FEYNMAN–KAC THEOREMS FOR GENERALIZED DIFFUSIONS

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Abstract. We find Feynman–Kac type representation theorems for generalized diffusions. To do this we need to establish existence, uniqueness and regularity results for equations with measure-valued coefficients.

1. Introduction

Generalized diffusions, also referred to as gap diffusions, provide a useful extension of the concept of one-dimensional diffusions: they allow jumps, but only to neighboring elements in the state space, and thus provide a unified framework for processes on discrete spaces as well as for processes on the real line, see [2], [3], [7], [8] and [10]. In the next section of the present paper we present generalized diffusions as time changes, using so-called speed measures, of Brownian motion, which is the customary way of defining them. We focus on generalized diffusions that also are local martingales.

We establish Kolmogorov backward equations for expected values $U(x,t)$ of functions of generalized diffusions at time t for processes starting at the point x. The backward equation takes the form

$$2m(dx)U_t(x,t) = U_{xx}(x,t)$$

where m denotes the speed measure. This equation is to be interpreted as an equality in the distributional sense. When the process is a regular diffusion, m is nothing but the multiplicative inverse of the diffusion coefficient and the equation reduces to the usual backward (heat) equation. We provide related Feynman–Kac theorems connecting solutions to the backward equation to expected values of functions of generalized diffusions and vice versa. We also study regularity of solutions to the backward equation. A consequence of our results is that (under suitable conditions) the measure U_{xx} is absolutely continuous with respect to the speed measure, which can be thought of as parabolic regularity in our present setting.

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The paper is organized as follows. In Section 2 we introduce the family of generalized diffusions under consideration, and we collect some known properties from existing literature. In Section 3 our main result, Theorem 3.2, is formulated. It states that there is a unique solution satisfying appropriate growth conditions of the backward equation corresponding to a generalized diffusion. Moreover, this solution coincides with the stochastic representation, thus establishing a Feynman–Kac type theorem in our setting. The proof of Theorem 3.2 is contained in Sections 4 and 5. Section 6 provides conditions under which certain properties of the initial condition are inherited by the solution, and Section 7 contains some examples. Finally, Section 8 contains a study of regularity of solutions.

2. Construction of generalized diffusions

In this section we construct generalized diffusions as time changes of Brownian motion, and we discuss some of their properties. Most of the contents of this section can also be found in [3], but for the convenience of the reader we include them here.

Let m be a nonnegative Borel measure on $\mathbb{R}$. Note that m is allowed to be (locally) infinite. We exclude the trivial case when $m = 0$. Let B be a Brownian motion with $B_0 = 0$ and let L^y_u be its local time at the point y up to time u. Then $B^x_t := x + B_t$ is a Brownian motion starting at x, and its local time at y is L^{y-x}_u. For a given starting point $x \in \mathbb{R}$ we define the increasing process

$$\Gamma^x_u := \int_{\mathbb{R}} L^{y-x}_u m(dy). \tag{1}$$

We note that $\Gamma^x_u \in [0, \infty]$, and we define its right-continuous inverse

$$A^x_t := \inf\{u : \Gamma^x_u > t\}. \tag{2}$$

Since m is non-zero, $\Gamma^x_u \to \infty$ as $u \to \infty$, so $A^x_t < \infty$. The process

$$X^x_t := B^x_t = x + B_{A^x_t} \tag{3}$$

will be called a generalized diffusion with speed measure m and starting point x.

Although $B^x_0 = x$, A^x_0 may be strictly positive and thus in general $X^x_0 \neq x$. Indeed, this is the case when x does not belong to the support of m, see [3], Lemma 3.3. We define $A^x_{0-} = 0$ and $X^x_{0-} = x + B_{A^x_{0-}} = x$ and thus we allow the possibility that $X^x_{0-} \neq X^x_0$.

We list some further properties of generalized diffusions.

(i) X is a local martingale for any initial point x if and only if $\operatorname{supp} m \cap (-\infty, -b] \neq \emptyset$ and $\operatorname{supp} m \cap [b, \infty) \neq \emptyset$ for all $b > 0$, see Theorem 7.3 in [3].

(ii) X is a martingale for any initial point x if and only if the speed measure m satisfies

$$\int_{-\infty}^{x} |y|\, m(dy) = \int_{x}^{\infty} |y|\, m(dy) = \infty \tag{4}$$

for any $x \in \mathbb{R}$, see Theorem 7.9 in [3].

(iii) If m_n is a sequence of speed measures that converges vaguely to m in the sense that

$$\int \phi\, dm_n \to \int \phi\, dm, \qquad \phi \in C_c^+(\mathbb{R}), \tag{5}$$

then the corresponding $A_t^{x,n} \to A_t^x$ and $X_t^{x,n} \to X_t^x$ a.s. as $n \to \infty$, for every x and $t > 0$, see Lemma 3.12 in [3].

(iv) If $dm(y) = \frac{dy}{\sigma^2(y)}$ for some continuous non-vanishing function σ, then X_t^x is a weak solution of

$$dX_t^x = \sigma(X_t^x)\, dW_t,$$

$X_0 = x$ (with W being Brownian motion), in which case X^x is a diffusion. Note that the speed measure m measures the inverse of speed rather than speed.

3. A Feynman–Kac type theorem for generalized diffusions

Throughout the rest of this article we assume that m is a locally finite and nonnegative Borel measure on $\mathbb{R}$ such that

$$\begin{cases} \operatorname{supp} m \cap (-\infty, -b] \neq \emptyset \\ \operatorname{supp} m \cap [b, \infty) \neq \emptyset \end{cases} \quad \text{for all } b > 0. \tag{6}$$

The corresponding generalized diffusion is then a local martingale by (i) above. Let

$$\Phi(x) := \begin{cases} 2\int_{[0,x)} y\, m(dy) & x \geq 0 \\ 2\int_{[x,0)} y\, m(dy) & x < 0 \end{cases} \tag{7}$$

and

$$\Psi(x) := \int_0^x \Phi(y)\, dy. \tag{8}$$

Then Φ is non-decreasing with $\Phi(0) = 0$, and Ψ is a non-negative convex function with left derivative Φ and second derivative $\Psi^{''}(x) = 2|x|m(dx)$ (in the distribution sense). Note that it follows from (6) that Ψ grows at least linearly as $|x| \to \infty$. Hence there is a constant C_1 such that, for all real x,

$$|x| \leq C_1(\Psi(x) + 1). \tag{9}$$

Definition 3.1. *Let g be a continuous function. We define $\overline{g}$ to be the function that agrees with g on the support of m, and which is affine outside the support.*

We now state our main result.

Theorem 3.2. *Suppose that m is a locally finite nonnegative Borel measure on $\mathbb{R}$ such that* (6) *holds. Let g be a continuous function such that $g(x) = o(\Psi(x))$ as $|x| \to \infty$. Then there exists a unique continuous function $U : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ such that $U(x, t) = o(\Psi(x))$ as $|x| \to \infty$ locally uniformly in t and such that*

$$2m(x)U_t(x, t) = U_{xx}(x, t) \qquad \text{for } t > 0 \tag{10}$$

holds in the sense of distributions with the initial values

$$U(x, 0) = \overline{g}(x). \tag{11}$$

Moreover, the function U is given by a stochastic representation

$$U(x, t) = \mathbb{E}g(X_t^x) = \mathbb{E}\overline{g}(X_t^x), \tag{12}$$

where X_t^x is the generalized diffusion with speed measure m and starting point x.

Remark. The distribution mU_t appearing in (10) is to be interpreted as the distribution derivative $(Um)_t$. (We regard here $m = m(x)$ as a distribution in $\mathbb{R} \times (0, \infty)$, independent of t.) Hence the partial differential equation (10) is equivalent to

$$-2 \iint U\varphi_t \, m(dx)\, dt = \iint U\varphi_{xx} \, dx\, dt \tag{13}$$

for all $\varphi \in \mathcal{D} = \mathcal{D}(\mathbb{R} \times (0, \infty))$, the space of smooth functions of compact support.

We shall see in Theorem 8.1 that the solution U actually has a continuous derivative U_t in $\mathbb{R} \times (0, \infty)$; for such functions, mU_t in (10) can, equivalently, also be interpreted in the usual sense as the product of the distribution $m(x)$ in $\mathbb{R} \times (0, \infty)$ and the continuous function U_t. We may still have to interpret U_{xx} in distribution sense, but it is equivalent to interpret (10) as an equation of distributions on $\mathbb{R}$, for every fixed $t > 0$.

