consequently, $\rho < \rho_+$ and thus $\rho < \rho_-$.

Finally, we use straightforward algebra and the fact that for $x \in [0,1]$, $\sqrt{1-x} \ge (1-x)/(1+x)$ and thus

$$\frac{1}{1+\sqrt{1-x}} \le \frac{1+x}{2}. \qquad \square$$

Proof of Theorem 2.1. Note that the assumptions and (2.4) imply that

$$|E(F_n)| \le S_3(F_n) = ns_3(F_n) = O(n).$$

Hence, by the discussion in Section 2, it suffices to consider $\tilde{G}(n, t; F_n)$.

The main idea is that we may collapse each component $\mathcal{C}_i(F_n)$ of F_n to a "supervertex" with *weight*

$$x_i = x_i^{(n)} := |\mathcal{C}_i(F_n)| = C_i(F_n). \tag{4.4}$$

The probability of an edge between $\mathcal{C}_i(F_n)$ and $\mathcal{C}_j(F_n)$ in $\tilde{G}(n, t; F_n)$ is, for $i \ne j$,

$$p_{ij}(t) = 1 - e^{-tx_ix_j/n}. \tag{4.5}$$

Hence, to obtain the distribution of component sizes in $\tilde{G}(n, t; F_n)$ we may instead consider the random graph H_n with $v = v(F_n)$ vertices having weights x_i given by (4.4) and edges added independently with probabilities p_{ij} given by (4.5); note that the size of a component in $\tilde{G}(n, t; F_n)$ is given by the weight of the corresponding component in H_n, i.e., the sum of the weights of the vertices in it.

The random graph H_n is an instance of the general random graph model studied in Bollobás, Janson and Riordan [5]; we will use results from [5], and therefore we show the relation in some detail.

We will actually consider a subsequence only, for technical reasons, and thus we at first obtain the result for this subsequence only. However, this means that if we start with any subsequence of the original sequence, there exists a subsubsequence where the result holds; this fact implies that the result actually holds for the full sequence by the subsubsequence principle, see e.g. [12, p. 12].

We have defined Z_n as the size of the component containing a random vertex in F_n. Let $\widehat{\nu}_n$ be the distribution of Z_n; thus $\widehat{\nu}_n$ is the probability measure on $\mathbb{Z}_+ := \{1, 2, \dots\}$ given by $\sum_i \frac{C_i}{n}\delta_{C_i}$. By (2.5), $\mathbb{E}\, Z_n = s_2(F_n) \leq s_3(F_n) = O(1)$, which implies that the sequence of random variables Z_n is tight, see e.g. [8, Section 5.8.3]. Consequently (see [8, Theorem 5.8.5]), we may select a subsequence such that Z_n converges in distribution to some random variable Z. Equivalently, $\widehat{\nu}_n$ converges (weakly) to some probability measure $\widehat{\mu}$ on $\mathbb{Z}_+$, where $\widehat{\mu}$ is the distribution of Z. Moreover, $\mathbb{E}\, Z_n^2 = s_3(F_n) = O(1)$, and thus [8, Theorem 5.4.2] Z_n are uniformly integrable; consequently [8, Theorem 5.5.8], $s_2(F_n) = \mathbb{E}\, Z_n \to \mathbb{E}\, Z$. We denote this limit by $\bar{s}_2$, and have thus

$$s_2(F_n) \to \bar{s}_2 = \mathbb{E}\, Z. \tag{4.6}$$

Let $\upsilon_k(F_n)$ be the number of components of order k in F_n and let ν_n be the measure on $\mathbb{Z}_+$ defined by

$$\nu_n\{k\} := \frac{\upsilon_k(F_n)}{n}.$$

Equivalently, $\nu_n := \frac{1}{n}\sum_{i=1}^{\upsilon} \delta_{C_i}$. The total mass of ν_n is thus $\nu_n(\mathbb{Z}_+) = \upsilon(F_n)/n \leq 1$. (In general, ν_n is not a probability measure.)

The total size of the components of order k in F_n is $k\upsilon_k(F_n)$, and thus

$$\widehat{\nu}_n\{k\} = \mathbb{P}(Z_n = k) = \frac{k\upsilon_k(F_n)}{n} = k\nu_n\{k\}.$$

Let μ be the measure on $\mathbb{Z}_+$ given by

$$\mu\{k\} := \widehat{\mu}\{k\}/k, \qquad k \geq 1.$$

Since we have $\widehat{\nu}_n\{k\} \to \widehat{\mu}\{k\}$, we also have

$$\nu_n\{k\} = \widehat{\nu}_n\{k\}/k \to \widehat{\mu}\{k\}/k = \mu\{k\}$$

for every $k \geq 1$. Moreover, if $f : \mathbb{Z}_+ \to \mathbb{R}$ is any bounded function, and $g(k) := f(k)/k$, then the convergence $\widehat{\nu}_n \to \widehat{\mu}$ implies

$$\int_{\mathbb{Z}_+} f(x)\,\mathrm{d}\nu_n(x) = \int_{\mathbb{Z}_+} g(x)\,\mathrm{d}\widehat{\nu}_n(x) \to \int_{\mathbb{Z}_+} g(x)\,\mathrm{d}\widehat{\mu}(x) = \int_{\mathbb{Z}_+} f(x)\,\mathrm{d}\mu(x).$$

Hence $\nu_n \to \mu$ weakly; in particular

$$\nu_n(A) \to \mu(A) \qquad \text{for every } A \subseteq \mathbb{Z}_+. \tag{4.7}$$

We let $(\mathbf{x}_n)_{n\geq 1}$ be the sequence $(C_1(F_n), \dots, C_{\upsilon_n}(F_n))$ of component sizes of F_n, where $\upsilon_n := \upsilon(F_n)$. We have just shown that the triple $\mathcal{V} := (\mathbb{Z}_+, \mu, (\mathbf{x}_n)_{n\geq 1})$ is a *generalized vertex space* in the sense of [5, p. 10]; in particular, the crucial condition [5, (2.4)] is our (4.7).

