VCG AUCTION MECHANISM COST EXPECTATIONS AND VARIANCES

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ABSTRACT. We consider Vickrey–Clarke–Groves (VCG) auctions for a very general combinatorial structure, in an average-case setting where item costs are independent, identically distributed uniform random variables. We prove that the expected VCG cost is at least double the expected nominal cost, and exactly double when the desired structure is a basis of a bridgeless matroid. In the matroid case we further show that, conditioned upon the VCG cost, the expectation of the nominal cost is exactly half the VCG cost, and we show several results on variances and covariances among the nominal cost, the VCG cost, and related quantities. As an application, we find the asymptotic variance of the VCG cost of the minimum spanning tree in a complete graph with random edge costs.

1. INTRODUCTION AND OUTLINE

We begin with a motivating example. We want to buy a specific structure in a graph, such as a spanning tree. Each edge has a separate owner, who has a minimum "true" cost for which he would sell the edge; this cost is private, known only to the owner. One obvious "mechanism" for choosing a spanning tree to buy is to ask each owner for the price of his edge (for example by a simple sealed-bid auction), and then choose the minimum (cheapest) spanning tree (MST) and pay the owners of the chosen edges the prices that they have demanded. The problem with this and many other mechanisms is that the owners have an incentive to lie about their true costs: an owner that claims a higher price will get more money, unless the price becomes so high that another spanning tree is chosen.

The Vickrey–Clarke–Groves (VCG) auction [Vic61, Cla71, Gro73] is designed to avoid this problem. By using a clever mechanism defined in Section 2 below, which typically will pay the owners of the chosen

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1

edges more than their claimed costs, an owner maximizes his profit by claiming his true cost of his edge (assuming that the owners act without collusion). Thus the selected structure will really be the cheapest one, and therefore the most efficient from society's point of view (in terms of using the least resources), although the price we pay for it is higher than the true cost. The main purpose of the present paper is to study this overpayment.

Examples are known for which the VCG mechanism results in arbitrarily large overpayment; see for example [AT07]. A typical situation may be better represented by an average-case setting. Previous work has shown that with independent, identically distributed uniform item costs, the expectation of the VCG auction price c^{VCG} is exactly 2 times that of the nominal cost c^* in a procurement auction for a minimum spanning tree, and larger by a factor asymptotically approaching 2 in auctions for a perfect matching in a complete bipartite graph, and a path in a complete graph [CFMS09].

A unifying explanation for these results was given by [AHW09], see Section 4. Using their argument, we show in Theorem 3.1 that for any VCG auction with independent, uniformly distributed item costs, $\mathbb{E}(c^{\text{VCG}}) \geq 2\mathbb{E}(c^*)$, and in Theorem 3.3 that in a matroid setting (including the minimum spanning tree example) $\mathbb{E}(c^{\text{VCG}}) = 2\mathbb{E}(c^*)$. In the matroid case, we also show further results. In particular, a matroidal development of the same ideas leads to Theorem 3.5, the much stronger result that $\mathbb{E}\left(c^* \mid c^{\text{VCG}}\right) = \frac{1}{2}c^{\text{VCG}}$ for the conditional expectation. This implies that $\text{Cov}(c^*, c^{\text{VCG}}) = \frac{1}{2}\text{Var}(c^{\text{VCG}})$, and Theorem 3.7 gives a variety of other variance relations.

Outline. Section 2 introduces VCG auctions, related notation, and the basic properties that we will draw on. Section 3 states the theorems just described, along with some related results. Proofs are given in Sections 4–7. Here we use the ideas of [AHW09] and our extension of them; we also use a very different type of matroidal argument, combined with a martingale method, to give a second proof of Theorem 3.3 and to prove parts of the variance results of Theorem 3.7.

In Section 8 we give some results for non-uniform cost distributions. Theorems 8.1 and 8.2 extend Theorems 3.1 and 3.5 to any distributions with decreasing density functions. In many probabilistic problems, exponential distributions yield simpler results than uniform distributions, but this is not the case here: Corollary 8.3 and Theorem 8.5 give particular results for the exponential distribution (which in this setting does not lead to equalities) and to a Beta distribution $B(\alpha, 1)$ (which does).

We give some examples illustrating our theorems in Section 9. In particular, Example 9.1 gives asymptotic results on the expectation and variance of the VCG cost of a MST in K_n with i.i.d. $U(0,1)$ edge costs: the expectation is $\sim 2\zeta(3)$ as shown by [CFMS09], and the variance is $\sim (24\zeta(4) - 18\zeta(3))/n$. The other examples are much simpler, and serve partly as counterexamples showing some ways in which our results cannot be improved.

2. VCG MECHANISM, PAYMENT, AND THRESHOLD

In order to define the VCG mechanism, we introduce some terminology. We consider, as in the example above, the case of a single buyer and multiple independent sellers of different items. (The buyer may also act as the auctioneer, or there may be a separate auctioneer, but in any case the two roles are equivalent for our purposes, and we will speak of the buyer.)

A *procurement* or *reverse* auction instance $I = (A, \mathbf{S}, c)$ consists of a set A of *items*, a *collection* $\mathbf{S}$ *of desired structures*, $\mathbf{S} \subseteq 2^A$, and a *cost function* $c \colon A \to \mathbb{R}_{\geq 0}$ giving the cost $c(a)$ of each item a. The buyer's goal is to obtain any structure $S \in \mathbf{S}$, paying as little as possible. For any $S \in \mathbf{S}$, the (nominal) cost of structure S is

$$c(S) := \sum_{a \in S} c(a) = \sum_{a \in A} c(a) \mathbf{1}[a \in S]. \tag{2.1}$$

We use the phrase *nominal cost* to distinguish it from the *VCG cost* defined below. Let c^* be the minimum cost,

$$c^* = c^*(I) := \min_{S \in \mathbf{S}} c(S), \tag{2.2}$$

and call any $S \in \mathbf{S}$ achieving this minimum a *minimum structure*. (Explicitly showing c^* as a function of the instance I will sometimes be important, for example in (2.3).)

The VCG mechanism purchases a minimum structure S^*, just as a simple auction would. (If there are several minimum structures, it chooses one arbitrarily. In the random setting that we will consider in our main results, the minimum structure is unique with probability 1, so there is no issue, but we will be precise about this throughout.) We say that the items $a \in S^*$ are *selected*. No payment is made for any item not selected. For each selected item, the owner is paid an amount at least as large as the nominal cost, and typically larger. This payment, the *VCG cost*, is defined as follows.

Given any instance I, for each item $a \in A$ define an instance I_a^0 to be identical to I except that $c(a) = 0$. Likewise, let I_a^∞ be identical to

I except that $c(a) = \infty$. Define the *incentive payment* for item a as

$$c^+(a) := c^*(I_a^\infty) - c^*(I) \geq 0, \tag{2.3}$$

and the *VCG threshold* for item a as

$$c^{\text{VCG}}(a; I) := c^*(I_a^\infty) - c^*(I_a^0). \tag{2.4}$$

(The names will soon be justified by (2.5) and (2.9)–(2.10).) The payment for item a under the VCG mechanism is given by:

$$\text{VCG payment to } a := \begin{cases} 0 & \text{if } a \notin S^* \\ c^{\text{VCG}}(a; I) = c(a) + c^+(a) & \text{if } a \in S^*, \end{cases} \tag{2.5}$$

with the implicit equality justified by (2.6). Note that by (2.4), the VCG threshold for a, and thus the VCG payment if a is selected, depends only on the costs of other items, not on $c(a)$.

Let S^* be a minimum structure. If $a \in S^*$ then $c^*(I) - c^*(I_a^0) = c(a)$, because if a is already part of the minimum structure S^*, then decreasing its cost to 0 means S^* is still a minimum structure, and the cost of S^* decreases by $c(a)$. Then, by (2.4) and (2.3),

$$\begin{aligned} c^{\text{VCG}}(a; I) &= c^*(I_a^\infty) - c^*(I_a^0) \\ &= [c^*(I) - c^*(I_a^0)] + [c^*(I_a^\infty) - c^*(I)] \\ &= c(a) + c^+(a) \tag{2.6} \\ &\geq c(a). \tag{2.7} \end{aligned}$$

If $a \notin S^*$, then $c^*(I_a^\infty) = c^*(I)$ so $c^+(a) = 0$; moreover, $c^*(I) \leq c^*(I_a^0) + c(a)$, and hence

$$c^{\text{VCG}}(a; I) = c^*(I_a^\infty) - c^*(I_a^0) = c^*(I) - c^*(I_a^0) \leq c(a). \tag{2.8}$$

Recapitulating, if S^* is a minimum structure,

$$a \in S^* \implies c^+(a) \geq 0 \text{ and } c^{\text{VCG}}(a; I) = c(a) + c^+(a) \geq c(a), \tag{2.9}$$

$$a \notin S^* \implies c^+(a) = 0 \text{ and } c^{\text{VCG}}(a; I) \leq c(a). \tag{2.10}$$

If S^* is a unique minimum structure, we can say a bit more. In this case, if $a \in S^*$, then $c^*(I_a^\infty) > c^*(I)$ so $c^+(a) > 0$ and there is strict inequality in (2.7). And if $a \notin S^*$, then $c^*(I_a^0) + c(a) > c(S^*) = c^*(I)$, and thus there is strict inequality in (2.8). That is, if S^* is unique then we cannot have equality in (2.9) nor in (2.10), so that

$$\begin{aligned} a \in S^* \iff c^+(a) > 0 &\iff c^{\text{VCG}}(a; I) \geq c(a) \\ &\iff c^{\text{VCG}}(a; I) > c(a). \end{aligned} \tag{2.11}$$

If follows that any minimum structure S^* contains every item a with $c(a) < c^{\text{VCG}}(a; I)$, and no item with $c(a) > c^{\text{VCG}}(a; I)$; this justifies

calling $c^{\mathrm{VCG}}(a; I)$ the VCG threshold. Furthermore, S^* is unique if and only if there are no items with $c(a) = c^{\mathrm{VCG}}(a; I)$. (It is easily seen that if there is any such item then there is at least one minimum S^* containing a and another without a. Recall that in our continuous random setting, minimum structures will be unique with probability 1.)