Remark. We can think of $g(x)$ as $u(x, 0-)$. Thus from time $0-$ to time 0, $u(x, t)$ changes from g to $\overline{g}$.

4. Uniqueness of solutions

In this section we prove uniqueness of solutions to (10)–(11). We begin with a maximum principle on bounded domains.

For a fixed $T > 0$ and $A > 0$, let $D = D_A := [-A, A] \times [0, T]$. Denote by $D^\circ = (-A, A) \times (0, T)$ the interior of D, and let $\partial D = [-A, A] \times \{0\} \cup \{-A, A\} \times [0, T]$ denote its parabolic boundary.

Lemma 4.1. *Suppose that $f \in C(D)$ and $f_t \in C(D^\circ)$. If $2f_t m \geq f_{xx}$ on D° (as distributions) and $f \geq 0$ on ∂D, then $f \geq 0$ on D.*

Proof. Let $\epsilon > 0$, and define

$$F(x, t) := f(x, t) + \epsilon(1 + t + A^2 - x^2). \tag{14}$$

Then $F \in C(D)$ and $F(x,t) \geq \epsilon$ on ∂D. Let $E := \{(x,t) \in D : F(x,t) \leq 0\}$. Then E is compact. Let

$$t_0 := \min\{t \geq 0 : (x,t) \in E \text{ for some } x \in [-A,A]\},$$

and suppose $t_0 < T$. Since $E \cap \partial D = \emptyset$ we have $t_0 > 0$. Take $x_0 \in [-A,A]$ with $(x_0,t_0) \in E$, i.e. $F(x_0,t_0) \leq 0$. Then $-A < x_0 < A$, so $(x_0,t_0) \in D^\circ$.

Note that $F_t = f_t + \epsilon \in C(D^\circ)$. If $t < t_0$, then $F(x_0,t) \geq 0 \geq F(x_0,t_0)$, so $F_t(x_0,t_0) \leq 0$. Consequently, $f_t(x_0,t_0) \leq -\epsilon$. By continuity of f_t, there exists a neighborhood $U \subset D^\circ$ of (x_0,t_0) such that $f_t < 0$ in U. Thus $f_t m \leq 0$ in $\mathcal{D}'(U)$, and $f_{xx} \leq 2 f_t m \leq 0$ in $\mathcal{D}'(U)$.

Let $\psi \in C_0^\infty(\mathbb{R}^2)$ with support in the unit ball B such that $\psi \geq 0$ and $\int \psi = 1$, and define $\psi_\eta(x) = \eta^{-2}\psi(x/\eta)$. Let V be a smaller neighborhood of (x_0,t_0), i.e. $\overline{V} \subset U$. Then, for η so small that $V + \eta B \subset U$, say $\eta < \eta_0$, we have that

$$(\psi_\eta * f)_{xx} = \psi_\eta * f_{xx} \leq 0$$

in $\mathcal{D}'(V)$. But $\psi_\eta * f \in C^\infty(V)$, so this means that $(\psi_\eta * f)_{xx} \leq 0$ as a continuous function, i.e. pointwise in V.

Choose h so small that $[x_0 - h, x_0 + h] \times \{t_0\} \subset V$. Then, for any $\eta < \eta_0$, $\psi_\eta * f(x,t_0)$ is concave on $[x_0 - h, x_0 + h]$, and thus

$$\psi_\eta * f(x_0 - h, t_0) + \psi_\eta * f(x_0 + h, t_0) \leq 2\psi_\eta * f(x_0,t_0).$$

Letting $\eta \to 0$ we have, since f is continuous, $\psi_\eta * f(x,t) \to f(x,t)$ for every $(x,t) \in V$, and thus

$$f(x_0 - h, t_0) + f(x_0 + h, t_0) \leq 2 f(x_0,t_0).$$

Consequently, using the definition (14),

$$\begin{aligned}
&F(x_0 - h, t_0) + F(x_0 + h, t_0) - 2F(x_0,t_0) \\
&\qquad = f(x_0 - h, t_0) + f(x_0 + h, t_0) - 2f(x_0,t_0) \\
&\qquad\qquad\qquad\qquad - \epsilon\big((x_0 - h)^2 + (x_0 + h)^2 - 2x_0^2\big) \\
&\qquad \leq -2\epsilon h^2 < 0.
\end{aligned}$$

However, by definition $F(x_0,t_0) \leq 0$ while $F(x,t) > 0$ for $x \in [-A,A]$ and $0 \leq t < t_0$, and thus by continuity we have $F(x,t_0) \geq 0$. Hence $F(x_0 - h, t_0) + F(x_0 + h, t_0) - 2F(x_0,t_0) \geq 0$. This is a contradiction, which shows that $t_0 < T$ is impossible. Consequently, either $t_0 = T$ or $E = \emptyset$. In both cases, $F(x,t) > 0$ for $(x,t) \in [-A,A] \times [0,T)$. By continuity $F \geq 0$ in D. Hence, by (14),

$$f(x,t) + \epsilon(1 + t + A^2 - x^2) \geq 0$$

in D. Letting $\epsilon \to 0$ finishes the proof. □

We next extend the maximum principle to an unbounded domain. Let $D_\infty := \mathbb{R} \times [0,T]$ and $D_\infty^\circ := \mathbb{R} \times (0,T)$.

Lemma 4.2. *Suppose that $f \in C(D_\infty)$ and $f_t \in C(D_\infty^\circ)$. Also assume that $2f_t m \ge f_{xx}$ on D_∞° (as distributions), $f \ge 0$ on $\mathbb{R} \times \{0\}$, and that $f(x,t) = o(\Psi(x))$ as $|x| \to \infty$ uniformly in $t \in [0,T]$. Then $f \ge 0$ on D.*

Proof. Let $\epsilon > 0$ and define, with C_1 as in (9),

$$h(x,t) := f(x,t) + \epsilon(1 + \Psi(x))e^{C_1 t}.$$

By the assumption $f(x,t) = o(\Psi(x))$, if A is large enough, then $h(x,t) \ge 0$ for $|x| \ge A$ and $t \in [0,T]$. Fix one such A. Then $h \ge 0$ on ∂D_A. Moreover, using (9),

$$\begin{aligned} h_{xx} &= f_{xx} + \epsilon\Psi_{xx}e^{C_1 t} = f_{xx} + \epsilon 2|x|me^{C_1 t} \\ &\le 2f_t m + \epsilon 2C_1(1 + \Psi(x))me^{C_1 t} = 2h_t m \end{aligned}$$

on D_A°. Hence Lemma 4.1 applies to h and yields $h \ge 0$ on D_A. Since we can choose A arbitrarily large, $h \ge 0$ on D_∞. Now, letting $\epsilon \to 0$ yields $f(x,t) \ge 0$ for $(x,t) \in D_\infty$. □

Lemma 4.3. *Suppose that $f \in C(D_\infty)$. Also assume that $2f_t m = f_{xx}$ on D_∞°, $f = 0$ on $\mathbb{R} \times \{0\}$, and that $f(x,t) = o(\Psi(x))$ as $|x| \to \infty$ uniformly in $t \in [0,T]$. Then $f = 0$ on D_∞.*

Proof. Define $F(x,t) = \int_0^t f(x,s)\,ds$. Then $F \in C(D_\infty)$ and $F_t(x,t) = f(x,t) \in C(D_\infty)$. We have

$$F_{xxt} = f_{xx} = 2f_t m = 2(fm)_t = 2(F_t m)_t.$$

Let $G := F_{xx} - 2F_t m \in \mathcal{D}'(D_\infty^\circ)$. Then $G_t = 0$, so $G(x,t) = h(x)$ for some distribution $h \in \mathcal{D}'(\mathbb{R})$.