We define the *kernel* κ on $\mathbb{Z}_+$ by

$$\kappa(x, y) := txy \tag{4.8}$$

(recall that t is fixed); the probability (4.5) of an edge in H_n between (super)vertices with weights x_i and x_j is thus $1-\exp(-\kappa(x_i,x_j)/n)$, which agrees with [5, (2.6)]. Hence, our random graph H_n is the graph denoted $G^{\mathcal{V}}(n,\kappa)$ in [5].

We further have, with $x_i = C_i(F_n)$, by (4.5),

$$\frac{1}{n}\mathbb{E}\, e(H_n) = \frac{1}{n}\sum_{1\le i<j\le v_n} p_{ij} = \frac{1}{n}\sum_{1\le i<j\le v_n}\bigl(1-\exp(-tx_ix_j/n)\bigr)$$
$$\le \frac{1}{n^2}\sum_{1\le i<j\le v_n} tx_ix_j \le \frac{t}{2}\left(\frac{1}{n}\sum_{i=1}^{v_n} x_i\right)^2 = \frac{t}{2},$$

and

$$\int_{\mathbb{Z}_+} x\,\mathrm{d}\mu(x) = \sum_{x=1}^{\infty} x\,\mathrm{d}\mu\{x\} = \sum_{x=1}^{\infty}\mathrm{d}\widehat{\mu}\{x\} = \widehat{\mu}(\mathbb{Z}_+) = 1 \tag{4.9}$$

(since $\widehat{\mu}$ is a probability measure on $\mathbb{Z}_+$); hence

$$\iint_{\mathbb{Z}_+^2} \kappa(x,y)\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y) = t\left(\int_{\mathbb{Z}_+} x\,\mathrm{d}\mu(x)\right)^2 = t \tag{4.10}$$

and

$$\frac{1}{n}\mathbb{E}\, e(H_n) \le \frac{1}{2}\iint_{\mathbb{Z}_+^2} \kappa(x,y)\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y).$$

Together with [5, Lemma 8.1], this shows that

$$\frac{1}{n}\mathbb{E}\, e(H_n) \to \frac{1}{2}\iint_{\mathbb{Z}_+^2} \kappa(x,y)\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y),$$

and thus, using also (4.10), the kernel κ is *graphical* [5, Definition 2.7].

We can now apply the results of [5]. The kernel $\kappa(x,y)$ is of the special type $\psi(x)\psi(y)$ (with $\psi(x) := t^{1/2}x$), which is the *rank 1* case studied in [5, Section 16.4], and it follows by [5, Theorem 3.1 and (16.8)] that H_n has a giant component if and only if $\|T_\kappa\| > 1$, where T_κ is the integral operator with kernel κ; in the rank 1 case T_κ has the norm, using also (4.6),

$$\|T_\kappa\| = \int_{\mathbb{Z}_+} \psi(x)^2\,\mathrm{d}\mu(x) = \int_{\mathbb{Z}_+} tx^2\,\mathrm{d}\mu(x) = \int_{\mathbb{Z}_+} tx\,\mathrm{d}\widehat{\mu}(x) = t\,\mathbb{E}\,Z = t\bar{s}_2.$$

Hence there is a phase transition at $t_{\mathsf{c}} := 1/\bar{s}_2$. We consider the cases $t \le t_{\mathsf{c}}$ and $t > t_{\mathsf{c}}$ separately.

4.1. **The (sub)critical case.** Consider first the case $t \le \bar{s}_2^{-1}$; then H_n thus has no giant component; more precisely,

$$C_1(H_n) = o_{\mathrm{p}}(n). \tag{4.11}$$

Recall, however, that we really are interested in the size of the largest component of $\tilde{G}(n,t;F_n)$, which is the same as the largest *weight* of a component in H_n. (Note also that the component with largest weight not necessarily is the component with largest number of vertices.) Nevertheless, the corresponding estimate follows easily: Let $A > 0$. Then the total weight of all vertices in H_n of weight larger than A is

$$\begin{aligned}\sum_i x_i \mathbf{1}[x_i > A] &= \sum_{k>A} k v_k(F_n) \le A^{-1} \sum_{k\ge 1} k^2 v_k(F_n) = A^{-1} S_2(F_n) \\ &= nA^{-1} s_2(F_n),\end{aligned}$$

and thus the weight of any component $\mathcal{C}$ in H_n is

$$\begin{aligned}\sum_{i\in\mathcal{C}} x_i &\le \sum_{i\in\mathcal{C}} x_i \mathbf{1}[x_i \le A] + \sum_i x_i \mathbf{1}[x_i > A] \le A|\mathcal{C}| + nA^{-1} s_2(F_n) \\ &\le AC_1(H_n) + nA^{-1} s_2(F_n).\end{aligned}$$

For any $\varepsilon > 0$, we may choose $A = A_n := \varepsilon^{-1} s_2(F_n)$ and find (since $A_n = O(1)$) w.h.p., using (4.11),

$$C_1(\tilde{G}(n, t; F_n)) = \sup_{\mathcal{C}} \sum_{i\in\mathcal{C}} x_i \le A_n C_1(H_n) + \varepsilon n \le 2\varepsilon n. \tag{4.12}$$

which proves (i) when $t \le 1/\bar{s}_2$.

