

ON THE TYPICAL STRUCTURE OF GRAPHS IN A MONOTONE PROPERTY

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ABSTRACT. Given a graph property \mathcal{P} , it is interesting to determine the typical structure of graphs that satisfy \mathcal{P} . In this paper, we consider monotone properties, that is, properties that are closed under taking subgraphs. Using results from the theory of graph limits, we show that if \mathcal{P} is a monotone property and r is the largest integer for which every r -colorable graph satisfies \mathcal{P} , then almost every graph with \mathcal{P} is close to being a balanced r -partite graph.

1. INTRODUCTION AND MAIN RESULTS

Given a graph property \mathcal{P} , it is natural to study the structure of a typical graph that satisfies \mathcal{P} . A graph property is *monotone* if it is closed under taking subgraphs and *hereditary* if it is closed under taking induced subgraphs. Thus, every monotone property is also hereditary. Many authors have studied the structure of typical graphs in various hereditary properties—see, e.g., [1, 3, 7, 8, 9, 11], as well as the survey [4]. In this note, we use results from graph limit theory to study the structure of a typical graph in a general monotone property.

Before stating our main result, let us recall certain basic notions of graph limit theory. For more details, see, e.g., [5, 6, 13], as well as the monograph [12]. Here, we simply recall that certain sequences of graphs are defined to be *convergent*. A convergent sequence has a limit, called a *graph limit*, which is unique if it exists.

Lovász and Szegedy [13] showed that a graph limit Γ may be represented by a *graphon*, a symmetric, measurable function $W : [0, 1]^2 \rightarrow [0, 1]$. (So, abusing notation slightly, we will sometimes write $G_n \rightarrow W$ if the sequence $\{G_n\}_{n=1}^\infty$ converges to the graph limit Γ represented by W .) More than one graphon may represent the same graph limit; we say that the graphons W_1 and W_2 are *equivalent*, and write $W_1 \cong W_2$, if they represent the same graph limit.

Let X_1, X_2, \dots be i.i.d. uniformly distributed random variables in $[0, 1]$. Given a graphon W , the *W -random graph* $G(n, W)$ is a graph with vertex set $[n]$ in which vertices i and j are adjacent with probability $W(X_i, X_j)$, independently of all other edges.

Let $h(x) = -x \log_2(x) - (1 - x) \log_2(1 - x)$ denote the binary entropy function. The *entropy* of a graphon W is

$$\text{Ent}(W) = \int_0^1 \int_0^1 h(W(x, y)) \, d\mu(x) d\mu(y),$$

where μ denotes the Lebesgue measure. As noted in [9], if $W_1 \cong W_2$, then $\text{Ent}(W_1) = \text{Ent}(W_2)$. In other words, entropy is a property of a graph limit, rather than of the graphon that represents it. Thus, we may define the entropy $\text{Ent}(\Gamma)$ of a graph limit Γ to be the entropy of any graphon that represents it.

Hatami, Janson, and Szegedy [9] posed the question of which graphons may arise as limits of sequences of graphs with a given property \mathcal{P} . In addition to the intrinsic

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interest of this question, it turns out that if \mathcal{P} is hereditary, then certain limits of sequences of graphs in \mathcal{P} (namely, those with maximum entropy) give a great deal of information about the number and typical structure of graphs in \mathcal{P} . (We do not distinguish between a graph property and the class of graphs with that property.) In order to state these results, we need to introduce more notation.

Let \mathcal{U}_n denote the set of unlabeled graphs on n vertices and let \mathcal{L}_n denote the set of labeled graphs with vertex set $[n]$. Given a graph property \mathcal{P} , we let $\mathcal{P}_n = \mathcal{P} \cap \mathcal{U}_n$ denote the set of unlabeled elements of \mathcal{P} with n vertices and let \mathcal{P}_n^L denote the set of labeled elements of \mathcal{P} with vertex set $[n]$. The function $n \mapsto |\mathcal{P}_n|$ is called the (unlabeled) *speed* of \mathcal{P} ; the labeled speed is defined similarly. Observe that

$$(1) \quad |\mathcal{P}_n| \leq |\mathcal{P}_n^L| \leq n! |\mathcal{P}_n|.$$

Given a graph property \mathcal{P} , we let $\widehat{\mathcal{P}}$ denote the set of graph limits of sequences in \mathcal{P} . We furthermore let $\widehat{\mathcal{P}}^*$ denote the set of elements of $\widehat{\mathcal{P}}$ of maximum entropy, i.e.,

$$\widehat{\mathcal{P}}^* = \left\{ \Gamma \in \widehat{\mathcal{P}} : \text{Ent}(\Gamma) = \max_{\Gamma' \in \widehat{\mathcal{P}}} \text{Ent}(\Gamma') \right\}.$$

We will also use these symbols to refer to the set of graphons (respectively, the set of maximum-entropy graphons) representing limits of sequences in \mathcal{P} . It is shown in [9] that if \mathcal{P} is hereditary (and not finite), then $\max_{\Gamma \in \widehat{\mathcal{P}}} \text{Ent}(\Gamma)$ is achieved—in other words, $\widehat{\mathcal{P}}^*$ is nonempty.

