RANDOM TREES WITH SUPEREXPONENTIAL BRANCHING WEIGHTS

SVANTE JANSON, THORDUR JONSSON, AND SIGURDUR ÖRN STEFÁNSSON

Abstract. We study rooted planar random trees with a probability distribution which is proportional to a product of weight factors w_n associated to the vertices of the tree and depending only on their individual degrees n. We focus on the case when w_n grows faster than exponentially with n. In this case the measures on trees of finite size N converge weakly as N tends to infinity to a measure which is concentrated on a single tree with one vertex of infinite degree. For explicit weight factors of the form $w_n = ((n-1)!)^\alpha$ with $\alpha > 0$ we obtain more refined results about the approach to the infinite volume limit.

1. Introduction

Random trees have been studied intensively by mathematicians and theoretical physicists in the last few decades. They have direct applications to many branches in science, they are essential in many mathematical models used by physicists and are a natural object to study from the point of view of pure mathematics.

The random trees we are concerned with here were originally called *simply generated trees* by probabilists [10]. Later the same tree ensembles were referred to as *random trees with a local action* by physicists and viewed as toy models in statistical mechanics and for some aspects of quantum gravity, see e.g. [2].

Simply generated trees with N vertices can be defined as follows: Let $(w_n)_{n\geq 1}$ be a sequence of nonnegative numbers which we will call *branching weights*. If T is a tree graph with vertex set $V(T)$ having N elements we define a probability distribution on the set of all such trees by the formula

$$\nu(T) = Z^{-1} \prod_{v\in V(T)} w_{\sigma(v)}, \tag{1.1}$$

where $\sigma(v)$ is the degree of the vertex v and Z is a normalization factor called partition function in physics. One is interested in typical properties of trees with respect to this measure, especially asymptotics for large N and the existence of a limiting measure as $N \to \infty$.

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A lot is known about such trees for "nice" branching weights as we review briefly below. In this paper we aim at complementing some of these results for weights w_n which grow faster than exponentially with n. In this case some of the formalism that has been used to study simply generated trees is not applicable any more as we will explain below. A physicist would say that the Grand partition function is divergent which normally is a signal of instability in a physical theory. We will indeed see that with superexponential branching weights one vertex becomes connected to all the other vertices in the infinite volume limit.

In the next section we give a more technical background and summarize our results. The final section contains detailed proofs.

2. Definitions and summary of results

We consider rooted planar trees with root r of degree 1. We let Γ_N be the set of trees with N edges and denote the set of finite and infinite trees by Γ. Vertices of infinite order are allowed and for such vertices the links pointing away from the root are ordered as $\mathbb{N}$, i.e. there is a leftmost edge pointing away from the root. The unique nearest neighbour of the root r will be denoted by s.

Remark 2.1. We include the root r just for convenience. It is equivalent to omit it and consider s as the root (now with arbitrary degree), with minor changes in the notation; N is then the number of vertices in the tree and the degree $\sigma(v)$ is replaced by $1+\sigma_+(v)$ where $\sigma_+(v)$ is the outdegree of v. It may be even more convenient to omit r but keep the pendant edge from s to r as an edge with one free endpoint; this point of view is used sometimes in the proofs below.

Remark 2.2. We can regard the set Γ as a set of subtrees of the infinite Ulam–Harris tree T_∞, which is the tree with vertex set $V(T_\infty)=\{r\}\cup\bigcup_{k=0}^\infty \mathbb{N}^k$, the set of all finite strings of natural numbers (and r), with $s=\emptyset$ (the empty string, so $\mathbb{N}^0=\{s\}$) and a vertex $v=v_1\cdots v_k$ having ancestor $v_1\cdots v_{k-1}$ when $k>0$. More precisely, Γ can be identified with the set of all rooted subtrees T of T_∞ such that if $v=v_1\cdots v_k$ is a vertex in T, then so is $v_1\cdots v_{k-1}i$ for every $i<v_k$. We call such subtrees of T_∞ *left subtrees* and more generally, we say that a tree $T'\in\Gamma$ is a left subtree of $T\in\Gamma$ if $V(T')\subseteq V(T)$.

We endow Γ with a metric d which is defined as follows: Let $T\in\Gamma$ and define $B_R(T)$ as the graph ball of radius R, centered on the root r in T. The *left ball* of radius R, $L_R(T)$, is defined as the maximal left subtree of $B_R(T)$ with vertices of degree no greater than R. The metric d is given by

$$d(T,T')=\inf\left\{\frac{1}{R+1}\;\middle|\;L_R(T)=L_R(T')\right\},\quad T,T'\in\Gamma. \tag{2.1}$$

Convergence in Γ, in the metric d, is equivalent to convergence of the degree $\sigma(v)$ for every $v \in V(T_\infty)$ (where we define $\sigma(v) = 0$ for $v \notin T$), see [7] for details.

To avoid trivialities we assume that the branching weights satisfy $w_1 \neq 0$ and $w_n \neq 0$ for at least some $n > 2$. We define the finite volume partition function

$$Z_N = \sum_{\tau \in \Gamma_N} \prod_{v \in V(\tau) \setminus \{r\}} w_{\sigma(v)} \tag{2.2}$$

and a probability distribution ν_N on Γ_N by

$$\nu_N(\tau) = Z_N^{-1} \prod_{v \in V(\tau) \setminus \{r\}} w_{\sigma(v)}. \tag{2.3}$$

This probability distribution describes a random tree T_N with N edges.

Let $\rho \geq 0$ be the radius of convergence of the generating function

$$g(z) = \sum_{n=0}^{\infty} w_{n+1} z^n \tag{2.4}$$

of the branching weights. A rescaling $w_n \mapsto ab^n w_n$ with $a, b > 0$ does not affect the distributions ν_N, and it is well-known and easy to see that if $\rho > 0$, we can by rescaling assume that (w_n) is a probability distribution, i.e. $\sum_0^\infty w_n = 1$. In that case, the random tree T_N with distribution ν_N is a Galton–Watson tree with offspring distribution $(w_{n+1})_{n=0}^\infty$, conditioned to have size N. If further $\lim_{z \nearrow \rho} zg'(z)/g(z) \geq 1$, then the distributions ν_N converge to the distribution of a random tree that is infinite, with all vertex degrees finite and exactly one infinite path, see further [1, 3, 4, 8]. The limiting measure describes an infinite critical Galton–Watson tree conditioned on nonextinction. On the other hand, in the subcritical case when $m = \lim_{z \nearrow \rho} zg'(z)/g(z) < 1$, then (at least under some technical conditions) the limit distribution still exists but now describes a random tree with exactly one vertex of infinite degree; the length of the path from r to this vertex has a geometric distribution with mean $1/(1-m)$; the rest of the tree can be described by a subcritical Galton–Watson process, see [7] for details.