4.2. **The supercritical case.** Suppose now that $t > \bar{s}_2^{-1}$.

By [5, Theorem 3.1], the size $C_1(H_n)$ of the largest component $\mathcal{C}_1$ of H_n satisfies

$$\frac{C_1(H_n)}{n} \xrightarrow{\mathrm{p}} \rho(\kappa) > 0.$$

Furthermore $C_2(H_n) = o_{\mathrm{p}}(n)$, and it follows by the same argument as for (4.12) above that the weigth of any component $\mathcal{C} \ne \mathcal{C}_1$ of H_n is at most

$$\max_{\mathcal{C}\ne\mathcal{C}_1} \sum_{i\in\mathcal{C}} x_i \le A_n C_2(H_n) + \varepsilon n \le 2\varepsilon n$$

w.h.p., and thus $o_{\mathrm{p}}(n)$. Since $\mathcal{C}_1$ has weight $\ge |\mathcal{C}_1| = \rho(\kappa)n + o_{\mathrm{p}}(n)$, it follows that w.h.p. the largest component $\mathcal{C}_1$ of H_n also has the largest weight, and thus corresponds to the largest component in $\tilde{G}(n, t; F_n)$, while $C_2(\tilde{G}(n, t; F_n)) = o_{\mathrm{p}}(n)$.

It remains to find the weight of $\mathcal{C}_1$. We first note that by [5, (2.13), Theorem 6.2 and (5.3)], $\rho(\kappa) = \int_{\mathbb{Z}_+} \rho_\kappa(x)\,\mathrm{d}\mu(x)$, where $\rho_\kappa(x)$ is the unique positive solution to

$$\rho_\kappa = \Phi_\kappa(\rho_\kappa) := 1 - e^{-T_\kappa \rho_\kappa}.$$

Since

$$T_\kappa \rho_\kappa(x) := \int_{\mathbb{Z}_+} \kappa(x, y)\rho_\kappa(y)\,\mathrm{d}\mu(y) = tx \int_{\mathbb{Z}_+} y\rho_\kappa(y)\,\mathrm{d}\mu(y),$$

we thus have

$$\rho_\kappa(x) = 1 - e^{-\rho t x}$$

with

$$\rho = \int_{\mathbb{Z}_+} x\rho_\kappa(x)\,\mathrm{d}\mu(x) = \int_{\mathbb{Z}_+} \rho_\kappa(x)\,\mathrm{d}\widehat{\mu}(x) = \int_{\mathbb{Z}_+} \left(1-e^{-\rho t x}\right)\,\mathrm{d}\widehat{\mu}(x). \quad (4.13)$$

To find the weight $w(\mathcal{C}_1)$ of $\mathcal{C}_1(H_n)$, we note that if $f(x) := x$, then $f : \mathbb{Z}_+ \to \mathbb{R}$ satisfies, using (4.9), $\frac{1}{n}\sum_i f(x_i) = \frac{1}{n}\sum_i x_i = |F_n|/n = 1 = \int f\,\mathrm{d}\mu$, and thus [5, Theorem 9.10] applies and yields

$$\frac{1}{n}w(\mathcal{C}_1) = \frac{1}{n}\sum_{i\in\mathcal{C}_1} x_i \xrightarrow{\mathrm{p}} \int_{\mathbb{Z}_+} x\rho_\kappa(x)\,\mathrm{d}\mu(x) = \rho. \quad (4.14)$$

Combining (4.13) and (4.14), we thus find that

$$|C_1(\tilde{G}(n,t;F_n))| = w(\mathcal{C}_1(H_n)) = \rho n + o_{\mathrm{p}}(n), \quad (4.15)$$

where ρ solves the equation (4.13), which also can be written

$$\rho = \mathbb{E}\left(1-e^{-\rho t Z}\right) = 1 - \mathbb{E}\, e^{-\rho t Z}. \quad (4.16)$$

Applying Lemma 4.1 to $Y := tZ$, we see that when $t > 1/\bar{s}_2 = 1/\mathbb{E}\,Z$, there is a unique $\rho > 0$ satisfying (4.16).

Further, in (ii), we may apply Lemma 4.1 also to $Y := tZ_n$; thus there indeed is a unique such ρ_n. Moreover, by Lemma 4.1(iv), $\rho_n \to \rho$. Hence, (4.15) yields

$$|C_1(\tilde{G}(n,t;F_n))| = \rho_n n + o_{\mathrm{p}}(n),$$

which proves (ii) when $t > 1/\bar{s}_2$.

We have shown the conclusions in (i) and (ii) when $t \le 1/\bar{s}_2$ and $t > 1/\bar{s}_2$, respectively. However, the statements use instead the slightly different conditions $t \le 1/s_2(F_n)$ and $t > 1/s_2(F_n)$. For (i), this is no problem: if $t \le 1/s_2(F_n)$ for infinitely many n, then $t \le 1/\bar{s}_2$ since we have assumed $s_2(F_n) \to \bar{s}_2$.

To complete the proof of (ii), however, we have to consider also the case $1/\bar{s}_2 \ge t > 1/s_2(F_n)$. If this holds (for a subsequence), then $\mathbb{E}(tZ_n) = ts_2(F_n) \le s_2(F_n)/\bar{s}_2 \to 1$, and thus $\rho_n \to 0$ by Lemma 4.1(iv). Since $t \le 1/\bar{s}_2$, (4.12) applies and shows that

$$|C_1(\tilde{G}(n,t;F_n))| = o_{\mathrm{p}}(n) = \rho_n n + o_{\mathrm{p}}(n), \quad (4.17)$$

so (ii) holds in this case too. This completes the proof of (i) and (ii).