In [9, Theorem 1.6], Hatami, Janson, and Szegedy showed that if a hereditary property \mathcal{P} has a single graph limit Γ of maximum entropy, then a typical element of \mathcal{P} is close to Γ (in terms of the standard cut metric on the space of graph limits).

Theorem 1. *Suppose that \mathcal{P} is a hereditary property and that $\max_{\Gamma \in \widehat{\mathcal{P}}} \text{Ent}(\Gamma)$ is attained by a unique graph limit $\Gamma_{\mathcal{P}}$. Then*

- (i) *if $G_n \in \mathcal{U}_n$ is a uniformly random unlabeled element of \mathcal{P}_n , then G_n converges in probability to $\Gamma_{\mathcal{P}}$ as $n \rightarrow \infty$;*
- (ii) *if $G_n \in \mathcal{L}_n$ is a uniformly random labeled element of \mathcal{P}_n^L , then G_n converges in probability to $\Gamma_{\mathcal{P}}$ as $n \rightarrow \infty$.*

Now we define a special class of graphons. All of these graphons will be defined on $[0, 1]^2$, rather than on $[0, 1]$; it is easy to see that this change is immaterial. Given $r \in \mathbb{N}$ and $i \in [r]$, let $I_i = [(i-1)/r, i/r)$ and let $E_r = \cup_{i \neq j} I_i \times I_j$. We also let $E_\infty = [0, 1]^2$. Given $r \in \mathbb{N} \cup \{\infty\}$, we let R_r denote the set of graphons W such that $W(x, y) = 1/2$ if $(x, y) \in E_r$ and $W(x, y) \in \{0, 1\}$ otherwise. It is easy to see that if $W \in R_r$, then

$$\text{Ent}(W) = \iint_{E_r} h(1/2) \, d\mu(x) d\mu(y) = \mu(E_r) = 1 - \frac{1}{r}.$$

For $r \in \mathbb{N}$ and $0 \leq s \leq r$, we let $W_{r,s}^*$ denote the graphon in R_r that equals 1 on $I_i \times I_i$ for $i \leq s$ and equals 0 on $I_i \times I_i$ for $s+1 \leq i \leq r$. Observe that R_∞ consists only of the graphon that equals 1/2 everywhere on $[0, 1]^2$; for notational convenience, we denote this graphon by $W_{\infty,0}^*$.

Given $r \in \mathbb{N}$ and $0 \leq s \leq r$, we let $\mathcal{C}(r, s)$ denote the class of graphs whose vertex sets can be partitioned into s (possibly empty) cliques and $r-s$ (possibly empty) independent sets. In particular, $\mathcal{C}(r, 0)$ is the class of r -colorable graphs. Observe that for each r and s , the class $\mathcal{C}(r, s)$ is hereditary, and that $\mathcal{C}(r, 0)$ is monotone.

It is shown in [9, Theorem 1.9] that if \mathcal{P} is a hereditary property, then the maximum entropy of an element of $\widehat{\mathcal{P}}$ takes one of countably many values, and furthermore that this value determines the asymptotic speed of \mathcal{P}_n .

Theorem 2. *If \mathcal{P} is a hereditary property, then there exists $r \in \mathbb{N} \cup \{\infty\}$ such that $\max_{\Gamma \in \widehat{\mathcal{P}}} \text{Ent}(\Gamma) = 1 - 1/r$ and such that every graph limit $\Gamma \in \widehat{\mathcal{P}}^*$ can be represented by a graphon $W \in R_r$. Moreover,*

$$|\mathcal{P}_n| = 2^{\left(1 - \frac{1}{r} + o(1)\right) \binom{n}{2}}.$$

Given a graph F , we say that a graph G is F -free if no subgraph of G is isomorphic to F . Given a (possibly infinite) family of graphs \mathcal{F} , we say that G is \mathcal{F} -free if it is F -free for every $F \in \mathcal{F}$. Observe that for any family \mathcal{F} , the class of \mathcal{F} -free graphs is monotone. (Conversely, every monotone class \mathcal{P} equals the class of \mathcal{F} -free graphs for some family \mathcal{F} —for example, $\mathcal{F} = \mathcal{U} \setminus \mathcal{P}$.) We write $\text{Forb}(\mathcal{F})$ for the class of \mathcal{F} -free graphs and write $\text{Forb}(F)$ when $\mathcal{F} = \{F\}$. Note in particular that $\text{Forb}(\emptyset)$ equals the class of all unlabeled finite graphs, which we denote by \mathcal{U} .

