FIRST CRITICAL PROBABILITY FOR A PROBLEM ON RANDOM ORIENTATIONS IN $G(n,p)$.

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Abstract. We study the random graph $G(n,p)$ with a random orientation. For three fixed vertices s,a,b in $G(n,p)$ we study the correlation of the events $\{a \to s\}$ and $\{s \to b\}$. We prove that asymptotically the correlation is negative for small p, $p < \frac{C_1}{n}$, where $C_1 \approx 0.3617$, positive for $\frac{C_1}{n} < p < \frac{2}{n}$ and up to $p = p_2(n)$. Computer aided computations suggest that $p_2(n) = \frac{C_2}{n}$, with $C_2 \approx 7.5$. We conjecture that the correlation then stays negative for p up to the previously known zero at $\frac{1}{2}$; for larger p it is positive.

1. Introduction

Let $G(n,p)$ be the random graph with n vertices where each edge has probability p of being present independent of the other edges. We further orient each present edge either way independently with probability $\frac{1}{2}$, and denote the resulting random directed graph by $\vec{G}(n,p)$. This version of orienting edges in a graph, random or not, is natural and has been considered previously in e.g. [1, 2, 3, 5].

Let a,b,s be three distinct vertices and define the events $A := \{a \to s\}$, that there exists a directed path in $\vec{G}(n,p)$ from a to s, and $B := \{s \to b\}$. In a previous paper, [2], we showed that, for fixed p, the correlation between A and B asymptotically is negative for $p < \frac{1}{2}$ and positive for $p > \frac{1}{2}$. Note that we take the covariance in the combined probability space of $G(n,p)$ and the orientation of edges, which is often referred to as the annealed case, see [2] for details. We say that a probability $p \in (0,1)$ is *critical* (for a given n) if the covariance $\mathrm{Cov}(A,B) = 0$. We have thus shown in [2] that there is a critical probability $\frac{1}{2} + o(1)$ for large n. (Moreover, this is the largest critical probability, since the covariance stays positive for all larger $p < 1$.) We also conjectured that for large n, there are in fact (at least) three critical probabilities when the covariance changed sign. Based on computer aided computations we guessed that the first two critical probabilities would be approximately $\frac{0.36}{n}$ and $\frac{7.5}{n}$. In this note we prove that there is a first critical probability of the conjectured order, where the covariance changes from negative to positive, and thus there must be at least three critical probabilities. Our theorem is as follows.

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Theorem 1.1. *With $p = \frac{2c}{n}$ and sufficiently large n, the covariance* $\mathrm{Cov}(A, B)$ *is negative for $0 < c < c_1$ and positive for $c_1 < c < 1$, where $c_1 \approx 0.180827$ is a solution to $(2-c)(1-c)^3 = 1$. Furthermore, for fixed c with $0 \le c < 1$,*

$$\mathrm{Cov}(A, B) = \bigl(1 - (2-c)(1-c)^3\bigr) \cdot \frac{c^3}{(1-c)^5} \cdot \frac{1}{n^3} + O\Bigl(\frac{1}{n^4}\Bigr). \tag{1.1}$$

In fact, the proof shows that (1.1) holds uniformly in $0 \le c \le c'$ for any $c' < 1$; moreover, we may (with just a little more care) for such c write the error term as $O(c^4 n^{-4})$. This implies that for large n, the critical $p \approx 2c_1/n$ is indeed the first critical probability, and that the covariance is negative for all smaller $p > 0$.

Remark 1.2. In a random orientation of any given graph G, it is a fact first observed by McDiarmid that $\mathbb{P}(a \to s)$ is equal to $\mathbb{P}(a \leftrightarrow s)$ in an edge percolation on the same graph with probability $1/2$ for each edge independently, see [5]. Hence the events A (and thus B) have the same probability as $\mathbb{P}(a \leftrightarrow s)$ in $G(n, p/2)$. With $p = 2c/n$ it is well known that for $c < 1$ this probability is $\frac{c}{(1-c)} n^{-1} + O(n^{-2})$, see e.g. [4]. Hence the covariance in (1.1) is of the order $O(\mathbb{P}(A)\,\mathbb{P}(B)/n)$.

The outline of the proof is as follows, see Sections 2 and 3 for details.

Let $p := 2c/n$, where $c < 1$. Let $X_A := \#\{a \to s\}$ be the number of paths from a to s in $\vec{G}(n, p)$ and $X_B := \#\{s \to b\}$. (In the proof below, for technical reasons, we actually only count paths that are not too long.) We first show that, in our range of p, the probability that $X_A \ge 2$ or $X_B \ge 2$ is small, and that we can ignore these events and approximate $\mathrm{Cov}(A, B)$ by $\mathrm{Cov}(X_A, X_B)$. The latter covariance is a double sum over pairs of possible paths (α, β), where α goes from a to s and β goes from s to b, and we show that the largest contribution comes from configurations of the following two types:

- **Type 1:** The two edges incident to s, i.e the last edge in α and the first edge in β, are the same but with opposite orientations; all other edges are distinct. See Figure 1.

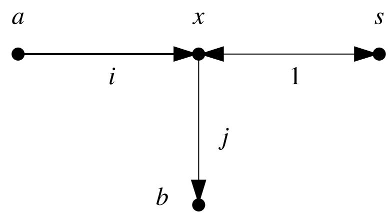

FIGURE 1. Configurations of Type 1 ($i, j \ge 0$, $i + j \ge 1$).

- **Type 2:** α and β contain a common subpath with the same orientation, but all other edges are distinct. See Figure 2.

If (α, β) is of Type 1, then α and β cannot both be paths in $\vec{G}(n, p)$, since they contain an edge with opposite orientations. Thus each such pair (α, β) gives a negative contribution to $\mathrm{Cov}(X_A, X_B)$. Pairs of Type 2, on the other hand, give a positive contribution. It turns out that both contributions are of the same order n^{-3}, see Lemmas 3.2 and 3.3, with constant factors depending on c such that the negative contribution

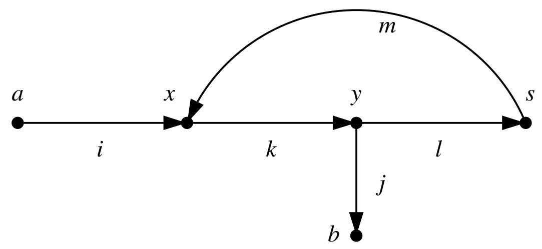

FIGURE 2. Configurations of Type 2 ($i, j \geq 0$, $k, l, m \geq 1$).

from Type 1 dominates for small c, and the positive contribution from Type 2 dominates for larger c.

Open problem 1.3. It would be interesting to find a method to compute also the second critical probability, which we in [2] conjectured to be approximately $\frac{7.5}{n}$. (The methods in the present paper apply only for $c < 1$.) Even showing that the covariance is negative when p is of the order $\frac{\log n}{n}$ is open. Moreover we conjecture that (for large n at least) there are only three critical probabilities, but that too is open.

2. Proof of Theorem 1.1

We give here the main steps in the proof of Theorem 1.1, leaving details to a sequence of lemmas in Section 3.

By a *path* we mean a directed path $\gamma = v_0 e_1 \cdots e_\ell v_\ell$ in the complete graph K_n. We use the conventions that a path is self-avoiding, i.e. has no repeated vertex, and that the length $|\gamma|$ of a path is the number of edges in the path.

We let Γ be the set of all such paths and let, for two distinct vertices v and w, Γ_{vw} be the subset of all paths from v to w.

If $\gamma \in \Gamma$, let I_γ be the indicator that γ is a path in $\vec{G}(n,p)$, i.e, that all edges in γ are present in $\vec{G}(n,p)$ and have the correct orientation there. Thus

$$\mathbb{E}\, I_\gamma = \mathbb{P}(I_\gamma = 1) = \left(\frac{p}{2}\right)^{|\gamma|} = \left(\frac{c}{n}\right)^{|\gamma|}. \tag{2.1}$$

Let I_A and I_B be the indicators of A and B. Note that the event A occurs if and only if $\sum_{\alpha \in \Gamma_{as}} I_\alpha \geq 1$, and similarly for B.