(iii) now follows easily from Lemma 4.1. We have, by (2.5), $\mathbb{E}(tZ_n) = ts_2 = 1 + \delta_n s_2$, $\mathbb{E}(tZ_n)^2 = t^2 s_3$ and $\mathbb{E}(tZ_n)^3 = t^3 s_4$. Hence,

$$\frac{\mathbb{E}(tZ_n)-1}{\mathbb{E}(tZ_n)^2} = \frac{\delta_n s_2}{t^2 s_3} = \frac{\delta_n s_2^3}{(1+\delta_n s_2)^2 s_3} > \delta_n \frac{s_2^3}{s_3}(1-2\delta_n s_2), \quad (4.18)$$

so the lower bound follows by (ii) and Lemma 4.1(ii).

For the upper bound we have by (4.18)

$$\frac{\mathbb{E}(tZ_n)-1}{\mathbb{E}(tZ_n)^2} < \delta_n \frac{s_2^3}{s_3},$$

and similarly

$$\frac{(\mathbb{E}(tZ_n)-1)\,\mathbb{E}(tZ_n)^3}{(\mathbb{E}(tZ_n)^2)^2}=\frac{\delta_n s_2 t^3 s_4}{t^4 s_3^2}=\frac{\delta_n s_2^2 s_4}{(1+\delta_n s_2)s_3^2}<\frac{\delta_n s_2^2 s_4}{s_3^2},$$

and the upper bound follows by Lemma 4.1(iii).

For (iv), we note that if $\liminf_n \delta_n > 0$, we can by ignoring some small n assume that $\inf_n \delta_n > 0$, and then the difference between the left-hand side and right-hand side in (4.18) is bounded below (since $1 \le s_2 \le s_3 = O(1)$); hence we can add some small $\eta > 0$ to the right hand side of (4.18) such that the inequality still holds for large n. Consequently,

$$C_1(\tilde{G}(n,t;F_n))/n \ge \delta_n \frac{s_2^3}{s_3}(1-2\delta_n s_2) - \eta + o_{\mathrm{p}}(1),$$

which implies that w.h.p.

$$C_1(\tilde{G}(n,t;F_n))/n \ge \delta_n \frac{s_2^3}{s_3}(1-2\delta_n s_2)$$

The upper bound follows in the same way. □

5. Proof of Theorems 3.2–3.5

Proof of Theorem 3.2. Define the functions

$$\begin{aligned}
f(t) &:= 1/\bar{s}_2(t),\\
g(t) &:= \bar{s}_3(t)/\bar{s}_2^3(t) = f^3(t)\bar{s}_3(t),\\
h(t) &:= \bar{s}_4(t)/\bar{s}_2^4(t) = f^4(t)\bar{s}_4(t).
\end{aligned}$$

The differential equations (3.2)–(3.4) then translate into, after simple calculations including some cancellations,

$$f'(t) = -\bar{x}_1^2(t)f^2(t) - \bigl(1-\bar{x}_1^2(t)\bigr), \tag{5.1}$$

$$g'(t) = 3\bar{x}_1^2(t)f^3(t) - 3\bar{x}_1^2(t)f(t)g(t), \tag{5.2}$$

$$h'(t) = 7\bar{x}_1^2(t)f^4(t) + 3\bigl(1-\bar{x}_1^2(t)\bigr)g^2(t)f^{-2}(t) - 4\bar{x}_1^2(t)f(t)h(t). \tag{5.3}$$

Consider first (5.1). The right hand side is locally Lipschitz in t and f, and thus there exists a unique solution with $f(0) = 1$ in some maximal interval $[0, t_f)$ with $t_f \le \infty$; if $t_f < \infty$ (which actually is the case, although we do not need this), $|f(t)| \to \infty$ as $t \nearrow t_f$. Since $0 < \bar{x}_1(t) < 1$ for all $t > 0$, and further $\bar{x}_1(t)$ is decreasing, $f'(t) \le -(1-\bar{x}_1^2(t)) < -c_0$, for some $c_0 > 0$ and all $t > 0.1$, say. Hence, $f(t)$ decreases and will hit 0 at some finite time $t_{\mathsf{c}} < t_f$. This means that $\bar{s}_2(t) = 1/f(t) \to \infty$ as $t \nearrow t_{\mathsf{c}}$, so (3.2) has a (unique) solution in $[0, t_{\mathsf{c}})$ but not further.

We have $f(t_{\mathsf{c}}) = 0$ and thus, by (5.1), $f'(t_{\mathsf{c}}) = -(1-\bar{x}_1^2(t_{\mathsf{c}})) < 0$. Consequently, defining $\rho := (1-\bar{x}_1^2(t_{\mathsf{c}}))^{-1} > 0$,

$$f(t) = \rho^{-1}(t_{\mathsf{c}}-t)\bigl((1+O(t_{\mathsf{c}}-t)\bigr), \qquad t \le t_{\mathsf{c}},$$

and thus

$$\bar{s}_2(t) = \frac{\rho}{t_\mathsf{c} - t}\big((1 + O(t_\mathsf{c} - t)\big), \qquad t < t_\mathsf{c},$$

as asserted.