The *coloring number* of a family of graphs \mathcal{F} is

$$\text{col}(\mathcal{F}) = \min_{F \in \mathcal{F}} \chi(F).$$

In particular, we define

$$(2) \quad \text{col}(\emptyset) = \infty.$$

Our main result says that if $\text{col}(\mathcal{F}) = r + 1$, then a typical element of $\text{Forb}(\mathcal{F})$ resembles a balanced r -partite graph in which cross-edges are present independently with probability $1/2$.

Theorem 3. *Let \mathcal{F} be a family of graphs and let $r = \text{col}(\mathcal{F}) - 1$. If $\mathcal{P} = \text{Forb}(\mathcal{F})$, then as n tends to ∞ , a sequence of uniformly random unlabeled (respectively, labeled) elements of \mathcal{P}_n (respectively, elements of \mathcal{P}_n^L) converges in probability to the graph limit Γ_r represented by $W_{r,0}^*$.*

Note that the quantity r in the statement of the theorem also equals the largest integer t for which every t -colorable graph is in $\text{Forb}(\mathcal{F})$.

It follows from Theorems 2 and 3 that if $\text{col}(\mathcal{F}) = r + 1$ then

$$(3) \quad |\text{Forb}(\mathcal{F})_n| = 2^{\left(1 - \frac{1}{r} + o(1)\right) \binom{n}{2}}.$$

Let us note that Balogh, Bollobás, and Simonovits [2] obtained a fairly sharp bound on the error term in (3).

Remark 4. The proof of Theorem 3 shows that if $r = \text{col}(\mathcal{F}) - 1$ and $\mathcal{P} = \text{Forb}(\mathcal{F})$, then $W_{r,0}^*$ is the unique maximum-entropy element of $\widehat{\mathcal{P}}$. For certain families \mathcal{F} , it is also possible to describe the set of all \mathcal{F} -free graph limits. For example, the set of limits of bipartite graphs is determined in [9, Example 2.1], and a very similar argument holds for r -partite graphs when $r \geq 3$. However, we know of no representation of the set of all \mathcal{F} -free graph limits for arbitrary \mathcal{F} .

Remark 5. Erdős, Frankl, and Rödl [7] showed that if $\chi(F) = r + 1$, then every F -free graph G may be made K_{r+1} -free by removing $o(n^2)$ edges from G . This result is similar in spirit to Theorem 3, but we see no direct implication: if $\{G_n\}_{n=1}^\infty$ is a sequence of uniformly random F -free graphs and $\{G'_n\}_{n=1}^\infty$ is the sequence of resulting K_{r+1} -free graphs, then the distribution of G'_n need not be uniform in $\text{Forb}(K_{r+1})_n$.

Remark 6. Theorem 3 says that if $\text{col}(\mathcal{F}) = r + 1$ then almost every (labeled or unlabeled) \mathcal{F} -free graph is close to a balanced r -partite graph. (Conversely, every r -partite graph is trivially \mathcal{F} -free.) In the case of labeled graphs, Prömel and Steger [14] proved a much stronger result for a specific class of monotone properties: they characterized the graphs F for which almost every labeled F -free

graph is $(\chi(F) - 1)$ -partite. Given a graph F , we say that $e \in E(F)$ is *critical* if $\chi(F - e) < \chi(F)$. Prömel and Steger showed that if $\chi(F) = r + 1$ then

$$|\text{Forb}(F)_n^L| = (1 + o(1)) |\mathcal{C}(r, 0)_n^L|$$

if and only if F contains a critical edge. They also showed that if F does not contain a critical edge, then there exists a constant $c_r > 0$ not depending on F such that

$$(4) \quad |\text{Forb}(F)_n^L| \geq c_r n |\mathcal{C}(r, 0)_n^L|$$

for all n sufficiently large. Theorem 3 shows that if \mathcal{F} is any family of graphs with $\text{col}(\mathcal{F}) = r + 1$, then $\text{Forb}(\mathcal{F})^L$ and $\mathcal{C}(r, 0)^L$ have roughly the same asymptotic speed. Note that this result does not contradict (4) when $\mathcal{F} = \{F\}$ and F does not contain a critical edge: if $\chi(F) = r + 1$, then, in view of (1) and (3), Theorem 3 implies the weaker statement that $|\text{Forb}(F)_n^L|$ and $|\mathcal{C}(r, 0)_n^L|$ differ by a factor of $2^{o(n^2)}$.