It will be convenient to restrict attention to paths that are not too long, so we introduce a cut-off $L := \log^2 n$ and let Γ_{vw}^L be the set of paths in Γ_{vw} of length at most L. Let

$$X_A := \sum_{\alpha \in \Gamma_{as}^L} I_\alpha \qquad \text{and} \qquad X_B := \sum_{\beta \in \Gamma_{sb}^L} I_\beta,$$

i.e, the numbers of paths in $\vec{G}(n,p)$ from a to s and from s to b, ignoring paths of length more than L.

Write $X_A = I'_A + X'_A$ and $X_B = I'_B + X'_B$, where I'_A and I'_B are the indicators for the events $X_A \geq 1$ and $X_B \geq 1$ respectively, so that

$$I'_A = \min(X_A, 1),$$
$$X'_A = (X_A - 1)_+ = \begin{cases} 0 & \text{if } X_A \leq 1, \\ X_A - 1 & \text{if } X_A > 1. \end{cases}$$

We have $I_A \geq I'_A$. Let $J_A := I_A - I'_A$ and $J_B := I_B - I'_B$. Thus

$$\operatorname{Cov}(A, B) = \operatorname{Cov}(I_A, I_B) = \operatorname{Cov}(I'_A, I'_B) + \operatorname{Cov}(I'_A, J_B) + \operatorname{Cov}(J_A, I_B). \tag{2.2}$$

We will show in Lemma 3.1 below that the last terms are small: $O(n^{-99})$. (The exponent 99 here and below can be replaced by any fixed number.)

Similarly, since $I'_A = X_A - X'_A$,

$$\operatorname{Cov}(I'_A, I'_B) = \operatorname{Cov}(X_A, X_B) - \operatorname{Cov}(X_A, X'_B) - \operatorname{Cov}(X'_A, X_B) + \operatorname{Cov}(X'_A, X'_B), \tag{2.3}$$

where Lemma 3.5 shows that the last three terms are $O(n^{-4})$. Hence, it suffices to compute

$$\operatorname{Cov}(X_A, X_B) = \operatorname{Cov}\Big(\sum_{\alpha \in \Gamma^L_{as}} I_\alpha, \sum_{\beta \in \Gamma^L_{sb}} I_\beta\Big) = \sum_{\alpha \in \Gamma^L_{as}} \sum_{\beta \in \Gamma^L_{sb}} \operatorname{Cov}(I_\alpha, I_\beta). \tag{2.4}$$

Lemmas 3.2 and 3.3 yield the contribution to this sum from pairs (α, β) of Types 1 and 2, and Lemma 3.4 shows that the remaining terms contribute only $O(n^{-4})$. Using (2.2)–(2.4) and the lemmas in Section 3 we thus obtain

$$\begin{aligned}
\operatorname{Cov}(A, B) &= \operatorname{Cov}(I'_A, I'_B) + O(n^{-99}) = \operatorname{Cov}(X_A, X_B) + O(n^{-4}) \\
&= \Big(-\frac{2c^3 - c^4}{(1-c)^2} + \frac{c^3}{(1-c)^5}\Big)\frac{1}{n^3} + O\Big(\frac{1}{n^4}\Big), \\
&= \frac{c^3}{(1-c)^5} \cdot \Big(1 - (2-c)(1-c)^3\Big)\frac{1}{n^3} + O\Big(\frac{1}{n^4}\Big),
\end{aligned}$$

which is (1.1).

The polynomial $1 - (2-c)(1-c)^3 = -c^4 + 5c^3 - 9c^2 + 7c - 1$ is negative for $c = 0$ and has two real zeros, for example because its discriminant is $-283 < 0$, see e.g. [6]; a numerical calculation yields the roots $c_1 \approx 0.180827$ and $c_2 \approx 2.380278$, which completes the proof. $\square$

3. Lemmas

We begin with some general considerations. We assume, as in Theorem 1.1, that $p = 2c/n$ and $0 \leq c < 1$.

Consider a term $\operatorname{Cov}(I_\alpha, I_\beta)$ in (2.4). Suppose that α and β have lengths ℓ_α and ℓ_β. Furthermore, suppose that β contains $\delta \geq 0$ edges not in α (ignoring the orientations) and that these form $\mu \geq 0$ subpaths of β that intersect α only at the endvertices. (We will use the notation $\beta \setminus \alpha$ for the set of (undirected) edges in β but not in α.) The number $\ell_{\alpha\beta}$ of edges common to α and β (again ignoring orientations) is thus $\ell_\beta - \delta$. By (2.1), $\mathbb{E}\, I_\alpha = (c/n)^{\ell_\alpha}$ and $\mathbb{E}\, I_\beta = (c/n)^{\ell_\beta}$.

(i) If α and β have no common edge, then I_α and I_β are independent and

$$\operatorname{Cov}(I_\alpha, I_\beta) = 0. \tag{3.1}$$

(ii) If all common edges have the same orientation in α and β, then

$$\operatorname{Cov}(I_\alpha, I_\beta) = \mathbb{E}(I_\alpha I_\beta) - \mathbb{E}\, I_\alpha\, \mathbb{E}\, I_\beta = \left(\frac{c}{n}\right)^{\ell_\alpha+\delta} - \left(\frac{c}{n}\right)^{\ell_\alpha+\ell_\beta}. \tag{3.2}$$

(iii) If some common edge has different orientations in α and β, then $\mathbb{E}(I_\alpha I_\beta) = 0$ and

$$\operatorname{Cov}(I_\alpha, I_\beta) = -\,\mathbb{E}\, I_\alpha\, \mathbb{E}\, I_\beta = -\left(\frac{c}{n}\right)^{\ell_\alpha+\ell_\beta}. \tag{3.3}$$

We denote the falling factorials by $(n)_\ell := n(n-1)\cdots(n-\ell+1)$. Note that the total number of paths of length ℓ in Γ_{vw} is $(n-2)_{\ell-1} := (n-2)\cdots(n-\ell)$, since the path is determined by choosing $\ell-1$ internal vertices in order, and all vertices are distinct.

Lemma 3.1. $\operatorname{Cov}(I'_A, J_B) = O(n^{-99})$ *and* $\operatorname{Cov}(J_A, I_B) = O(n^{-99})$.

Proof. J_A is the indicator of the event that there is a path in $\vec{G}(n,p)$ from a to s, and that every such path has length $> L = \log^2 n$. Thus,

$$0 \le J_A \le \sum_{\alpha\in\Gamma_{as},\, |\alpha|>L} I_\alpha$$

and thus, using (2.1) and the fact that there are $(n-2)_{\ell-1} \le n^{\ell-1}$ paths of length ℓ in Γ_{as},

$$0 \le \mathbb{E}\, J_A \le \sum_{\alpha\in\Gamma_{as},\, |\alpha|>L} \left(\frac{c}{n}\right)^{|\alpha|} \le \sum_{\ell=L}^{\infty} n^{\ell-1}\left(\frac{c}{n}\right)^{\ell} \le \sum_{\ell=L}^{\infty} c^\ell = O(c^L) = O(n^{-99}).$$

Since $J_A, I_\beta \in [0,1]$,

$$|\operatorname{Cov}(J_A, I_B)| \le \mathbb{E}(J_A I_B) + \mathbb{E}\, J_A\, \mathbb{E}\, I_B \le 2\,\mathbb{E}\, J_A = O(n^{-99}).$$

Similarly, $|\operatorname{Cov}(I'_A, J_B)| = O(n^{-99})$. □

Lemma 3.2. *Pairs of Type 1 contribute* $-\frac{1}{n^3}\frac{2c^3-c^4}{(1-c)^2} + O(\frac{1}{n^4})$ *to the covariance* $\operatorname{Cov}(X_A, X_B)$.