Next, treating $\bar{x}_1(t)$ and f as known functions, (5.2) is a linear differential equation in g. An integrating factor is

$$G(t) := 3\int_0^t \bar{x}_1^2(u)f(u)\,\mathrm{d}u, \tag{5.4}$$

and then the unique solution in $[0, t_f)$ is given by

$$g(t) = e^{-G(t)} + 3e^{-G(t)}\int_0^t e^{G(u)}\bar{x}_1^2(u)f^3(u)\,\mathrm{d}u. \tag{5.5}$$

Hence (3.3) has the unique solution $g(t)\bar{s}_2^3(t)$, $t \in [0, t_\mathsf{c})$, with $g(t)$ given by (5.5). Note that $g(t) > 0$ for $t \le t_\mathsf{c}$.

Let $\beta := g(t_\mathsf{c}) > 0$. By (5.2), $g'(t_\mathsf{c}) = 0$, and thus, for $t < t_\mathsf{c}$, $g(t) = \beta + O(t_\mathsf{c} - t)^2$, and

$$\bar{s}_3(t) = \beta\bar{s}_2^3(t)\big(1 + O(t_\mathsf{c} - t)^2\big) = \frac{\beta\alpha^3}{(t_\mathsf{c} - t)^3}\big((1 + O(t_\mathsf{c} - t)\big),$$

Finally we consider (5.3). Here the right-hand side is singular at t_c because of the factor $f^{-2}(t)$ in the second term, so we modify h and consider

$$h_1(t) := h(t) - 3g^2(t)\bar{s}_2(t) = h(t) - 3g^2(t)f^{-1}(t),$$

which satisfies the differential equation

$$\begin{aligned} h_1'(t) &= 7\bar{x}_1^2(t)f^4(t) - 18\bar{x}_1^2(t)g(t)f^2(t) + 15\bar{x}_1^2(t)g^2(t) - 4\bar{x}_1^2(t)f(t)h(t) \\ &= 7\bar{x}_1^2(t)f^4(t) - 18\bar{x}_1^2(t)g(t)f^2(t) + 3\bar{x}_1^2(t)g^2(t) - 4\bar{x}_1^2(t)f(t)h_1(t). \end{aligned}$$

Again, this is a linear differential equation, with a unique solution in $[0, t_f)$. We leave the explicit form to the reader, since we need only that $h_1(t) = O(1)$ for $t \le t_\mathsf{c}$, which yields that for $t \in [0, t_\mathsf{c})$,

$$\begin{aligned} \bar{s}_4(t) &= h(t)\bar{s}_2^4(t) = 3g^2(t)\bar{s}_2^5(t) + h_1(t)\bar{s}_2^4(t) \\ &= 3\beta^2\bar{s}_2^5(t) + O\big(\bar{s}_2^4(t)\big). \end{aligned}$$ □

Proof of Theorem 3.3. For $k = 2$, this is, as said above, proved in [14, Theorems 1.1 and 4.3]. We prove the extension by the same method (with somewhat different notation).

Let, for a vertex $v \in G$, $\mathcal{C}(v)$ be the component of G containing the vertex v, and $C(v) := |\mathcal{C}(v)|$.

For a given graph G, let G^+ be the random graph obtained by adding one random edge by the Bohman–Frieze rule; we assume that the edge was chosen from the pair $e_1 = \{v_1, w_1\}$ and $e_2 = \{v_2, w_2\}$. If the added edge is $\{v, w\}$ (which thus is either $\{v_1, w_1\}$ or $\{v_2, w_2\}$), and further $\mathcal{C}(v) \ne \mathcal{C}(w)$, then, by (2.1),

$$S_k(G^+) - S_k(G) = \big(C(v) + C(w)\big)^k - C(v)^k - C(w)^k, \tag{5.6}$$

while $S_k(G^+) - S_k(G) = 0$ if $\mathcal{C}(v) = \mathcal{C}(w)$. We define

$$\Delta_k^* = \Delta_k^*(G; v, w) := \bigl(C(v) + C(w)\bigr)^k - C(v)^k - C(w)^k. \tag{5.7}$$

Hence,

$$\begin{aligned}\mathbb{E}\bigl(S_k(G^+) - S_k(G) - \Delta_k^*\bigr) &= -\,\mathbb{E}\bigl(\Delta_k^* \mathbf{1}[\mathcal{C}(v) = \mathcal{C}(w)]\bigr)\\ &= -\,\mathbb{E}\bigl((2^k - 2) C(v)^k \mathbf{1}[\mathcal{C}(v) = \mathcal{C}(w)]\bigr)\end{aligned}$$

and thus

$$\begin{aligned}&|\,\mathbb{E}\bigl(S_k(G^+) - S_k(G) - \Delta_k^*\bigr)| \le 2^k\,\mathbb{E}\bigl(C(v)^k \mathbf{1}[\mathcal{C}(v) = \mathcal{C}(w)]\bigr)\\ &\quad\le 2^k\,\mathbb{E}\bigl(C(v_1)^k \mathbf{1}[\mathcal{C}(v_1) = \mathcal{C}(w_1)]\bigr) + 2^k\,\mathbb{E}\bigl(C(v_2)^k \mathbf{1}[\mathcal{C}(v_2) = \mathcal{C}(w_2)]\bigr)\\ &\quad= \frac{2^{k+1}}{n}\,\mathbb{E}\,C(v_1)^{k+1} \le 2^{k+1}\frac{C_1(G)^{k+1}}{n}.\end{aligned}$$

In particular, if $C_1(G) = O(\log n)$, then

$$|\,\mathbb{E}\bigl(S_k(G^+) - S_k(G) - \Delta_k^*\bigr)| = O\Bigl(\frac{\log^{k+1} n}{n}\Bigr) = o(1). \tag{5.8}$$

Expanding (5.7), we have

$$\Delta_2^* = 2C(v)C(w), \tag{5.9}$$

$$\Delta_3^* = 3C(v)^2 C(w) + 3C(v)C(w)^2, \tag{5.10}$$

$$\Delta_4^* = 4C(v)^3 C(w) + 6C(v)^2 C(w)^2 + 4C(v)C(w)^3. \tag{5.11}$$

The Bohman–Frieze rule is to take $\{v, w\} = \{v_1, w_1\}$ if $C(v_1) = C(w_1) = 1$. The probability of this is $x_1(G)^2$, and in this case $\Delta_k^* = 2^k - 2$.