2. PROOF OF THEOREM 3.

Lemma 7. *Let \mathcal{P} be a monotone property and let $W \in \widehat{\mathcal{P}}$. If W' is a graphon such that $W' \leq W$ pointwise, then $W' \in \widehat{\mathcal{P}}$.*

Proof. Consider the sequences of random graphs $\{G(n, W)\}_{n=1}^\infty$ and $\{G(n, W')\}_{n=1}^\infty$. Since $W' \leq W$ pointwise, a standard argument shows that the two sequences can be coupled so that for each n , $G(n, W') \subseteq G(n, W)$ almost surely. It is shown in [10, Theorem 3.1] that if $W \in \widehat{\mathcal{P}}$ then, for each n , $G(n, W) \in \mathcal{P}$ almost surely. It follows that for each n , we almost surely have $G(n, W') \in \mathcal{P}$, as well. Finally, it is shown in [5, Theorem 4.5] that $G(n, W') \rightarrow W'$ almost surely as $n \rightarrow \infty$, which implies that $W' \in \widehat{\mathcal{P}}$, as claimed. \square

Now we prove our main result, Theorem 3.

Proof of Theorem 3. We begin by showing that, up to equivalence of graphons, $\widehat{\mathcal{P}}$ contains a unique element of maximum entropy. By Theorem 2, there exists $t \in \mathbb{N} \cup \{\infty\}$ such that $\widehat{\mathcal{P}}^* \subseteq R_t$ up to equivalence of graphons.

First, suppose that $t < \infty$. Observe that if $W \in \widehat{\mathcal{P}}^* \cap R_t$, then $W \geq W_{t,0}^*$ pointwise, which by Lemma 7 implies that $W_{t,0}^* \in \widehat{\mathcal{P}}^*$. We claim that, up to equivalence of graphons, $W_{t,0}^*$ is in fact the only maximum-entropy element of $\widehat{\mathcal{P}}$. Indeed, let $W' \in \widehat{\mathcal{P}}^* \cap R_t$ and suppose that $\mu(W' = 1) > 0$. But then Lemma 7 implies that $\min\{W', 1/2\}$ is a graphon in $\widehat{\mathcal{P}}$ with entropy strictly larger than $1 - 1/t$, which contradicts the definition of t . Thus, $W' = W_{t,0}^*$ a.e.; in particular, $W' \cong W_{t,0}^*$. If $t = \infty$, then (up to equivalence) $\widehat{\mathcal{P}}^*$ must consist of the sole element of R_∞ , that is, the graphon $W_{\infty,0}^*$.

Now we show that $t = \text{col}(\mathcal{F}) - 1 = r$. Suppose that $t < \infty$. It is observed in [9, Remark 1.10] that if \mathcal{P} is hereditary and $0 \leq s \leq r < \infty$, then $W_{r,s}^* \in \widehat{\mathcal{P}}$ if and only if $\mathcal{C}(r, s) \subseteq \mathcal{P}$. By the definition of $\text{col}(\mathcal{F})$, it is easy to see that if $u \leq r$, then $\mathcal{C}(u, 0) \subseteq \mathcal{P}$ and hence $W_{u,0}^* \in \widehat{\mathcal{P}}$. On the other hand, \mathcal{F} contains some element of $\mathcal{C}(r + 1, 0)$, which implies that $\mathcal{C}(r + 1, 0) \not\subseteq \mathcal{P}$. This implies that $W_{u,0}^* \notin \widehat{\mathcal{P}}$ when $u \geq r + 1$, and hence that $t = r$.

If $t = \infty$, then we claim that $\mathcal{P} = \text{Forb}(\emptyset) = \mathcal{U}$; the conclusion then follows from (2). Suppose to the contrary that \mathcal{P} does not contain some graph F . Then $\mathcal{C}(\chi(F), 0) \not\subseteq \mathcal{P}$, which implies that $W_{\chi(F),0}^* \notin \widehat{\mathcal{P}}$. However, $W_{\chi(F),0}^* \leq W_{\infty,0}^*$ pointwise, so Lemma 7 implies that $W_{\infty,0}^* \notin \widehat{\mathcal{P}}$, which is a contradiction.

Finally, since \mathcal{P} is a hereditary property, it follows from Theorem 1 that a uniformly random (labeled or unlabeled) element of \mathcal{P} converges in probability to Γ_r , as claimed. \square

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