In the present paper we are interested in the case when the radius of convergence $\rho = 0$. Note that then there is no Galton–Watson interpretation. We prove in Section 3 weak convergence, as $N \to \infty$, of the measures ν_N (in the topology generated by d) in this case too, under certain conditions on the weights. The result can be seen as a natural limiting case of the result in [7] as $m \to 0$; the resulting limit tree is in this case non-random, and is simply an infinite star.

Theorem 2.3. *If the branching weights satisfy*

$$\frac{w_{n+1}}{w_n} \xrightarrow[n \to \infty]{} \infty \tag{2.5}$$

then the measures ν_N viewed as probability measures on Γ, converge weakly to the probability measure that is concentrated on the single tree which has $\sigma(s) = \infty$ and all other vertices of degree one.

Furthermore, we obtain stronger convergence results for certain explicit choices of weights. In the language of statistical mechanics these results give an explicit description of the finite size effects.

Theorem 2.4. *For the branching weights $w_2 = \lambda$ and $w_n = (n-1)!$, $n \neq 2$, the partition function satisfies*

$$\frac{Z_N}{e^{\lambda}(N-1)!} \to 1 \tag{2.6}$$

and we have

$$N - \sigma(s) \xrightarrow{d} \mathrm{Pois}(\lambda) \tag{2.7}$$

as $N \to \infty$. Moreover, the tree T_N consists of r, s, and $\sigma(s) - 1$ branches attached to s; with probability tending to 1, $N - \sigma(s)$ of these branches have size 2 and all other have size 1 (i.e., they contain a single leaf only).

Note that in the limit $N \to \infty$, the branches of size 2 disappear to infinity, so we do not see them in the limit given by Theorem 2.3.

Theorem 2.5. *Let the branching weights be $w_n = ((n-1)!)^{\alpha}$, where $0 < \alpha < 1$. Then the partition function satisfies*

$$Z_N = \bigl((N-1)!\bigr)^{\alpha} \exp\bigl(O(N^{1-\alpha})\bigr) = \exp\bigl(\alpha N \log(N) - \alpha N + O(N^{1-\alpha})\bigr). \tag{2.8}$$

Furthermore, with probability tending to 1, the random tree T_N has the following properties, with $K = \lfloor 1/\alpha \rfloor$:

(i) $\sigma(s) = N - O(N^{1-\alpha})$.
(ii) *All vertices except s have degrees $\leq K + 1$.*
(iii) *All subtrees attached to s have sizes $\leq K + 1$.*

Moreover, let $X_{i,N}$ be the number of vertices of degree i in T_N and let

$$n_i = i!^{\alpha} N^{1-i\alpha}. \tag{2.9}$$

(iv) *If $1 \leq i < 1/\alpha$, then $n_i \to \infty$ as $N \to \infty$ and*

$$\frac{X_{i+1,N}}{n_i} \xrightarrow{\mathrm{p}} 1. \tag{2.10}$$

If $i = 1/\alpha = K$ (which occurs only when $1/\alpha$ is an integer), then $n_K = K!^{\alpha}$ is constant and

$$X_{K+1,N} \xrightarrow{\mathrm{d}} \mathrm{Pois}(n_K). \tag{2.11}$$

With these branching weights, the asymptotic distributions of the numbers $X_{i,N}$ of vertices of different degrees are Gaussian, except in the Poisson case when (2.11) applies.

Theorem 2.6. *Let $w_n = ((n-1)!)^\alpha$ with $0 < \alpha < 1$ as in Theorem 2.5. Then there exist numbers $n_i^* = n_i^*(N) = \bigl(1+o(1)\bigr)n_i$, $1 \le i < 1/\alpha$, with n_i given by (2.9), such that, as $N \to \infty$,*

$$\frac{X_{i+1,N} - n_i^*}{\sqrt{n_i}} \xrightarrow{\mathrm{d}} \mathcal{N}(0,1), \qquad 1 \le i < 1/\alpha, \tag{2.12}$$

$$X_{i+1,N} \xrightarrow{\mathrm{d}} \mathrm{Pois}(n_i), \qquad i = K = 1/\alpha. \tag{2.13}$$

Moreover, these hold jointly for all $i \le K$, with independent limits.

More precisely, for each $i < 1/\alpha$,

$$n_i^* = n_i\bigl(1 - (1-i\alpha)N^{-\alpha} + O\bigl(N^{-2\alpha}\bigr)\bigr) + O(1). \tag{2.14}$$

In particular, when α is not too small, we have the explicit limits

$$\frac{X_{2,N} - N^{1-\alpha}}{N^{(1-\alpha)/2}} \xrightarrow{\mathrm{d}} \mathcal{N}(0,1), \qquad 1 > \alpha > \tfrac{1}{3}, \tag{2.15}$$

$$\frac{X_{2,N} - (N^{1-\alpha} - (1-\alpha)N^{1-2\alpha})}{N^{(1-\alpha)/2}} \xrightarrow{\mathrm{d}} \mathcal{N}(0,1), \qquad 1 > \alpha > \tfrac{1}{5}, \tag{2.16}$$

$$\frac{X_{3,N} - 2^\alpha N^{1-2\alpha}}{N^{(1-2\alpha)/2}} \xrightarrow{\mathrm{d}} \mathcal{N}(0,2^\alpha), \qquad \tfrac{1}{2} > \alpha > \tfrac{1}{4}, \tag{2.17}$$

$$\frac{X_{3,N} - (2^\alpha N^{1-2\alpha} - (1-2\alpha)2^\alpha N^{1-3\alpha})}{N^{(1-2\alpha)/2}} \xrightarrow{\mathrm{d}} \mathcal{N}(0,2^\alpha), \qquad \tfrac{1}{2} > \alpha > \tfrac{1}{6}, \tag{2.18}$$

$$\frac{X_{4,N} - 6^\alpha N^{1-3\alpha}}{N^{(1-3\alpha)/2}} \xrightarrow{\mathrm{d}} \mathcal{N}(0,6^\alpha), \qquad \tfrac{1}{3} > \alpha > \tfrac{1}{5}, \tag{2.19}$$

$$\frac{X_{4,N} - (6^\alpha N^{1-3\alpha} - (1-3\alpha)6^\alpha N^{1-4\alpha})}{N^{(1-3\alpha)/2}} \xrightarrow{\mathrm{d}} \mathcal{N}(0,6^\alpha), \qquad \tfrac{1}{3} > \alpha > \tfrac{1}{7}. \tag{2.20}$$

For smaller α, it is possible to obtain further terms in the expansion of n_i^*, and thus explicit forms of the asymptotic mean of $X_{i+1,N}$. However, this approach seems to become more and more difficult as α becomes smaller.