Fix $\phi \in \mathcal{D}(\mathbb{R})$, and let $\psi \in \mathcal{D}(0,\epsilon)$ with $\psi \ge 0$ and $\int \psi = 1$. Then, with C depending on ϕ only,

$$\begin{aligned} |\langle F_{xx}(x,t), \phi(x)\psi(t)\rangle| &= \left| \iint F(x,t)\phi_{xx}(x)\psi(t)\,dx\,dt \right| \\ &\le C \sup_{0 \le t \le \epsilon, x \in \operatorname{supp}\phi} |F(x,t)| \\ &\le C\epsilon \sup_{0 \le t \le \epsilon, x \in \operatorname{supp}\phi} |f(x,t)| = o(\epsilon) \end{aligned}$$

as $\epsilon \to 0$. Furthermore,

$$\begin{aligned} |\langle F_t m(x,t), \phi(x)\psi(t)\rangle| &= \left| \iint f(x,t)\phi(x)\psi(t)\,m(dx)\,dt \right| \\ &\le C \sup_{0 \le t \le \epsilon, x \in \operatorname{supp}\phi} |f(x,t)| = o(1) \end{aligned}$$

as $\epsilon \to 0$. Hence $|\langle G, \phi(x)\psi(t)\rangle| = o(1)$ as $\epsilon \to 0$. But

$$\langle G, \phi(x)\psi(t)\rangle = \langle h(x), \phi(x)\psi(t)\rangle = \langle h, \phi\rangle \int \psi(t)\,dt = \langle h, \phi\rangle.$$

Hence, letting $\epsilon \to 0$, $\langle h, \phi \rangle = 0$. Since $\phi \in \mathcal{D}(\mathbb{R})$ is arbitrary, $h = 0$ and thus $G = 0$. Consequently, $F_{xx} = 2F_t m$ on D°_∞. Moreover, $F(x,0) = 0$ and

$$|F(x,t)| \le T \sup_{0\le t\le T} |f(x,t)| = o(\psi(x))$$

as $|x| \to \infty$. Hence Lemma 4.2 applies to F and shows that $F \ge 0$ on D_∞. Moreover, Lemma 4.2 also applies to $-F$ and yields $-F \ge 0$ on D_∞. Hence $F = 0$ on D_∞, which implies that $f = 0$ on D_∞. □

Proof of uniqueness in Theorem 3.2. Assume that U_1 and U_2 both solve (10)–(11) with the same initial condition $\overline{g}$, and that $U_1 = o(\Psi(x))$ and $U_2 = o(\Psi(x))$ as $|x| \to \infty$ locally uniformly in t. Applying Lemma 4.3 to the function $f := U_1 - U_2$ shows that $f = 0$ on D_∞, so $U_1 = U_2$. □

5. Existence of solutions

In this section we prove the existence claim of Theorem 3.2. Indeed, we show that the function U given by stochastic representation in (12) solves (10)–(11) and is $o(\Psi(x))$ as $|x| \to \infty$.

Throughout this section we assume that g is continuous on $\mathbb{R}$ with $g(x) = o(\Psi(x))$ as $|x| \to \infty$. (Sometimes this assumption is further strengtened by assuming that g and some of its derivatives are bounded.)

Lemma 5.1. *There exists a constant C_1 such that for any x and $t \ge 0$,*

$$\mathbb{E}\Psi(X^x_t) \le (\Psi(x) + 1)e^{C_1 t}.$$

Remark. The constant C_1 can be chosen as the constant appearing in (9).

Proof. As in the proof of Theorem 7.9 in [3], the Itô–Tanaka formula may be employed to show that, for any $r \ge 0$,

$$\mathbb{E}\Psi(B^x_{A^x_r\wedge H}) \le \Psi(x) + \int_0^r \mathbb{E}|B^x_{A^x_t\wedge H}|\,dt \tag{15}$$

for any exit time $H := \inf\{t : B^x_t \notin (-a,a)\}$ with $|x| \le a$. Inserting (9) yields

$$\mathbb{E}|B^x_{A^x_r\wedge H}| \le C_1\mathbb{E}\Psi(B^x_{A^x_r\wedge H}) + C_1 \le C_1\Psi(x) + C_1 + C_1\int_0^r \mathbb{E}|B^x_{A^x_t\wedge H}|\,dt,$$

so Gronwall's lemma [11, Appendix 1] yields

$$\mathbb{E}|B^x_{A^x_t\wedge H}| \le (C_1\Psi(x) + C_1)e^{C_1 t}.$$

Inserting this into (15) gives

$$\mathbb{E}\Psi(B^x_{A^x_r\wedge H}) \le (\Psi(x) + 1)e^{C_1 r}.$$

By Fatou's lemma, letting $a \to \infty$ and thus $H \to \infty$,

$$\mathbb{E}\Psi(X^x_r) = \mathbb{E}\Psi(B^x_{A^x_r}) \le (\Psi(x) + 1)e^{C_1 r}.$$ □

Lemma 5.2. *If $K \subset \mathbb{R}$ is a compact set and $T > 0$, then the set of random variables $\{g(X^x_t) : (x,t) \in K \times [0,T]\}$ is uniformly integrable.*

Proof. By Lemma 5.1, $\mathbb{E}\Psi(X_t^x) \le C$ for $(x,t) \in K \times [0,T]$ and some $C < \infty$. Since $g(x) = o(\Psi(x))$ as $|x| \to \infty$, it follows that the set of random variables $\{g(X_t^x)\}$ with $(x,t) \in K \times [0,T]$ is uniformly integrable, see e.g. [5, Theorem 5.4.3 and its proof]. □

To prove the existence part of Theorem 3.2 we consider the function $U(x,t) := \mathbb{E}g(X_t^x)$. Note that $g(X_t^x) = \overline{g}(X_t^x)$ a.s. since $X_t^x \in \operatorname{supp} m$ (see [3, Lemma 3.1]), so $U(x,t) = \mathbb{E}g(X_t^x) = \mathbb{E}\overline{g}(X_t^x)$.

Lemma 5.3. *The function $U(x,t) = \mathbb{E}g(X_t^x)$ is continuous and $U(x,0) = \overline{g}(x)$. Furthermore, $U(x,t) = o(\Psi(x))$ as $|x| \to \infty$ locally uniformly in $t \ge 0$.*

Proof. Consider a sequence of points $(x_n,t_n) \in \mathbb{R}\times[0,\infty)$ such that $(x_n,t_n) \to (x,t)$ as $n \to \infty$. We may assume, for notational simplicity, that $x = 0$. By (1),

$$\Gamma_u^{x_n} = \int_{\mathbb{R}} L_u^y \, m(x_n + dy) = \int_{\mathbb{R}} L_u^y \, m_n(dy), \tag{16}$$

where m_n is the translated measure defined by $m_n(S) := m(S + x_n)$ for Borel sets $S \subseteq \mathbb{R}$. If we further define $\nu_n = t_n^{-1} m_n$, then, by (2),

$$A_{t_n}^{x_n} = \inf\Big\{u : \int_{\mathbb{R}} L_u^y \, m_n(dy) > t_n\Big\} = \inf\Big\{u : \int_{\mathbb{R}} L_u^y \, \nu_n(dy) > 1\Big\}. \tag{17}$$

(Note that this holds also in the case $t_n = 0$, when the measure ν_n only takes the values 0 and ∞.)

Since m is locally finite, $m_n \to m$ vaguely, see (5), and thus $\nu_n \to \nu := t^{-1} m$ vaguely. Hence, (17) and its analogue for (x,t) imply by (iii) in Section 2 that $A_{t_n}^{x_n} \to A_t^x$ a.s. Hence, a.s.,

$$X_{t_n}^{x_n} = x_n + B_{A_{t_n}^{x_n}} \to x + B_{A_t^x} = B_{A_t^x} = X_t^x,$$

and thus $g(X_{t_n}^{x_n}) \to g(X_t^x)$. Taking the expectations we obtain, using Lemma 5.2, that

$$U(x_n,t_n) = \mathbb{E}g(X_{t_n}^{x_n}) \to \mathbb{E}g(X_t^x) = U(x,t),$$

which shows the continuity of $U(x,t)$.

For $t = 0$ we have by [3, Lemma 3.3] that if $x \in \operatorname{supp} m$, then $A_0^x = 0$ and thus $X_0^x = x$ a.s., so $U(x,0) = g(x)$, while if $x \notin \operatorname{supp} m$, then A_0^x is a.s. the first time B_t^x hits $\operatorname{supp} m$; hence, if $x \in (a,b)$ where (a,b) is a component of the complement of $\operatorname{supp} m$, then $U(x,0) = \frac{x-a}{b-a} g(a) + \frac{b-x}{b-a} g(b) = \overline{g}(x)$. Thus $U(x,0) = \overline{g}(x)$ in both cases.