As said above, the VCG payment for a selected item a depends only on the costs of other items, not on $c(a)$. This leads to the *truthfulness* property of the VCG mechanism stated earlier: it is always optimal for the owner of item a to set its declared price, $c(a)$, equal to the item's true value to him, call it $c^{\mathrm{priv}}(a)$. If $c^{\mathrm{priv}}(a) < c^{\mathrm{VCG}}(a; I)$ then the owner will be happy to receive payment of $c^{\mathrm{VCG}}(a; I)$ and therefore should declare some price $c(a) < c^{\mathrm{VCG}}(a; I)$. If on the other hand $c^{\mathrm{priv}}(a) > c^{\mathrm{VCG}}(a; I)$, then the owner would rather not sell, and therefore should declare some price $c(a) > c^{\mathrm{VCG}}(a; I)$. In either case, setting $c(a) = c^{\mathrm{priv}}(a)$ will work. Since the owner does not know $c^{\mathrm{VCG}}(a; I)$, declaring a cost $c(a) \neq c^{\mathrm{priv}}(a)$ merely risks the loss of a profitable sale (if $c^{\mathrm{priv}}(a) < c^{\mathrm{VCG}}(a; I) < c(a)$) or a disadvantageous sale (if $c^{\mathrm{priv}}(a) > c^{\mathrm{VCG}}(a; I) > c(a)$).

Let us now look at the total amount paid by the VCG mechanism, which is the sum over all selected items, items $a \in S^*$, of the payment for a given by (2.5). Let us start with the incentive payments $c^+(a)$. We use the common notation that $x_+ := x$ if $x > 0$, and 0 otherwise. Define the *VCG overpayment* for any minimum structure S^* by

$$c^+(S^*) := \sum_{a \in S^*} c^+(a)$$

$$= \sum_{a \in A} c^+(a) = \sum_{a \in A} \left(c^{\mathrm{VCG}}(a; I) - c(a)\right)_+ =: c^+(I). \quad (2.12)$$

The second inequality relies on (2.10), and the third on (2.10) and (2.9). The final quantity $c^+(I)$ is well-defined because the middle expressions do not depend on S^*. That is, the VCG overpayment is a function of the instance: it is the same for any minimum structure S^*. Similarly, define the *total VCG cost* by

$$c^{\mathrm{VCG}}(S^*) := \sum_{a \in S^*} c^{\mathrm{VCG}}(a; I) = \sum_{a \in S^*} \left(c(a) + c^+(a)\right)$$

$$= c^* + c^+(S^*) = c^* + c^+(I) =: c^{\mathrm{VCG}}(I); \quad (2.13)$$

again, $c^{\mathrm{VCG}}(I)$ is well-defined because neither $c^+(S^*)$ nor c^* depends on S^*.

3. Main results

Our main results concern the expectations of the costs c^* and c^{VCG} when the item costs are independent random values. In particular, we are interested in the case when the costs are uniformly distributed on some intervals $[0, d_a]$; in this case we have the following general result.

Theorem 3.1. *In the general VCG setting, if the costs $c(a)$ are independent uniform random variables, $c(a) \sim U(0, d_a)$, then*

$$\mathbb{E}(c^{\mathrm{VCG}}) \geq 2\mathbb{E}(c^*). \tag{3.1}$$

That is, the expected VCG cost is at least twice as large as the nominal cost of the minimum structure: the VCG mechanism is overpaying significantly. While this is the natural way of thinking of the conclusion, at a technical level our proofs, and results such as Theorem 3.5 below, suggest that it is perhaps better thought of as $\mathbb{E}(c^*) \leq \frac{1}{2}\mathbb{E}(c^{\mathrm{VCG}})$.

We obtain further results in a *matroid* setting where $\mathbf{S}$ is the family of *bases* of a matroid M on *ground set* A. We recall that there are many equivalent definitions of matroids and many examples of different types; see e.g. [Wel76, Whi86]. One important example of the matroid case is the motivating spanning-tree example at the beginning of the paper, which translates to matroid language as follows. (All the ideas in the paper can be appreciated by thinking of just this example, but working with matroids gives greater generality, and highlights the restricted set of properties needed.)

Example 3.2. G is a connected graph. M, the *graph matroid* (or *cycle matroid*) of G, has *ground set* $A = E(G)$, the edges of G, and its bases are the spanning trees of G. A minimum-cost structure is then a minimum-cost basis, namely a minimum-cost spanning tree (MST).

Let M be a matroid with ground set A. For any subset $S \subseteq A$, let $r(S)$ denote the *rank* of S. An element $a \in S$ is called a *bridge* in S if $r(S \setminus a) < r(S)$. (When M is a graph matroid, this definition conforms with that of a bridge in a graph.) Let

$$\beta(S) := |\{a \in S : r(S \setminus a) < r(S)\}| \tag{3.2}$$

be the number of bridges in S. The bridges of M are those of its ground set A, and M is bridgeless if there is no such element. (We will use the terms "element" and "item" interchangeably, favoring element in the matroid context, and item in the auction context.)

For bridgeless matroids using a common uniform distribution of the item costs, we have exact equality in Theorem 3.1. (We have to exclude bridges, since a bridge belongs to every minimum structure and thus

its VCG payment is ∞, so the total VCG cost is infinite for a matroid with bridges, c.f. (3.5) below.)

Theorem 3.3. *For a bridgeless matroid with costs $c(a)$ i.i.d. uniform random variables, $c(a) \sim U(0, 1)$,*

$$\mathbb{E}(c^{\text{VCG}}) = 2\mathbb{E}(c^*).$$

Theorem 3.3's special case of a MST in a complete graph was proved by [CFMS09]. That paper also showed that the expected VCG cost is asymptotically (but not exactly) equal to twice the nominal cost for a minimum-cost path between a pair of vertices in a complete graph, and for a minimum-cost assignment in a complete bipartite graph. These are special cases of Theorem 3.1 but not of Theorem 3.3, and the asymptotic equality is discussed further in Remark 4.2.

Proofs of the theorems above are given in Section 4. Section 5 gives a second proof of Theorem 3.3 using the following formulas of independent interest for the nominal and VCG costs in the matroid case. For $t \geq 0$ let

$$A(t) := \{a \in A \colon c(a) \leq t\} \tag{3.3}$$

be the set of items of cost at most t. (Recall that costs are by definition non-negative.)

Lemma 3.4. *Let M be a matroid with arbitrary (random or non-random) costs $c(a)$. Then*

$$c^*(M) = \int_0^\infty \big(r(A) - r(A(t))\big)\, dt \tag{3.4}$$

$$c^{\text{VCG}}(M) = c^*(M) + \int_0^\infty \beta(A(t))\, dt. \tag{3.5}$$

Theorem 3.3 concerns the overall expectations of c^{VCG} and c^*. In Section 6 we use a matroidal extension of the ideas in the first proof of Theorem 3.3 to show the following result on the conditional expectation of c^* given c^{VCG}; note that it is a considerable strengthening of Theorem 3.3 (which follows by taking the expectation).

Theorem 3.5. *For a bridgeless matroid with costs $c(a)$ i.i.d. uniform random variables, $c(a) \sim U(0, 1)$,*

$$\mathbb{E}\big(c^* \mid c^{\text{VCG}}\big) = \tfrac{1}{2} c^{\text{VCG}}. \tag{3.6}$$

As an immediate corollary, we obtain the following formula for some mixed moments of c^* and c^{VCG}.

Corollary 3.6. *For a bridgeless matroid with costs $c(a)$ i.i.d. uniform random variables, $c(a) \sim U(0,1)$, and any real $m \geq 0$,*

$$\mathbb{E}\big(c^*(c^{\mathrm{VCG}})^m\big) = \tfrac{1}{2}\mathbb{E}\big((c^{\mathrm{VCG}})^{m+1}\big). \tag{3.7}$$

Proof. Multiply (3.6) by $(c^{\mathrm{VCG}})^m$ and take the expectation. $\qquad\square$

These results do not generalize to higher powers of c^*. Example 9.3 shows that, given c^{VCG}, the conditional distribution of c^* is not determined, although by Theorem 3.5 its conditional expectation is. The same example shows that $\mathbb{E}((c^*)^2)$ and $\mathbb{E}((c^{\mathrm{VCG}})^2)$ are not in fixed proportion, i.e., Theorem 3.3 does not generalize to the second moment; see also Theorem 7.1. Similarly, while the power of c^{VCG} in Corollary 3.6 is arbitrary, it can be seen from the example that there is no similar formula involving the second or higher power of c^*. Nevertheless, there are some general relations between the variances and covariances of c^* and c^{VCG}, and we prove the following in Section 7. The proof will use Theorem 7.1 (itself proved with the first of our two methods for proving Theorem 3.3) and Lemma 7.3 (proved with the second method).

Theorem 3.7. *For a bridgeless matroid with costs $c(a)$ i.i.d. uniform random variables, $c(a) \sim U(0,1)$,*

$$\mathrm{Cov}(c^*, c^{\mathrm{VCG}}) = \frac{1}{2}\mathrm{Var}(c^{\mathrm{VCG}}). \tag{3.8}$$

$$\mathrm{Var}(c^{\mathrm{VCG}}) = 4\,\mathrm{Var}(c^*) - \mathrm{Var}(c^{\mathrm{VCG}} - 2c^*), \tag{3.9}$$

$$\mathrm{Var}(c^*) = \int_0^1 \int_0^1 \mathrm{Cov}\big(r(A(s)), r(A(t))\big)\,ds\,dt, \tag{3.10}$$

$$\mathrm{Var}\big(c^{\mathrm{VCG}} - 2c^*\big) = \int_0^1 \big(r(A) - \mathbb{E}r(A(t))\big)\,2t\,dt \tag{3.11}$$

$$= \int_0^1 \mathbb{E}(\beta(A(t)))\,t\,dt. \tag{3.12}$$

The proof of Theorem 3.5 uses the following very general results of independent interest; the proofs are given in Section 6. We extend the notations I_a^0 and I_a^∞ from Section 2. Given an instance I and two disjoint sets of items F and E, let $I_{F,E}^{0,\infty}$ be the instance I modified so that all items in F have cost 0 and all items in E have cost ∞; if E or F is empty we write just I_F^0 and I_E^∞, respectively. In analogy with (2.4), for any independent set F and any $a \in F$, define an *extended VCG threshold* by

$$c^{\mathrm{Xvcg}}(F, a) := c^*(I_{F\setminus\{a\},a}^{0,\infty}) - c^*(I_F^0). \tag{3.13}$$

Thus $c^{\mathrm{VCG}}(a) = c^{\mathrm{Xvcg}}(\{a\}, a)$, and $c^{\mathrm{Xvcg}}(F, a)$ extends several key properties of the simple VCG threshold. One is immediate from the definition: for $a \in F$, $c^{\mathrm{Xvcg}}(F, a)$ has value independent of the costs on F, just as $c^{\mathrm{VCG}}(a)$ is independent of the cost of a. Other extended properties follow from the lemma below.