Remark 2.7. The proof of (2.12) shows also the stronger result that the joint distribution of $(X_{i+1,N})_{i=1}^K$ can be approximated by the joint distribution of independent Poisson random variables $Y_{i,N} \sim \mathrm{Pois}(n_i^*)$, in the sense that the total variation distance tends to 0 as $N \to \infty$:

$$\frac{1}{2}\sum_{m_1,\dots,m_K} \Bigl|\mathbb{P}(X_{i+1,N} = m_i,\, \forall i) - \mathbb{P}(Y_{i,N} = m_i,\, \forall i)\Bigr| \to 0. \tag{2.21}$$

Remark 2.8. The estimate (2.8) of the partition function can be improved to

$$Z_N = (N-1)!^\alpha \exp\left(N^{1-\alpha} + \Bigl(2^\alpha - \frac{1-\alpha}{2}\Bigr)N^{1-2\alpha} + O\bigl(N^{1-3\alpha}\bigr) + o(1)\right). \tag{2.22}$$

In particular, if $1 > \alpha > \frac{1}{2}$, then $Z_N = (N-1)!^\alpha \exp\bigl(N^{1-\alpha} + o(1)\bigr)$.

Again it seems possible, but more complicated, to obtain further terms in the exponent.

Remark 2.9. It is straightforward to show, using the same methods as in the proof of Theorem 2.4, that when $\alpha > 1$

$$Z_N = (N-1)!^{\alpha}(1+o(1)) \tag{2.23}$$

and all the branches which are attached to s have size 1, with a probability which tends to 1 as $N \to \infty$. In this case the leading contribution to the partition function comes only from the Boltzmann factor of the vertex s, i.e. $w_{\sigma(s)}$. The case $\alpha = 1$ is a marginal case when larger branches start to appear and their entropy adds a contribution to the partition function which appears in the associated exponential.

3. Proofs of theorems

In this section we state and prove a few lemmas and prove Theorems 2.3–2.6. In the following we will always assume that the branching weights satisfy the condition in Equation (2.5). Define

$$Z(N,n) = \sum_{d_1+\cdots+d_N=n} \prod_{i=1}^{N} w_{d_i+1}. \tag{3.1}$$

By Lagrange's inversion formula [7, 6] (or by a combinatorial argument, see [5, 9, 11]), it holds that

$$Z_N = \frac{1}{N} Z(N, N-1). \tag{3.2}$$

More generally the partition function for an ordered forest of m trees with a total number of edges N is

$$Z_N^{(m)} = \frac{m}{N} Z(N, N-m). \tag{3.3}$$

Lemma 3.1. *For every $\epsilon > 0$ there exists a $C_\epsilon < \infty$ such that for all N and n*

$$Z(N,n) \le \epsilon Z(N, n+1) + C_\epsilon^N. \tag{3.4}$$

Proof. Consider a finite sequence $d_1, \dots, d_N$ for which $\sum_i d_i = n$. Let i^* be the smallest index such that $d_{i^*} = \max_i d_i$. Define a sequence

$$d_i^* = \begin{cases} d_i + 1 & \text{if } i = i^*, \\ d_i & \text{otherwise.} \end{cases} \tag{3.5}$$

Note that $d_{i^*}^*$ is the unique maximum in (d_i^*), so (d_i) can be recovered from (d_i^*) and the map $(d_i) \mapsto (d_i^*)$ is injective.

Let $\epsilon > 0$ be given. Choose a number A_ϵ such that $w_i/w_{i+1} < \epsilon$ if $i \ge A_\epsilon$. Then

$$\sum_{\substack{d_1+\cdots+d_N=n \\ \max_i d_i > A_\epsilon}} \prod_{i=1}^{N} w_{d_i+1} \le \epsilon \sum_{d_1^*+\cdots+d_N^*=n+1} \prod_{i=1}^{N} w_{d_i^*+1} \le \epsilon Z(N, n+1) \tag{3.6}$$

and, crudely,

$$\sum_{\substack{d_1+\cdots+d_N=n\\ \max_i d_i\le A_\epsilon}} \prod_{i=1}^{N} w_{d_i+1} \le \left(\sum_{i=0}^{A_\epsilon} w_{i+1}\right)^N . \tag{3.7}$$

Taking $C_\epsilon = \sum_{i=0}^{A_\epsilon} w_{i+1}$ completes the proof. □

Lemma 3.2. *As $N\to\infty$, $\sigma(s) \xrightarrow{p} \infty$.*

Proof. It suffices to show that

$$\nu_N(\sigma(s)=k) \to 0 \tag{3.8}$$

for every fixed $k\ge 1$, since $\nu_N(\sigma(s)\ge m) = 1-\sum_{k=1}^{m-1}\nu_N(\sigma(s)=k)$. If the vertex s, in a tree with N edges, has degree $k+1$, then removing s and r but leaving all edges from s to its children as pendant edges, cf. Remark 2.1, leaves a forest with k trees and $N-1$ edges. Therefore, using (3.2) and (3.3),

$$\nu_N(\sigma(s)=k+1) = \frac{N}{N-1} k w_{k+1} \frac{Z(N-1,N-k-1)}{Z(N,N-1)}. \tag{3.9}$$

Let $\epsilon>0$ be given. Use Lemma 3.1 k times to get

$$Z(N-1,N-k-1) \le \epsilon^k Z(N-1,N-1) + kC_\epsilon^{N-1} \tag{3.10}$$

and note that

$$Z(N,N-1) \ge w_1 Z(N-1,N-1). \tag{3.11}$$

Since the branching weights satisfy (2.5), $Z(N-1,N-1)\ge w_N w_1^{N-2}$ grows super exponentially and in particular $Z(N-1,N-1)\ge (2C_\epsilon)^{N-1}$ for N large enough. Therefore

$$\nu_N(\sigma(s)=k+1) \le \frac{N}{N-1} k w_{k+1}(w_1^{-1}\epsilon^k + k2^{-N+1}) \tag{3.12}$$

and (3.8) follows since ϵ is arbitrary. □

Lemma 3.3. *For any $N\ge 1$ and $n\ge 0$*

$$\sum_{\ell=0}^{N} \ell w_{\ell+1} Z(N-1,n-\ell) = \frac{n}{N} Z(N,n). \tag{3.13}$$

Proof.

$$\sum_{\ell=0}^{n} \ell w_{\ell+1} Z(N-1,n-\ell) = \sum_{d_1+\cdots+d_{N-1}+\ell=n} \ell w_{\ell+1} \prod_{i=1}^{N-1} w_{d_i+1}$$

$$= \sum_{d_1+\cdots+d_N=n} d_N \prod_{i=1}^{N} w_{d_i+1}. \tag{3.14}$$

By symmetry we can replace d_N in front of the product by any d_j, $j=1,\dots,N$. Summing over j then gives the desired result. □

Lemma 3.4. *Assume $N > 1$ and let s_1 be the first child of s. If $L \geq 1$ and $k \geq L$, then*

$$\nu_N\big(L+1 \leq \sigma(s_1) \leq k+1 \mid \sigma(s) = k+1\big) \leq \frac{2}{L}. \tag{3.15}$$

Proof. If $\sigma(s) = k+1 \geq 2$ and $\sigma(s_1) = \ell+1 \geq 1$ then removing the vertices r, s and s_1, again leaving pendant edges, leaves a forest with $k+\ell-1$ trees and $N-2$ edges. Therefore (assuming $N \geq 3$),

$$\begin{aligned}\nu_N(\sigma(s) &= k+1, \sigma(s_1) = \ell+1) \\ &= \frac{N(k+\ell-1)w_{k+1}w_{\ell+1}}{N-2}\frac{Z(N-2, N-1-k-\ell)}{Z(N,N-1)}.\end{aligned} \tag{3.16}$$