Proof. Let the path α from a to s consist of $i+1$ edges, where the last edge is the first in the path β of length $j+1$ from s to b, see Figure 1. The paths must not share any more edges, but could have more common vertices. Here $i, j \ge 0$ and $i + j \ge 1$ since $a \ne b$. Let $R_{i,j}$ be the number of such pairs of paths, for given i and j. If $j \ge 1$, the paths are determined by the choice of i distinct vertices for α and then $j-1$ distinct vertices for β; if $j = 0$, then $i \ge 1$ and the paths are determined by the choice of $i-1$ distinct vertices for α. Order is important so, for $i, j \le L$, with a minor modification if $j = 0$,

$$(n-2)_i \cdot (n-3)_{j-1} \ge R_{i,j} \ge (n-2)_{i+j-1},$$

Thus $R_{i,j} = n^{i+j-1}\Bigl(1+O\Bigl(\frac{(i+j)^2}{n}\Bigr)\Bigr)$ and summing over all such pairs (α,β) gives by (3.3) a contribution to $\operatorname{Cov}(X_A,X_B)$ of

$$\begin{aligned}
-\sum_{i+j\geq 1} R_{i,j}\Bigl(\frac{c}{n}\Bigr)^{i+j+2} &= -\sum_{\substack{i+j\geq 1\\ i,j\leq L}} n^{i+j-1}\Bigl(1+O\Bigl(\frac{(i+j)^2}{n}\Bigr)\Bigr)\Bigl(\frac{c}{n}\Bigr)^{i+j+2}\\
&= -\sum_{\substack{i+j\geq 1\\ i,j\leq L}} c^{i+j+2}n^{-3} + \sum_{i+j\geq 1} O\bigl((i+j)^2\bigr)c^{i+j+2}n^{-4}\\
&= -n^{-3}\Bigl(2\sum_{j\geq 1} c^{j+2} + \sum_{i,j\geq 1} c^{i+j+2} + O(c^L)\Bigr) + O\bigl(n^{-4}\bigr)\\
&= -n^{-3}\left(\frac{2c^3}{1-c}+\frac{c^4}{(1-c)^2}\right) + O\bigl(n^{-4}\bigr) = -n^{-3}\cdot\frac{2c^3-c^4}{(1-c)^2} + O\bigl(n^{-4}\bigr). \qquad \square
\end{aligned}$$

Lemma 3.3. *Type 2 pairs contribute* $\frac{1}{n^3}\cdot\frac{c^3}{(1-c)^5}+O\bigl(\frac{1}{n^4}\bigr)$ *to the covariance* $\operatorname{Cov}(X_A,X_B)$.

Proof. A pair (α,β) of paths of Type 2 must contain a directed cycle containing s, from which there are $m\geq 1$ edges to a vertex x to which there is a directed path of length $i\geq 0$ from a. The cycle continues from x with $k\geq 1$ edges to a vertex y, which connects to b via $j\geq 0$ edges. The cycle is completed by $l\geq 1$ edges from y to s, see Figure 2. By (3.2), then

$$\operatorname{Cov}(I_\alpha,I_\beta) = \Bigl(\frac{c}{n}\Bigr)^{i+j+k+l+m}\left(1-\Bigl(\frac{c}{n}\Bigr)^k\right). \tag{3.4}$$

Let $R_{i,j,k,l,m}$ be the number of such pairs (α,β) with given i,j,k,l,m. The path α is determined by $i+k+l-1$ distinct vertices and given α, if $j\geq 1$, then the path β is determined by choosing $m+j-2$ vertices; if $j=0$ then b lies on α, so α is determined by choosing $i+k+l-2$ vertices, and then β is determined by choosing $m-1$ further vertices. Reasoning as in the proof of Lemma 3.2 we have

$$R_{i,j,k,l,m} = n^{i+j+k+l+m-3}\Bigl(1+O\Bigl(\frac{(i+j+k+l+m)^2}{n}\Bigr)\Bigr).$$

Due to our cut-off, we have to have $i+k+l\leq L$ and $j+k+m\leq L$, but we may for simplicity here allow also paths α,β with lengths larger than L; the contribution below from pairs with such α or β is $O(c^L)=O(n^{-99})$. Summing over all possible configurations gives

$$\begin{aligned}
\sum_{i,j\geq 0,\,k,l,m\geq 1} & R_{i,j,k,l,m}\Bigl(\frac{c}{n}\Bigr)^{i+j+k+l+m}\cdot\left(1-\Bigl(\frac{c}{n}\Bigr)^k\right)\\
&= \frac{1}{n^3}\cdot\sum_{i,j\geq 0,\,k,l,m\geq 1} c^{i+j+k+l+m}\cdot\left(1-\Bigl(\frac{c}{n}\Bigr)^k\right) + O\Bigl(\frac{1}{n^4}\Bigr)\\
&= \frac{1}{n^3}\cdot\frac{c^3}{(1-c)^5} + O\Bigl(\frac{1}{n^4}\Bigr). \qquad \square
\end{aligned}$$

Lemma 3.4. *The sum* $\sum|\operatorname{Cov}(I_\alpha,I_\beta)|$ *over all pairs* (α,β) *with* $\alpha\in\Gamma^L_{as}$, $\beta\in\Gamma^L_{sb}$ *and* (α,β) *not of Type 1 or 2 is* $O\bigl(n^{-4}\bigr)$.

Proof. Consider pairs (α,β) with some given ℓ_α,δ,μ. The path α, which has $\ell_\alpha-1$ interior vertices, may be chosen in $\le n^{\ell_\alpha-1}$ ways. The 2μ endvertices of the μ subpaths of $\beta\setminus\alpha$ are either b or lie on α, and given α, these may be chosen (in order) in $\le(\ell_\alpha+2)^{2\mu}$ ways. The $\delta-\mu$ internal vertices in the subpaths can be chosen in $\le n^{\delta-\mu}$ ways. They can be distributed in $\binom{\delta-1}{\mu-1}$ (interpreted as 1 if $\mu=\delta=0$) ways over the subpaths. The path β is determined by these endvertices, the sequence of $\delta-\mu$ interior vertices in the subpaths between these endvertices and which vertices belong to which subpath; hence the total number of choices of β is $\le\binom{\delta-1}{\mu-1}(\ell_\alpha+2)^{2\mu}n^{\delta-\mu}$.

For each such pair (α,β), we have by (3.1)–(3.3) $|\mathrm{Cov}(I_\alpha,I_\beta)|\le(c/n)^{\ell_\alpha+\delta}$. Consequently, the total contribution to $\sum|\mathrm{Cov}(I_\alpha,I_\beta)|$ from the paths with given ℓ_α,δ,μ is at most

$$\binom{\delta-1}{\mu-1}(\ell_\alpha+2)^{2\mu}n^{\ell_\alpha-1+\delta-\mu}\left(\frac{c}{n}\right)^{\ell_\alpha+\delta}=\binom{\delta-1}{\mu-1}(\ell_\alpha+2)^{2\mu}c^{\ell_\alpha+\delta}n^{-\mu-1}. \tag{3.5}$$

We consider several different cases and show that each case yields a contribution $O\big(n^{-4}\big)$, noting that we may assume that $\ell_\beta>\delta$, since otherwise α and β are edge-disjoint, and thus $\mathrm{Cov}(I_\alpha,I_\beta)=0$ by (3.1).

(i) $\mu\ge4$: Using that $\binom{\delta-1}{\mu-1}\le\delta^{\mu-1}\le L^\mu$, and summing (3.5) over $\delta\ge0$ and $\ell_\alpha\le L$, yields for a fixed μ a contribution

$$\le(L+2)^{3\mu}(1-c)^{-2}n^{-\mu-1}, \tag{3.6}$$

and the sum of these for $\mu\ge4$ is

$$O\big(L^{12}n^{-5}\big)=O\big(n^{-5}\log^{24}n\big)=O\big(n^{-4}\big). \tag{3.7}$$

(ii) $\mu=3$: Using that, with $\mu=3$, $\binom{\delta-1}{\mu-1}=\binom{\delta-1}{2}\le\delta^2$, and summing (3.5) over all $\ell_\alpha,\delta\ge0$ yields a contribution of at most

$$\sum_{\ell_\alpha,\delta\ge0}\delta^2(\ell_\alpha+2)^6c^{\ell_\alpha+\delta}n^{-4}\le\sum_{\ell_\alpha\ge0}(\ell_\alpha+2)^6c^{\ell_\alpha}\sum_{\delta\ge0}\delta^2c^\delta n^{-4}=O\big(n^{-4}\big). \tag{3.8}$$

It remains to consider $\mu\le2$.

(iii) $\mu=0$: In this case, $\beta\subset\alpha$, and thus $\delta=0$ and $\ell_\alpha>\ell_\beta$ (because $a\ne b$). Given ℓ_α and ℓ_β, we can choose β in $\le n^{\ell_\beta-1}$ ways and then α in $\le n^{\ell_\alpha-\ell_\beta-1}$ ways; for each choice (3.3) applies since the edges in β have opposite orientations in α, and thus the contribution to $\sum|\mathrm{Cov}(I_\alpha,I_\beta)|$ is at most

$$n^{\ell_\beta-1+\ell_\alpha-\ell_\beta-1}\left(\frac{c}{n}\right)^{\ell_\alpha+\ell_\beta}=c^{\ell_\alpha+\ell_\beta}n^{-\ell_\beta-2}. \tag{3.9}$$

If $\ell_\beta=1$, then (α,β) is of Type 1, see Figure 1 ($j=0$). Since we have excluded such pairs, we may thus assume that $\ell_\beta\ge2$. Summing (3.9) over $\ell_\alpha>\ell_\beta\ge2$ yields $O\big(n^{-4}\big)$.