For the final claim we note that for any $\epsilon > 0$ there exists C_ϵ such that

$$|g(x)| \le \epsilon\Psi(x) + C_\epsilon,$$

and then by Lemma 5.1, for $0 \le t \le T$,

$$\mathbb{E}|g(X_t^x)| \le \epsilon\mathbb{E}\Psi(X_t^x) + C_\epsilon \le \epsilon e^{C_1 T}\Psi(x) + \epsilon e^{C_1 T} + C_\epsilon;$$

since ϵ is arbitrary, this implies that $\mathbb{E}|g(X_t^x)|/\Psi(x) \to 0$ uniformly for $0 \le t \le T$ as $|x| \to \infty$ and thus $\Psi(x) \to \infty$. □

In view of Lemma 5.3, it merely remains to prove that U satisfies $2mU_t = U_{xx}$ in the sense of distributions. (Recall that this means (13); we will use this form of the equation below, usually without comment.) This is done below by a series of approximations.

Lemma 5.4. *Assume that g is bounded and that m_n is a sequence of speed measures converging vaguely to m, see (5), and let $X^{x,n}$ and X^x be the corresponding generalized diffusions. Then $U_n(x,t) := \mathbb{E}g(X_t^{x,n}) \to \mathbb{E}g(X_t^x) =: U(x,t)$ as $n \to \infty$, for any x and $t > 0$.*

Proof. By (iii) in Section 2, $X_t^{x,n} \to X_t^x$ almost surely as $n \to \infty$. The result then follows by the continuity of g and bounded convergence. $\square$

Lemma 5.5. *Assume that $m(dx) \geq \epsilon\, dx$ for some $\epsilon > 0$, and that g, g' and g'' are bounded. Then $U(x,t)$ satisfies* (10).

Proof. First note that if m has a density which is regular enough (for the sake of simplicity, say C^1 with a bounded derivative), and bounded away from 0, then X is the weak solution of a stochastic differential equation, and by the standard Feynman–Kac theorem (see for example [4, Theorem 6.5.3]), U is the unique bounded classical solution of the initial value problem (10)–(11). In particular, see (13),

$$-2\iint U\varphi_t\, m(dx)\, dt = \iint U\varphi_{xx}\, dx\, dt \tag{18}$$

for all $\varphi \in \mathcal{D}$.

Now let m be as specified in the lemma, i.e. $m(dx) \geq \epsilon\, dx$ for some $\epsilon > 0$, and let m_n be a sequence of measures with regular densities such that $m_n(dx) \geq \epsilon\, dx$ for all n and such that m_n converges to m vaguely. (Such a sequence can be constructed as convolutions $m_n := \psi_n * m$ with a suitable sequence of regularising kernels $\psi_n \in \mathcal{D}(\mathbb{R})$ in the usual way.) Denote by $X_t^{x,n}$ the corresponding generalized diffusion, and let $U_n(x,t) = \mathbb{E}g(X_t^{x,n})$. Since m_n has a regular density, say $dm_n(y) = \frac{dy}{\sigma_n^2(y)}$ where σ_n is C^1 and bounded, $X_t^{x,n}$ is a weak solution of the stochastic differential equation

$$dY_t = \sigma_n(Y_t)\, dW_t \tag{19}$$

with $Y_0 = x$ (see (iv) in Section 2). Now, let $Y_t^{x,n}$ denote the *strong* solution of (19) for some given Brownian motion W (a unique strong solution exists since σ is C^1). Then, by weak uniqueness, $Y_t^{x,n}$ and $X_t^{x,n}$ coincide in law, so $U_n(x,t) = \mathbb{E}g(X_t^{n,x}) = \mathbb{E}g(Y_t^{n,x})$. Furthermore, since σ_n is bounded, $Y_t^{x,n}$ is a martingale, and by a comparison result for one-dimensional diffusions (see [11, Theorem IX.3.7]) we have $Y_t^{x,n} \leq Y_t^{y,n}$ if $x < y$. Consequently, if $x < y$, then

$$\begin{aligned}|U_n(y,t) - U_n(x,t)| &\leq \mathbb{E}|g(Y_t^{n,y}) - g(Y_t^{n,x})| \leq C\mathbb{E}|Y_t^{n,y} - Y_t^{n,x}| \\ &= C\mathbb{E}(Y_t^{n,y} - Y_t^{n,x}) = C(y-x),\end{aligned}$$

where C is a Lipschitz constant of g. Consequently, U_n is Lipschitz continuous in x uniformly in n.

Let D be a global bound for $|g''|$. We claim that

$$|U_n(x,t) - g(x)| \leq \frac{D}{2\epsilon} t. \tag{20}$$

To see this, consider the function

$$f(x,t) = U_n(x,t) - g(x) + \frac{D}{2\epsilon} t.$$

Then f is a supersolution, i.e. it satisfies

$$\begin{cases} 2m_n f_t = 2m_n (U_n)_t + (D/\epsilon) m_n \geq (U_n)_{xx} + D \geq f_{xx}, & t > 0, \\ f(x,0) = U_n(x,0) - g(x) = 0, \end{cases}$$

so Lemma 4.2 yields $f \geq 0$. Consequently, $U_n(x,t) \geq g(x) - \frac{D}{2\epsilon} t$. Similarly, the function $U_n(x,t) - g(x) - \frac{D}{2\epsilon} t$ is a subsolution, so $U_n(x,t) \leq g(x) + \frac{D}{2\epsilon} t$, which finishes the proof of (20).

Next, using the Markov property and (20) we find that

$$\begin{aligned} |U_n(x,t+h) - U_n(x,t)| &= \left|\mathbb{E}\left[g(X_{t+h}^{x,n}) - g(X_t^{x,n})\right]\right| \\ &= \left|\mathbb{E}\left[U_n(X_h^{x,n}, t) - g(X_t^{x,n})\right]\right| \leq \frac{D}{2\epsilon} h \end{aligned}$$

for $h > 0$. It follows from this, together with the uniform Lipschitz continuity in x proven above, that $(x,t) \mapsto U_n(x,t)$ is Lipschitz continuous uniformly in n. Consequently, the convergence $U_n(x,t) \to U(x,t)$ guaranteed by Lemma 5.4 is uniform on any compact subset of $\mathbb{R} \times (0, \infty)$.

As noted in (18),

$$0 = 2 \iint U_n \varphi_t \, m_n(dx) dt + \iint U_n \varphi_{xx} \, dx \, dt$$

for $\varphi \in \mathcal{D}$. By bounded convergence,

$$\iint U_n \varphi_{xx} \, dx \, dt \to \iint U \varphi_{xx} \, dx \, dt$$

as $n \to \infty$. Moreover,

$$\begin{aligned} \iint U_n \varphi_t \, m_n(dx) dt &= \iint (U_n - U) \varphi_t \, m_n(dx) dt + \iint U \varphi_t \, m_n(dx) dt \\ &\to \iint U \varphi_t \, m(dx) dt \end{aligned}$$

as $n \to \infty$ since $U_n \to U$ uniformly on $\operatorname{supp} \varphi$ and $m_n \to m$ vaguely. Consequently,

$$0 = 2 \iint U \varphi_t \, m(dx) dt + \iint U \varphi_{xx} \, dx \, dt,$$

for any $\varphi \in \mathcal{D}$, so U is a solution of (13) and thus (10). □

Lemma 5.6. *Assume that g, g' and g'' are bounded. Then $U(x,t)$ satisfies (10).*

Proof. For a given speed measure m, let $m_n(dx) = m(dx) + n^{-1}\,dx$ and let U and U_n be the corresponding stochastic representations. By Lemma 5.4, $U_n \to U$ pointwise on $\mathbb{R} \times (0, \infty)$ as $n \to \infty$. By Lemma 5.5,

$$0 = 2 \iint U_n \varphi_t \, m_n(dx)dt + \iint U_n \varphi_{xx} \, dx \, dt$$

for any $\varphi \in \mathcal{D}$. Here

$$\begin{aligned}\iint U_n \varphi_t \, m_n(dx)dt &= \iint U_n \varphi_t \, m(dx)dt + \frac{1}{n} \iint U_n \varphi_t \, dx \, dt \\ &\to \iint U \varphi_t \, m(dx)dt\end{aligned}$$

as $n \to \infty$ by bounded convergence and the fact that the functions U_n are uniformly bounded (by $\sup |g|$). Similarly,

$$\iint U_n \varphi_{xx} \, dx \, dt \to \iint U \varphi_{xx} \, dx \, dt$$

as $n \to \infty$. It follows that

$$0 = 2 \iint U \varphi_t \, m(dx)dt + \iint U \varphi_{xx} \, dx \, dt$$

for any $\varphi \in \mathcal{D}$, which finishes the proof. □

Lemma 5.7. *Assume that g is bounded. Then $U(x, t)$ satisfies* (10).