By (3.9) and (3.16),

$$\begin{aligned}\nu_N(\sigma(s_1) &= \ell+1 \mid \sigma(s) = k+1) \\ &= \frac{(N-1)(k+\ell-1)w_{\ell+1}}{(N-2)k}\frac{Z(N-2, N-1-k-\ell)}{Z(N-1,N-1-k)}.\end{aligned} \tag{3.17}$$

By Lemma 3.3,

$$\begin{aligned}\sum_{\ell \geq L} w_{\ell+1} Z(N-2, N-1-k-\ell) &\leq \frac{1}{L}\sum_{\ell \geq 0} \ell w_{\ell+1} Z(N-2, N-1-k-\ell) \\ &= \frac{1}{L}\frac{N-1-k}{N-1} Z(N-1, N-1-k).\end{aligned} \tag{3.18}$$

Hence, (3.17) implies

$$\sum_{\ell=L}^{k} \nu_N(\sigma(s_1) = \ell+1 \mid \sigma(s) = k+1) \leq \frac{N-1-k}{N-2}\frac{2}{L} \leq \frac{2}{L}. \tag{3.19}$$

□

Lemma 3.5. *As $N \to \infty$, $\nu_N(\sigma(s_1) = 1) \to 1$.*

Proof. Fix $L > 1$ and an ℓ such that $1 \leq \ell < L$. Note that when $\ell \geq 1$ the formula (3.16) is symmetric in k and ℓ. Therefore

$$\begin{aligned}\nu_N(\sigma(s_1) = \ell+1) &= \sum_{k=1}^{\infty} \nu_N(\sigma(s) = k+1, \sigma(s_1) = \ell+1) \\ &= \nu_N(\sigma(s) = \ell+1, \sigma(s_1) \geq 2) \leq \nu_N(\sigma(s) = \ell+1)\end{aligned} \tag{3.20}$$

and thus $\nu_N(\sigma(s_1)=\ell+1)\to 0$ as $N\to\infty$ by Lemma 3.2. Next, Lemma 3.4 implies

$$\begin{aligned}\nu_N(L+1\le\sigma(s_1)\le\sigma(s))\\ =\sum_k \nu_N(L+1\le\sigma(s_1)\le k+1\mid\sigma(s)=k+1)\nu_N(\sigma(s)=k+1)\le\frac{2}{L}.\end{aligned}\tag{3.21}$$

Thus

$$\begin{aligned}\limsup_{N\to\infty}\nu_N(2\le\sigma(s_1)\le\sigma(s))&\le\limsup_{N\to\infty}\left(\sum_{\ell=1}^{L-1}\nu_N(\sigma(s_1)=\ell+1)+\frac{2}{L}\right)\\ &=\frac{2}{L}.\end{aligned}\tag{3.22}$$

Since L is arbitrary, $\nu_N(2\le\sigma(s_1)\le\sigma(s))\to 0$ as $N\to\infty$. By the symmetry of (3.16) in k and ℓ we also find that

$$\nu_N(2\le\sigma(s)\le\sigma(s_1))=\nu_N(2\le\sigma(s_1)\le\sigma(s))\to 0\tag{3.23}$$

as $N\to\infty$. Finally, since $\sigma(s)\ge 2$, we have

$$\nu_N(\sigma(s_1)\ge 2)\le\nu_N(2\le\sigma(s)\le\sigma(s_1))+\nu_N(2\le\sigma(s_1)\le\sigma(s))\to 0\tag{3.24}$$

as $N\to\infty$. □

Proof of Theorem 2.3. Let $R>0$. By Lemma 3.2, $\sigma(s)\xrightarrow{\text{p}}\infty$, so $\nu_N\big(\sigma(s)\ge R\big)\to 1$. Given that $\sigma(s)\ge R$, denote the first $R-1$ children of s by $s_1,\dots,s_{R-1}$. Then by Lemma 3.5 and symmetry $\nu_N\big(\sigma(s_i)=1,\,\sigma(s)\ge R\big)\to 1$ for every $i\le R$ and thus we find that

$$\nu_N(\sigma(s)\ge R,\,\sigma(s_1)=\cdots=\sigma(s_{R-1})=1)\to 1\tag{3.25}$$

as $N\to\infty$. Since R is arbitrary, the result follows from the definition of the topology on R, cf. the comment below (2.1). □

Proof of Theorem 2.4. First, we establish an upper bound on Z_N. Consider Equation (3.2) for Z_N. For a given sequence (d_i), let m_j denote the number of indices i for which $d_i=j$ where $j=0,\dots,N-1$. Instead of summing over (d_i) we sum over (m_j). For a given sequence (m_j) there are $\binom{N}{m_0,\dots,m_{N-1}}$ sequences (d_i) and therefore, since $w_1=1$,

$$\begin{aligned}\frac{Z_N}{(N-1)!}&=\sum_{\substack{m_0+\cdots+m_{N-1}=N\\ m_1+2m_2+\cdots+(N-1)m_{N-1}=N-1}}\prod_{i=0}^{N-1}\frac{w_{i+1}^{m_i}}{m_i!}\\ &=\sum_{m_1+2m_2+\cdots+(N-1)m_{N-1}=N-1}\frac{1}{\Big(N-\sum_{j=1}^{N-1}m_j\Big)!}\prod_{i=1}^{N-1}\frac{w_{i+1}^{m_i}}{m_i!}.\end{aligned}\tag{3.26}$$

Denote the maximum vertex degree by M and fix a number $L \geq 2$. By Lemma 3.2, it is sufficient to consider the case $M > L$. That contribution to (3.26) can be estimated by shifting $m_{M-1} \to m_{M-1} + 1$ which yields the upper bound

$$\begin{aligned} &\sum_{\substack{m_1+2m_2+\cdots+(M-1)m_{M-1} \\ =N-M,\ M>L}} \frac{(M-1)!}{\left(M + \sum_{j=1}^{M-1}(j-1)m_j - 1\right)!(m_{M-1}+1)} \prod_{i=1}^{M-1} \frac{w_{i+1}^{m_i}}{m_i!} \\ &\qquad \leq \sum_{\substack{m_1+2m_2+\cdots+(M-1)m_{M-1}=N-M \\ M>L}} \frac{\lambda^{m_1}}{m_1!} \prod_{i=2}^{M-1} \frac{(i!/M^{i-1})^{m_i}}{m_i!} \\ &\qquad \leq \exp\left(\lambda + \sum_{i=2}^{\infty} \frac{i!}{(i \vee L)^{i-1}}\right) \end{aligned} \tag{3.27}$$

where $A \vee B$ denotes the maximum of A and B. The last expression converges to e^{λ} when $L \to \infty$ by dominated convergence.