The opposite case $\{v, w\} = \{v_2, w_2\}$, which we denote by $\mathcal{E}_2$, has probability $1 - x_1(G)^2$. Conditioning on this case places us basically in the well-studied Erdős–Rényi regime. That is, v and w are uniform and independent, and thus for any k and ℓ,

$$\begin{aligned}\mathbb{E}\bigl(C(v)^k C(w)^l \mid \mathcal{E}_2\bigr) &= \frac{1}{n^2}\sum_{v,w} C(v)^k C(w)^\ell = \frac{1}{n^2}\sum_i C_i^{k+1}\sum_j C_j^{\ell+1}\\ &= s_{k+1}(G) s_{\ell+1}(G).\end{aligned}$$

Hence, (5.9)–(5.11) yield

$$\begin{aligned}\mathbb{E}\,\Delta_2^* &= 2x_1^2(G) + (1 - x_1(G)^2)\cdot 2s_2(G)^2,\\ \mathbb{E}\,\Delta_3^* &= 6x_1^2(G) + (1 - x_1(G)^2)\cdot 6s_2(G)s_3(G),\\ \mathbb{E}\,\Delta_4^* &= 14x_1^2(G) + (1 - x_1(G)^2)\cdot\bigl(8s_2(G)s_4(G) + 6s_3(G)^2\bigr).\end{aligned}$$

By (5.8), we thus have, for $k = 2, 3, 4$ and provided $C_1(G) = O(\log n)$,

$$\begin{aligned}\mathbb{E}\bigl(S_k(G^+) - S_k(G)\bigr) &= \mathbb{E}\,\Delta_k^* + O\bigl(\log^{k+1} n/n\bigr)\\ &= 2f_k\bigl(x_1(G), s_2(G), s_3(G), s_4(G)\bigr) + O\bigl(\log^{k+1} n/n\bigr),\end{aligned} \tag{5.12}$$

with

$$f_2(x_1, s_2, s_3, s_4) := x_1^2 + (1 - x_1^2)s_2^2,$$

$$f_3(x_1,s_2,s_3,s_4) := 3x_1^2 + 3(1-x_1^2)s_2s_3,$$
$$f_4(x_1,s_2,s_3,s_4) := 7x_1^2 + (1-x_1^2)\big(4s_2s_4 + 3s_3^2\big).$$

Similarly, as shown in [14],

$$\mathbb{E}\big(n_1(G^+) - n_1(G)\big) = 2f_1\big(x_1(G), s_2(G), s_3(G), s_4(G)\big) + O\big(1/n\big), \quad (5.13)$$

where (the variables s_2, s_3, s_4 are redundant here)

$$f_1(x_1,s_2,s_3,s_4) := -x_1^2 - (1-x_1^2)x_1.$$

Consider the vector-valued random process

$$X_i := \big(x_1(\mathsf{BF}_i), s_2(\mathsf{BF}_i), s_3(\mathsf{BF}_i), s_4(\mathsf{BF}_i)\big),$$

and let $\mathcal{F}_i = \sigma(X_0,\dots,X_i)$ be the σ-field describing the history up to time i. Further, let $\Phi := (f_1,f_2,f_3,f_4) : \mathbb{R}^4 \to \mathbb{R}^4$. Using this notation, (5.12)–(5.13) yield

$$\mathbb{E}\big(n(X_{i+1} - X_i) \mid \mathcal{F}_i\big) - 2\Phi(X_i) = O\big(\log^5 n/n\big), \quad (5.14)$$

uniformly in $i \le tn/2$, provided $C_1(\mathsf{BF}_i) = O(\log n)$.

By [14, Theorem 1.1], there exists a constant c' (depending on t) such that w.h.p. $C_1(\mathsf{BF}_i) \le c'\log n$ for all $i \le tn/2$. As in [14], we avoid the problem when $C_1(\mathsf{BF}_i) > c'\log n$ by defining $X_0^* = X_0 = (1,1,1,1)$, $X_{i+1}^* = X_{i+1}$ when $C_1(\mathsf{BF}_i) \le c'\log n$ and $X_{i+1}^* = X_i^* + \frac{2}{n}\Phi(X_i^*)$ otherwise. Then w.h.p. $X_i^* = X_i$ for all $i \le tn/2$, so we can just as well consider X_i^*. We have, by (5.14) but now without side condition, for all $i \le tn/2$,

$$\mathbb{E}\big(n(X_{i+1}^* - X_i^*) \mid \mathcal{F}_i\big) = 2\Phi(X_i^*) + O(\log^5 n/n)$$

and also, for some c'', from (5.6) and $|n_1(G^+) - n_1(G)| \le 2$,

$$|X_{i+1}^* - X_i^*| \le c'' \log^4 n/n.$$

The differential equation method in the form of Spencer and Wormald [14, Theorem 4.1], which is taken from Wormald [15, Theorem 5.1], now applies (with $Y(i) = nX_i^*$) and the result follows; note that the differential equations (3.1)–(3.4) can be written $\varphi'(t) = \Phi(\varphi(t))$ with $\varphi = (\bar{x}_1, \bar{s}_2, \bar{s}_3, \bar{s}_4)$, where further $\varphi(0) = (1,1,1,1) = X_0 = X_0^*$. □

Proof of Theorem 3.5. We may assume that δ is small, since the result is trivial for $\delta \ge \delta_0 > 0$ if we choose K large enough. In particular, we assume $\delta < 1$.