(iv) $\mu\in\{1,2\}$ *and α and β have some common edge with opposite orientations*: In this case, (3.3) applies, and $\binom{\delta-1}{\mu-1}\le\delta\le\ell_\beta$. Thus, if we let $\ell_{\alpha\beta}=\ell_\beta-\delta\ge1$ be the number of common edges in α and β, then the total contribution to $\sum|\mathrm{Cov}(I_\alpha,I_\beta)|$ for given

$\ell_\alpha, \ell_\beta, \mu, \ell_{\alpha\beta}$ (which determine $\delta = \ell_\beta - \ell_{\alpha\beta}$) is at most, in analogy with (3.5) but using (3.3),

$$\ell_\beta(\ell_\alpha+2)^{2\mu} n^{\ell_\alpha-1+\delta-\mu}\Big(\frac{c}{n}\Big)^{\ell_\alpha+\ell_\beta} = (\ell_\alpha+2)^{2\mu}\ell_\beta\, c^{\ell_\alpha+\ell_\beta} n^{-1-\ell_{\alpha\beta}-\mu}. \tag{3.10}$$

For fixed μ, the sum of (3.10) over $\ell_\alpha, \ell_\beta \geq 1$ and $\ell_{\alpha\beta} \geq 3-\mu$ is $O\big(n^{-4}\big)$, so we only have to consider $1 \leq \ell_{\alpha\beta} \leq 2-\mu$. In this case we must have $\mu = 1$ and $\ell_{\alpha\beta} = 1$ (and $\binom{\delta-1}{\mu-1} = 1$); thus α and β have exactly one common edge, which is adjacent to one of the endvertices of β. If the common edge is adjacent to s, we have a pair (α,β) of Type 1, see Figure 1; we may thus assume that the common edge is not adjacent to s. Then, $\ell_\beta \geq 2$ and the common edge is adjacent to b, which implies $b \in \alpha$. Given ℓ_α, the number of paths α that pass through b is $(\ell_\alpha-1)(n-3)_{\ell_\alpha-2}$, since b may be any of the $\ell_\alpha - 1$ interior vertices. The choice of α fixes the last interior vertex of β (as the successor of b in α), and the remaining $\ell_\beta - 2$ interior vertices may be chosen in $\leq n^{\ell_\beta-2}$ ways. The total contribution from this case is thus at most

$$(\ell_\alpha-1)n^{\ell_\alpha-2+\ell_\beta-2}\Big(\frac{c}{n}\Big)^{\ell_\alpha+\ell_\beta} = (\ell_\alpha-1)c^{\ell_\alpha+\ell_\beta}n^{-4}, \tag{3.11}$$

and summing over ℓ_α and ℓ_β we again obtain $O\big(n^{-4}\big)$.

(v) $\mu \in \{1,2\}$ *and all common edges in* α *and* β *have the same orientation*: The edge in β at s does not belong to α (since it would have opposite orientation there), so one of the μ subpaths of β outside α begins at s. If $\mu = 1$, or if $\mu = 2$ and $b \notin \alpha$, then (α,β) is of Type 2, see Figure 2 ($j = 0$ and $j \geq 1$, respectively). We may thus assume that $\mu = 2$ and $b \in \alpha$. As in case (iv), given ℓ_α, we may choose α in $(\ell_\alpha-1)(n-3)_{\ell_\alpha-2} \leq \ell_\alpha n^{\ell_\alpha-2}$ ways. The $\mu = 2$ subpaths of β outside α have 4 endvertices belonging to α; one is s and the others may be chosen in $\leq \ell_\alpha^3$ ways. For any such choice, the remaining $\delta - 2$ vertices of β may be chosen in $\leq n^{\delta-2}$ ways. The total contribution for given ℓ_α and δ is thus, using (3.2), at most

$$\ell_\alpha^4 n^{\ell_\alpha-2+\delta-2}\Big(\frac{c}{n}\Big)^{\ell_\alpha+\delta} = \ell_\alpha^4 c^{\ell_\alpha+\delta}n^{-4}, \tag{3.12}$$

and summing over all ℓ_α, δ we obtain $O\big(n^{-4}\big)$. $\square$

Lemma 3.5. *With notation as before, we have* $\operatorname{Cov}(X_A, X'_B) = \operatorname{Cov}(X'_A, X_B) = O(n^{-4})$ *and* $\operatorname{Cov}(X'_A, X'_B) = O(n^{-4})$.

Proof. We only need to consider paths in Γ^L, which is assumed throughout the proof. Define $Y_A := \binom{X_A}{2}$, the number of pairs of (distinct) paths from a to s, and similarly $Y_B := \binom{X_B}{2}$. Then $0 \leq X'_A \leq Y_A$ and $0 \leq X'_B \leq Y_B$. Let $Y'_A := Y_A - X'_A$ and $Y'_B := Y_B - X'_B$. Then $Y'_A = 0$ unless $X_A \geq 3$.

Further, let $Z_A := \binom{X_A}{3}$, the number of triples of (distinct) paths from a to s. Then $0 \leq Y'_A \leq Z_A$.

To show that $\operatorname{Cov}(X_A, X'_B) = \operatorname{Cov}(X'_A, X_B) = O(n^{-4})$, we write $\operatorname{Cov}(X'_A, X_B) = \operatorname{Cov}(Y_A - Y'_A, X_B) = \operatorname{Cov}(Y_A, X_B) - \operatorname{Cov}(Y'_A, X_B)$. Here, $\operatorname{Cov}(Y'_A, X_B) = \mathbb{E}(Y'_A X_B) - \mathbb{E}(Y'_A)\cdot\mathbb{E}(X_B)$, where $\mathbb{E}(Y'_A X_B) \leq \mathbb{E}(Z_A X_B)$, which we will show is $O(n^{-4})$. Further we will show that $\mathbb{E}(X_A) = \mathbb{E}(X_B) = O(n^{-1})$ and that $\mathbb{E}(Y'_A) \leq \mathbb{E}(Z_A) = O(n^{-3})$, so that

$\mathrm{Cov}(Y'_A, X_B) = O(n^{-4})$. Finally we will show that $\mathrm{Cov}(Y_A, X_B) = O(n^{-4})$ finishing the proof of the first part of the lemma.

For the second part we write $\mathrm{Cov}(X'_A, X'_B) = \mathbb{E}(X'_A X'_B) - \mathbb{E}(X'_A) \cdot \mathbb{E}(X'_B)$. We prove that $\mathbb{E}(X'_A X'_B) \le \mathbb{E}(Y_A Y_B) = O(n^{-4})$ and that $\mathbb{E}(X'_A) = \mathbb{E}(X'_B) \le \mathbb{E}(Y_A) = O(n^{-2})$, which finishes the proof.