Lemma 3.8. *Consider a bridgeless matroid with ground set A and arbitrary costs $c(a)$, and let S^* be a minimum basis. Then*

$$F \subseteq S^* \implies (\forall a \in F): c^{\mathrm{VCG}}(a) = c^{\mathrm{Xvcg}}(F, a). \qquad (3.14)$$

Furthermore,

$$F \subseteq S^* \implies (\forall a \in F): c(a) \leq c^{\mathrm{Xvcg}}(F, a), \qquad (3.15)$$

$$F \subseteq S^* \impliedby (\forall a \in F): c(a) < c^{\mathrm{Xvcg}}(F, a), \qquad (3.16)$$

and if all $2^{|A|}$ sums of costs over different sets of elements are distinct,

$$F \subseteq S^* \iff (\forall a \in F): c(a) \leq c^{\mathrm{Xvcg}}(F, a). \qquad (3.17)$$

By (3.14), if $a \in F \subseteq S^*$, the extended VCG threshold is equal to the simple one. The key properties of the simple VCG threshold are (2.9)–(2.11); (3.15)–(3.17) give analogous properties for the extended VCG threshold.

Lemma 3.8 shows that for $F \subseteq S^*$, the VCG payments $c^{\mathrm{VCG}}(a)$ for the items in F depend only on the costs of the items *not* in F. (For this conclusion, we get the strongest result if we choose $F = S^*$.) This is a substantial (and non-obvious) extension of the observation after (2.5) that the payment $c^{\mathrm{VCG}}(a)$ does not depend on $c(a)$.

A possibly interesting implication is that collusion among a set F of owners will not affect their payments, as long as $F \subseteq S^*$: price-fixing can only affect the payments if some of its participants are not amongst the auction winners.

Theorem 3.9. *Consider a bridgeless matroid with ground set A and costs $c(a)$ independent continuous random variables, and let $F \subseteq A$ be an independent set. Condition on the costs of all items not in F (which by (3.13) determines $c^{\mathrm{Xvcg}}(F, a)$ for all $a \in F$), and on the event $F \subseteq S^*$ (which by (3.14) determines $c^{\mathrm{VCG}}(a) = c^{\mathrm{Xvcg}}(F, a)$ for all $a \in F$). Then the conditional distributions of $c(a)$, $a \in F$, are independent with $c(a)$ having the same distribution as the conditioned random variable $\left(c(a) \mid c(a) \leq c^{\mathrm{VCG}}(a)\right)$, where we regard $c^{\mathrm{VCG}}(a) = c^{\mathrm{Xvcg}}(a)$ as a constant.*

The final statement means, more formally, that if the conditioning in the theorem yields a value $c^{\mathrm{VCG}}(a) = c^{\mathrm{Xvcg}}(a) = x$, then the conditional

distribution of $c(a)$ equals the distribution of $\big(c(a) \mid c(a) \leq x\big)$ (where we do not condition on anything else).

Note that this theorem does not assume uniform distributions: the item costs may follow any continuous distributions, which need not be identical. We give some applications to non-uniform cost distributions in Section 8.

4. A simple proof of Theorems 3.1 and 3.3

In this section we give a proof of Theorem 3.1 which also yields Theorem 3.3. This argument was sketched in [AHW09]; since it has not appeared in print, we give it in full here, with kind permission of the authors.

Proof of Theorem 3.1. Since the distribution of each $c(a)$ is continuous, almost surely $c(a) \neq c^{\mathrm{VCG}}(a; I)$ and thus S^* is unique and (2.11) holds. The total VCG cost is thus by (2.13), a.s.,

$$c^{\mathrm{VCG}} = \sum_{a \in S^*} c^{\mathrm{VCG}}(a; I) = \sum_{a \in A} c^{\mathrm{VCG}}(a; I)\mathbf{1}\big[c(a) \leq c^{\mathrm{VCG}}(a; I)\big] \quad (4.1)$$

and the expected total VCG cost is

$$\mathbb{E}\big(c^{\mathrm{VCG}}\big) = \sum_{a \in A} \mathbb{E}\Big(c^{\mathrm{VCG}}(a; I)\mathbf{1}\big[c(a) \leq c^{\mathrm{VCG}}(a; I)\big]\Big). \quad (4.2)$$

By the method of conditional expectations, the expected contribution for item a is

$$\begin{aligned}
\mathbb{E}\big(c^{\mathrm{VCG}}&(a; I)\mathbf{1}\big[c(a) \leq c^{\mathrm{VCG}}(a; I)\big]\big) \\
&= \mathbb{E}_{A \setminus a}\big(\mathbb{E}_a\big(c^{\mathrm{VCG}}(a; I)\mathbf{1}\big[c(a) \leq c^{\mathrm{VCG}}(a; I)\big] \mid A \setminus a\big)\big) \quad (4.3)
\end{aligned}$$

where in the latter expression the inner (conditional) expectation E_a means the expectation over the cost distribution for item a conditioned upon the values of the costs for all items *except* a (with slight redundancy, we also indicate this as a conditioning on $A \setminus a$), and the outer expectation $E_{A \setminus a}$ means the expectation over the distribution of the costs for all other items. Taking advantage of the independence of the item costs and recalling that $c^{\mathrm{VCG}}(a; I)$ does not depend on $c(a)$, looking at the inner expectation above we have

$$\begin{aligned}
\mathbb{E}_a\big(c^{\mathrm{VCG}}&(a; I)\mathbf{1}\big[c(a) \leq c^{\mathrm{VCG}}(a; I)\big] \mid A \setminus a\big) \\
&= c^{\mathrm{VCG}}(a; I)\,\mathrm{Pr}_a\big(c(a) \leq c^{\mathrm{VCG}}(a; I) \mid A \setminus a\big). \quad (4.4)
\end{aligned}$$

Similarly, for the total nominal cost c^* we have almost surely

$$c^* = \sum_{a \in A} c(a) \mathbf{1}\big[c(a) \le c^{\mathrm{VCG}}(a; I)\big], \tag{4.5}$$

$$\mathbb{E}c^* = \sum_{a \in A} \mathbb{E}\Big(c(a) \mathbf{1}\big[c(a) \le c^{\mathrm{VCG}}(a; I)\big]\Big), \tag{4.6}$$

where the expected contribution for item a is

$$\mathbb{E}_{A \setminus a}\big(\mathbb{E}_a\big(c(a) \mathbf{1}\big[c(a) \le c^{\mathrm{VCG}}(a; I)\big] \mid A \setminus a\big)\big). \tag{4.7}$$

Taking advantage of the independence of the item costs, and that a has uniform distribution $U(0, d_a)$, in the inner expectation in (4.7) we have

$$\mathbb{E}_a\big(c(a) \mathbf{1}\big[c(a) \le c^{\mathrm{VCG}}(a; I)\big] \mid A \setminus a\big)$$
$$= \mathbb{E}_a\big(c(a) \mid c(a) \le c^{\mathrm{VCG}}(a; I)\big) \Pr_a\big(c(a) \le c^{\mathrm{VCG}}(a; I) \mid A \setminus a\big)$$
$$= \tfrac{1}{2} \min\big\{d_a, c^{\mathrm{VCG}}(a; I)\big\} \Pr_a\big(c(a) \le c^{\mathrm{VCG}}(a; I) \mid A \setminus a\big). \tag{4.8}$$

Immediately, each term given by (4.8) is at most half that of the same term in (4.4), and thus the expectation (4.7) is at most half the expectation (4.3); correspondingly the sum of the expectations in (4.6) at most half that of (4.2). That is, the expected nominal cost is at most half the expected VCG cost. □

First proof of Theorem 3.3. In this special case of Theorem 3.1, all $d_a = 1$. Moreover, by the matroid basis exchange property, since a is not a bridge, if S is any basis containing a, there exists $b \ne a$ such that $S \setminus \{a\} \cup \{b\}$ is a basis, and it follows from (2.4) that

$$c^{\mathrm{VCG}}(a; I) \le 1. \tag{4.9}$$

Hence, the minimum in (4.8) is achieved by $c^{\mathrm{VCG}}(a; I)$, each term in (4.8) is exactly half that in (4.4), and thus the proof above of Theorem 3.1 yields equality. □

Remark 4.1. The proof of Theorem 3.3 shows that equality holds in Theorem 3.1 if and only if $c^{\mathrm{VCG}}(a; I) \le d_a$ a.s., for every $a \in A$. This is not generally true outside the matroid setting, even if all d_a are equal; see Example 9.5 and [CFMS09]. As pointed out by [AHW09], if we know *a priori* that all item costs are bounded by 1, we could use a modified VCG auction where the VCG payment for each item is capped to be at most 1; this is equivalent to introducing fictitious "shadow copies" with costs 1 of all items. For this modified auction mechanism, the argument above shows that equality holds in (3.1) [AHW09].

Remark 4.2. The MST result of [CFMS09] is the special case of Theorem 3.3 given by Example 3.2. We will not rederive the results of [CFMS09] for shortest paths and minimum-cost matchings, but they may be thought of as following from the reasoning of Remark 4.1: in these settings the VCG costs are not deterministically capped at 1, but they are less than 1 asymptotically almost surely, leading to the asymptotic equalities found in [CFMS09].

5. A MATROIDAL PROOF OF THEOREM 3.3

In this section we give a second proof of Theorem 3.3 using properties specific to matroids, and introducing arguments useful later. We begin by proving Lemma 3.4.