Proof. Let $g_n := \psi_n * g$ where ψ_n is a sequence of regularising kernels in $\mathcal{D}(\mathbb{R})$. Then each g_n satisfies the conditions of Lemma 5.6, and thus the corresponding $U_n(x, t) := \mathbb{E} g_n(X_t^x)$ satisfies (10), i.e.

$$0 = 2 \iint U_n \varphi_t \, m(dx)dt + \iint U_n \varphi_{xx} \, dx \, dt. \tag{21}$$

Moreover, $g_n(x) \to g(x)$ for every x, and thus by bounded convergence $U_n(x, t) \to U(x, t)$ for every x and t. Hence, bounded convergence applied to (21) shows that U satisfies (10). □

Completion of the proof of Theorem 3.2. Let $g = o(\Psi(x))$ as $|x| \to \infty$, and let $g_n := (g \wedge n) \vee (-n)$ be the function g truncated at n and $-n$. Denote by $U_n(x, t) = \mathbb{E} g_n(X_t^x)$ the corresponding stochastic representations. Then $U_n \to U$ pointwise as $n \to \infty$ by dominated convergence. Moreover, by Lemma 5.7 we have

$$0 = 2 \iint U_n \varphi_t \, m(dx)dt + \iint U_n \varphi_{xx} \, dx \, dt \tag{22}$$

for any $\varphi \in \mathcal{D}$. Since $|U_n(x, t)| \le \mathbb{E}|g(X_t^x)|$, Lemma 5.2 implies that the functions U_n are locally bounded uniformly in n. Thus bounded convergence applied to (22) shows that U satisfies (10). □

6. Properties of the solution

In this section we study monotonicity, Lipschitz continuity and convexity of the function U. Let m be a given speed measure and g be a given continuous function with $g(x) = o(\Psi(x))$ as $|x| \to \infty$.

Theorem 6.1 (Monotonicity)**.** *If g is non-decreasing, then also $U(x,t)$ is non-decreasing in x for any fixed $t \geq 0$.*

Proof. First let m_n be a sequence of measures with positive and regular densities such that m_n converges to m vaguely (such a sequence can be constructed as convolutions $m_n := \psi_n * (m + \frac{1}{n}\lambda)$ with a suitable sequence of regularising kernels $\psi_n \in \mathcal{D}(\mathbb{R})$ in the usual way). Let $U_n(x,t) = \mathbb{E}g(X_t^{n,x})$ where X^n is the generalized diffusion with speed measure m_n. By property (iii) in Section 2, $X_t^{n,x} \to X_t^x$ almost surely as $n \to \infty$. As in the proof of Lemma 5.5, the comparison result for one-dimensional diffusions yields that $U_n(x,t)$ is increasing in x.

Assume first that g is bounded. Then (12) implies, by bounded convergence, that $U_n(x,t) \to U(x,t)$ as $n \to \infty$. Consequently, $U(x,t)$ is increasing in x and the theorem is proved in the case of bounded g.

In the general case, let $g_M := (g \wedge M) \vee (-M)$. Then the theorem follows by letting $M \to \infty$ and using dominated convergence. $\square$

Theorem 6.2 (Lipschitz continuity)**.** *Suppose that the speed measure m satisfies* (4) *so that the corresponding generalized diffusion X is a martingale. If g is Lipschitz continuous with some constant C, i.e. $|g(x) - g(y)| \leq C|x-y|$ for all $x, y \in \mathbb{R}$, then so is $x \mapsto U(x,t)$ for every $t \geq 0$.*

Proof. The martingale condition (4) implies that $\Phi(x) \to \pm\infty$ as $x \to \pm\infty$, and thus $x = o(\Psi(x))$. If U is a solution to (10)–(11), then $Cx \pm U(x,t)$ are solutions to (10) with the initial values $Cx \pm \overline{g}(x)$. Since $Cx \pm g(x)$ are non-decreasing, Theorem 6.1 shows that $Cx \pm U(x,t)$ are non-decreasing, and thus $|U(x,t) - U(y,t)| \leq C|x-y|$. $\square$

Theorem 6.3 (Convexity)**.** *Assume that m satisfies* (4) *so that the corresponding generalized diffusion X is a martingale. If g is convex, then the function $U(x,t) = \mathbb{E}g(X_t^x)$ is convex in x for any fixed $t \geq 0$.*

Remark. Preservation of convexity has been widely studied in the financial mathematics literature, see [6] and the references therein. Note that Theorem 6.3 includes the case of general martingale diffusions since we have no pointwise growth condition on the diffusion coefficents at infinity.

Proof. Without loss of generality we assume that the measure m has no point mass at zero (this can be achieved by translation). First approximate Ψ with smooth convex functions Ψ_n such that

- $\Psi_n \geq \Psi$
- $\Psi_n \geq |x|^3/n$ for $|x| \geq n$
- $\Psi_n(x) = \Psi_n(0) + |x|^3$ for $|x| \leq 1/n$

- $\Psi_n \to \Psi$ pointwise as $n \to \infty$
- the measure m_n defined by $m_n(dx) = \frac{1}{2|x|}\Psi_n''(x)\,dx$ has a strictly positive density,

and let X^n be the corresponding generalized diffusion. Then $m_n \to m$ vaguely as $n \to \infty$, so $X_t^{n,x} \to X_t^x$ almost surely by (iii) in Section 2. By (9), $|x| \le C_1(\Psi_n(x)+1)$ holds with the same constant C_1 uniformly in n. Using the arguments of Lemmas 5.1 and 5.2 yields that $\{g(X_t^{n,x})\}$ is uniformly integrable in n provided that $g(x) = o(\Psi(x))$ as $|x| \to \infty$. Consequently, $U_n(x,t) := \mathbb{E}g(X_t^{n,x}) \to \mathbb{E}g(X_t^x) =: U(x,t)$ pointwise as $n \to \infty$.

Now, if g is convex, then $U_n(x,t)$ is convex in x for each fixed t, see for example [6]. Since the pointwise limit of a sequence of convex functions is convex, the result follows. □

7. Examples

In this section we consider a few explicit examples of generalized diffusions and the corresponding Feynman-Kac type theorems.

7.1. **Brownian motion with a sticky point.**

In this section we study the particular case in which $m = \delta + \lambda$, where δ is a Dirac measure at 0 and λ is the Lebesgue measure. The corresponding generalized diffusion X then behaves like a Brownian motion outside the point 0, which is called a sticky point for X (see [1] for a study of sticky Brownian motion).

For a given continuous initial condition $g(x) = o(|x|^3)$ we write $g = g_e + g_o$ as the sum of an even and an odd function, where

$$g_e(x) := (g(x) + g(-x))/2$$

and

$$g_o(x) := (g(x) - g(-x))/2.$$

Now, consider the classical initial boundary value problem

$$\begin{cases} u_t = \frac{1}{2}u_{xx} & (x,t) \in (0,\infty)^2 \\ u = g_e & (x,t) \in [0,\infty) \times \{0\} \\ u_t = u_x & (x,t) \in \{0\} \times (0,\infty). \end{cases}$$

By standard parabolic theory, this problem admits a unique solution in our class, compare [9, Section V.4]. (Alternatively, for suitable g_e, we can use the transformation $w(x,t) = u(x,t) - \int_0^t u_x(x,s)\,ds$, which solves $w_t = \frac{1}{2}w_{xx} - g_{e,x}(x)$ with boundary values $w(x,0) = g_e(x)$ and $w(0,t) = g_e(0)$, so $w_t(0,t) = 0$; we omit the details.) Similarly, let v be the unique solution of the initial boundary value problem

$$\begin{cases} v_t = \frac{1}{2}v_{xx} & (x,t) \in (0,\infty)^2 \\ v = g_o & (x,t) \in [0,\infty) \times \{0\} \\ v = 0 & (x,t) \in \{0\} \times (0,\infty). \end{cases}$$

Then the function

$$U(x,t) := u(|x|,t) + \frac{x}{|x|} v(|x|,t)$$

solves (10)–(11). Consequently,

$$U(x,t) = \mathbb{E} g(X_t^x).$$

Example. Let $g(x) = |x| + x^2$. Then $g_e = g$ and $U(x,t) = |x| + x^2 + t$.