(i) $\mathbb{E}(X_A) = O(n^{-1})$:

Let α denote an arbitrary path from a to s (in Γ^L) with length $l \ge 1$. Then,

$$\mathbb{E}(X_A) = \mathbb{E}\left(\sum_{\alpha} I_\alpha\right) = \sum_{\alpha} \mathbb{E}(I_\alpha) \le \sum_{l=1}^{L} n^{l-1}\left(\frac{c}{n}\right)^l \le \frac{c}{1-c} \cdot n^{-1} = O(n^{-1}).$$

(ii) $\mathbb{E}(Y_A) = O(n^{-2})$:

Let α_1 and α_2, with lengths l_1 and l_2 be two distinct paths from a to s. Further, let $\delta = |\alpha_2 \setminus \alpha_1|$ be the number of edges in α_2 not in α_1, which form $\mu > 0$ subpaths of α_2 with no interior vertices in common with α_1. The number of choices for α_2 is (compare the proof of Lemma 3.4) at most $n^{\delta-\mu}(l_1+1)^{2\mu}\binom{\delta-1}{\mu-1}$, which gives

$$\begin{aligned}\mathbb{E}(Y_A) = \sum_{\alpha_1 \ne \alpha_2} \mathbb{E}(I_{\alpha_1} I_{\alpha_2}) &\le \sum_{l_1,\delta,\mu} n^{l_1-1}\left(\frac{c}{n}\right)^{l_1} n^{\delta-\mu}(l_1+1)^{2\mu}\binom{\delta-1}{\mu-1} \cdot \left(\frac{c}{n}\right)^{\delta} \\ &= \sum_{l_1,\delta,\mu} n^{-\mu-1}(l_1+1)^{2\mu} c^{l_1+\delta}\binom{\delta-1}{\mu-1}.\end{aligned}$$

Case 1: $\mu \ge 2$.
Here, $(l_1+1)^{2\mu} \le (L+1)^{2\mu}$, $\binom{\delta-1}{\mu-1} \le (\delta-1)^{\mu-1} \le \delta^\mu \le L^\mu$, so that the terms are at most $n^{-\mu-1}c^{l_1+\delta}(L+1)^{3\mu}$. Summing over l_1 and δ gives at most $\frac{c^2}{(1-c)^2}(L+1)^{3\mu}n^{-\mu-1}$, which summed for $\mu \ge 2$ is $O(L^6 n^{-3}) = O(n^{-3}\log^{12} n) = O(n^{-2})$.
Case 2: $\mu = 1$.
Here, $\binom{\delta-1}{\mu-1} = 1$, and

$$\sum_{l_1,\delta} \mathbb{E}(I_{\alpha_1} I_{\alpha_2}) \le n^{-2} \sum_{l_1 \ge 1}(l_1+1)^2 c^{l_1} \sum_{\delta \ge 1} c^\delta = O(n^{-2}).$$

(iii) $\mathbb{E}(Z_A) = O(n^{-3})$:

We have

$$\mathbb{E}(Z_A) = \sum_{\alpha_1,\alpha_2,\alpha_3} \mathbb{E}(I_{\alpha_1} I_{\alpha_2} I_{\alpha_3}),$$

where α_1, α_2 and α_3 denote three distinct paths from a to s.

Let l_1 denote the length of α_1, let $\delta_2 = |\alpha_2 \setminus \alpha_1|$ be the number of edges in α_2 not in α_1 forming $\mu_2 > 0$ subpaths of α_2 intersecting α_1 only at the endvertices, and let $\delta_3 = |\alpha_3 \setminus (\alpha_1 \cup \alpha_2)|$ be the number of edges in α_3 not in α_1 or α_2 forming $\mu_3 \ge 0$ subpaths of α_3 whose interior vertices are not in α_1 or α_2. Note that $\mu_3 = 0$ is possible if $\mu_2 \ge 2$, as then α_3 can be formed by one part from α_1 and one part from α_2; however, if $\mu_2 = 1$ then $\mu_3 \ge 1$. Hence, $\mu_2 + \mu_3 \ge 2$.

If all common edges of the three paths have the same direction, $\mathbb{E}(I_{\alpha_1}I_{\alpha_2}I_{\alpha_3}) = \left(\frac{c}{n}\right)^{l_1+\delta_2+\delta_3}$, otherwise it is 0, so we need only consider paths with the same direction. The number of choices for α_2 is, as in (ii), at most $n^{\delta_2-\mu_2}\cdot(l_1+1)^{2\mu_2}\cdot\binom{\delta_2-1}{\mu_2-1}$ and the number of choices for α_3 is at most $n^{\delta_3-\mu_3}\cdot(l_1+\delta_2-\mu_2+1)^{2\mu_3}\cdot\binom{\delta_3-1}{\mu_3-1}\cdot 2^{\mu_2}$, where the last factor is an upper bound for the possible number of choices between segments of α_1 and α_2. Thus, with summation over $l_1\geq 1, \delta_2\geq\mu_2\geq 1, \delta_3\geq\mu_3\geq 0$, with $\mu_2+\mu_3\geq 2$,

(3.13)
$$\begin{aligned}\sum\mathbb{E}(I_{\alpha_1}I_{\alpha_2}I_{\alpha_3}) &\leq \sum n^{l_1-1}\cdot n^{\delta_2-\mu_2}\cdot(l_1+1)^{2\mu_2}\cdot\tbinom{\delta_2-1}{\mu_2-1}\cdot\\ &\quad\cdot n^{\delta_3-\mu_3}\cdot(l_1+\delta_2-\mu_2+1)^{2\mu_3}\cdot\tbinom{\delta_3-1}{\mu_3-1}\cdot 2^{\mu_2}\cdot\left(\frac{c}{n}\right)^{l_1+\delta_2+\delta_3}\\ &= \sum n^{-\mu_2-\mu_3-1}\cdot(l_1+1)^{2\mu_2}\cdot\tbinom{\delta_2-1}{\mu_2-1}\cdot(l_1+\delta_2-\mu_2+1)^{2\mu_3}\cdot\tbinom{\delta_3-1}{\mu_3-1}\cdot 2^{\mu_2}\cdot c^{l_1+\delta_2+\delta_3}.\end{aligned}$$

Case 1: $\mu_2+\mu_3\geq 3$.
Here, $(l_1+1)^{2\mu_2}\leq(L+1)^{2\mu_2}$, $\binom{\delta_2-1}{\mu_2-1}\leq L^{\mu_2}$, $(l_1+\delta_2-\mu_2+1)^{2\mu_3}\leq(2L+1)^{2\mu_3}\leq(L+1)^{3\mu_3}$ (assuming as we may $L\geq 4$), $\binom{\delta_3-1}{\mu_3-1}\leq L^{\mu_3}$ and $2^{\mu_2}\leq L^{\mu_2}$, so that the sum over l_1,δ_2,δ_3 is at most

$$n^{-\mu_2-\mu_3-1}\cdot(L+1)^{4\mu_2+4\mu_3}\cdot\sum c^{l_1+\delta_2+\delta_3}\leq(1-c)^{-3}\cdot n^{-\mu_2-\mu_3-1}\cdot(L+1)^{4(\mu_2+\mu_3)}. \tag{3.14}$$

Summing over μ_2 and μ_3, with $\mu_2+\mu_3\geq 3$ gives

$$O(n^{-4}\cdot L^{12}) = O(n^{-4}\log^{24}n) = O(n^{-3}).$$

Case 2: $\mu_2+\mu_3=2$.
Here, $(\mu_2,\mu_3)\in\{(2,0),(1,1)\}$, so that $(l_1+1)^{2\mu_2}\leq(l_1+1)^4$, $\binom{\delta_2-1}{\mu_2-1}\leq\delta_2$, $(l_1+\delta_2-\mu_2+1)^{2\mu_3}\leq(l_1+\delta_2)^2$, $\binom{\delta_3-1}{\mu_3-1}=1$ and $2^{\mu_2}\leq 4$, so that summing over l_1,δ_2,δ_3 and $\mu_2+\mu_3=2$ gives at most

$$2\cdot 4\cdot n^{-3}\cdot\sum_{l_1,\delta_2,\delta_3}(l_1+1)^4\cdot\delta_2\cdot(l_1+\delta_2)^2\cdot c^{l_1+\delta_2+\delta_3} = O(n^{-3}).$$

(iv) $\mathbb{E}(Z_A\cdot X_B) = O(n^{-4})$:

$\mathbb{E}(Z_A\cdot X_B) = \sum\mathbb{E}(I_{\alpha_1}I_{\alpha_2}I_{\alpha_3}I_\beta)$, where α_1, α_2 and α_3 are three distinct paths from a to s and β is a path from s to b. We need only consider paths where all common edges have the same direction, as $\mathbb{E}(I_{\alpha_1}I_{\alpha_2}I_{\alpha_3}I_\beta)=0$ otherwise.

As in (iii) the three α paths are described by $l_1,\delta_2,\mu_2,\delta_3,\mu_3$. Let $\delta_4 := |\beta\setminus(\alpha_1\cup\alpha_2\cup\alpha_3)|$ be the number of edges in β, not in any of the α paths, and let these form μ_4 subpaths of β whose endvertices lie on $\alpha_1,\alpha_2,\alpha_3$ but share no other vertices with those paths. The number of choices for the α paths are the same as in (iii) and given those, and δ_4,μ_4, the β path can be chosen in at most $n^{\delta_4-\mu_4}\cdot(l_1+\delta_2-\mu_2+\delta_3-\mu_3+1)^{2\mu_4}\cdot\binom{\delta_4-1}{\mu_4-1}\cdot 3^{2(\mu_2+\mu_3)}$ ways, where the last factor is a crude upper bound for the number of ways β can choose different sections from the α paths, as there are at most $2(\mu_2+\mu_3)$ vertices where a choice can be made and there are at most 3 possible choices at each of these. Clearly, $\mathbb{E}(I_{\alpha_1}I_{\alpha_2}I_{\alpha_3}I_\beta) = (\frac{c}{n})^{l_1+\delta_2+\delta_3+\delta_4}$ since all common edges have the same direction.

