PATTERNS IN RANDOM PERMUTATIONS AVOIDING THE PATTERN 321

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Abstract. We consider a random permutation drawn from the set of **321**-avoiding permutations of length n and show that the number of occurrences of another pattern σ has a limit distribution, after scaling by $n^{m+\ell}$ where m is the length of σ and ℓ is the number of blocks in it. The limit is not normal, and can be expressed as a functional of a Brownian excursion.

1. Introduction

Let $\mathfrak{S}_n$ be the set of permutations of $[n] := \{1, \dots, n\}$, and $\mathfrak{S}_* := \bigcup_{n\geqslant 1} \mathfrak{S}_n$. If $\sigma = \sigma_1 \cdots \sigma_m \in \mathfrak{S}_m$ and $\pi = \pi_1 \cdots \pi_n \in \mathfrak{S}_n$, then an *occurrence* of σ in π is a sequence $(i_1, \dots, i_m)$ with $1 \leqslant i_1 < \cdots < i_m \leqslant n$, such that the subsequence $\pi_{i_1} \cdots \pi_{i_m}$ has the same order as σ, i.e., $\pi_{i_j} < \pi_{i_k} \iff \sigma_j < \sigma_k$ for all $j, k \in [m]$. We let $n_\sigma(\pi)$ be the number of occurrences of σ in π, and note that

$$\sum_{\sigma\in\mathfrak{S}_m} n_\sigma(\pi) = \binom{n}{m}, \tag{1.1}$$

for every $\pi \in \mathfrak{S}_n$. For example, an inversion is an occurrence of **21**, and thus $n_{\mathbf{21}}(\pi)$ is the number of inversions in π.

Remark 1.1. It is often natural to think of an occurence as the subsequence $\pi_{i_1} \cdots \pi_{i_m}$ rather than the corresponding sequence of indices $i_1, \cdots, i_m$. However, in the present paper we use in formal arguments the definition above with indices.

We say that π *avoids* another permutation τ if $n_\tau(\pi) = 0$; otherwise, π *contains* τ. Let

$$\mathfrak{S}_n(\tau) := \{\pi \in \mathfrak{S}_n : n_\tau(\pi) = 0\}, \tag{1.2}$$

the set of permutations of length n that avoid τ. We also let $\mathfrak{S}_*(\tau) := \bigcup_{n=1}^\infty \mathfrak{S}_n(\tau)$ be the set of τ-avoiding permutations of arbitrary length.

The classes $\mathfrak{S}_*(\tau)$ of τ-avoiding permutations have been studied for a long time, see e.g. Knuth [35, Exercise 2.2.1-5], Simion and Schmidt [46], Billey, Jockusch and Stanley [8]. One classical problem is to enumerate the sets

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$\mathfrak{S}_n(\tau)$, either exactly or asymptotically, see Bóna [9, Chapters 4–5]. We note here only the fact that for any τ with $|\tau| = 3$, $\mathfrak{S}_n(\tau)$ has the same size

$$|\mathfrak{S}_n(\tau)| = C_n := \frac{1}{n+1}\binom{2n}{n}, \tag{1.3}$$

the n-th Catalan number, see e.g. [35, Exercises 2.2.1-4,5], [46], [47, Exercise 6.19ee,ff], [9, Corollary 4.7]. (The situation for $|\tau| \geqslant 4$ is more complicated.)

The general problem that concerns us is to take a fixed permutation τ and let $\boldsymbol{\pi}_{\tau,n}$ be a uniformly random τ-avoiding permutation, i.e., a uniformly random element of $\mathfrak{S}_n(\tau)$, and then study the distribution of the random variable $n_\sigma(\boldsymbol{\pi}_{\tau,n})$ for some other fixed permutation σ. (Only σ that are τ-avoiding are interesting, since otherwise $n_\sigma(\boldsymbol{\pi}_{\tau,n}) = 0$.) One instance of this problem was studied already by Robertson, Wilf and Zeilberger [45], who gave a generating function for $n_{\mathbf{123}}(\boldsymbol{\pi}_{\mathbf{132},n})$. The exact distribution of $n_\sigma(\boldsymbol{\pi}_{\tau,n})$ for a given n was studied numerically in [32], where higher moments and mixed moments are calculated for small n for several cases ($\tau = \mathbf{132}$, $\mathbf{123}$ and $\mathbf{1234}$; several σ with $|\sigma| = 3$).