Next we establish a corresponding lower bound on Z_N. Consider the contribution to (3.26) from terms for which the only nonzero elements in the sequence (m_i) are m_0, m_1 and $m_k = 1$ where $k \geq 2$ is arbitrary; thus $m_0 = k$, $m_1 = N - k - 1$ and $m_k = 1$. These terms provide the following lower bound of (3.26)

$$\sum_{k=2}^{N-1} w_{k+1} \frac{1}{k!} \frac{w_2^{N-1-k}}{(N-1-k)!} = \sum_{\ell=0}^{N-3} \frac{\lambda^{\ell}}{\ell!} \to e^{\lambda} \tag{3.28}$$

as $N \to \infty$. This and (3.27) prove (2.6).

To complete the proof, note that the probability that T_N has $\sigma(s) = N-j$ and that exactly j of the $\sigma(s) - 1 = N - j - 1$ branches attached to s have size 2 and all others size 1 is, assuming $N > 2j$ and using (2.6),

$$\frac{1}{Z_N}\binom{N-j-1}{j} w_1^{N-j-1} w_2^{j} w_{N-j} = \frac{1}{Z_N}\binom{N-j-1}{j} \lambda^j (N-j-1)! \to \frac{\lambda^j}{j!} e^{-\lambda}. \tag{3.29}$$

These limits sum to 1 and yield the Pois(λ) distribution in (2.7). □

Proof of Theorem 2.5. Consider the weights $w_{n+1} = n!^{\alpha}$. Write, again by (3.2),

$$Z_N = \frac{1}{N} \sum_{d_1+\cdots+d_N=N-1} \prod_{i=1}^{N} d_i!^{\alpha}. \tag{3.30}$$

We get the lower bound

$$Z_N \geq (N-1)!^{\alpha} \tag{3.31}$$

by considering only the terms in Z_N with one $d_i = N-1$, and all others 0 (i.e., stars).

Define $Z_N(k, \epsilon)$ as the contribution to Z_N when precisely $k \geq 0$ vertices have degree greater than $\epsilon(N-1)$ where ϵ is some small positive number.

First consider the case when $k = 0$. Let $(d_i)_{i=1}^N$ be a sequence which satisfies $d_i \leq \epsilon(N-1)$ for all i. Using the simple relation

$$N!\,M! \leq (N+1)!\,(M-1)!, \qquad \text{for } N \geq M-1 \tag{3.32}$$

we can distribute and add the smallest elements in $(d_i)_{i=1}^N$ to the larger ones until each of them reaches $\epsilon(N-1)$. Thus we obtain the upper bound, using Stirling's formula,

$$\prod_{i=1}^N d_i!^\alpha \leq \lceil \epsilon(N-1) \rceil !^{\alpha/\epsilon} \leq C_1 N^{2\alpha/\varepsilon} (N-1)!^\alpha \epsilon^{\alpha N} \tag{3.33}$$

where $C_1 > 0$ is a number independent of N (but, as other constants below, it may depend on α and ε). Therefore,

$$Z_N(0,\epsilon) \leq C_1 N^{2\alpha/\varepsilon}(N-1)!^\alpha \epsilon^{\alpha N} \binom{2N-2}{N-1} \leq C_1 N^{2\alpha/\varepsilon}(N-1)!^\alpha \epsilon^{\alpha N} 2^{2N} \tag{3.34}$$

which is negligible compared to (3.31) as $N \to \infty$ for ϵ small enough.

Next consider the case when two or more of the d_i are larger than $\epsilon(N-1)$, i.e. when $k \geq 2$ in $Z_N(k,\epsilon)$. Clearly, $k < 1/\epsilon$. Denote the d_i which are greater than $\epsilon(N-1)$ by $d_{i_1}, \ldots, d_{i_k}$ and let $D_j = d_{i_j}$. The indices i_j can be chosen in $\binom{N}{k}$ ways. We will now lump together all the D_i into a single one, i.e. we define

$$D = D_1 + \cdots + D_k.$$

For each D, there are at most $\binom{D+k-1}{k-1}$ choices of $D_1, \ldots, D_k$. Note that, with $D_* = \lceil \varepsilon(N-1) \rceil$, using $D_i \geq D_*$ and Stirling's formula again,

$$\frac{D_1! \cdots D_k!}{D!} \leq \frac{D_*^k}{(kD_*)!} \leq C_2 N^k \left(\frac{1}{k}\right)^{k\epsilon N} \tag{3.35}$$

where $C_2 > 0$ is independent of N. Thus, we get the upper bound

$$\begin{aligned}
\sum_{2 \leq k \leq 1/\epsilon} Z_N(k,\epsilon) \leq C_2^\alpha \sum_{2 \leq k \leq 1/\epsilon} \binom{N}{k} N^{\alpha k} \left(\frac{1}{k}\right)^{\alpha k \epsilon N} & \\
\times \sum_{\substack{D+d_1+\cdots+d_{N-k}=N-1 \\ D>\epsilon(N-1),\ d_i \leq \epsilon(N-1),\ \forall i}} \binom{D+k-1}{k-1} D!^\alpha \prod_{i=1}^{N-k} d_i!^\alpha & \\
\leq C_3 N^{3/\epsilon} \left(\frac{1}{2}\right)^{2\alpha\epsilon N} Z_N(1,\epsilon). &
\end{aligned} \tag{3.36}$$

where $C_3 > 0$ is independent of N. This estimate, together with (3.34), shows that the main contribution to Z_N for N large comes from $Z_N(1,\epsilon)$.

Finally, we consider $Z_N(1,\epsilon)$. Using the representation as in (3.26) we have, writing $L = \lfloor \varepsilon(N-1)\rfloor$ for convenience,

$$\frac{Z_N(1,\epsilon)}{(N-1)!} = \sum_{D=L+1}^{N-1} \sum_{m_1+2m_2+\cdots+Lm_L=N-1-D} \frac{D!^\alpha}{(N-1-\sum_{j=1}^L m_j)!} \prod_{i=1}^{L} \frac{i!^{\alpha m_i}}{m_i!} \tag{3.37}$$

where $D+1$ denotes the degree of the large vertex and m_i denotes the number of vertices which have degree $i+1$. Consider one term in this sum and let $\widetilde{D} = D + \sum_{i=K+1}^{L} i m_i$, adding the outdegrees of all vertices which have degree greater than $K+1$ to the large vertex, where we recall that $K = \lfloor 1/\alpha \rfloor$. Then

$$\widetilde{D}! \geq D! \cdot D^{\sum_{i=K+1}^L i m_i} \geq D! \cdot L^{\sum_{i=K+1}^L i m_i} \tag{3.38}$$

and

$$\Bigl(N-1-\sum_{i=1}^{K} m_i\Bigr)! \leq \Bigl(N-1-\sum_{i=1}^{L} m_i\Bigr)! \cdot N^{\sum_{i=K+1}^L m_i} \tag{3.39}$$