Note that $\mu_4 \ge 1$ for non-zero terms as otherwise the first edge in β from s would be the last edge in one of the α paths, and therefore would have opposite direction. Further, $\mu_2 \ge 1$, $\mu_3 \ge 0$, but $\mu_2+\mu_3 \ge 2$ as $\mu_2 = 1, \mu_2 = 0$ would imply that $\alpha_3 = \alpha_1$ or $\alpha_3 = \alpha_2$.

Summing over $l_1 \ge 1$, $\mu_2 \ge 1$, $\delta_2 \ge \mu_2$, $\mu_3 \ge 0$, $\delta_3 \ge \mu_3$, $\mu_4 \ge 1$ and $\delta_4 \ge \mu_4$ with $\mu_2+\mu_3 \ge 2$ gives at most

(3.15)

$$\begin{aligned}&\sum n^{l_1-1}\cdot n^{\delta_2-\mu_2}\cdot(l_1+1)^{2\mu_2}\cdot\tbinom{\delta_2-1}{\mu_2-1}\cdot n^{\delta_3-\mu_3}\cdot(l_1+\delta_2-\mu_2+1)^{2\mu_3}\cdot\tbinom{\delta_3-1}{\mu_3-1}\cdot 2^{\mu_2}\cdot\\ &\quad\cdot n^{\delta_4-\mu_4}\cdot(l_1+\delta_2-\mu_2+\delta_3-\mu_3+1)^{2\mu_4}\cdot\tbinom{\delta_4-1}{\mu_4-1}\cdot 3^{2(\mu_2+\mu_3)}\cdot\left(\frac{c}{n}\right)^{l_1+\delta_2+\delta_3+\delta_4}\\ &\quad=\sum n^{-\mu_2-\mu_3-\mu_4-1}\cdot(l_1+1)^{2\mu_2}\cdot\tbinom{\delta_2-1}{\mu_2-1}\cdot(l_1+\delta_2-\mu_2+1)^{2\mu_3}\cdot\tbinom{\delta_3-1}{\mu_3-1}\cdot 2^{\mu_2}\cdot\\ &\quad\cdot(l_1+\delta_2-\mu_2+\delta_3-\mu_3+1)^{2\mu_4}\cdot\tbinom{\delta_4-1}{\mu_4-1}\cdot 3^{2(\mu_2+\mu_3)}\cdot c^{l_1+\delta_2+\delta_3+\delta_4}.\end{aligned}$$

Case 1: $\mu_2+\mu_3+\mu_4 \ge 4$.
Here, using the same type of estimates as in (iii) and summing over $l_1, \delta_2, \delta_3, \delta_4$ gives at most

$$n^{-\mu_2-\mu_3-\mu_4-1}\cdot(L+1)^{7\mu_2+7\mu_3+4\mu_4}\sum c^{l_1+\delta_2+\delta_3+\delta_4} \le (1-c)^{-4}n^{-\mu_2-\mu_3-\mu_4-1}\cdot(L+1)^{7(\mu_2+\mu_3+\mu_4)},$$

which summed over $\mu_2+\mu_3+\mu_4 \ge 4$ is

$$O(n^{-5}\cdot L^{28}) = O(n^{-5}\cdot\log^{56} n) = O(n^{-4}).$$

Case 2: $\mu_2+\mu_3+\mu_4 = 3$.
Here, $(\mu_2,\mu_3,\mu_4) \in \{(2,0,1),(1,1,1)\}$ so that $(l_1+1)^{2\mu_2} \le (l_1+1)^4$, $\binom{\delta_2-1}{\mu_2-1} \le \delta_2$, $(l_1+\delta_2-\mu_2+1)^{2\mu_3} \le (l_1+\delta_2)^2$, $\binom{\delta_3-1}{\mu_3-1} = \binom{\delta_4-1}{\mu_4-1} = 1$, $2^{\mu_2} \le 4$, $(l_1+\delta_2-\mu_2+\delta_3-\mu_3+1)^{2\mu_4} \le (l_1+\delta_2+\delta_3)^2$ and $3^{2(\mu_2+\mu_3)} = 3^4 = 81$, so that the sum over $l_1,\delta_2,\delta_3,\delta_4$ is finite and the total contribution is $O(n^{-4})$.

(v) $\mathbb{E}(Y_A\cdot Y_B) = O(n^{-4})$:

$\mathbb{E}(Y_A\cdot Y_B) = \sum\mathbb{E}(I_{\alpha_1}I_{\alpha_2}I_{\beta_3}I_{\beta_4})$, where α_1 and α_2 are two distinct paths from a to s and β_3 and β_4 are two distinct paths from s to b. As above, we need only consider paths where all common edges have the same direction. As before, α_1 and α_2 are described by $l_1 = |\alpha_1| \ge 1$, $\delta_2 = |\alpha_2\setminus\alpha_1| \ge 1$, the number of edges in α_2 not in α_1, and $\mu_2 \ge 1$, the number of subpaths they form that intersect α_1 in (and only in) the endvertices. Then β_3 is described by $\delta_3 = |\beta_3\setminus(\alpha_1\cup\alpha_2)|$, the number of edges in β_3 not in α_1 or α_2, and μ_3, the number of subpaths they form with no interior vertices in common with α_1,α_2. Similarly, β_4 is described by $\delta_4 = |\beta_3\setminus(\alpha_1\cup\alpha_2\cup\beta_3)| \ge 0$, the number of edges in β_4 not in α_1, α_2 or β_3 and $\mu_4 \ge 0$, the number of subpaths they form which intersect $\alpha_1,\alpha_2,\beta_3$ in (and only in) the endvertices. Note that $\mu_3 \ge 1$ for every non-zero term, as otherwise the first edge in β_3 from s would be the last edge in one of the α paths, and therefore would have opposite direction.

The number of choices for the α paths are the same as in (ii) and given those, and $\delta_3,\mu_3,\delta_4,\mu_4$, the β paths can be chosen in at most $n^{\delta_3-\mu_3}\cdot(l_1+\delta_2-\mu_2+1)^{2\mu_3}\cdot\binom{\delta_3-1}{\mu_3-1}\cdot 2^{\mu_2}\cdot n^{\delta_4-\mu_4}\cdot(l_1+\delta_2-\mu_2+\delta_3-\mu_3+1)^{2\mu_4}\cdot\binom{\delta_4-1}{\mu_4-1}\cdot 3^{2(\mu_2+\mu_3)}$, where the last factor is an upper bound for the number of ways β_4 can choose different sections from the α paths and β_3.

When all common edges have the same direction, $\mathbb{E}(I_{\alpha_1} I_{\alpha_2} I_{\beta_3} I_{\beta_4}) = (\frac{c}{n})^{l_1+\delta_2+\delta_3+\delta_4}$. Summing over $l_1 \geq 1$, $\mu_2 \geq 1$, $\delta_2 \geq \mu_2$, $\mu_3 \geq 1$, $\delta_3 \geq \mu_3$, $\mu_4 \geq 0$ and $\delta_4 \geq \mu_4$ gives at most

$$\begin{aligned}
&\sum n^{l_1-1} \cdot n^{\delta_2-\mu_2} \cdot (l_1+1)^{2\mu_2} \cdot \tbinom{\delta_2-1}{\mu_2-1} \cdot n^{\delta_3-\mu_3} \cdot (l_1+\delta_2-\mu_2+1)^{2\mu_3} \cdot \tbinom{\delta_3-1}{\mu_3-1} \cdot 2^{\mu_2} \cdot \\
&\quad \cdot n^{\delta_4-\mu_4} \cdot (l_1+\delta_2-\mu_2+\delta_3-\mu_3+1)^{2\mu_4} \cdot \tbinom{\delta_4-1}{\mu_4-1} \cdot 3^{2(\mu_2+\mu_3)} \cdot \left(\frac{c}{n}\right)^{l_1+\delta_2+\delta_3+\delta_4} \\
&\quad = \sum n^{-\mu_2-\mu_3-\mu_4-1} \cdot (l_1+1)^{2\mu_2} \cdot \tbinom{\delta_2-1}{\mu_2-1} \cdot (l_1+\delta_2-\mu_2+1)^{2\mu_3} \cdot \tbinom{\delta_3-1}{\mu_3-1} \cdot 2^{\mu_2} \cdot \\
&\quad \cdot (l_1+\delta_2-\mu_2+\delta_3-\mu_3+1)^{2\mu_4} \cdot \tbinom{\delta_4-1}{\mu_4-1} \cdot 3^{2(\mu_2+\mu_3)} \cdot c^{l_1+\delta_2+\delta_3+\delta_4}.
\end{aligned}$$

We sum the same terms as in (3.15), so the sum over all terms with $\mu_4 \geq 1$ is $O(n^{-4})$ by the estimates in part (iv). Hence it suffices to consider the terms with $\mu_4 = 0$ and thus $\delta_4 = 0$.