We are mainly interested in asymptotics of the distribution of $n_\sigma(\boldsymbol{\pi}_{\tau,n})$. and of its moments, as $n \to \infty$, for some fixed τ and σ. The case $\tau = \mathbf{132}$ and arbitrary σ were studied in detail in [30]. In the present paper we study the case $\tau = \mathbf{321}$. Together with obvious symmetries, these two cases cover all cases where τ has length $|\tau| = 3$. (Note that the cases with $|\tau| = 2$ are trivial.) The cases with $|\tau| \geqslant 4$ seem much more difficult, and are left as challenging open problems to the readers.

The expectation $\mathbb{E}\, n_\sigma(\boldsymbol{\pi}_{\tau,n})$, or equivalently, the total number of occurences of σ in all τ-avoiding permutations, has been treated in a number of papers for various cases, beginning with Bóna [11; 13] (with $\tau = \mathbf{132}$). In particular, for the $\mathbf{321}$-avoiding permutations studied in the present paper, Cheng, Eu and Fu [17] gave an exact formula for the total number of inversions (occurences of $\mathbf{21}$), and Homberger [27] gave generating functions for the total number of occurences of σ in $\mathfrak{S}_n(\mathbf{321})$ for all σ with $|\sigma| \leqslant 3$ and as a consequence asymptotic formulas as $n \to \infty$ for these numbers. (The results in [27] are really stated for $\mathfrak{S}_n(\mathbf{123})$, which is equivalent.) These results in [17] and [27] imply (after correcting some typos in [27]), in our notation,

$$\mathbb{E}\, n_{\mathbf{21}}(\boldsymbol{\pi}_{\mathbf{321},n}) \sim \frac{\sqrt{\pi}}{4} n^{3/2}, \tag{1.4}$$

$$\mathbb{E}\, n_{\mathbf{231}}(\boldsymbol{\pi}_{\mathbf{321},n}) = \mathbb{E}\, n_{\mathbf{312}}(\boldsymbol{\pi}_{\mathbf{321},n}) \sim \frac{1}{4} n^2, \tag{1.5}$$

$$\mathbb{E}\, n_{\mathbf{132}}(\boldsymbol{\pi}_{\mathbf{321},n}) = \mathbb{E}\, n_{\mathbf{213}}(\boldsymbol{\pi}_{\mathbf{321},n}) \sim \frac{\sqrt{\pi}}{8} n^{5/2}, \tag{1.6}$$

$$\mathbb{E}\, n_{\mathbf{123}}(\boldsymbol{\pi}_{\mathbf{321},n}) \sim \binom{n}{3} \sim \frac{1}{6} n^3. \tag{1.7}$$

Moreover, the equivalence given by Cheng, Eu and Fu [17] between $n_{\mathbf{21}}(\boldsymbol{\pi}_{\mathbf{321},n})$ and the number of certain squares under a Catalan path implies by standard results for the area under the equivalent Dyck paths that, as $n \to \infty$,

$$n^{-3/2} n_{\mathbf{21}}(\boldsymbol{\pi}_{\mathbf{321},n}) \xrightarrow{\mathrm{d}} 2^{-1/2} \int_0^1 \mathbf{e}(x)\,\mathrm{d}x, \tag{1.8}$$

where the limit random variable is, up to a constant factor, the area under a Brownian excursion $\mathbf{e}$ (see e.g. [29] for many other results on this random area). See also the related expressions for the distribution of $n_{\mathbf{21}}(\boldsymbol{\pi}_{\mathbf{321},n})$ in Chen, Mei and Wang [16].

Apart from (1.8), we do not know any previous result on asymptotic distributions of $n_\sigma(\boldsymbol{\pi}_{\mathbf{321},n})$ beyond the expectations in (1.4)–(1.7).

Our main result is the following, using the notion of blocks defined in Subsection 2.1 below. The proof is given in Section 4 below, and is based on results for **321**-avoiding permutations by Hoffman, Rizzolo and Slivken [24; 25].