Example. Let $g(x) = |x| + x^2 + 2\cos x - \sin|x|$. Again, $g_e = g$, and $U(x,t) = |x| + t + x^2 + e^{-t/2}(2\cos x - \sin|x|)$. Consequently, $U(x,0)$ is C^2, but for all positive t the solution U has a kink at $x = 0$.

Remark. Recall that solutions of parabolic equations with positive Hölder continuous diffusion coefficient gain two spatial derivatives. Thus starting with continuous initial data, the solution is twice continuously differentiable in space for any positive t. The first example above shows that there is no such general gain in regularity at points where the speed measure is singular with respect to Lebesgue measure. In the second example the initial data is twice continuously differentiable, but the solution is only Lipschitz in space. Thus, in this case regularity is even lost.

7.2. **Brownian motion skipping an interval.** Now, let the speed measure m be Lebesgue measure on $\mathbb{R} \setminus (-1,1)$ and 0 on $(-1,1)$. The corresponding generalized diffusion X behaves like a Brownian motion outside $(-1,1)$, and it spends no time in $(-1,1)$.

Again, write $g = g_e + g_o$ with g_e and g_o as above. Let u and v be the unique solutions (of order $o(|x^3|)$) of the problems

$$\begin{cases} u_t = \frac{1}{2} u_{xx} & (x,t) \in (1,\infty) \times (0,\infty) \\ u = g_e & (x,t) \in [1,\infty) \times \{0\} \\ u_x = 0 & (x,t) \in \{1\} \times (0,\infty) \end{cases}$$

and

$$\begin{cases} v_t = \frac{1}{2} v_{xx} & (x,t) \in (1,\infty) \times (0,\infty) \\ v = g_o & (x,t) \in [1,\infty) \times \{0\} \\ v_x = v & (x,t) \in \{1\} \times (0,\infty), \end{cases}$$

respectively. Then the function

$$U(x,t) := \begin{cases} u(|x|,t) + \frac{x}{|x|} v(|x|,t) & |x| \ge 1 \\ u(1,t) + v(1,t)x & |x| < 1 \end{cases}$$

solves (10)–(11). Consequently,

$$U(x,t) = \mathbb{E} g(X_t^x).$$

Example. Consider the initial value $g(x) = \overline{g}(x) = \max\{|x|, 1\}$. Using the recipe above, one finds that the unique solution (of order $o(|x|^3)$) of (10)–(11) is given by

$$U(x,t) = \begin{cases} |x| - 2(|x|-1) \int_{-\infty}^{-(|x|-1)/\sqrt{t}} \phi(y)\, dy + 2\sqrt{t}\phi((|x|-1)/\sqrt{t}), & |x| > 1, \\ 1 + \sqrt{2t/\pi}, & |x| \le 1, \end{cases}$$

where

$$\phi(y) := \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$$

is the density of the standard normal distribution. It is straightforward to check that $x \mapsto U(x,t)$ is C^1 for $t > 0$, but it fails to be C^2 since

$$U_{xx}(1+,t) = U_{xx}(-1-,t) = \sqrt{\frac{2}{\pi t}} \neq 0 = U_{xx}(1-,t) = U_{xx}(-1+,t).$$

8. Regularity

Theorem 8.1. *Let the assumptions in Theorem 3.2 hold. Then the function U solving* (10)–(11) *is infinitely differentiable in t; moreover, U and all its derivatives with respect to t and are locally Lipschitz on $\mathbb{R} \times (0,\infty)$.*

In view of the examples in Section 7.1, this regularity result is sharp. Together with the remark after Theorem 3.2, Theorem 8.1 yields the following.

Corollary 8.2. *The distribution U_{xx} is a measure on $\mathbb{R} \times (0,\infty)$ that is absolutely continuous with respect to $m \times \lambda$. Moreover, for each fixed $t > 0$, U_{xx} is a measure on $\mathbb{R}$ that is absolutely continuous with respect to m.*

Remark. The result in Corollary 8.2 can be viewed as parabolic regularity in our setting. Indeed, two spatial derivatives in regularity are gained by the solution, when we measure regularity with respect to m.

Corollary 8.3. *Assume that m is absolutely continuous with respect to Lebesgue measure λ. Then $x \mapsto \frac{\partial^k}{\partial t^k} U(x,t)$ is C^1 for all $t > 0$ and $k \geq 0$. Moreover, if the Radon–Nikodym derivative $\frac{dm}{d\lambda}$ is continuous, then $x \mapsto \frac{\partial^k}{\partial t^k} U(x,t)$ is C^2 for all $t > 0$ and $k \geq 0$.*

Remark. The example in Section 7.2 illustrates the sharpness of Corollary 8.3.

To prove Theorem 8.1, let m be a locally finite nonnegative Borel measure on $\mathbb{R}$ such that (6) holds, and let $g(x) = o(\Psi(x))$ as $|x| \to \infty$. Let, as in the proof of Theorem 6.3, Ψ_n be a sequence of smooth convex functions satisfying

- $\Psi_n \geq \Psi$
- $\Psi_n(x) = \Psi_n(0) + |x|^3$ for $|x| \leq 1/n$
- $\Psi_n \to \Psi$ pointwise as $n \to \infty$
- the measure m_n defined by $m_n(dx) = \frac{1}{2|x|} \Psi_n''(x)\, dx$ has a strictly positive density,

and let U_n be the corresponding solutions to (10)–(11) for the measure m_n. Lemma 5.1 then holds with the same constant C_1 independent of n, for the corresponding generalized diffusion X^n, which implies that U_n is locally bounded uniformly in n. Consequently, the following lemma holds.

Lemma 8.4. *For every rectangle* $[a,b]\times[t_1,t_2]\subset\mathbb{R}\times(0,\infty)$ *we have*

$$\sup_n \int_a^b \int_{t_1}^{t_2} U_n^2(x,t)\,dt\,m_n(dx)<\infty. \tag{23}$$

Classical regularity theory implies that each U_n is smooth on $\mathbb{R}\times(0,\infty)$. We have the following L^2-estimates for derivatives of U_n.

Lemma 8.5. *We have*

$$\sup_n \int_a^b \int_{t_1}^{t_2} \left(\frac{\partial^k}{\partial t^k}U_n(x,t)\right)^2 dt\,m_n(dx)<\infty \tag{24}$$

and

$$\sup_n \int_a^b \int_{t_1}^{t_2} \left(\frac{\partial^{k+1}}{\partial x\partial t^k}U_n(x,t)\right)^2 dt\,dx<\infty \tag{25}$$

for all $k\geq 0$ *and any rectangle* $[a,b]\times[t_1,t_2]\subset\mathbb{R}\times(0,\infty)$.

Proof. We assume, somewhat more generally, that U_n is any sequence of solutions to (10), for the measure m_n, such that (23) holds for any rectangle.