Proof of Lemma 3.4. We regard item a as arriving at time $c(a)$. Let $S \in \mathbf{S}$ be a structure (here, a basis), and select the elements of S as they arrive. At time t, we thus have selected the set $S \cap A(t)$ of items. Since the structure S is a basis, $|S| = r(A)$ and also $S \cap A(t)$ is an independent subset of $A(t)$, whence $|S \cap A(t)| \leq r(A(t))$. Consequently,

$$
\begin{aligned}
c(S) = \sum_{a \in S} c(a) = \sum_{a \in S} \int_0^\infty \mathbf{1}[t < c(a)] \, dt &= \int_0^\infty \sum_{a \in S} \mathbf{1}[t < c(a)] \, dt \\
&= \int_0^\infty \left(|S| - \sum_{a \in S} \mathbf{1}[t \geq c(a)] \right) dt \\
&= \int_0^\infty \left(|S| - |S \cap A(t)| \right) dt \geq \int_0^\infty \left(r(A) - r(A(t)) \right) dt. \quad (5.1)
\end{aligned}
$$

Now, suppose S to be the greedily chosen basis, that is, the one produced by the following algorithm: take the items in order of arrival (increasing cost), breaking ties arbitrarily, and add an item to S if it is independent of all items previously seen (thus independent of the items already in S). By construction, S is independent, and at each time $t \geq 0$, $|S \cap A(t)| = r(A(t))$, since every new item that increases the rank of $A(t)$ is added to S. Letting $t \to \infty$ yields $|S| = r(A)$, and thus S is a basis, so $S \in \mathbf{S}$. Furthermore, there is equality in (5.1) for this basis S. This establishes (3.4) of Lemma 3.4. (It also shows that S is a minimum structure S^*, reproving the well-known fact that the greedy algorithm is optimal for matroids. Indeed this property characterizes matroids, a fact known as the Rado-Edmonds theorem; see [Rad57, Gal68, Edm71].)

For the VCG payment, note that the incentive payment $c^+(a)$ of an item a by the definition (2.3) is $c^*(M_a^\infty) - c^*(M)$, where M_a^∞ is the

matroid M with the cost $c(a)$ changed to ∞. This changes the set $A(t)$ to $A(t) \setminus a$, for every $t < \infty$, and thus (3.4) yields

$$
\begin{aligned}
c^+(a) &= c^*(M_a^\infty) - c^*(M) \\
&= \int_0^\infty \Big(r(A) - r(A(t) \setminus a) \Big)\, dt - \int_0^\infty \Big(r(A) - r(A(t)) \Big)\, dt \\
&= \int_0^\infty \Big(r(A(t)) - r(A(t) \setminus a) \Big)\, dt.
\end{aligned}
\tag{5.2}
$$

Summing over a we obtain by (2.12) the total VCG overpayment

$$
c^{\mathrm{VCG}} - c^* = \sum_{a \in A} c^+(a) = \int_0^\infty \sum_{a \in A} \big(r(A(t)) - r(A(t) \setminus a) \big)\, dt.
\tag{5.3}
$$

By definition, $r(A(t)) - r(A(t) \setminus a) = 1$ if a is a bridge in $A(t)$, and 0 otherwise, and thus, recalling the definition (3.2),

$$
\sum_{a \in A} \big(r(A(t)) - r(A(t) \setminus a) \big) = \beta(A(t)),
\tag{5.4}
$$

the number of bridges in $A(t)$. Combining (5.3) and (5.4) yields (3.5). $\qquad\square$

Remark 5.1. If we change the cost of a to 0, the set $A(t)$ is changed to $A(t) \cup \{a\}$ for every $t > 0$. Applying (3.4) to I_a^∞ and I_a^0 yields, similarly to (5.2), a formula for the VCG threshold for an item:

$$
c^{\mathrm{VCG}}(a; I) = \int_0^\infty \Big(r(A(t) \cup \{a\}) - r(A(t) \setminus a) \Big)\, dt.
\tag{5.5}
$$

Remark 5.2. In the case of a graph matroid defined by a connected graph G as in Example 3.2, for any $F \subseteq A$, $r(F) = n - \kappa(F)$, where n is the number of vertices in G and $\kappa(F)$ is the number of components of the subgraph F of G with vertex set $V(G)$ and edge set F. Since G is connected, $r(A) = n - 1$. Hence (3.4) can be written

$$
c^*(M) = \int_0^\infty \big(\kappa(A(t)) - 1 \big)\, dt.
\tag{5.6}
$$

This formula has been used by [Jan95] and [CFI$^+$12] to study the cost of the MST in a complete graph K_n with random i.i.d. edge costs. See further Example 9.1.

We turn to the case of random costs with the uniform distribution $U(0, 1)$.

Lemma 5.3. *Where each element $a \in A$ has cost $c(a) \sim U(0,1)$ independently,*

$$\mathbb{E}(\beta(A(t))) = t\frac{d}{dt}\mathbb{E}(r(A(t))), \qquad 0 < t < 1.$$

Proof. We argue informally. (See also Section 7 below, specifically the second proof of (3.11)–(3.12).) Run the process $A(t)$ backwards in time, from $t = 1$ to $t = 0$. At time t, all elements in $A(t)$ have distribution $U(0,1)$ conditioned upon being less than t, which is to say distribution $U(0,t)$. It follows that in the (backward) time interval $(t, t - dt)$, each element of $A(t)$ is lost with probability $(1/t)dt$. In particular, this holds for each of the $\beta(A(t))$ bridges, whose deletion would reduce the rank. Thus the expected decrease in $r(A(t))$ is $\beta(A(t))t^{-1}dt$. Going forward in time, this means

$$\frac{d}{dt}\mathbb{E}(r(A(t))) = t^{-1}\mathbb{E}(\beta(A(t))). \qquad \square$$

Second proof of Theorem 3.3. Since M is bridgeless, $\beta(A(t)) = \beta(A) = 0$ for $t > 1$, and thus by (3.5), Lemma 5.3, integration by parts and (3.4),

$$
\begin{aligned}
\mathbb{E}c^{\text{VCG}} - \mathbb{E}c^* &= \int_0^1 \mathbb{E}\beta(A(t))\,dt = \int_0^1 t\frac{d}{dt}\mathbb{E}(r(A(t)))\,dt \\
&= \Big[t\big(\mathbb{E}r(A(t))\big)\Big]_{t=0}^1 - \int_0^1 \mathbb{E}(r(A(t)))\,dt \\
&= r(A) - \int_0^1 \mathbb{E}(r(A(t)))\,dt \\
&= \int_0^1 \big(r(A) - \mathbb{E}r(A(t))\big)\,dt = \mathbb{E}c^*. \qquad (5.7)
\end{aligned}
$$

$\square$

6. Conditional expectations and distributions

In this section we prove Lemma 3.8 and Theorem 3.9, then present two corollaries and prove Theorem 3.5.

Proof of Lemma 3.8. We first prove (3.14). We assume for simplicity that all $2^{|A|}$ sums of costs over different sets of elements are distinct. Note that this can always be achieved by adding small independent random increments $\Delta c(a)$ to the costs. (If S^* is a minimum structure, we may for example take $\Delta c(a) \sim U(0, \varepsilon/|S^*|)$ for $a \in S^*$ and $\Delta c(a) \sim U(\varepsilon, 2\varepsilon)$ for $a \notin S^*$, for a small $\varepsilon > 0$, so that S^* is still a minimum

structure.) Since both sides of the equality in (3.14) are continuous functions of the item costs, the general case follows by continuity.

Define $B(F, E)$ to be a minimum-cost basis in $I^{0,\infty}_{F\setminus E, E}$, so that

$$c(B(F, E)) = c^*(I^{0,\infty}_{F\setminus E, E}) + c(F \setminus E). \tag{6.1}$$

We will only use this definition where F is an independent set and $E \subseteq F$, and we think of $B(F, E)$ as a cheapest basis forced to avoid E but to include the rest of F. For convenience we write $B(F, \{a\})$ as $B(F, a)$. Note that $B(F, E)$ has finite cost if and only if there exists a basis avoiding E, i.e. if $r(A \setminus E) = r(A)$. By our assumption above, $B(F, E)$ is unique when it has finite cost; in particular, $S^* = B(\emptyset, \emptyset)$ is unique.

We claim that

$$F \subseteq S^* \implies (\forall a \in F)(\exists b \in A)\colon B(\{a\}, a) = B(F, a) = (S^* \setminus a) \cup \{b\}. \tag{6.2}$$

Informally, this says that if F is optimal, then any single-element exclusion perturbs its best completion by just one further element. To see this, let $a \in F \subseteq S^*$ and consider constructing $B = B(\{a\}, a)$ by the greedy algorithm. (Since the matroid is bridgeless, $A \setminus \{a\}$ includes a basis, so $B(\{a\}, a)$ has finite cost.) This goes identically to the greedy construction of S^* until element a is reached: a is added to S^* but not to B. Then, every element accepted to S^* is accepted to B, but eventually some b not chosen for S^* is chosen for B; denote by S_i^* and B_i the two collections of elements at this point. Then $S_i^* \cup \{b\} = B_i \cup \{a\}$ is dependent, and

$$r(B_i) = |B_i| = |S_i^*| = r(S_i^*) = r(S_i^* \cup \{b\}) = r(B_i \cup \{a\}). \tag{6.3}$$

Thus $r(S_i^*) = r(S_i^* \cup \{b\})$ and $r(B_i) = r(B_i \cup \{a\})$, and it follows by the submodular property of the rank that for any subset $H \subseteq A$,

$$r(B_i \cup H) = r(B_i \cup \{a\} \cup H) = r(S_i^* \cup \{b\} \cup H) = r(S_i^* \cup H). \tag{6.4}$$

Since the greedy algorithm selects the elements that increase the rank of the set selected so far, it follows from (6.4) that from now on, exactly the same elements will be selected for $B = B(\{a\}, a)$ and S^*. Thus $B(\{a\}, a) = (S^* \setminus a) \cup \{b\}$. The minimization for $B(\{a\}, a)$ is a relaxation of that for $B(F, a)$, yet we have shown that $B(\{a\}, a) \cap F = [(S^* \setminus a) \cup \{b\}] \cap F = F \setminus a$, therefore $B(\{a\}, a) = B(F, a)$, which completes the proof of (6.2).