Theorem 1.2. *Let σ be a fixed **321**-avoiding permutation. Let $m := |\sigma|$, suppose that σ has ℓ blocks of lengths $m_1, \dots, m_\ell$, and let w_σ be the positive constant defined in* (3.3)*. Then, as $n \to \infty$,*

$$n_\sigma(\boldsymbol{\pi}_{\mathbf{321},n})/n^{(m+\ell)/2} \xrightarrow{\mathrm{d}} W_\sigma \tag{1.9}$$

for a positive random variable W_σ that can be represented as

$$W_\sigma = w_\sigma \int_{0<t_1<\dots<t_\ell<1} \mathbf{e}(t_1)^{m_1-1} \cdots \mathbf{e}(t_\ell)^{m_\ell-1}\,\mathrm{d}t_1 \cdots \mathrm{d}t_\ell, \tag{1.10}$$

where the random function $\mathbf{e}(t)$ is a Brownian excursion.

Moreover, the convergence (1.9) *holds jointly for any set of σ, with W_σ given by* (1.10) *with the same $\mathbf{e}$ for all σ.*

All moments of W_σ are finite, and all moments (including mixed moments) converge in (1.9)*. In particular,*

$$\mathbb{E}[n_\sigma(\boldsymbol{\pi}_{\mathbf{321},n})] \sim \mathbb{E}[W_\sigma] n^{(m+\ell)/2}, \tag{1.11}$$

$$\operatorname{Var}[n_\sigma(\boldsymbol{\pi}_{\mathbf{321},n})] \sim \operatorname{Var}[W_\sigma] n^{m+\ell}. \tag{1.12}$$

Example 1.3. Let $\sigma = \mathbf{21}$. Then $w_{\mathbf{21}} = 2^{-1/2}$ by Example 3.3; hence (1.9)–(1.10), with $\ell = 1$ and $m_1 = m = 2$, yield a new proof of (1.8).

We note two special cases when the multiple integral in (1.10) reduces to a single integral.

Example 1.4. If σ is indecomposable, i.e., has only one block (see Subsection 2.1), (1.10) yields

$$W_\sigma = w_\sigma \int_0^1 \mathbf{e}(t)^{m-1}\,\mathrm{d}t. \tag{1.13}$$

The special case $m = 2$ (i.e., $\sigma = \mathbf{21}$) yields, as said in Example 1.3, the Brownian excursion area in (1.8), which has been intensely studied, see e.g. [29] and the references there.

The case $m = 3$ (i.e., $\mathbf{231}$ or $\mathbf{312}$) yields the random variable $\int_0^1 \mathbf{e}(t)^2\,\mathrm{d}t$, which has been studied before by Nguyen The [40]; among other results he found a simple formula for the Laplace transform, which as noted in [30, Example 7.17] shows that the limit W_σ in this case, ignoring the constant factor w_σ, has the distribution denoted $S_{3/2}$ by Biane, Pitman and Yor [7].

The integral in (1.13) for a general m has been studied by Richard [43].

Example 1.5. If all blocks have the same size $m_1 = \cdots = m_\ell$, then, by symmetry, (1.10) yields

$$W_\sigma = \frac{w_\sigma}{\ell!}\biggl(\int_0^1 \mathbf{e}(t)^{m_1-1}\,\mathrm{d}t\biggr)^\ell. \tag{1.14}$$

Cf. Example 1.4 (the special case $\ell = 1$), and see again [29; 40; 43]. In particular, if all blocks have size 2, then W_σ is a constant times a power of the Brownian excursion area.

Theorem 1.2 should be compared to the similar result for $\mathbf{132}$-avoiding permutations in [30, Theorem 2.1], where also the limiting distributions can be expressed using a Brownian excursion, but in general in a much more complicated way, see [30, Section 7]. (At least, we do not know any simpler descriptions of those limit variables, although it is conceivable that such might exist.) In particular, the limits in (1.13) appear also as limits for $\mathbf{132}$-avoiding permutations, see [30, Examples 7.6–7.8].