Let $\varepsilon := \delta^{2/3} > \delta$. We stop the process at $t_{\mathsf{c}} - \varepsilon$, and let $F := \mathsf{BF}(t_{\mathsf{c}} - \varepsilon)$. We then let the process evolve to $t_{\mathsf{c}} + \delta$ by adding $(\varepsilon+\delta)n/2$ further edges according to the Bohman–Frieze rule. Actually, for convenience, we add instead a random number of edges with a Poisson distribution $\mathrm{Po}\big((\varepsilon + \delta)n/2\big)$; this will not affect our asymptotic results (by the same standard argument as for comparing the different models in Section 2). We denote the resulting graph by $\widetilde{\mathsf{BF}}(t_{\mathsf{c}} + \delta)$.

By Theorems 3.3 and 3.2, for $k = 2, 3, 4$, and with a_k as in Theorem 3.2,

$$s_k(F) = \bar{s}_k(t_{\mathsf{c}} - \varepsilon) + o_{\mathrm{p}}(1) = \frac{a_k}{\varepsilon^{2k-3}}\big(1 + O(\varepsilon)\big) + o_{\mathrm{p}}(1).$$

Since $|o_{\mathrm{p}}(1)| \le \varepsilon$ w.h.p., we thus have w.h.p.

$$s_k(F) = \frac{a_k}{\varepsilon^{2k-3}}\big(1 + O(\varepsilon)\big). \tag{5.15}$$

(This means that there exists a constant c, not depending on ε or n, such that (5.15) holds with the error term $O(\varepsilon) \in [-c\varepsilon, c\varepsilon]$ w.h.p.) Similarly, $x_1(F) \xrightarrow{\mathrm{p}} \bar{x}_1(t_{\mathsf{c}} - \varepsilon) = \bar{x}_1(t_{\mathsf{c}}) + O(\varepsilon)$, so w.h.p. $x_1(F) = \bar{x}_1(t_{\mathsf{c}}) + O(\varepsilon)$.

We fix F (i.e., we condition on F) and assume that (5.15) holds together with $x_1(F) = \bar{x}_1(t_{\mathsf{c}}) + O(\varepsilon)$ (for some fixed implicit constant c in the $O(\varepsilon)$; we have just shown that this holds w.h.p. provided c is chosen large enough).

We cannot directly apply Theorem 2.1 since the graph evolves by the Bohman–Frieze evolution and not by the Erdős–Rényi evolution. Nevertheless, we can approximate and find upper and lower bounds of the graphs where we can apply Theorem 2.1; the idea is that we consider the Erdős–Rényi edges separately as an Erdős–Rényi evolution.

For a lower bound, let V_1 be the set of isolated vertices in F and consider only the pairs of edges $e_1 = \{v_1, w_1\}$, $e_2 = \{v_2, w_2\}$ where $v_1 \notin V_1$ or $w_1 \notin V_1$. Since the graphs BF_ℓ in the continued process contain F, the vertices v_1 and w_1 are not both isolated in the current BF_ℓ, and thus $e_2 = (v_2, w_2)$ is added, and these are independent Erdős–Rényi edges, i.e., uniformly chosen. The number of such Erdős–Rényi edges is $\mathrm{Po}\big((1 - x_1(F)^2)\varepsilon + \delta)n/2\big)$, since each time we add an edge, the probability of it being of this type is $1-(|V_1|/n)^2 = 1 - x_1(F)^2$. (Note that we ignore some Erdős–Rényi edges in order to avoid unpleasant dependencies.)

Call the resulting graph $H^- \subseteq \mathsf{BF}(t_{\mathsf{c}} + \delta)$. Then Theorem 2.1(iv) applies to H^-, with

$$t = \big(1 - x_1(F)^2\big)(\varepsilon + \delta) = \big(1 - \bar{x}_1(t_{\mathsf{c}})^2 + O(\varepsilon)\big)(\varepsilon + \delta)$$

and, recalling (5.15) and $\alpha = \big(1 - \bar{x}_1^2(t_{\mathsf{c}})\big)^{-1}$,

$$\delta_n = t - 1/s_2(F) = \big(1 - \bar{x}_1(t_{\mathsf{c}})^2\big)(\varepsilon + \delta) - \alpha^{-1}\varepsilon + O(\varepsilon^2) = \big(1 - \bar{x}_1(t_{\mathsf{c}})^2\big)\delta + O(\varepsilon^2), \tag{5.16}$$

which yields w.h.p., using again (5.15),

$$\begin{aligned}
\frac{C_1(\widetilde{\mathsf{BF}}(t_{\mathsf{c}} + \delta))}{n} &\ge \frac{C_1(H^-)}{n} \ge 2\delta_n \frac{s_2(F)^3}{s_3(F)}\big(1 - 2\delta_n s_2(F)\big) \\
&= 2\left(\big(1 - \bar{x}_1(t_{\mathsf{c}})^2\big)\delta + O(\varepsilon^2)\right) \frac{\alpha^3\varepsilon^{-3}}{\beta\alpha^3\varepsilon^{-3}} \left(1 + O(\varepsilon) + O\left(\frac{\delta + \varepsilon^2}{\varepsilon}\right)\right) \\
&= \frac{2}{\alpha\beta}\delta\left(1 + O(\varepsilon) + O(\delta/\varepsilon) + O(\varepsilon^2/\delta)\right) \\
&= \frac{2}{\alpha\beta}\delta\left(1 + O(\delta^{1/3})\right) = \gamma\delta + O(\delta^{4/3}),
\end{aligned} \tag{5.17}$$

with our choice $\varepsilon = \delta^{2/3}$ (which is optimal in this estimate).