Thus

$$\frac{D!^\alpha}{(N-1-\sum_{i=1}^L m_i)!} \leq \frac{\widetilde{D}!^\alpha \, N^{\sum_{i=K+1}^L m_i}}{(N-1-\sum_{i=1}^K m_i)! \, L^{\alpha \sum_{i=K+1}^L i m_i}} \tag{3.40}$$

and

$$\frac{Z_N(1,\epsilon)}{(N-1)!} \leq \sum_{\widetilde{D}=L+1}^{N-1} \sum_{m_1+\cdots+Km_K=N-1-\widetilde{D}} \sum_{m_{K+1},\dots,m_L\geq 0} \frac{\widetilde{D}!^\alpha}{(N-1-\sum_{j=1}^K m_j)!} \prod_{i=1}^{K} \frac{i!^{\alpha m_i}}{m_i!} \prod_{i=K+1}^{L} \left(\frac{Ni!^\alpha}{L^{i\alpha}}\right)^{m_i} \frac{1}{m_i!}. \tag{3.41}$$

We have

$$\sum_{m_{K+1},\dots,m_L} \prod_{i=K+1}^{L} \left(\frac{Ni!^\alpha}{L^{i\alpha}}\right)^{m_i} \frac{1}{m_i!} = \prod_{i=K+1}^{L} \exp\left(\frac{Ni!^\alpha}{L^{i\alpha}}\right) = \exp\left(\sum_{i=K+1}^{L} \frac{Ni!^\alpha}{L^{i\alpha}}\right). \tag{3.42}$$

Let, using $L = \lfloor \varepsilon(N-1)\rfloor > \varepsilon N/2$ (assuming N large),

$$a_i = \frac{Ni!^\alpha}{L^{i\alpha}} \leq \frac{2^{i\alpha} i!^\alpha}{\varepsilon^{i\alpha}} N^{1-i\alpha}. \tag{3.43}$$

Noting that $a_{i+1}/a_i = ((i+1)/L)^\alpha \le 1$ for $i < L$, we find

$$\sum_{i=K+1}^{L} a_i \le (K+1)a_{K+1} + Na_{2K+2} = O\left(N^{1-(K+1)\alpha}\right) + O\left(N^{2-(2K+2)\alpha}\right) \\ = o(1) \tag{3.44}$$

and thus from (3.41),

$$\frac{Z_N(1,\epsilon)}{(N-1)!} \le \bigl(1+o(1)\bigr) \sum_{\widetilde{D}=L+1}^{N-1} \sum_{m_1+\dots+Km_K=N-1-\widetilde{D}} \frac{\widetilde{D}!^\alpha}{(N-1-\sum_{j=1}^K m_j)!} \prod_{i=1}^{K} \frac{i!^{\alpha m_i}}{m_i!}. \tag{3.45}$$

The sum here is just the sum in (3.37) with $m_i = 0$ for $i > K$, so we have shown that $Z_N(1,\varepsilon)$ is dominated by such terms. Recalling (3.34) and (3.36) we see that

$$\frac{Z_N}{(N-1)!} = \bigl(1+o(1)\bigr) \sum_{m_1+\dots+Km_K<N-L-1} \frac{(N-1-\sum_{j=1}^K jm_j)!^\alpha}{(N-1-\sum_{j=1}^K m_j)!} \prod_{i=1}^{K} \frac{i!^{\alpha m_i}}{m_i!} \tag{3.46}$$

and that Z_N is dominated by trees having exactly one vertex of degree $> \varepsilon(N-1)$ and all other vertices having degrees $\le K+1$.

By Lemma 3.2, the contribution from trees with $\sigma(s) \le K+1$ is negligible, so it suffices to consider the case when the unique vertex with high degree is s, which proves (ii).

To obtain the more precise results in (i) and (iv), fix $i \le K$, fix m_j for $j \ne i$, and denote the summand in (3.46) by $b(m_i)$. Increasing m_i by 1 decreases $D = N - 1 - \sum_{j=1}^K jm_j$ by i and, assuming still $D > L$ and recalling the definition of n_i in (2.9),

$$\frac{b(m_i+1)}{b(m_i)} \le NL^{-i\alpha} \frac{i!^\alpha}{m_i+1} \le C_4 \frac{N^{1-i\alpha} i!^\alpha}{m_i+1} = C_4 \frac{n_i}{m_i+1} \ . \tag{3.47}$$

If $m_i \ge \lfloor 2C_4 n_i \rfloor$, this ratio is less than 1/2. In particular,

$$\sum_{m_i \ge 3C_4 n_i} b(m_i) \le 2b(\lfloor 3C_4 n_i \rfloor) \le 2^{2-C_4 n_i} b(\lfloor 2C_4 n_i \rfloor). \tag{3.48}$$

If $i < 1/\alpha$, then $n_i \to \infty$ as $N \to \infty$. Summing over all m_j, $j \ne i$, we see that the contribution to Z_N from $m_i \ge 3C_4 n_i$ is negligible, so we may assume that $m_i < 3C_4 n_i$. In the exceptional case $i = 1/\alpha$, we obtain by the same argument that we may assume $m_i < \log N$, say. In particular, since $n_i = O(N^{1-i\alpha}) = O(N^{1-\alpha})$, we see that we may assume $\sigma(s) = D + 1 = N - \sum_{j=1}^K jm_j = N - O(N^{1-\alpha})$, which proves (i).

For the remaining terms, we now may use $D = N - o(N)$ to improve (3.47) to

$$\frac{b(m_i+1)}{b(m_i)} = (1+o(1))N \cdot N^{-i\alpha} \frac{i!^\alpha}{m_i+1} = (1+o(1)) \frac{n_i}{m_i+1}. \tag{3.49}$$

Assume $i < 1/\alpha$ and let $\delta > 0$. We can repeat the argument above, using (3.49) instead of (3.47) and $(1+\delta/2)n_i$ instead of $2C_4 n_i$, and conclude that the terms with $m_i \geq (1+\delta)n_i$ are negligible. Similarly, (3.49) implies also that the terms with $m_i \leq (1-\delta)n_i$ are negligible. Hence, Z_N is dominated by terms with $(1-\delta)n_i < m_i < (1+\delta)n_i$. Since $X_{i+1,N} = m_i$, this proves (iv) for $i < 1/\alpha$.

If $1/\alpha$ is an integer and $i = K = 1/\alpha$, then it follows from (3.49) in the same way that m_K is stochastically bounded and that $\nu_N\{m_K = m + 1\}/\nu_N\{m_K = m\} \to n_K/(m+1)$ for every m, which implies that $m_K \xrightarrow{\mathrm{d}} \operatorname{Pois}(n_K)$, completing the proof of (iv).