Remark 1.6. The results obtained here for random $\mathbf{321}$-avoiding permutations, and in [30] for random $\mathbf{132}$-avoiding permutations, are very different from the non-restricted case of uniformly random permutations in $\mathfrak{S}_n$: it is well-known that if $\boldsymbol{\pi}$ is a uniformly random permutation in $\mathfrak{S}_n$, then $n_\sigma(\boldsymbol{\pi})$ has an asymptotic normal distribution as $n \to \infty$ for every fixed permutation σ, and that (as a consequence) $n_\sigma(\boldsymbol{\pi})$ is concentrated around its mean in the sense that $n_\sigma(\boldsymbol{\pi})/\mathbb{E}[n_\sigma(\boldsymbol{\pi})] \xrightarrow{\mathrm{p}} 1$ as $n \to \infty$. See Bóna [10; 12] and Janson, Nakamura and Zeilberger [32].

The moment convergence in Theorem 1.2 yields the asymptotic formula (1.11) for the expectation, involving the constant

$$\mathbb{E}\,W_\sigma = w_\sigma \int_{0<t_1<\cdots<t_\ell<1} \mathbb{E}\bigl[\mathbf{e}(t_1)^{m_1-1}\cdots\mathbf{e}(t_\ell)^{m_\ell-1}\bigr]\,\mathrm{d}t_1\cdots\mathrm{d}t_\ell. \tag{1.15}$$

We do not know any general formula for this integral, but it can be computed in many cases, and often higher moments too, see Section 5. In particular, we obtain the following:

Corollary 1.7. *If $\sigma \in \mathfrak{S}_m(\mathbf{321})$ is an indecomposable $\mathbf{321}$-avoiding permutation, then, as $n \to \infty$, with w_σ given by* (3.3)*,*

$$\mathbb{E}[n_\sigma(\boldsymbol{\pi}_{\mathbf{321},n})] \sim (\mathbb{E}\,W_\sigma)n^{(m+\ell)/2} = w_\sigma 2^{-(m-1)/2}\Gamma\bigl(\tfrac{m+1}{2}\bigr)n^{(m+1)/2}. \tag{1.16}$$

Similarly, (1.12) *holds with*

$$\operatorname{Var} W_\sigma = w_\sigma^2 2^{1-m}\Bigl(\frac{2(m-1)!}{m}\Bigl(1-\frac{m!^2}{(2m)!}\Bigr)-\Gamma\Bigl(\frac{m+1}{2}\Bigr)^2\Bigr). \tag{1.17}$$

Corollary 1.8. *If σ has two blocks, of lengths m_1 and m_2, then* (1.11) *holds with $\ell = 2$ and*

$$\mathbb{E}\, W_\sigma = w_\sigma 2^{-m/2}\frac{m}{m_1 m_2}\Bigl(1-\frac{m_1!\,m_2!}{m!}\Bigr)\Gamma\Bigl(\frac{m}{2}\Bigr), \tag{1.18}$$

where $m = |\sigma| = m_1 + m_2$ and w_σ is given by (3.3)*.*

In particular, in the cases $\sigma = \mathbf{21}$, $\mathbf{231}$, $\mathbf{312}$, Corollary 1.7 yields, using the values of w_σ in Example 3.3, the asymptotics in (1.4)–(1.5) obtained from [17] and [27]. Similarly, (1.6) follows from Corollary 1.8 (or by the method in Example 5.3), and (1.7) follows trivially from (1.11) since (1.10) yields $W_{\mathbf{123}} = 1/6$.

Remark 1.9. The general problem can be generalized to permutations avoiding a given set of permutations. Define, extending (1.2), $\mathfrak{S}_n(\tau_1,\dots,\tau_k) := \bigcap_{i=1}^k \mathfrak{S}_n(\tau_i)$, and let $\boldsymbol{\pi}_{\tau_1,\dots,\tau_k;n}$ be a uniformly random permutation in the set $\mathfrak{S}_n(\tau_1,\dots,\tau_k)$. The size $|\mathfrak{S}_n(\tau_1,\dots,\tau_k)|$ was found for all cases with $k \geqslant 2$ and all $|\tau_i| = 3$ by Simion and Schmidt [46]; we give some simple results on the asymptotic distribution of $n_\sigma(\boldsymbol{\pi}_{\tau_1,\dots,\tau_k;n})$ for these cases in [31]. Somewhat surprisingly, there are cases with an asymptotic normal distribution similar to the one for random unrestricted permutations (see Remark 1.6), and thus quite different from the limiting distributions for $n_\sigma(\boldsymbol{\pi}_{\tau,n})$ for a single τ with $|\tau| = 3$ in the present paper and [30].