Next, recalling (6.1),

$$c^*(I_a^0) = c(B(\{a\}, \emptyset)) - c(a) \quad \text{and} \quad c^*(I_a^\infty) = c(B(\{a\}, a)). \tag{6.5}$$

Furthermore, if $a \in F \subseteq S^*$, then $B(\{a\}, \emptyset) = B(F, \emptyset) = S^*$, and (6.2) shows that $B(\{a\}, a) = B(F, a)$. Thus by the VCG threshold definition (2.4), (6.5), (6.1) again, and (3.13),

$$
\begin{aligned}
c^{\mathrm{VCG}}(a) &= c^*(I_a^\infty) - c^*(I_a^0) = c(B(\{a\}, a)) - [c(B(\{a\}, \emptyset)) - c(a)] \\
&= c(B(F, a)) - [c(B(F, \emptyset)) - c(a)] \\
&= c^*(I_{F\setminus\{a\}, a}^{0,\infty}) - c^*(I_F^0) \\
&= c^{\mathrm{Xvcg}}(F, a).
\end{aligned}
$$

This proves (3.14).

We next prove (3.17), under the hypothesis that all $2^{|A|}$ sums of costs over different sets of elements are distinct. (This hypothesis is natural and convenient, but could be weakened.)

We first claim that

$$
F \subseteq S^* \iff (\forall a \in F): c(B(F, a)) \geq c(B(F, \emptyset)). \qquad (6.6)
$$

The forward implication is trivial: $B(F, a)$ is a basis, S^* is a minimum-cost basis, and $B(F, \emptyset)$ is a basis minimized over a set of possibilities including S^*. The backward implication, through the contrapositive, says informally that if F is not contained within the minimum-cost basis then excluding some single element a improves it, with $c(B(F, a)) < c(B(F, \emptyset))$. To prove this, note that by assumption, all element costs are distinct. Sort both $B = B(F, \emptyset)$ and S^* in order of increasing cost with elements $b_1, b_2, \ldots$ and $s_1, s_2, \ldots$ and write $B_i := \{b_1, \ldots, b_i\}$ and $S_i^* := \{s_1, \ldots, s_i\}$. Let i be the first index for which $s_i \neq b_i$. Since S^* can be constructed by the greedy algorithm it must be that $c(s_i) \leq c(b_i)$, and since all element costs are distinct, $c(s_i) < c(b_i)$. Adding s_i to B creates a circuit C_1, at least one of whose elements b_j must have index $j \geq i$ (or else $C_1 \subseteq B_{i-1} \cup \{s_i\} = S_{i-1}^* \cup \{s_i\} = S_i^*$, contradicting independence of S^*). Thus $c(s_i) < c(b_i) \leq c(b_j)$, and $B' = B \cup \{s_i\} \setminus b_j$ has lower cost than B. B' is also a basis. (If not, it has a circuit $C_2 \ni s_i$, thus $B \cup \{s_i\}$ has circuits C_1 containing s_i and b_j, and C_2 containing s_i but not b_j, and by the circuit exchange axiom has a circuit C_3 with $C_3 \subseteq C_1 \cup C_2 \setminus s_i \subseteq B$, contradicting independence of B.) It cannot be that $b_j \in B \setminus F$, for then $B' \supseteq F$ which would contradict minimality of $B = B(F, \emptyset)$. Thus $b_j \in F$ and we have shown that $c(B(F, b_j)) \leq c(B') < c(B(F, \emptyset))$, which completes the proof of (6.6).

By (6.1) and definition (3.13), observation (6.6) is equivalent to (3.17).

We have proved (3.17) under the hypothesis that all sums of costs over different sets of elements are distinct. This immediately implies (3.15) and (3.16) under the same hypothesis. Moreover, the continuity

argument at the beginning of the proof applies to (3.15) and (3.16), which shows that they hold for arbitrary costs, concluding the proof of Lemma 3.8. (Note that the continuity argument does not apply to (3.17).) □

Proof of Theorem 3.9. Fix F and condition on the costs $c|_{A\setminus F}$. (I.e., sample these costs first.) As noted after (3.13), for $a \in F$ this determines $c^{\mathrm{Xvcg}}(a) = c^*(I_{F\setminus a, a}^{0,\infty}) - c^*(I_F^0)$.

By hypothesis, the costs $c(a)$ are random with independent continuous distributions, and thus almost surely $c(a) \neq c^{\mathrm{Xvcg}}(a)$ for every $a \in F$. This remains true when we condition also on $F \subseteq S^*$ since this is an event of positive probability. By (3.15)–(3.16), and $c(a) \neq c^{\mathrm{Xvcg}}(a)$ a.s. as just observed, this is the same as further conditioning on $c(a) \leq c^{\mathrm{Xvcg}}(F, a)$ for all $a \in F$, and thus the conditioned distributions of $c(a)$, $a \in F$, are independent and given by separately conditioning each $c(a)$ on $c(a) \leq c^{\mathrm{Xvcg}}(F, a)$. □

Remark 6.1. Since the resulting conditional distribution in Theorem 3.9 depends only on the event $F \subseteq S^*$ and the VCG costs $c^{\mathrm{VCG}}(a)$, we obtain (by the definition of conditional expectation) the same result by conditioning on these only.

In the special case of i.i.d. uniform costs, Theorem 3.9 can be stated as follows.

Corollary 6.2. *Consider a bridgeless matroid with ground set A and i.i.d. costs $c(a) \sim U(0, 1)$, and let $F \subseteq A$ be an independent set. Then, conditioning on the event $F \subseteq S^*$ and on the costs $c(a)$, $a \notin F$, the VCG costs $c^{\mathrm{VCG}}(a)$, $a \in F$, are determined by (3.14) and the conditional distributions of $c(a)$, $a \in F$, are independent with $c(a) \sim U(0, c^{\mathrm{VCG}}(a))$.*

The same holds if we instead condition on $F \subseteq S^*$ and the values $c^{\mathrm{VCG}}(a)$, $a \in F$.

Proof. Theorem 3.9 shows that the conditional distribution of $c(a)$ is uniform with $c(a) \sim U(0, \min(c^{\mathrm{VCG}}(a), 1))$. Furthermore, by (4.9), or as a consequence of (6.2), we have $c^{\mathrm{VCG}}(a) \leq 1$. Hence, the conditional distribution is $U(0, c^{\mathrm{VCG}}(a))$. □

In the case when $F = \{a\}$ is a singleton, Corollary 6.2 says that for any item a, conditioned on $a \in S^*$ and $c^{\mathrm{VCG}}(a)$, the distribution of $c(a)$ is uniform on $(0, c^{\mathrm{VCG}}(a))$. This was seen already in the proof of Theorems 3.1 and 3.3 in Section 4; Corollary 6.2 is thus a considerable generalization of this fact to several costs.

The other extreme case is the case when F is a basis. In this case, $F \subseteq S^*$ is equivalent to $F = S^*$, and Corollary 6.2 reduces to the following.

Corollary 6.3. *Consider a bridgeless matroid with costs $c(a)$ i.i.d. uniform random variables, $c(a) \sim U(0,1)$. Conditioned on a minimum structure S^* and VCG payments $c^{\mathrm{VCG}}(a)$, $a \in S^*$, for its elements, the costs $c(a)$, $a \in S^*$, of the elements in the minimum structure are independent with $c(a) \sim U(0, c^{\mathrm{VCG}}(a))$.* □

Proof of Theorem 3.5. It follows immediately from Corollary 6.3 that conditioned on the minimum structure S^* and on the VCG payments $c^{\mathrm{VCG}}(a)$ for all $a \in S^*$, the conditional expectation of $c^* = \sum_{a \in S^*} c(a)$ equals $\frac{1}{2} \sum_{a \in S^*} c^{\mathrm{VCG}}(a) = \frac{1}{2} c^{\mathrm{VCG}}$. Hence the same holds also if we condition only on c^{VCG}. □

Remark 6.4. In Corollary 6.3, conditioning on the individual VCG values $c^{\mathrm{VCG}}(a)$, we obtain the conditional *distribution* of the costs $c(a)$, $a \in S^*$, and thus that of their sum c^*. In Theorem 3.5, conditioning on the total VCG cost c^{VCG} but not on the individual values $c^{\mathrm{VCG}}(a)$, we obtain only a formula for the conditional *expectation* of c^*; there is no general formula for the conditional distribution of c^* given c^{VCG}, as is seen in Example 9.3.

7. VARIANCES AND COVARIANCES

In this section, we continue to consider a bridgeless matroid with costs $c(a)$ i.i.d. uniform random variables, $c(a) \sim U(0,1)$, and will prove Theorem 3.7.

First, we use Corollary 6.3 to obtain a formula for $\mathrm{Var}(c^*)$.