Furthermore, (3.47) implies, for all m_i such that $D > L$,

$$\frac{b(m_i)}{b(0)} \leq \frac{(C_4 n_i)^{m_i}}{m_i!}. \tag{3.50}$$

Using this for each $i \leq K$, we see that the general summand in (3.46) is at most $\prod_{i=1}^K \frac{(C_4 n_i)^{m_i}}{m_i!}$, and thus (3.46) yields

$$\frac{Z_N}{(N-1)!} \leq (1+o(1)) \frac{(N-1)!^\alpha}{(N-1)!} \sum_{m_1,\dots,m_K} \prod_{i=1}^K \frac{(C_4 n_i)^{m_i}}{m_i!} \tag{3.51}$$

and

$$\frac{Z_N}{(N-1)!^\alpha} \leq (1+o(1)) \prod_{i=1}^K \exp(C_4 n_i) = \exp\Bigl(\sum_{i=1}^K C_4 n_i + o(1)\Bigr), \tag{3.52}$$

which proves (2.8).

Finally, we show (iii). If τ is a tree in Γ_N such that all vertices except s have degrees $\leq K+1$, but some branch attached to s has more than $K+1$ vertices, pick the first such branch and find, by breadth-first search, say, a subtree τ_0 of that branch with exactly $K+2$ vertices. Rearrange the edges inside τ_0 so that τ_0 is replaced by a star with center adjacent to s; this produces a vertex of degree $K+2$. Let $\tau' \in \Gamma_N$ be the result of making this change inside τ. We have changed the degree of (at most) $K+2$ vertices, and since all old and new degrees are at most $2K+1$, the weights of τ and τ' differ by at most a constant factor. Furthermore, τ' has exactly one vertex of degree $K+2$, and thus only a bounded number of trees τ can produce the same τ'. Consequently,

$$\begin{aligned} &\mathbb{P}(T_n \text{ has a branch of size } > K+1) \\ &\qquad\qquad \leq C_5 \mathbb{P}(T_n \text{ has a vertex} \neq s \text{ of degree } > K+1), \end{aligned} \tag{3.53}$$

and this probability tends to 0 by (ii). ∎

Proof of Theorem 2.6. Recall that Z_N is given by (3.46), and that the significant terms have $m_i = \big(1+o(1)\big)n_i = O(N^{1-i\alpha})$, except when $i = 1/\alpha$.

Let us first note that if $1/\alpha$ is an integer and $i = K = 1/\alpha$, then, see the proof of Theorem 2.5, (3.49) implies that $X_{K+1,N} = m_K$ has an asymptotic Poisson distribution $\mathrm{Pois}(n_K)$, which further is asymptotically independent of $X_{i,N}$, $i \le K$; furthermore, $\sum_{m_K} b(m_K) = \exp(n_K + o(1))b(0)$. In the sequel we thus assume $m_K = 0$ and sum only over m_i, $i < K$, when $i = K = 1/\alpha$; we omit the trivial modifications below in this case.

Define, for a fixed $\eta \in (0,1)$, $V = \prod_{i=1}^n [(1-\eta)n_i, (1+\eta)n_i]$. In the sequel we consider only $(m_i)_1^K \in V$; recall that it suffices to sum over such (m_i) in (3.46). For more compact notation, write

$$A = \sum_{i=1}^K m_i \qquad \text{and} \qquad B = \sum_{i=1}^K i m_i. \tag{3.54}$$

Note that A and B are $O(N^{1-\alpha})$. Use Stirling's approximation on the first factor in the sum in (3.46) to get

$$\begin{aligned}
&\frac{(N-1-B)!^\alpha}{(N-1-A)!} \\
&= \sqrt{\frac{(2\pi(N-1-B))^\alpha}{2\pi(N-1-A)}} \left(\frac{N-1-B}{e}\right)^{\alpha(N-1-B)} \left(\frac{e}{N-1-A}\right)^{N-1-A} (1+O(N^{-1})) \\
&= \sqrt{(2\pi(N-1))^{\alpha-1}} \left(\frac{N-1}{e}\right)^{(\alpha-1)(N-1)} (N-1)^{A-\alpha B} \\
&\qquad \times \exp\left\{\alpha B - A + \alpha\,(N-1-B)\log\left(1-\frac{B}{N-1}\right)\right. \\
&\qquad\qquad \left. -(N-1-A)\log\left(1-\frac{A}{N-1}\right)\right\}(1+O(N^{-\alpha})) \\
&= (N-1)!^{\alpha-1}(N-1)^{A-\alpha B}\exp\left\{\sum_{j=2}^K \frac{\alpha B^j - A^j}{j(j-1)(N-1)^{j-1}}\right\}(1+o(1)) \\
&= (N-1)!^{\alpha-1} N^{A-\alpha B}\exp\left\{\sum_{j=2}^K \frac{\alpha B^j - A^j}{j(j-1)N^{j-1}}\right\}(1+o(1)),
\end{aligned} \tag{3.55}$$

where in the last step we expanded the logarithms and kept only powers of A and B which contribute for large N, and then approximated $N-1$ by N.

Hence, (3.46) yields, using Stirling's formula again,

$$\frac{Z_N}{(N-1)!^\alpha} = \bigl(1+o(1)\bigr) \sum_{(m_i)\in V} \exp\left\{ \sum_{j=2}^K \frac{\alpha B^j - A^j}{j(j-1)N^{j-1}} \right\} \prod_{i=1}^K \frac{N^{m_i - i\alpha m_i} i!^{\alpha m_i}}{m_i!}$$
$$= \sum_{(m_i)\in V} \exp\bigl(f(m_1,\dots,m_K) + o(1)\bigr),$$

where

$$f(m_1,\dots,m_K) = \sum_{i=1}^K \Bigl((1-\alpha i) m_i \log N + \alpha m_i \log(i!) - m_i \log m_i$$
$$+ m_i - \frac12 \log(2\pi m_i) \Bigr) + \sum_{j=2}^K \frac{\alpha B^j - A^j}{j(j-1)N^{j-1}}. \tag{3.56}$$

Regard f as a function of real variables $m_1,\dots,m_K$. Then, for $m_1,\dots,m_K \in V$, which entails $A, B = O(N^{1-\alpha})$,

$$\frac{\partial f}{\partial m_i} = (1-\alpha i)\log N + \alpha \log(i!) - \log m_i - \frac{1}{2m_i} + \sum_{j=2}^K \frac{\alpha i B^{j-1} - A^{j-1}}{(j-1)N^{j-1}}$$
$$= \log n_i - \log m_i - \frac{1}{2m_i} + \frac{\alpha i B - A}{N} + O\bigl(N^{-2\alpha}\bigr)$$
$$= \log n_i - \log m_i - \frac{1}{2m_i} - \frac{(1-i\alpha)m_1}{N} + O\bigl(N^{-2\alpha}\bigr)$$
$$= \log n_i - \log m_i + o(1) \tag{3.57}$$

and, similarly,

$$\frac{\partial^2 f}{\partial m_i \partial m_j} = -\frac{\delta_{ij}}{m_i} + O\Bigl(\frac{\delta_{ij}}{m_i m_j}\Bigr) + O\Bigl(\frac1N\Bigr). \tag{3.58}$$