In the present paper we study only the numbers n_σ of occurences of some pattern in $\boldsymbol{\pi}_{\tau,n}$. There is also a number of papers studing other properties of random τ-avoiding permutations. Some examples, in addition to those mentioned above, are consecutive patterns [6]; descents and the major index [5]; number of fixed points [44; 21; 22; 39; 26]; position of fixed points [39; 26]; exceedances [21; 22]; longest increasing subsequence [19]; shape and distribution of individual values π_i [37; 38; 24].

2. Preliminaries

2.1. **Compositions and decompositions of permutations.** If $\sigma \in \mathfrak{S}_m$ and $\tau \in \mathfrak{S}_n$, their *composition* $\sigma * \tau \in \mathfrak{S}_{m+n}$ is defined by letting τ act on $[m+1, m+n]$ in the natural way; more formally, $\sigma * \tau = \pi \in \mathfrak{S}_{m+n}$ where $\pi_i = \sigma_i$ for $1 \leqslant i \leqslant m$, and $\pi_{j+m} = \tau_j + m$ for $1 \leqslant j \leqslant n$. It is easily seen that $*$ is an associative operation that makes $\mathfrak{S}_*$ into a semigroup (without unit, since we only consider permutations of length $\geqslant 1$). We say that a permutation $\pi \in \mathfrak{S}_*$ is *decomposable* if $\pi = \sigma * \tau$ for some $\sigma, \tau \in \mathfrak{S}_*$, and *indecomposable* otherwise; we also call an indecomposable permutation a *block*. Equivalently, $\pi \in \mathfrak{S}_n$ is decomposable if and only if $\pi : [m] \to [m]$ for some $1 \leqslant m < n$. See e.g. [18, Exercise VI.14].

It is easy to see that any permutation $\pi \in \mathfrak{S}_*$ has a unique decomposition $\pi = \pi_1 * \cdots * \pi_\ell$ into indecomposable permutations (blocks) $\pi_1, \dots, \pi_\ell$ (for some, unique, $\ell \geqslant 1$); we call these the *blocks of* π.

An *inversion* in a permutation π is an occurence (i,j) of the pattern **21**. Given a permutation $\pi \in \mathfrak{S}_n$, its *inversion graph* Γ_π is the graph with vertex set $[n]$, and an edge ij for every inversion (i,j) in π. (This is the same as the intersection graph of the set of line segments $[(i,0),(\pi_i,1)] \subset \mathbb{R}^2$. The graphs isomorphic to Γ_π for some permutation π are known as *permutation graphs*, see e.g. [14].)

It is easy to see that the connected components of the inversion graph Γ_π are precisely the blocks of π; in particular, π is indecomposable if and only if Γ_π is connected, see [36].

2.2. **321-avoiding permutations.** Given any permutation $\pi \in \mathfrak{S}_n$, let

$$E_+ = E_+(\pi) := \{i \in [n] : \pi_i > i\}, \tag{2.1}$$

$$E_- = E_-(\pi) := [n] \setminus E_+(\pi) = \{i \in [n] : \pi_i \leqslant i\}. \tag{2.2}$$

Thus E_+ and E_- form a partition of $[n]$. E_+ is known as the set of *exceedances* of π.

It is well-known that a permutation π is **321**-avoiding if and only if π is the union of two increasing subsequences, and in particular, if $\pi \in \mathfrak{S}_*(\mathbf{321})$, then the subsequences with indices in E_+ and E_- are increasing. (This is easy to see directly; it also follows from the BJS bijection in Subsection 2.3.) In other words, if $i < j$ and $i,j \in E_+$ or $i,j \in E_-$, then $\pi_i < \pi_j$. Furthermore, if $i < j$ and $i \in E_-$, $j \in E_+$, then $\pi_i \leqslant i < j < \pi_j$. Consequently:

$$\text{If } (i,j) \text{ is an inversion in } \pi \in \mathfrak{S}_*(\mathbf{321}), \text{ then } i \in E_+(\pi) \text{ and } j \in E_-(\pi). \tag{2.3}$$