For an upper bound, note that w.h.p. at most $(\varepsilon+\delta)n \le 2\varepsilon n$ edges are added to F, so at most $4\varepsilon n$ vertices are hit, and thus during the process from F to $\widetilde{\mathsf{BF}}(t_{\mathsf{c}}+\delta)$,

$$x_1 \ge x_1(F) - 4\varepsilon = \bar{x}_1(t_{\mathsf{c}}) - O(\varepsilon).$$

Hence we add w.h.p. at most

$$\bigl(1-(\bar{x}_1(t_{\mathsf{c}})-O(\varepsilon))^2\bigr)(\varepsilon+\delta)n/2 = \bigl(1-\bar{x}_1(t_{\mathsf{c}})^2+O(\varepsilon)\bigr)(\varepsilon+\delta)n/2$$

Erdős–Rényi edges. We also add a number of non-Erdős–Rényi edges, all joining two isolated vertices (or being loops). They may depend on the Erdős–Rényi edges already chosen, but we avoid this dependency by being generous and adding the edge $e_1 = (v_1, w_1)$ in each round whenever both v_1 and w_1 are isolated in F and neither is an endpoint of an already added non-Erdős–Rényi edge. (We add e_2 by the same Bohman–Frieze rule as before, so we may now sometimes add both e_1 and e_2.)

Let c_1 be a large constant and let H^+ be the graph obtained from F by adding $2\varepsilon n$ (to be on the safe side) non-Erdős–Rényi edges in this way, together with $\bigl(1-\bar{x}_1(t_{\mathsf{c}})^2+c_1\varepsilon\bigr)(\varepsilon+\delta)n/2$ Erdős–Rényi edges, independent of each other and of the non-Erdős–Rényi edges. We conclude that, if c_1 is chosen large enough, we may couple H^+ with the Bohman–Frieze process such that w.h.p. $\widetilde{\mathsf{BF}}(t_{\mathsf{c}}+\delta) \subseteq H^+$.

Since the two types of edges are added independently, we may further add all non-Erdős–Rényi edges first. Let F_1 be F together with all non-Erdős–Rényi edges. There are $2\varepsilon n$ such edges, and each joins two isolated vertices and changes S_k by 2^k-2 (or by 0 if the edge is a loop). Hence, for every $k \le 4$, by (5.15),

$$s_k(F_1) = s_k(F) + O(\varepsilon) = \frac{a_k}{\varepsilon^{2k-3}}\bigl(1+O(\varepsilon)\bigr). \tag{5.18}$$

Since H^+ is obtained by adding the Erdős–Rényi edges to F_1, Theorem 2.1 applies with

$$t = \bigl(1-\bar{x}_1(t_{\mathsf{c}})^2+c_1\varepsilon\bigr)(\varepsilon+\delta)$$

and

$$\delta_n = t - 1/s_2(F_1) = \bigl(1-\bar{x}_1(t_{\mathsf{c}})^2\bigr)(\varepsilon+\delta) - \alpha^{-1}\varepsilon + O(\varepsilon^2) = \bigl(1-\bar{x}_1(t_{\mathsf{c}})^2\bigr)\delta + O(\varepsilon^2), \tag{5.19}$$

the same estimate as was obtained in (5.16). We use the upper bound in Theorem 2.1(iv). By (5.18) and (5.19),

$$\delta_n \frac{s_2(F_1)^2 s_4(F_1)}{s_3(F_1)^2} = \delta_n \frac{a_2^2 a_4/\varepsilon^7}{a_3^2/\varepsilon^6}\bigl(1+O(\varepsilon)\bigr) = O\Bigl(\frac{\delta_n}{\varepsilon}\Bigr) = O\Bigl(\frac{\delta}{\varepsilon}\Bigr) = O\bigl(\delta^{1/3}\bigr).$$

Hence Theorem 2.1(iv) applies (for small δ) and yields, w.h.p.,

$$\begin{aligned} n^{-1}C_1(\widetilde{\mathsf{BF}}(t_{\mathsf{c}}+\delta)) \le n^{-1}C_1(H^+) &\le 2\delta_n \frac{s_2(F_1)^3}{s_3(F_1)}\bigl(1+O(\delta^{1/3})\bigr) \\ &= 2(1-\bar{x}_1(t_{\mathsf{c}})^2)\delta\frac{\alpha^3\varepsilon^{-3}}{\beta\alpha^3\varepsilon^{-3}}\bigl(1+O(\varepsilon^2/\delta+\varepsilon+\delta^{1/3})\bigr) \\ &= \frac{2(1-\bar{x}_1(t_{\mathsf{c}})^2)}{\beta}\delta\bigl(1+O(\delta^{1/3})\bigr) = \gamma\delta + O(\delta^{4/3}). \end{aligned}$$

This and the corresponding lower bound (5.17) yield the result. □

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