Case 1: $\mu_2 + \mu_3 \geq 4$, $\mu_4 = 0$.
Here, each term is $3^{2(\mu_2+\mu_3)}$ times the corresponding term in (3.13). Hence, the estimates in (iii) show that, cf. (3.14), summing over l_1, δ_2, δ_3 gives at most

$$(1-c)^{-3} n^{-\mu_2-\mu_3-1} \cdot (L+1)^{6(\mu_2+\mu_3)},$$

which summed over $\mu_2 + \mu_3 \geq 4$ is

$$O(n^{-5} \cdot L^{24}) = O(n^{-5} \cdot \log^{48} n) = O(n^{-4}).$$

Case 2: $\mu_2 + \mu_3 = 3$, $\mu_4 = 0$.
Here, $\mu_2, \mu_3 \leq 2$ so that $(l_1+1)^{2\mu_2} \leq (l_1+1)^4$, $\binom{\delta_2-1}{\mu_2-1} \leq \delta_2$, $(l_1+\delta_2-\mu_2+1)^{2\mu_3} = (l_1+\delta_2)^4$, $\binom{\delta_3-1}{\mu_3-1} \leq \delta_3$, $2^{\mu_2} \leq 4$, and $3^{2(\mu_2+\mu_3)} = 3^6 = 729$, so that the sum over l_1, δ_2, δ_3 is $O(n^{-\mu_2-\mu_3-1})$ and the contribution is $O(n^{-4})$.

Case 3: $\mu_2 + \mu_3 = 2$, $\mu_4 = 0$.
This can only occur if $\mu_2 = \mu_3 = 1$. Thus, β_3 starts with an edge not in any of the α paths and, as this is its only excursion it must end up at one of the α paths and follow it to b (if β_3 were to go straight to b without coinciding with any of the α paths then β_4 would have to do the same, so that $\beta_3 = \beta_4$). β_4 must start as β_3 until it encounters an α path and must have the possibility to chose a different path to b than β_3 along the α paths. This means that both α paths must pass through b and that they only differ somewhere between a and b. Thus, see Figure 3, there must be three vertices x (possibly $x = a$), y (possibly $y = x$) and z (possibly $z = b$) between a and b, so that both α paths pass in order a, x, y, z, b, s, and both β paths pass in order s, x, y, z, b. Both the two α paths and the two β paths follow different subpaths between y and z. Let the number of edges between a and x be $i \geq 0$, between x and y be $j \geq 0$, between y and z be $k \geq 1$ and $l \geq 1$ for the two possibilities (with $k + l \geq 3$), between z and b be $m \geq 0$, between s and x be $r \geq 1$ and between b and s be $t \geq 1$.

Then, $\mathbb{E}(I_{\alpha_1} I_{\alpha_2} I_{\beta_3} I_{\beta_4}) = \left(\frac{c}{n}\right)^{i+j+k+l+m+r+t}$ and the number of possibilities is at most $2n^{i+j+k+l+m+r+t-4}$, so that the sum over i, j, k, l, m, r, t is $O(n^{-4})$.

(vi) $\mathrm{Cov}(Y_A, X_B) = O(n^{-4})$:

$$|\mathrm{Cov}(Y_A, X_B)| = |\sum_{\alpha_1 \neq \alpha_2} \sum_{\beta} \mathrm{Cov}(I_{\alpha_1} \cdot I_{\alpha_2}, I_\beta)| \leq \sum_{\alpha_1 \neq \alpha_2} \sum_{\beta} |\mathrm{Cov}(I_{\alpha_1} \cdot I_{\alpha_2}, I_\beta)|,$$

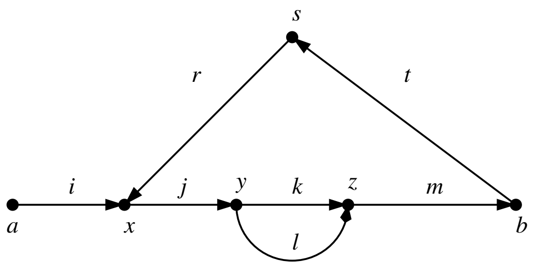

FIGURE 3. Configurations for *Case 3* of **(v)**: $\mu_2+\mu_3+\mu_4=2$.

where

$$\mathrm{Cov}(I_{\alpha_1}\cdot I_{\alpha_2}, I_\beta)=\mathbb{E}(I_{\alpha_1}\cdot I_{\alpha_2}\cdot I_\beta)-\mathbb{E}(I_{\alpha_1}\cdot I_{\alpha_2})\cdot\mathbb{E}(I_\beta),$$

which is 0 if α_1 and α_2 have a common edge with opposite directions, or if β has no edge in common with the α paths.

Let as above α_1 have length l_1, α_2 have δ_2 edges not in α_1 forming μ_2 subpaths of α_2 intersecting α_1 in (and only in) the endvertices. Let also β have length l_β with δ_3 edges not in α_1 or α_2 forming μ_3 subpaths of β intersecting α_1,α_2 in (and only in) the endvertices. Then, if all common edges of β and $\alpha_1\cup\alpha_2$ have the same direction,

$$|\mathrm{Cov}(I_{\alpha_1}\cdot I_{\alpha_2}, I_\beta)|=\left|\left(\frac{c}{n}\right)^{l_1+\delta_2+\delta_3}-\left(\frac{c}{n}\right)^{l_1+\delta_2+l_\beta}\right|\le\left(\frac{c}{n}\right)^{l_1+\delta_2+\delta_3},$$

and if β has at least one common edge in opposite direction,

$$|\mathrm{Cov}(I_{\alpha_1}\cdot I_{\alpha_2}, I_\beta)|=\left(\frac{c}{n}\right)^{l_1+\delta_2+l_\beta}\le\left(\frac{c}{n}\right)^{l_1+\delta_2+\delta_3.}$$

The number of ways of choosing α_1, α_2 and β is at most, as in (iii) above,

$$n^{l_1-1}\cdot n^{\delta_2-\mu_2}\cdot(l_1+1)^{2\mu_2}\cdot\tbinom{\delta_2-1}{\mu_2-1}\cdot n^{\delta_3-\mu_3}\cdot(l_1+\delta_2-\mu_2+1)^{2\mu_3}\cdot\tbinom{\delta_3-1}{\mu_3-1}\cdot 4^{2\mu_2}.$$

The last factor is $4^{2\mu_2}$ in this case as β can have opposite direction in the common subpaths. If there is a crossing between α_1 and α_2 there may be 4 choices for β and there are at most $2\mu_2$ such vertices. Thus,

$$\begin{aligned}\sum_{\alpha_1\neq\alpha_2}\sum_{\beta}|\mathrm{Cov}(I_{\alpha_1}\cdot I_{\alpha_2}, I_\beta)|&\le\sum_{l_1,\mu_2,\delta_2,\mu_2,\delta_3}n^{l_1-1}\cdot n^{\delta_2-\mu_2}\cdot(l_1+1)^{2\mu_2}\cdot\tbinom{\delta_2-1}{\mu_2-1}\cdot\\&\cdot n^{\delta_3-\mu_3}\cdot(l_1+\delta_2-\mu_2+1)^{2\mu_3}\cdot\tbinom{\delta_3-1}{\mu_3-1}\cdot 4^{2\mu_2}\cdot\left(\frac{c}{n}\right)^{l_1+\delta_2+\delta_3}\\&\le\sum n^{-\mu_2-\mu_3-1}\cdot(l_1+1)^{2\mu_2}\cdot\tbinom{\delta_2-1}{\mu_2-1}\cdot(l_1+\delta_2-\mu_2+1)^{2\mu_3}\cdot\tbinom{\delta_3-1}{\mu_3-1}\cdot 4^{2\mu_2}\cdot c^{l_1+\delta_2+\delta_3}.\end{aligned}$$

Here, $l_1\ge 1$, $\mu_2\ge 1$, $\delta_2\ge\mu_2$, $\mu_3\ge 0$ and $\delta_3\ge\mu_3$. Note that the terms in the final sum are the same as in (3.13), except that 2^{μ_2} is replaced by $4^{2\mu_2}$.