Theorem 7.1. *For a bridgeless matroid with costs $c(a)$ i.i.d. uniform random variables, $c(a) \sim U(0,1)$,*

$$\mathrm{Var}\big(c^*\big) = \frac{1}{4}\,\mathrm{Var}\big(c^{\mathrm{VCG}}\big) + \frac{1}{4}\,\mathbb{E}\Big(\sum_{a \in S^*} c(a)^2\Big) \tag{7.1}$$

and

$$\mathbb{E}\big((c^*)^2\big) = \frac{1}{4}\,\mathbb{E}\big((c^{\mathrm{VCG}})^2\big) + \frac{1}{4}\,\mathbb{E}\Big(\sum_{a \in S^*} c(a)^2\Big). \tag{7.2}$$

Proof. Let $\mathcal{G}$ denote the σ-field generated by S^* and the VCG costs $c^{\mathrm{VCG}}(a)$, $a \in S^*$. (More formally, $\mathcal{G}$ is generated by the random variables $\big\{\mathbf{1}[a \in S^*]\,, \mathbf{1}[a \in S^*]\, c^{\mathrm{VCG}}(a) \colon a \in A\big\}$. Recall that S^* is a.s. unique.) By Corollary 6.3, conditioned on $\mathcal{G}$, the random variables $c(a)$

for $a \in S^*$ are independent with $c(a) \sim U(0, c^{\mathrm{VCG}}(a))$. Hence, by the law of total variance, recalling $c^* = \sum_{a \in S^*} c(a)$,

$$
\begin{aligned}
\mathrm{Var}(c^*) &= \mathrm{Var}\big(\mathbb{E}(c^* \mid \mathcal{G})\big) + \mathbb{E}\big(\mathrm{Var}(c^* \mid \mathcal{G})\big) \\
&= \mathrm{Var}\Big(\sum_{a \in S^*} \mathbb{E}(c(a) \mid \mathcal{G})\Big) + \mathbb{E}\Big(\sum_{a \in S^*} \mathrm{Var}\big(c(a) \mid \mathcal{G})\big)\Big) \\
&= \mathrm{Var}\Big(\sum_{a \in S^*} \tfrac{1}{2} c^{\mathrm{VCG}}(a)\Big) + \mathbb{E}\Big(\sum_{a \in S^*} \tfrac{1}{12}(c^{\mathrm{VCG}}(a))^2\Big) \\
&= \mathrm{Var}\Big(\frac{1}{2} c^{\mathrm{VCG}}\Big) + \frac{1}{12}\mathbb{E}\Big(\sum_{a \in S^*}(c^{\mathrm{VCG}}(a))^2\Big). \qquad (7.3)
\end{aligned}
$$

The same conditioning gives

$$
\mathbb{E}\Big(\sum_{a \in S^*} c(a)^2\Big) = \mathbb{E}\Big(\mathbb{E}\Big(\sum_{a \in S^*} c(a)^2 \mid \mathcal{G}\Big)\Big) = \frac{1}{3}\mathbb{E}\Big(\sum_{a \in S^*}(c^{\mathrm{VCG}}(a))^2\Big), \quad (7.4)
$$

and (7.1) follows from (7.3) and (7.4).

Finally, (7.2) follows from (7.1) using Theorem 3.3. $\square$

Remark 7.2. An alternative (but related) proof is obtained by writing $c^* = \sum_{a \in A} \mathbf{1}[a \in S^*] c(a)$ and computing mixed second moments $\mathbb{E}\big(\mathbf{1}[a, b \in F] c(a)c(b)\big)$ of the terms in this sum by conditioning and using Corollary 6.2 with $F = \{a, b\}$. This method can also be used to obtain similar (but more complicated) formulas for $\mathbb{E}(c^*)^3$ and higher moments.

We use also some of the ideas in the second proof of Theorem 3.3 (Section 5), to establish the following formula related to Lemma 3.4.

Lemma 7.3. *For a bridgeless matroid with costs $c(a)$ i.i.d. uniform random variables,*

$$
\mathbb{E}\Big(\sum_{a \in S^*} c(a)^2\Big) = \int_0^1 \big(r(A) - \mathbb{E}r(A(t))\big)\, 2t\, dt = \int_0^1 \mathbb{E}(\beta(A(t)))\, t\, dt.
$$

Proof. S^* is a.s. unique, and is then given by the greedy algorithm. In this case we have, arguing as for (5.1),

$$
\begin{aligned}
\sum_{a \in S^*} c(a)^2 &= \sum_{a \in S^*} \int_0^\infty \mathbf{1}[t < c(a)]\, 2t\, dt = \int_0^\infty \big(|S^*| - |S^* \cap A(t)|\big)\, 2t\, dt \\
&= \int_0^\infty \big(r(A) - r(A(t))\big)\, 2t\, dt. \qquad (7.5)
\end{aligned}
$$

(Note that (7.5) holds for a matroid with arbitrary costs $c(a)$, even if there is not a unique minimum structure, as long as S^* is chosen by the greedy algorithm.)

The first equality in the lemma follows by taking the expectation, and the second follows by an integration by parts, like that in (5.7). $\quad\square$

Proof of Theorem 3.7. The theorem consists of equations (3.8)–(3.12). We begin by noting the special case $m = 1$ of Corollary 3.6:

$$\mathbb{E}\big(c^* c^{\mathrm{VCG}}\big) = \tfrac{1}{2}\mathbb{E}\big((c^{\mathrm{VCG}})^2\big). \tag{7.6}$$

Equation (3.8) follows from (7.6) and Theorem 3.3.

By (3.8), $\mathrm{Cov}(2c^* - c^{\mathrm{VCG}}, c^{\mathrm{VCG}}) = 0$, and thus

$$\mathrm{Var}(2c^*) = \mathrm{Var}(c^{\mathrm{VCG}}) + \mathrm{Var}(2c^* - c^{\mathrm{VCG}}), \tag{7.7}$$

which yields (3.9).

Next, (3.10) follows directly from Lemma 3.4's equation (3.4). (The arithmetic is simplified using that $\mathrm{Var}(X) = \mathrm{Cov}(X, X)$ and that $\mathrm{Cov}(X, Y)$ is a bilinear form.) This does not use our assumptions on the distribution of the costs, so (3.10) holds for any cost distribution, except that in general one has to integrate to ∞.

Finally, (7.7) and Theorem 7.1's equation (7.1) yield

$$\mathrm{Var}\big(c^{\mathrm{VCG}} - 2c^*\big) = 4\,\mathrm{Var}\big(c^*\big) - \mathrm{Var}\big(c^{\mathrm{VCG}}\big) = \mathbb{E}\Big(\sum_{a \in S^*} c(a)^2\Big), \tag{7.8}$$

and thus (3.11)–(3.12) follow by Lemma 7.3. $\quad\square$

We also want to point out an alternative proof of (3.11)–(3.12) using martingale theory and the ideas in Section 5.

Second proof of (3.11)–(3.12). The argument in the proof of Lemma 5.3 really shows that

$$\mathcal{M}(x) := r(A(1 - x)) - r(A) + \int_{1-x}^1 \beta(A(t))\,\frac{dt}{t}, \qquad 0 \le x \le 1, \tag{7.9}$$

is a (continuous-time) martingale on $[0, 1]$. (With respect to the σ-fields $\mathcal{F}_x$ generated by $\{c(a) \vee (1 - x)\}$, i.e., by the item arrivals after time $1 - x$. We may modify $\mathcal{M}$ to be right-continuous to conform with standard theory; this makes no difference below and will be ignored.) Note that $\mathcal{M}(0) = 0$, and thus $\mathbb{E}\mathcal{M}(x) = 0$, which by (7.9) leads to a (perhaps more formal) version of the proof of Lemma 5.3.

By (7.9), the martingale $\mathcal{M}(x)$ has finite variation, with jumps $\Delta\mathcal{M}(x) = \Delta r(A(1 - x))$ that are (a.s.) all -1. Hence the quadratic variation $[\mathcal{M}, \mathcal{M}]_x$ of the martingale is given by

$$[\mathcal{M}, \mathcal{M}]_x = \sum_{0 \le y \le x} (\Delta\mathcal{M}(y))^2 = \sum_{0 \le y \le x} (-\Delta r(A(1 - y)))$$
$$= r(A) - r(A(1 - x)), \tag{7.10}$$

and thus

$$\mathrm{Var}(\mathcal{M}(x)) = \mathbb{E}[\mathcal{M}, \mathcal{M}]_x = r(A) - \mathbb{E}r(A(1 - x)). \qquad (7.11)$$

(See e.g. [Pro05, Section II.6, in particular Corollary 3] or [Kal02, Theorem 26.6].) Furthermore, since $\mathcal{M}$ is a martingale, if $0 \leq x \leq y \leq 1$ then

$$\mathrm{Cov}(\mathcal{M}(x), \mathcal{M}(y)) = \mathrm{Cov}(\mathcal{M}(x), \mathcal{M}(x)) = r(A) - \mathbb{E}r(A(1 - x)). \qquad (7.12)$$

Thus, integrating over $x, y \in [0, 1]$,

$$\mathrm{Var}\Big(\int_0^1 \mathcal{M}(x)\,dx \Big) = \int_0^1 \int_0^1 \mathrm{Cov}\big(\mathcal{M}(x), \mathcal{M}(y)\big)\,dx\,dy$$

$$= 2 \int_0^1 \int_x^1 \big(r(A) - \mathbb{E}r(A(1 - x))\big)\,dy\,dx$$

$$= 2 \int_0^1 (1 - x)\big(r(A) - \mathbb{E}r(A(1 - x))\big)\,dx. \qquad (7.13)$$

By (7.9),

$$\int_0^1 \mathcal{M}(x)\,dx = \int_0^1 \big(r(A(1 - x)) - r(A)\big)\,dx + \int_0^1 \int_{1-x}^1 \beta(A(t))\,\frac{dt}{t}\,dx$$

$$= \int_0^1 \big(r(A(t)) - r(A)\big)\,dt + \int_0^1 \beta(A(t)) \int_{1-t}^1 dx\,\frac{dt}{t}$$

$$= \int_0^1 \big(r(A(t)) - r(A)\big)\,dt + \int_0^1 \beta(A(t))\,dt$$

$$= -c^* + (c^{\mathrm{VCG}} - c^*)$$

by (3.4)–(3.5). Hence $\mathrm{Var}\big(c^{\mathrm{VCG}} - 2c^*\big) = \mathrm{Var}\big(\int_0^1 \mathcal{M}(x)\,dx\big)$, and (3.11) follows from (7.13). As said earlier, (3.12) follows from (3.11) by integration by parts. $\qquad \square$

8. NON-UNIFORM COST DISTRIBUTIONS

We have so far, with a few exceptions (notably Theorem 3.9), assumed that the item costs are independent and uniformly distributed $U(0, 1)$. However, we can also derive some related results for random costs with other distributions (not even necessarily identical), still assuming that the costs of different items are independent.

For the next theorem, we say that a positive random variable has a *(weakly) decreasing density function* if it is absolutely continuous with a density function $f(x)$ on $(0, \infty)$ satisfying $f(x) \geq f(y)$ when $0 < x < y < \infty$, and a *strictly decreasing density function* if furthermore

there exists $B \leq \infty$ such that $f(x) > f(y)$ when $0 < x < y < B$ and $f(x) = 0$ for $x > B$.

Theorem 3.1 extends to this setting.

Theorem 8.1. *In the general VCG setting, if the costs $c(a)$ are independent random variables with decreasing density functions, then*

$$\mathbb{E}(c^*) \leq \tfrac{1}{2}\mathbb{E}(c^{\mathrm{VCG}}). \tag{8.1}$$

If furthermore every $c(a)$ has a strictly decreasing density function, then the inequality (8.1) is strict, provided $\mathbb{E}(c^*) < \infty$.