2.3. **Dyck paths and the BJS bijection.** A *Dyck path* of length $2n \geqslant 0$ is a mapping $\gamma : \{0,\dots,2n\} \to \mathbb{Z}$ such that $\gamma(0) = \gamma(2n) = 0$, $\gamma(x) \geqslant 0$ for every x, and $|\gamma(x+1) - \gamma(x)| = 1$ for all $x \in \{0,\dots,2n-1\}$. We identify a Dyck path with the corresponding continuous function $\gamma : [0,2n] \to \mathbb{R}$ obtained by linear interpolation. Let $\mathfrak{D}_{2n}$ be the set of Dyck paths of length $2n$. It is well-known that $|\mathfrak{D}_{2n}| = C_n$, the n-th Catalan number in (1.3), see e.g. [47, Exercise 6.19(i)].

We use, as [24; 25], a bijection between $\mathfrak{D}_{2n}$ and $\mathfrak{S}_n(\mathbf{321})$, i.e., between Dyck paths of length $2n$ and **321**-avoiding permutations of length n; the bijection is known as the BJS bijection after Billey, Jockusch and Stanley [8] and can be described as follows. (See also [15] for more on this and on other bijections between $\mathfrak{D}_{2n}$ and $\mathfrak{S}_n(\mathbf{321})$.)

Fix a Dyck path $\gamma \in \mathfrak{D}_{2n}$, and let m be the number of increases (or decreases) in γ. Let $a_i \geqslant 1$ be the length of the i-th run of increases, and let $d_i \geqslant 1$ be the length of the i-th run of decreases in γ. Let, for $0 \leqslant i \leqslant m$, $A_i := \sum_{j=1}^i a_j$ and $D_i := \sum_{j=1}^i d_j$; let $\mathcal{A} := \{A_i : 1 \leqslant i \leqslant m-1\}$, $\mathcal{A}_1 := \{A_i + 1 : 1 \leqslant i \leqslant m-1\}$, $\mathcal{D} := \{D_i : 1 \leqslant i \leqslant m-1\}$, $\mathcal{A}_1^{\mathsf{c}} := [n] \setminus \mathcal{A}_1$, and $\mathcal{D}^{\mathsf{c}} := [n] \setminus \mathcal{D}$. Finally, define the permutation $\pi_\gamma \in \mathfrak{S}_n$ as the unique permutation with $\pi : \mathcal{D} \to \mathcal{A}_1$, and therefore $\pi : \mathcal{D}^{\mathsf{c}} \to \mathcal{A}_1^{\mathsf{c}}$, such that π is increasing on $\mathcal{D}$ and on $\mathcal{D}^{\mathsf{c}}$. (In particular, $\pi_\gamma(D_i) = A_i{+}1$ for $1 \leqslant i \leqslant m{-}1$.)

Then, $\gamma \to \pi_\gamma$ is a bijection of $\mathfrak{D}_n$ onto $\mathfrak{S}_n(\mathbf{321})$, see e.g. [8; 15]. Moreover [24, Lemma 2.1],

$$E_+(\pi_\gamma) = \mathcal{D}(\gamma), \qquad E_-(\pi_\gamma) = \mathcal{D}^{\mathsf{c}}(\gamma). \tag{2.4}$$

We define also, as in [24],

$$y_i := A_i - D_i = \gamma(A_i + D_i). \tag{2.5}$$

2.4. **Brownian excursion.** A (normalized) Brownian excursion $\mathbf{e}(t)$ is a random continuous function on $[0,1]$ that can be defined as a Brownian motion $B(t)$ conditioned on $B(1) = B(0) = 0$ and $B(t) \geqslant 0$, $t \in [0,1]$; since this means conditioning on an event of probability zero, the conditioning has to be interpreted with some care, e.g. as a suitable limit. See also [42, Chapter XII] for an alternative definition.

The distribution of the Brownian excursion $\mathbf{e}$ has also several other descriptions; for example, $\mathbf{e}$ has the same distribution as a Bessel bridge of dimension 3 over $[0,1]$, see e.g. [42, Theorem XII.(4.2)] and thus also as the absolute value of a 3-dimensional Brownian bridge, i.e.,

$$\mathbf{e}(t) \overset{\mathrm{d}}{=} \sqrt{\mathbf{b}_1(t)^2 + \mathbf{b}_2(t)^2 + \mathbf{b}_3(t)^2}, \qquad t \in [0,1], \tag{2.6}$$

where $\mathbf{b}_1, \dots \mathbf{b}_3$ are independent Brownian bridges.