To simplify the notation, we suppress the dependence on n and write $u=U_n$ and $\mu=m_n$, respectively. By construction, m_n has a smooth density, which we also denote by μ. If $\psi\in C_c^2(\mathbb{R})$, then using (10) and integrations by parts we find that

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}}\psi u^2\,\mu(dx) &= \int_{\mathbb{R}}2\psi u u_t\,\mu(dx)=\int_{\mathbb{R}}\psi u u_{xx}\,dx\\ &= -\int_{\mathbb{R}}\psi u_x^2\,dx-\int_{\mathbb{R}}\psi_x u u_x\,dx\\ &= -\int_{\mathbb{R}}\psi u_x^2\,dx+\frac12\int_{\mathbb{R}}\psi_{xx}u^2\,dx,\end{aligned} \tag{26}$$

and thus

$$\int_{\mathbb{R}}\psi u_x^2\,dx=\frac12\int_{\mathbb{R}}\psi_{xx}u^2\,dx-\frac{d}{dt}\int_{\mathbb{R}}\psi u^2\,\mu(dx). \tag{27}$$

If $0<t_0<t_2$, integrating (27) for $t\in(t_0,t_2)$ gives

$$\begin{aligned}\int_{t_0}^{t_2}\int_{\mathbb{R}}\psi u_x^2\,dx\,dt &= \frac12\int_{t_0}^{t_2}\int_{\mathbb{R}}\psi_{xx}u^2\,dx\,dt+\int_{\mathbb{R}}\psi(x)u^2(x,t_0)\,\mu(dx)\\ &\quad -\int_{\mathbb{R}}\psi(x)u^2(x,t_2)\,\mu(dx).\end{aligned} \tag{28}$$

Assume further that $\psi\geq 0$ and integrate again for $t_0\in(t_1/2,t_1)$ with $t_1<t_2$ to obtain

$$\frac{t_1}{2}\int_{t_1}^{t_2}\int_{\mathbb{R}}\psi u_x^2\,dx\,dt\leq\frac{t_1}{4}\int_{t_1/2}^{t_2}\int_{\mathbb{R}}|\psi_{xx}|u^2\,dx\,dt+\int_{t_1/2}^{t_1}\int_{\mathbb{R}}\psi u^2\,\mu(dx)\,dt \tag{29}$$

Let $a<b$. We choose $\psi=\psi_n$ depending on n as follows. Find $a_1<a_2<a-1$ and $a_4>a_3>b+1$ with $a_1,a_2,a_3,a_4\in\operatorname{supp}m$ and $a_2-a_1>3$,

$a_4 - a_3 > 3$. Denote by I_1, I_2, I_3, I_4 the disjoint open intervals in $\mathbb{R} \setminus [a, b]$ such that $I_j = (a_j - 1, a_j + 1)$. Then $m(I_j) > 0$ for $j = 1, 2, 3, 4$, and for n large $\mu(I_j) > \frac{1}{2} m(I_j)$. Define ψ by

$$\psi_{xx} = \begin{cases} \mu/\mu(I_1) & \text{in } I_1 \\ -\mu/\mu(I_2) & \text{in } I_2 \\ -\gamma\mu/\mu(I_3) & \text{in } I_3 \\ \gamma\mu/\mu(I_4) & \text{in } I_4 \\ 0 & \text{elsewhere} \end{cases} \tag{30}$$

with $\psi(x) = 0$ for $x \leq a_1 - 1$ and $\gamma > 0$ to be chosen. Then

$$\psi_x = \begin{cases} 0 & \text{on } (-\infty, a_1 - 1) \\ 1 & \text{on } (a_1 + 1, a_2 - 1) \\ 0 & \text{on } (a_2 + 1, a_3 - 1) \\ -\gamma & \text{on } (a_3 + 1, a_4 - 1) \\ 0 & \text{on } (a_4 + 1, \infty) \end{cases}$$

Hence ψ is constant and larger than $a_2 - 1 - (a_1 + 1) \geq 1$ on $[a, b] \subset (a_2 + 1, a_3 - 1)$, smaller than $a_2 + 1 - (a_1 - 1) = a_2 - a_1 + 2$ everywhere, and by choosing a suitable γ we have $\psi = 0$ on $(a_4 + 1, \infty)$. Then $\psi \in C_c^1(\mathbb{R})$, and ψ_{xx} is bounded and continuous everywhere but at a finite number of points. By an approximation argument (or extending (26)), the inequality (29) holds also for such functions. Moreover, γ is uniformly bounded for n large. Thus, for large n, $|\psi_{xx}| \leq C\mu 1_{[a_0, a_5]}$ and $\psi \leq C 1_{[a_0, a_5]}$ with $a_0 := a_1 - 1$, $a_5 := a_4 + 1$ and C independent of n. Thus (29) implies, by our assumption (23),

$$\frac{t_1}{2} \int_{t_1}^{t_2} \int_a^b u_x^2 \, dx \, dt \leq C \int_{t_1/2}^{t_2} \int_{a_0}^{a_5} u^2 \mu(dx) \, dt + C \int_{t_1/2}^{t_1} \int_{a_0}^{a_5} u^2 \, \mu(dx) \, dt \leq C,$$

i.e. for any $[a, b] \times [t_1, t_2] \subset \mathbb{R} \times (0, \infty)$,

$$\int_{t_1}^{t_2} \int_a^b u_x^2 \, dx \, dt \leq C \tag{31}$$

with C independent of n (but depending on a, b, t_1, t_2). This is (25) for $k = 0$.

Again, let $\psi \in C_c^2(\mathbb{R})$ with $\psi \geq 0$. Then

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} \psi u_x^2 \, dx &= 2 \int_{\mathbb{R}} \psi u_x u_{xt} \, dx = -2 \int_{\mathbb{R}} (\psi u_x)_x u_t \, dx \\ &= -2 \int_{\mathbb{R}} \psi u_{xx} u_t \, dx - 2 \int_{\mathbb{R}} \psi_x u_x u_t \, dx \\ &= -4 \int_{\mathbb{R}} \psi u_t^2 \, \mu(dx) - \int_{\mathbb{R}} \frac{\psi_x u_x u_{xx}}{\mu} \, dx \\ &= -4 \int_{\mathbb{R}} \psi u_t^2 \, \mu(dx) + \frac{1}{2} \int_{\mathbb{R}} \Big(\frac{\psi_x}{\mu} \Big)_x u_x^2 \, dx \end{aligned}$$

or

$$4\int_{\mathbb{R}} \psi u_t^2 \,\mu(dx) = -\frac{d}{dt}\int_{\mathbb{R}} \psi u_x^2\,dx + \frac{1}{2}\int_{\mathbb{R}} \Big(\frac{\psi_x}{\mu}\Big)_x u_x^2\,dx. \tag{32}$$

Integrate for $t \in (t_0, t_2)$ to obtain

$$\begin{aligned} 4\int_{t_0}^{t_2}\int_{\mathbb{R}} \psi u_t^2\,\mu(dx)\,dt &= \int_{\mathbb{R}} \psi u_x^2(x,t_0)\,dx - \int_{\mathbb{R}} \psi u_x^2(x,t_2)\,dx \\ &\quad + \frac{1}{2}\int_{t_0}^{t_2}\int_{\mathbb{R}} \Big(\frac{\psi_x}{\mu}\Big)_x u_x^2\,dx\,dt. \end{aligned} \tag{33}$$

Integrating once more for $t_0 \in (t_1/2, t_1)$ yields

(34)

$$2t_1\int_{t_1}^{t_2}\int_{\mathbb{R}} \psi u_t^2\,\mu(dx)\,dt \le \int_{t_1/2}^{t_1}\int_{\mathbb{R}} \psi u_x^2\,dx\,dt + \frac{t_1}{4}\int_{t_1/2}^{t_2}\int_{\mathbb{R}} \left|\Big(\frac{\psi_x}{\mu}\Big)_x\right| u_x^2\,dx\,dt.$$

We choose again $\psi = \psi_n$ depending on n. This time, with $a < b$ given and notations as above, let $\varphi_1 \in C_c^\infty(I_1)$ and $\varphi_4 \in C_c^\infty(I_4)$ with $\varphi_1, \varphi_4 \ge 0$ and $\varphi_1(a_1), \varphi_4(a_4) > 0$. Then $\int \varphi_j\,\mu(dx) \to \int \varphi_j\,m(dx) > 0$ for $j = 1, 4$ as $n \to \infty$. Define $\psi = \psi_n \in C_c^\infty(\mathbb{R})$ by

$$\psi_x = \begin{cases} \frac{\varphi_1\mu}{\int \varphi_1\mu(dx)} & \text{on } I_1 \\ -\frac{\varphi_4\mu}{\int \varphi_4\mu(dx)} & \text{on } I_4 \\ 0 & \text{elsewhere.} \end{cases} \tag{35}$$

Then $\psi = 1$ on $[a_1+1, a_4-1] \supset [a,b]$. Moreover, $0 \le \psi \le 1_{[a_0,a_5]}$ and $|(\frac{\psi_x}{\mu})_x| \le C1_{[a_0,a_5]}$ (for n large) so (34) implies

$$2t_1\int_{t_1}^{t_2}\int_a^b u_t^2\,\mu(dx)\,dt \le \int_{t_1/2}^{t_1}\int_{a_0}^{a_5} u_x^2\,dx\,dt + C\int_{t_1/2}^{t_2}\int_{a_0}^{a_5} u_x^2\,dx\,dt.$$

Consequently, (31) yields, for any $[a,b]\times[t_1,t_2] \subset \mathbb{R}\times(0,\infty)$,

$$\int_{t_1}^{t_2}\int_a^b u_t^2\,\mu(dx)\,dt \le C \tag{36}$$

with C independent of n.