Proof. As long as the costs are independent, the proof of Theorem 3.1 in Section 4 up to (4.7) applies to any continuous distributions. If we condition on the costs of all items except a, which determines $c^{\mathrm{VCG}}(a)$, and then condition further on the event $c(a) \leq c^{\mathrm{VCG}}(a)$, then the conditional distribution of $c(a)$ has a decreasing density function with support in $[0, c^{\mathrm{VCG}}(a)]$, and thus the conditional expectation of $c(a)$ is at most $\tfrac{1}{2}c^{\mathrm{VCG}}(a)$, with strict inequality if the density function is strictly decreasing. It follows that the expectation (4.7) is at most half the expectation (4.3), and the result follows by summing over a as in the proof of Theorem 3.1. □

Likewise, in the matroid case, the corresponding result for conditional expectations Theorem 3.5 generalizes as the theorem below, now with an inequality where Theorem 3.5 had equality.

Theorem 8.2. *For a bridgeless matroid with costs $c(a)$ that are independent random variables with decreasing density functions,*

$$\mathbb{E}\bigl(c^* \mid c^{\mathrm{VCG}}\bigr) \leq \tfrac{1}{2}c^{\mathrm{VCG}}. \tag{8.2}$$

If furthermore every $c(a)$ has a strictly decreasing density function, then the inequality (8.2) is strict.

Proof. Let F be a basis in the matroid and condition on $S^* = F$, and on the VCG costs $c^{\mathrm{VCG}}(a)$, $a \in F$. (Recall from Theorem 3.9 that the VCG costs depend only on the item costs outside of F.) By Theorem 3.9 and our hypothesis, the conditional distribution of each $c(a)$ has a decreasing density function with support in $[0, c^{\mathrm{VCG}}(a)]$, and thus the conditional expectation of $c(a)$ is at most $\tfrac{1}{2}c^{\mathrm{VCG}}(a)$, with strict inequality if the density function is strictly decreasing. The result follows by summing over a and relaxing the conditioning as in the proof of Theorem 3.5. □

Note that while the unconditional inequality (8.1) holds in the general VCG setting, the conditional inequality (8.2) does not always hold

in the non-matroid case, even for $U(0, 1)$ costs; this will be shown in Example 9.5.

One case where these theorems apply (with strict inequalities) is when the costs have independent exponential distributions. The exponential distribution is discussed further in Example 9.4.

Corollary 8.3. *In the general VCG setting, if the costs $c(a)$ are independent random variables with exponential distributions (possibly with different means), then* (8.1) *holds with strict inequality.*

Moreover, for a bridgeless matroid with such costs, (8.2) *holds with strict inequality.* □

Remark 8.4 ([AHW09])**.** In the general VCG setting, suppose that each item a exists in infinitely many copies $a_1, a_2, \ldots$, with costs $c(a_1) < c(a_2) < \ldots$ given by the points of a Poisson process on $(0, \infty)$ with constant intensity $\lambda(a)$. Only the cheapest copy a_1 may be selected, and its cost has an exponential distribution, so the nominal cost is the same as with exponentially distributed costs. However, the existence of further copies may reduce the VCG cost, since $c^{\mathrm{VCG}}(a_1) \leq c(a_2)$. If we condition on the costs of all copies of all items, except for a_1, then $c(a_1) \sim U(0, c(a_2))$ and since $c^{\mathrm{VCG}}(a_1) \leq c(a_2)$, we obtain $\mathbb{E}c^* = \frac{1}{2}\mathbb{E}c^{\mathrm{VCG}}$ as in the first proof of Theorem 3.3; see also Remark 4.1.

As noted in the introduction (and in [AHW09]), simple expressions result when the item costs are i.i.d. with Beta distribution $B(\alpha, 1)$, i.e., with density function $\alpha x^{\alpha-1}$ on $(0, 1)$, for some $\alpha > 0$. (Note that $\alpha = 1$ is the uniform case $U(0, 1)$, and that the densities are decreasing for $\alpha \leq 1$ but not for $\alpha > 1$.)

Theorem 8.5. *In the general VCG setting, if the costs $c(a)$ are independent $B(\alpha, 1)$ random variables, with $\alpha > 0$, then*

$$\mathbb{E}(c^*) \leq \frac{\alpha}{\alpha + 1}\mathbb{E}(c^{\mathrm{VCG}}). \tag{8.3}$$

For a bridgeless matroid with these item costs, equality holds and, moreover,

$$\mathbb{E}\big(c^* \mid c^{\mathrm{VCG}}\big) = \frac{\alpha}{\alpha + 1}c^{\mathrm{VCG}}. \tag{8.4}$$

Proof. We argue again as in the proofs of Theorems 8.1 and 8.2. In the present case, the conditional distribution of $c(a)$ given $c(a) \leq v$ has support on $[0, \min(v, 1)]$, where it has density proportional to x^α. It follows that

$$\mathbb{E}\big(c(a) \mid c(a) \leq v\big) = \frac{\alpha}{\alpha + 1}\min(v, 1) \tag{8.5}$$

for every $v > 0$, and the result follows as in the proofs above. □

If Theorem 3.1 casts doubt on the practicality of a VCG auction, with its factor-of-two expected overpayment, Theorem 8.5 may be taken more hopefully: for Beta-distributed costs, with large α, the expected overpayment is small.

9. EXAMPLES

Example 9.1 (Minimum spanning tree). Our motivating example (see also Example 3.2) was the minimum-cost spanning tree in the complete graph K_n with i.i.d. $U(0,1)$ edge costs. This object has received extensive study. It was shown by Frieze [Fri85] that its expected cost (in our language, the expected nominal cost) satisfies

$$\mathbb{E}c^* \to \zeta(3) \qquad \text{as } n \to \infty; \tag{9.1}$$

see [CFI+12] for a recent sharper result.

Furthermore, it was shown in [Jan95] that $n^{1/2}(c^* - \mathbb{E}c^*) \xrightarrow{\text{d}} N(0, \sigma^2)$, as $n \to \infty$, where, by Wästlund [Wäs05] (see also [JW06]), $\sigma^2 = 6\zeta(4) - 4\zeta(3)$. It can be verified by using estimates in the proof in [Jan95] that the variance converges too, i.e.,

$$\operatorname{Var}(c^*) \sim \frac{\sigma^2}{n} = \frac{6\zeta(4) - 4\zeta(3)}{n} = \frac{1.68571\ldots}{n}. \tag{9.2}$$

Our theorems now yield the expectation and variance of the VCG cost. (The expectation (9.3) was found more directly by [CFMS09].)

Theorem 9.2. *For the minimum spanning tree in a complete graph with i.i.d. $U(0,1)$ edge weigths,*

$$\mathbb{E}c^{\text{VCG}} = 2\mathbb{E}c^* \to 2\zeta(3), \tag{9.3}$$

$$\operatorname{Var}(c^{\text{VCG}}) \sim \frac{24\zeta(4) - 18\zeta(3)}{n} = \frac{4.33873\ldots}{n}. \tag{9.4}$$

Proof. The expectation (9.3) follows directly from Theorem 3.3 and (9.1). For the variance, by (3.11) and Remark 5.2,

$$\operatorname{Var}(c^{\text{VCG}} - 2c^*) = \int_0^1 \big(r(A) - \mathbb{E}r(A(t))\big)\, 2t\, dt \tag{9.5}$$

$$= \int_0^1 2t\big(\mathbb{E}\kappa(A(t)) - 1\big)\, dt. \tag{9.6}$$

The latter integral can be estimated using minor modifications of estimates in [CFI+12] (the main terms come from counting tree components; we omit the details), which yields

$$\operatorname{Var}(c^{\text{VCG}} - 2c^*) = \int_0^1 2t\big(\mathbb{E}\kappa(A(t)) - 1\big)\, dt \sim \frac{2\zeta(3)}{n}. \tag{9.7}$$

Finally, (9.4) follows from (9.2) and (9.7) by (3.9).   □

Example 9.3 (Uniform matroid). As a very simple example, consider the uniform matroid $U_{n,k}$: it has n elements (items) and every subset with exactly k elements is a basis. Our objective is thus simply to buy any k items, and obviously the minimum structure consists of the k cheapest items. This simple example can easily be analyzed directly, without the general theory above. We assume $1 \le k \le n - 1$ so that $U_{n,k}$ is bridgeless.

If we order the items as $a_{(1)}, \ldots, a_{(n)}$ in order of increasing cost (for simplicity assuming that there are no ties), the minimum structure is $S^* = \{a_{(1)}, \ldots, a_{(k)}\}$. Thus

$$c^* = \sum_{i=1}^{k} c(a_{(i)}). \tag{9.8}$$

If $a \in S^*$, then in the instance I_a^∞, we select $a_{(k+1)}$ instead of a; thus the incentive payment (2.3) is $c(a_{(k+1)}) - c(a)$ and the VCG payment for a is, by (2.5), $c(a_{(k+1)})$. Hence

$$c^{\text{VCG}} = kc(a_{(k+1)}). \tag{9.9}$$

(In fact, it is easily seen that $c^{\text{VCG}}(a; I) = c(a_{(k+1)})$ if $a \in S^*$ and $c^{\text{VCG}}(a; I) = c(a_{(k)})$ if $a \notin S^*$.)

If the costs $c(a_i) = U_i$ are i.i.d. $U(0, 1)$, we thus have

$$c^* = \sum_{i=1}^{k} U_{(i)}, \tag{9.10}$$

$$c^{\text{VCG}} = kU_{(k+1)}, \tag{9.11}$$

where $U_{(i)}$, the ith order statistic of $\{U_j\}_1^n$, is well known to have Beta distribution $B(i, n + 1 - i)$ and mean $\mathbb{E}U_{(i)} = i/(n + 1)$. Thus

$$\mathbb{E}c^* = \sum_{i=1}^{k} \frac{i}{n + 1} = \frac{k(k + 1)}{2(n + 1)}, \tag{9.12}$$

$$\mathbb{E}c^{\text{VCG}} = \frac{k(k + 1)}{n + 1}, \tag{9.13}$$

in accordance with Theorem 3.3.