2.5. **Some notation.** λ_d denotes d-dimensional Lebesgue measure.

For typographical reasons, we sometimes write $\pi(i)$ for π_i.

We say that an event $\mathcal{E}_n$ (depending on n, e.g. through $\boldsymbol{\pi}_{\mathbf{321},n}$) holds *with high probability* if $\mathbb{P}(\mathcal{E}_n) \to 1$ as $n \to \infty$, and *with very high probability* if $\mathbb{P}(\mathcal{E}_n) = 1 - O(e^{-n^c})$ for some $c > 0$; note that the latter implies $\mathbb{P}(\mathcal{E}_n) = 1 - O(n^{-C})$ for any $C > 0$.

We let c and C, possibly with subscripts, denote unspecified positive constants that may depend on σ; they may vary between different occurrences.

3. The parameter w_σ

Let σ be a $\mathbf{321}$-avoiding permutation.

First, assume that σ is a block with $m = |\sigma| > 1$. In this case, let Π_σ be the set of all vectors $(x_2, \dots, x_m) \in [0,\infty)^{m-1}$ such that, with $x_1 = 0$,

- (i) $0 = x_1 \leqslant x_2 \leqslant \dots \leqslant x_m$;
- (ii) If $i < j$, $i \in E_+(\sigma)$ and $j \in E_-(\sigma)$, then
  - (a) if $\sigma_i < \sigma_j$, then $x_j \geqslant x_i + 1$;
  - (b) if $\sigma_i > \sigma_j$, then $x_j \leqslant x_i + 1$.

By (2.3), (ii)(b) applies whenever (i,j) is an inversion in σ. Hence, $|x_i - x_j| \leqslant 1$ whenever ij is an edge in the inversion graph Γ_σ, and since the inversion graph is connected (because σ is assumed to be a block), it follows that

$$|x_i| = |x_i - x_1| \leqslant m - 1 \tag{3.1}$$

for every $i \leqslant m$. Consequently, the set Π_σ is bounded, and since it is defined as an intersection of closed half-planes, Π_σ is compact and a polytope. It is also easy to see that Π_σ has a nonempty interior Π_σ°, obtained by taking strict inequalities in (i)–(ii). Let $v_\sigma := \lambda_{m-1}(\Pi_\sigma)$, the volume of the polytope Π_σ; thus $0 < v_\sigma < \infty$.

Next, for a **321**-avoiding block σ, let

$$w_\sigma := \begin{cases} 2^{(|\sigma|-3)/2} v_\sigma, & \sigma \text{ is a block with } |\sigma| > 1, \\ 1, & |\sigma| = 1. \end{cases} \tag{3.2}$$

Finally, for an arbitrary **321**-avoiding permutation σ with blocks $\sigma^1, \dots, \sigma^\ell$, define

$$w_\sigma := \prod_{i=1}^{\ell} w_{\sigma^i}. \tag{3.3}$$

Example 3.1. For $\sigma = \mathbf{21}$, we only have to consider the case $i = 1$, $j = 2$ for (ii) in the definition of Π_σ; in this case (ii)(b) applies, and yields $x_2 \leqslant 1$. Together with (i) we obtain $0 \leqslant x_2 \leqslant 1$, so $\Pi_{21} = [0, 1]$, and

$$v_{\mathbf{21}} = 1. \tag{3.4}$$

For both $\sigma = \mathbf{231}$ and $\sigma = \mathbf{312}$, we similarly obtain $\Pi_\sigma : \{(x_2, x_3) : 0 \leqslant x_2 \leqslant x_3 \leqslant 1\}$. Thus

$$v_{\mathbf{231}} = v_{\mathbf{312}} = \tfrac{1}{2}. \tag{3.5}$$

Similarly, elementary calculations show that for the 5 blocks in $\mathfrak{S}_4(\mathbf{321})$,

$$v_{\mathbf{2341}} = v_{\mathbf{2413}} = v_{\mathbf{3142}} = v_{\mathbf{3412}} = v_{\mathbf{4123}} = \tfrac{1}{6}. \tag{3.6}$$