V is compact and f continuous, so f attains its maximum in V at some point $\mathbf{n}^* = (n_1^*,\dots,n_K^*) \in V$. By (3.57), for large N, $\frac{\partial f}{\partial m_i} > 0$ when $m_i = (1-\eta)n_i$ and $\frac{\partial f}{\partial m_i} < 0$ when $m_i = (1+\eta)n_i$, so the maximum is not attained on the boundary of V, i.e. $|n_i^* - n_i| < \eta n_i$. Consequently, by (3.57),

$$0 = \frac{\partial f}{\partial m_i}(\mathbf{n}^*) = \log n_i - \log n_i^* + o(1) \tag{3.59}$$

and thus $n_i^* = \bigl(1+o(1)\bigr)n_i$. A Taylor expansion of f at $\mathbf{n}^*$ yields, using (3.59) and (3.58), for $\mathbf{m} = (m_1,\dots,m_K) \in V$,

$$f(\mathbf{m}) = f(\mathbf{n}^*) - \frac12 \sum_{i=1}^K \left(\frac{(m_i - n_i^*)^2}{n_i^*} + O\Bigl(\frac{|m_i - n_i^*|^2 + |m_i - n_i^*|^3}{n_i^2} \Bigr) \right)$$
$$+ O\Bigl(\frac{|\mathbf{m} - \mathbf{n}^*|^2}{N}\Bigr) \tag{3.60}$$

Choosing η small enough, this implies first (for large N)

$$f(\mathbf{m}) \le f(\mathbf{n}^*) - \frac{1}{3}\sum_{i=1}^{K} \frac{(m_i - n_i^*)^2}{n_i^*}, \tag{3.61}$$

which implies that it suffices to consider terms in (3.56) with, say, $|m_i - n_i^*| < n_i^{1/2}\log N$; let $V_1 \subset V$ be the set of such $\mathbf{m}$. For such terms, (3.60) yields

$$f(\mathbf{m}) = f(\mathbf{n}^*) - \frac{1}{2}\sum_{i=1}^{K} \frac{(m_i - n_i^*)^2}{n_i^*} + o(1), \tag{3.62}$$

and thus by (3.56), letting $\beta = f(\mathbf{n}^*)$ be the maximum value of f on V,

$$\frac{Z_N}{(N-1)!^\alpha} = \bigl(1+o(1)\bigr) \sum_{(m_i)\in V_1} \exp\Bigl(\beta - \frac{1}{2}\sum_{i=1}^{K} \frac{(m_i - n_i^*)^2}{n_i^*} + o(1)\Bigr). \tag{3.63}$$

Since each term here corresponds to the case $X_{i+1,N} = m_i$, $i = 1, \dots, K$, and $n_i^* = \bigl(1+o(1)\bigr)n_i$, (2.12) follows. Furthermore, (3.63) also yields the Poisson approximation result in Remark 2.7, since the Poisson probabilities $\mathbb{P}(Y_{i,N} = m_i, \forall i)$ can easily be approximated by the same Gaussian as in (3.63); we omit the details.

In order to obtain more precise estimates of n_i^*, we go back to (3.57) and refine (3.59) to

$$0 = \frac{\partial f}{\partial m_i}(\mathbf{n}^*) = \log n_i - \log n_i^* - \frac{(1-i\alpha)n_1^*}{N} + O\bigl(N^{-2\alpha} + N^{i\alpha-1}\bigr) \tag{3.64}$$

which yields

$$\log\frac{n_i^*}{n_i} = -\frac{(1-i\alpha)n_1^*}{N} + O\bigl(N^{-2\alpha} + N^{i\alpha-1}\bigr) \tag{3.65}$$

and thus, recalling $n_1^*/N = O(N^{-\alpha})$,

$$\frac{n_i^*}{n_i} = 1 - \frac{(1-i\alpha)n_1^*}{N} + O\bigl(N^{-2\alpha} + N^{i\alpha-1}\bigr) \tag{3.66}$$

Taking $i = 1$ we find $n_1^*/n_1 = 1 + O(N^{-\alpha})$, and thus $n_1^* - n_1 = O(N^{1-2\alpha})$, so (3.66) yields

$$\frac{n_i^*}{n_i} = 1 - \frac{(1-i\alpha)n_1}{N} + O\bigl(N^{-2\alpha} + N^{i\alpha-1}\bigr), \tag{3.67}$$

establishing (2.14).

We obtain (2.15)–(2.20) from (2.12) and (2.14) by checking that in each case, the omitted terms in the numerator are of smaller order than the denominator. □

Finally, to evaluate the partition function, we approximate the sum in (3.63) by a Gaussian integral and obtain

$$\frac{Z_N}{(N-1)!^\alpha} = \big(1+o(1)\big)e^\beta \prod_{i=1}^K \sqrt{2\pi n_i^*} = e^{\beta+o(1)} \prod_{i=1}^K \sqrt{2\pi n_i}. \tag{3.68}$$

We have $\beta = f(\mathbf{n}^*)$. Further, (3.67) shows $n_i^* - n_i = O\big(n_i N^{-\alpha}\big) = O\big(N^{1-2\alpha}\big)$, and it follows from (3.60) that, with $\mathbf{n} = (n_1, \dots, n_K)$,

$$f(\mathbf{n}) = f(\mathbf{n}^*) + O\big(N^{1-3\alpha}\big) = \beta + O\big(N^{1-3\alpha}\big), \tag{3.69}$$

so it remains to evaluate $f(\mathbf{n})$. For $\mathbf{m} = \mathbf{n}$, the final sum in (3.56) is

$$\frac{\alpha B^2 - A^2}{2N} + O\big(N^{1-3\alpha}\big) = \frac{(\alpha-1)n_1^2}{2N} + O\big(N^{1-3\alpha}\big), \tag{3.70}$$

and thus, after some cancellations,

$$f(\mathbf{n}) = \sum_{i=1}^K \Big(n_i - \frac{1}{2}\log(2\pi n_i)\Big) - \frac{(1-\alpha)n_1^2}{2N} + O\big(N^{1-3\alpha}\big). \tag{3.71}$$

Hence, (3.68) yields, with (3.69) and (3.71) and recalling $n_1 = N^{1-\alpha}$,

$$\frac{Z_N}{(N-1)!^\alpha} = \exp\left(\sum_{i=1}^K n_i - \frac{1-\alpha}{2}N^{1-2\alpha} + O\big(N^{1-3\alpha}\big) + o(1)\right). \tag{3.72}$$

We substitute n_1 and n_2 from (2.9) and drop n_i for $i \geq 3$, which yields (2.22).

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Department of Mathematics, Uppsala University, PO Box 480, SE-751 06 Uppsala, Sweden

E-mail address: svante.janson@math.uu.se

URL: http://www2.math.uu.se/∼svante/

The Science Institute, University of Iceland, Dunhaga 3, 107 Reykjavik, Iceland

E-mail address: thjons@raunvis.hi.is

NORDITA, Roslagstullsbacken 23, SE-106 91 Stockholm, Sweden

E-mail address: sigste@nordita.org