Case 1: $\mu_2+\mu_3\ge 4$.

Here, using the same estimates as in (iii), see (3.14), the sum over l_1,δ_2,δ_3 is, for $L\ge 16$, at most

$$(1-c)^{-3}\cdot n^{-\mu_2-\mu_3-1}\cdot(L+1)^{4(\mu_2+\mu_3)}.$$

Summing over $\mu_2+\mu_3\ge 4$ gives $O(n^{-5}\cdot L^{16})=O(n^{-5}\log^{32}n)=O(n^{-4})$.

Case 2: $\mu_2 + \mu_3 = 3$.
Here, $(\mu_2, \mu_3) \in \{(3,0),(2,1),(1,2)\}$ and $(l_1+1)^{2\mu_2} \le (l_1+1)^6$, $\binom{\delta_2-1}{\mu_2-1} \le \delta_2^2$, $(l_1+\delta_2-\mu_2+1)^{2\mu_3} \le (l_1+\delta_2)^4$, $\binom{\delta_3-1}{\mu_3-1} \le \delta_3$ and $4^{2\mu_2} \le 4^6 = 4096$. Summing over $l_1, \delta_2, \mu_2, \delta_3, \mu_3$ gives at most

$$3n^{-4} \sum_{l_1,\delta_2,\delta_3} 4096 \cdot (l_1+1)^6 \cdot \delta_2^2 \cdot (l_1+\delta_2)^4 \cdot \delta_3 \cdot c^{l_1+\delta_2+\delta_3} = O(n^{-4}).$$

Case 3: $\mu_2 = \mu_3 = 1$.
We need only consider the situation when β has at least one edge in common with $\alpha_1 \cup \alpha_2$, as otherwise the covariance is 0.
Subcase 3.1: At least one common edge has opposite direction.
$|\mathrm{Cov}(I_{\alpha_1} \cdot I_{\alpha_2}, I_\beta)| = c^{l_1+\delta_2+l_\beta} \cdot n^{-l_1-\delta_2-l_\beta}$. Here, $l_\beta \ge 2$, as $l_\beta = 1$ would imply that $\mu_3 = 0$. Further, $l_1+\delta_2 \ge 3$, as otherwise $\alpha_1 = \alpha_2$. Let $l_{\alpha\beta} = |\beta \cap (\alpha_1 \cup \alpha_2)| = l_\beta - \delta_3 \ge 1$. Then, estimating the number of possible choices of the paths as above,

$$\begin{aligned}
&\sum_{l_1,\delta_2,\delta_3,l_\beta} |\mathrm{Cov}(I_{\alpha_1} \cdot I_{\alpha_2}, I_\beta)| \\
&\qquad \le \sum n^{l_1-1} \cdot n^{\delta_2-1} \cdot (l_1+1)^2 \cdot n^{\delta_3-1} \cdot (l_1+\delta_2)^2 \cdot 2 \cdot c^{l_1+\delta_2+l_\beta} \cdot n^{-l_1-\delta_2-l_\beta} \\
&\qquad = 2 \cdot \sum_{l_1,\delta_2,\delta_3,l_{\alpha\beta}} (l_1+1)^2 \cdot (l_1+\delta_2)^2 \cdot c^{l_1+\delta_2+\delta_3+l_{\alpha\beta}} \cdot n^{-3-l_{\alpha\beta}} = O(n^{-4}).
\end{aligned}$$

Subcase 3.2: All common edges have the same direction.
The first edge of β, from s, must be disjoint with $\alpha_1 \cup \alpha_2$. Let β start with $i \ge 1$ disjoint steps and then join one of the α paths, α_1 say, for a further $j \ge 1$ steps to b. Further, let α_1 have $k \ge 0$ steps before joining β and ending with l steps from b to s. As before, α_2 is determined by two vertices on α_1 and $\delta_2 - 1$ exterior vertices giving at most $(l_1+1)^2 \cdot n^{\delta_2-1}$ possibilities. Further, β can join either of the α paths, and may then do an excursion along the other path, giving at most 4 possibilities. Then, as $l_1 = k+j+l$,

$$\begin{aligned}
&\sum |\mathrm{Cov}(I_{\alpha_1} \cdot I_{\alpha_2}, I_\beta)| \\
&\quad \le 4 \cdot \sum_{i\ge1} \sum_{k\ge0} \sum_{j\ge1} \sum_{l\ge1} \sum_{\delta_2\ge1} n^{i-1} \cdot n^{k+j+l-2} \cdot (l_1+1)^2 \cdot n^{\delta_2-1} \cdot \left(\frac{c}{n}\right)^{i+k+j+l+\delta_2} \\
&\quad = 4n^{-4} \cdot \sum_{i,k,j,l,\delta_2} (k+j+l+1)^2 \cdot c^{i+k+j+l+\delta_2} = O(n^{-4}).
\end{aligned}$$

Case 4: $\mu_3 = 0$, $\mu_2 \in \{1,2\}$.
$\mu_3 = 0$ implies that $\beta \subset (\alpha_1 \cup \alpha_2)$, so that the first edge in β has opposite direction in $\alpha_1 \cup \alpha_2$. Furthermore, at least one of the α paths, α_1 say, must pass through b, so that $l_1 \ge 2$. α_2 can be chosen in at most $(l_1+1)^{2\mu_2} \cdot n^{\delta_2-\mu_2}$ ways and there are at most 2^{μ_2} ways for β to choose between the α paths, giving at most $n^{l_1-2} \cdot (l_1+1)^{2\mu_2} \cdot n^{\delta_2-\mu_2} \cdot 2^{\mu_2} \le 4 \cdot (l_1+1)^4 \cdot n^{l_1+\delta_2-\mu_2-2}$ ways of choosing α_1, α_2 and β. The covariance is $-\left(\frac{c}{n}\right)^{l_1+\delta_2+l_\beta}$.

Summing over $l_1 \geq 2$, $\mu_2 = 1, 2$, $\delta_2 \geq \mu_2$ and $l_\beta \geq 1$ gives

$$\sum |\mathrm{Cov}(I_{\alpha_1} \cdot I_{\alpha_2}, I_\beta)| \leq 4 \sum (l_1+1)^4 \cdot n^{l_1+\delta_2-\mu_2-2} \cdot \left(\frac{c}{n}\right)^{l_1+\delta_2+l_\beta}$$
$$= 4 \sum (l_1+1)^4 \cdot c^{l_1+\delta_2+l_\beta} \cdot n^{-\mu_2-l_\beta-2} = O(n^{-4}),$$

which finishes the proof. □

References

[1] Sven Erick Alm and Svante Linusson, A counter-intuitive correlation in a random tournament, *Combin. Probab. Comput.* **20** (2011), no. 1, 1–9.

[2] Sven Erick Alm, Svante Janson and Svante Linusson, Correlations for paths in random orientations of $G(n,p)$ and $G(n,m)$, *Random Structures Algorithms* **39** (2011), no. 4, 486–506.

[3] Geoffrey R. Grimmett, Infinite paths in randomly oriented lattices, *Random Structures Algorithms* **18** (2001), no. 3, 257–266.

[4] Svante Janson and Malwina Luczak, Susceptibility in subcritical random graphs. *J. Math. Phys.* **49**:12 (2008), 125207.

[5] Colin McDiarmid, General percolation and random graphs, *Adv. in Appl. Probab.* **13** (1981), 40–60.

[6] Heinrich Weber, *Lehrbuch der Algebra*, Zweite Auflage, Erster Band. Friedrich Vieweg und Sohn, Braunschweig (1898).

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