We have used (23) to show that (31) and (36) hold. Since $u_t = \frac{\partial U_n}{\partial t}$ is another solution of the equation (with $m = m_n$), the estimates (24) and (25) hold by induction for all $k \ge 0$. $\square$

Lemma 8.6. *For each fixed integer $k \ge 0$,*

$$\sup_n \int_a^b \left(\frac{\partial^k}{\partial t^k} U_n(x,t)\right)^2 m_n(dx) < \infty \tag{37}$$

and

$$\sup_n \int_a^b \left(\frac{\partial^{k+1}}{\partial x \partial t^k} U_n(x,t)\right)^2 dx < \infty \tag{38}$$

uniformly in $t \in [t_3, t_4]$, for every rectangle $[a,b]\times[t_3,t_4] \subset \mathbb{R}\times(0,\infty)$.

Proof. For simplicity of notation we consider the case $k = 0$ (the general case being completely analogous). Fix $a < b$, let ψ be defined by (30) and with compact support, and fix $[t_3, t_4] \subset (0, \infty)$. For any $t_1, t_2 \in [t_3, t_4]$, (28) together with (24) and (25) yield

$$\begin{aligned}
&\left| \int_{\mathbb{R}} \psi(x) u^2(x, t_1)\, \mu(dx) - \int_{\mathbb{R}} \psi(x) u^2(x, t_2)\, \mu(dx) \right| \\
&\qquad \le \int_{t_3}^{t_4} \int_{\mathbb{R}} \psi u_x^2 \, dx\, dt + \frac{1}{2} \int_{t_3}^{t_4} \int_a^b |\psi_{xx}| u^2 \, dx\, dt \\
&\qquad \le C \int_{t_3}^{t_4} \int_{a_0}^{a_5} u_x^2 \, dx\, dt + C \int_{t_3}^{t_4} \int_{a_0}^{a_5} u^2 \, \mu(dx)\, dt \le C.
\end{aligned}$$

Since further, by (24) again,

$$\int_{t_3}^{t_4} \int_{\mathbb{R}} \psi u^2 \, \mu(dx)\, dt \le C \int_{t_3}^{t_4} \int_{a_0}^{a_5} u^2 \, \mu(dx)\, dt \le C,$$

it follows that, uniformly for $t \in [t_3, t_4]$,

$$\int_{\mathbb{R}} \psi u^2 \, \mu(dx) \le C$$

and thus, for any $[a, b]$ and $[t_3, t_4] \subset (0, \infty)$ and every $t \in [t_3, t_4]$

$$\int_a^b u^2(x, t) \mu(dx) \le C, \tag{39}$$

which is (37) for $k = 0$.

Similarly, inserting ψ defined as in (35) and with compact support in (33) gives, for any $t_1, t_2 \in [t_3, t_4]$,

$$\begin{aligned}
&\left| \int_{\mathbb{R}} \psi(x) u_x^2(x, t_1)\, dx - \int_{\mathbb{R}} \psi(x) u_x^2(x, t_2)\, dx \right| \\
&\qquad \le 4 \int_{t_3}^{t_4} \int_{\mathbb{R}} \psi u_t^2 \, \mu(dx)\, dt + \int_{t_3}^{t_4} \int_{\mathbb{R}} \left| \left(\frac{\psi_x}{\mu} \right)_x \right| u_x^2 \, dx\, dt \\
&\qquad \le C \int_{t_3}^{t_4} \int_{a_0}^{a_5} u_t^2 \, \mu(dx)\, dt + \int_{t_3}^{t_4} \int_{a_0}^{a_5} u_x^2 \, dx\, dt \le C.
\end{aligned}$$

Since further, by (25),

$$\int_{t_3}^{t_4} \int_{\mathbb{R}} \psi u_x^2 \, dx\, dt \le \int_{t_3}^{t_4} \int_{a_0}^{a_5} u_x^2 \, dx\, dt \le C,$$

it follows that for every $t \in [t_3, t_4]$,

$$\int_{\mathbb{R}} \psi u_x^2(x, t)\, dx \le C.$$

Consequently,

$$\int_a^b u_x^2(x, t)\, dx \le C \tag{40}$$

uniformly in n and $t \in [t_3, t_4]$, for every $[a, b] \times [t_3, t_4] \subset \mathbb{R} \times (0, \infty)$, so (38) holds for $k = 0$. □

Lemma 8.7. *The functions $\frac{\partial^k U_n}{\partial t^k}$ and $\frac{\partial^{k+1} U_n}{\partial x \partial t^k}$ with $k \geq 0$ are locally bounded on $\mathbb{R} \times (0, \infty)$, uniformly in n. Thus U_n and all its time derivatives are locally Lipschitz on $\mathbb{R} \times (0, \infty)$, uniformly in n.*

Proof. First note that it follows from (38) that $u := U_n$ is locally Hölder(1/2)-continuous in the spatial variable, uniformly in n. Indeed, to see this, note that by the Cauchy–Schwarz inequality

$$|u(y,t) - u(x,t)| = \left| \int_x^y u_z(z,t)\, dz \right| \leq C\sqrt{|y - x|} \tag{41}$$

uniformly in $x, y \in [a, b]$ and $t \in [t_3, t_4]$, for every rectangle $[a, b] \times [t_3, t_4] \subset \mathbb{R} \times (0, \infty)$.

Now, let $I = [a, b] \subset \mathbb{R}$ be a given non-empty interval. By (6), we may increase I so that $\eta := m(I^\circ) > 0$. It follows from the vague convergence of $\mu = m_n$ to m that $\mu(I) \geq \eta/2$ for sufficiently large n. Pick $x_0 \in I$ and $t_3, t_4 \in (0, \infty)$, and note that for $t \in (t_3, t_4)$ the local Hölder continuity of u implies that

$$|u(x,t) - u(x_0,t)| \leq C\sqrt{b - a}$$

for all $x \in I$. It therefore follows from (37) that

$$|u(x_0,t)| \leq C\sqrt{b - a} + \sqrt{2C/\eta}$$

for n sufficiently large. Consequently, u is locally bounded uniformly in n. The case of time derivatives of u is completely analogous.

Similarly, the Cauchy–Schwarz inequality applied to $u_x = \frac{\partial U_n}{\partial x}$ yields

$$\begin{aligned} |u_x(y,t) - u_x(x,t)| &= \left| \int_x^y u_{zz}(z,t)\, dz \right| = \left| \int_x^y 2u_t(z,t)\mu(dz) \right| \\ &\leq 2 \left(\int_x^y u_t^2(z,t)\, \mu(dz) \right)^{1/2} (\mu(x,y))^{1/2} \\ &\leq C\, (\mu(a,b))^{1/2} \leq C, \end{aligned} \tag{42}$$

where the last inequality holds since $\limsup_{n\to\infty} \mu(a, b) \leq m[a, b] < \infty$. Note that (42) holds uniformly in $x, y \in [a, b]$ and $t \in [t_3, t_4]$, for every rectangle $[a, b] \times [t_3, t_4] \subset \mathbb{R} \times (0, \infty)$. Together with (38) for $k = 0$, this gives the desired local bound of $u_x = \frac{\partial U_n}{\partial x}$. The case of time derivatives of u_x is completely analogous. □

Proof of Theorem 8.1. Since $m_n \to m$ vaguely as $n \to \infty$, $X_t^{n,x} \to X_t^x$ almost surely by (iii) in Section 2. Arguing as in the proof of Theorem 6.3, $U_n(x,t) = \mathbb{E}g(X_t^{n,x}) \to \mathbb{E}g(X_t^x) = U(x,t)$ pointwise as $n \to \infty$. By Lemma 8.7 and the Arzela–Ascoli theorem, U and its partial derivatives with respect to time exist and they are locally Lipschitz continuous. □

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