In this simple case, we can study the conditional distribution of c^* given c^{VCG} in detail. It is well known that, conditioned on $U_{(k+1)}$, the distribution of $(U_{(i)})_{i=1}^{k}$ equals the distribution of the order statistics of k independent uniform random variables on the interval $(0, U_{(k+1)})$.

Hence,

$$\big((U_{(i)})_{i=1}^k \mid U_{(k+1)}\big) \stackrel{\mathrm{d}}{=} (V_{(i)}U_{(k+1)})_{i=1}^k,$$

where $(V_i)_1^k$ also are i.i.d. $U(0,1)$ random variables, independent of $(U_j)_1^n$. Consequently, (9.10)–(9.11) yield

$$\big(c^* \mid c^{\mathrm{VCG}}\big) \stackrel{\mathrm{d}}{=} S_k U_{(k+1)} = \frac{S_k}{k} c^{\mathrm{VCG}}, \tag{9.14}$$

where $S_k := \sum_1^k V_i$ is independent of c^{VCG}. Since $\mathbb{E}(S_k/k) = \frac{1}{2}$, this is in accordance with Theorem 3.5. It also shows that the conditional distribution of c^* given c^{VCG} may vary (although its expectation is always $\frac{1}{2}c^{\mathrm{VCG}}$). For example, if $k = 1$ then c^* is uniform on $(0, c^{\mathrm{VCG}})$, while if k is large, c^* is concentrated close to $\frac{1}{2}c^{\mathrm{VCG}}$. In particular, $\mathbb{E}(c^*)^2$ is not determined by $\mathbb{E}(c^{\mathrm{VCG}})^2$, and there is no analogue of Theorem 3.3 for the second moment. Likewise there is no analogue of Corollary 3.6 for higher moments of c^* without c^* appearing on the right-hand side; this is clear for $m = 0$, and counterexamples can be generated for larger values of m.

The variances are also easily computed, using that for $1 \le i \le j \le n$, $\mathbb{E}(U_{(i)} \mid U_{(j)}) = \frac{i}{j}U_{(j)}$. We find that, for $1 \le i \le j \le n$,

$$\mathbb{E}(U_{(i)}U_{(j)}) = \frac{i(j+1)}{(n+1)(n+2)} \tag{9.15}$$

$$\mathrm{Cov}(U_{(i)}, U_{(j)}) = \frac{i(n+1-j)}{(n+1)^2(n+2)}, \tag{9.16}$$

and straightforward calculations from (9.10), (9.11) yield

$$\mathrm{Var}(c^*) = \frac{k(k+1)(4nk+2n+k+2-3k^2)}{12(n+1)^2(n+2)}, \tag{9.17}$$

$$\mathrm{Var}(c^{\mathrm{VCG}}) = \frac{k^2(k+1)(n-k)}{(n+1)^2(n+2)}. \tag{9.18}$$

From (3.8) this allows us to obtain $\mathrm{Cov}(c^*, c^{\mathrm{VCG}})$; we can also calculate it too directly and verify equality. By (3.9), $\mathrm{Var}(c^{\mathrm{VCG}} - 2c^*) = k(k+1)(k+2)/3(n+1)(n+2)$, and this can be checked against its calculation by (3.11)–(3.12).

In particular, if $n \to \infty$ and $1 \ll k \ll n$, then $\mathrm{Var}(c^*)/\mathrm{Var}(c^{\mathrm{VCG}}) \to 1/3$, and more precisely the covariance matrix of (c^*, c^{VCG}) is asymptotic to

$$\frac{k^3}{n^2}\begin{pmatrix} 1/3 & 1/2 \\ 1/2 & 1 \end{pmatrix}. \tag{9.19}$$

If $n \to \infty$ with $k \ge 1$ fixed, we have instead $\mathrm{Var}(c^*)/\mathrm{Var}(c^{\mathrm{VCG}}) \to (2k+1)/(6k)$.

The latter limit demonstrates that $\mathrm{Var}(c^*)$ and $\mathrm{Var}(c^{\mathrm{VCG}})$ are not in fixed proportion, so there is not a direct analogue of Theorem 3.3 for variance.

Example 9.4 (Uniform matroid, exponential distribution). We continue to consider the uniform matroid $\mathsf{U}_{n,k}$ as in Example 9.3, but now consider costs $c(a)$ that are i.i.d. and exponential $\mathrm{Exp}(1)$. Let $X_{(i)} = c(a_{(i)})$, the ith smallest cost. It is well-known that the increments $X_{(i)} - X_{(i-1)}$ (where $X_{(0)} = 0$) are independent random variables with $X_{(i)} - X_{(i-1)} \sim \mathrm{Exp}(1/(n-i-1))$. Consequently,

$$\mathbb{E}X_{(i)} = \sum_{j=1}^{i} \frac{1}{n-j+1} \tag{9.20}$$

and thus by (9.8)–(9.9),

$$\mathbb{E}c^* = \sum_{j=1}^{k} \frac{k-j+1}{n-j+1} = \sum_{j=1}^{k} \frac{j}{n-k+j}, \tag{9.21}$$

$$\mathbb{E}c^{\mathrm{VCG}} = k\sum_{j=1}^{k+1} \frac{1}{n-j+1}. \tag{9.22}$$

Not only are these not a factor of 2 apart, as we already knew from Corollary 8.3, but there doesn't seem to be any simple general relation.

If for simplicity we consider the case $k = 1$, where our aim just is to select the cheapest item, it follows from Theorem 3.9 that the conditional distribution of c^* given $c^{\mathrm{VCG}} = v$ has density $e^{-x}/(1 - e^{-v})$ on $(0, v)$. A simple calculation yields

$$\mathbb{E}\big(c^* \mid c^{\mathrm{VCG}} = v\big) = \frac{1 - e^{-v} - ve^{-v}}{1 - e^{-v}} = \frac{e^v - 1 - v}{e^v - 1}. \tag{9.23}$$

This is always $< v/2$, as shown by Corollary 8.3, with asymptotic equality as $v \to 0$.

Example 9.5 (Paths). [CFMS09] considered the minimum-cost path between two given vertices in K_n with i.i.d. $\mathrm{Exp}(1)$ costs of the edges. They showed that $\mathbb{E}c^{\mathrm{VCG}} \sim 2\mathbb{E}c^*$ as $n \to \infty$. It is not hard to show that the same holds with i.i.d. $U(0, 1)$ costs. In these cases, however, there is not equality in (3.1). (This is not a matroid case, so Theorem 3.3 does not apply.)

For a simple explicit example, consider $n = 3$. Denoting the three edges in K_3 by a_1, a_2, a_3 and their costs by X_1, X_2, X_3, and let the two given vertices be those joined by $\{a_1\}$ and $\{a_2, a_3\}$, so that

$$c^* = \min\{X_1, X_2 + X_3\}. \tag{9.24}$$

It follows from this and (2.4) that $c^{\text{VCG}}(a_1) = X_2 + X_3$, $c^{\text{VCG}}(a_2) = (X_1 - X_3)_+$ and $c^{\text{VCG}}(a_3) = (X_1 - X_2)_+$, and thus

$$c^{\text{VCG}} = \begin{cases} X_2 + X_3, & \text{if } X_1 \leq X_2 + X_3, \\ 2X_1 - X_2 - X_3, & \text{if } X_1 \geq X_2 + X_3. \end{cases} \tag{9.25}$$

Thus,

$$c^{\text{VCG}} = \max(X_2 + X_3, \, 2X_1 - X_2 - X_3). \tag{9.26}$$

Simple calculations, which we omit, show that in the $U(0,1)$ case, $\mathbb{E}c^* = 11/24$ while $\mathbb{E}c^{\text{VCG}} = 13/12$ (a check on these is that from (9.26), $2c^* + c^{\text{VCG}} = 2X_1 + X_2 + X_3$, thus $2\mathbb{E}c^* + \mathbb{E}c^{\text{VCG}} = 2$), showing strict inequality in (3.1). Similarly, in the $\text{Exp}(1)$ case, $\mathbb{E}c^* = 3/4$ while $\mathbb{E}c^{\text{VCG}} = 5/2$, with strict inequality in (8.1) as shown in Corollary 8.3.

Moreover, still for $n = 3$ and with $U(0,1)$ costs, the joint density of c^* and c^{VCG} in $[0,1] \times [0,2]$ can easily be calculated from (9.24)–(9.26) to be

$$f(x,y) = y\mathbf{1}[x \leq y \leq 1] + (2-y)\mathbf{1}[1 < y \leq 2] + \frac{x}{2}\mathbf{1}[x \leq \min(y, 2-y)]. \tag{9.27}$$

For example, $f(x,1) = 1 + x/2$, $x \in [0,1]$, and thus the conditional distribution of $(c^* \mid c^{\text{VCG}} = 1)$ has density $(4 + 2x)/5$ and conditional expectation $\mathbb{E}(c^* \mid c^{\text{VCG}} = 1) = 8/15 > 1/2$, showing that the inequality (8.2) does not necessarily hold in the non-matroid setting. More generally, it follows from (9.27) that the density of c^{VCG} is

$$f_2(y) = \begin{cases} \frac{5}{4}y^2 & \text{if } 0 \leq y \leq 1, \\ (2-y) + \frac{1}{4}(2-y)^2 & \text{if } 1 \leq y \leq 2, \end{cases} \tag{9.28}$$

and the conditional expectation $\mathbb{E}(c^* \mid c^{\text{VCG}})$ is

$$\mathbb{E}(c^* \mid c^{\text{VCG}} = y) = \begin{cases} \frac{2y^3/3}{f_2(y)} = \frac{8}{15}y & \text{if } 0 \leq y \leq 1, \\ \frac{\frac{1}{2}(2-y) + \frac{1}{6}(2-y)^3}{f_2(y)} = \frac{6 + 2(2-y)^2}{12 + 3(2-y)} & \text{if } 1 \leq y \leq 2. \end{cases}$$

In this example, $\mathbb{E}(c^* \mid c^{\text{VCG}})$ is a *decreasing* function of c^{VCG} in the interval $c^{\text{VCG}} \in [1, 6 - \sqrt{19}]$, which shows that the behaviour of the conditional expectation can be quite complicated and unexpected in the general non-matroid case.

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