However, for longer blocks, v_σ depends not only on the length $|\sigma|$. For example, omitting the calculations,

$$v_{\mathbf{23451}} = v_{\mathbf{51234}} = \tfrac{1}{24}, \qquad v_{\mathbf{24153}} = \tfrac{2}{24}, \tag{3.7}$$

$$v_{\mathbf{234561}} = v_{\mathbf{612345}} = \tfrac{1}{120}, \qquad v_{\mathbf{315264}} = \tfrac{5}{120}. \tag{3.8}$$

Problem 3.2. Based on these and other similar examples, we conjecture that for every block $\sigma \in \mathfrak{S}_*(\mathbf{321})$, $v_\sigma = \nu_\sigma/(|\sigma|-1)!$ for some integer $\nu_\sigma \geqslant 1$. Prove this! Moreover, if this holds, find a combinatorial interpretation of ν_σ.

Example 3.3. The values for v_σ in Example 3.1 yield by (3.2)–(3.3)

$$w_{\mathbf{21}} = 2^{-1/2} v_{\mathbf{21}} = 1/\sqrt{2}, \tag{3.9}$$

$$w_{\mathbf{231}} = w_{\mathbf{312}} = 1/2, \tag{3.10}$$

$$w_{\mathbf{132}} = w_{\mathbf{213}} = w_{\mathbf{1}} w_{\mathbf{21}} = 1/\sqrt{2}, \tag{3.11}$$

$$w_{\mathbf{123}} = w_{\mathbf{1}} w_{\mathbf{1}} w_{\mathbf{1}} = 1. \tag{3.12}$$

As said above, (3.9)–(3.12) combine with Corollary 1.7 and (1.15) to yield (1.4)–(1.7); furthermore, (3.9) and Theorem 1.2 yield (1.8).

4. Proof of Theorem 1.2

The proof of Theorem 1.2 is rather long, and will be interspersed with several lemmas.

Suppose that $\sigma \in \mathfrak{S}_*(\mathbf{321})$ is fixed and that $\pi \in \mathfrak{S}_n(\mathbf{321})$ (for a large n) Consider first the case when σ is a block.

Lemma 4.1. *Suppose that $\sigma \in \mathfrak{S}_m(\mathbf{321})$ is a block with $m = |\sigma| > 1$. If $\pi \in \mathfrak{S}_n(\mathbf{321})$ and $1 \leqslant k_1 < \cdots < k_m \leqslant n$, then $\mathbf{k} := (k_1, \dots, k_m)$ is an occurrence of σ in π if and only if:*

(i) *$k_i \in E_+(\pi)$ for $i \in E_+(\sigma)$;*
(ii) *$k_i \in E_-(\pi)$ for $i \in E_-(\sigma)$;*
(iii) *if $i < j$ with $i \in E_+(\sigma)$ and $j \in E_-(\sigma)$, then*

$$\pi_{k_i} > \pi_{k_j} \iff \sigma_i > \sigma_j. \tag{4.1}$$

Proof. Note first that by definition, $\mathbf{k}$ is an occurrence of σ if and only if (4.1) holds for every pair (i, j) with $1 \leqslant i < j \leqslant m$; the point of (iii) is that we only have to check this for certain pairs (i, j).

$\Longrightarrow$: Suppose that $\mathbf{k}$ is an occurrence of σ.

Let $i \in E_+(\sigma)$. Since σ is a block, its inversion graph Γ_σ is connected. Hence there is an inversion (i, j) for some $j > i$ or an inversion (j, i) for some $j < i$, but the latter is impossible when $i \in E_+$ by (2.3). Consequently there is an inversion (i, j) in σ, and then (k_i, k_j) must be an inversion in π; in particular, $k_i \in E_+(\pi)$ by (2.3). Hence (i) holds.

The proof of (ii) is similar.

Finally, (4.1) holds, as noted above, for all pairs (i, j) with $i < j$.

$\Longleftarrow$: Conversely, suppose that (i)–(iii) hold, and let $i < j$. If $i, j \in E_+(\sigma)$, then $k_i, k_j \in E_+(\pi)$ by (i), and thus (2.3) implies that both $\sigma_i < \sigma_j$ and $\pi_{k_i} < \pi_{k_j}$; hence (4.1) holds in this case. Similarly, (4.1) holds if $i, j \in E_-(\sigma)$, or if $i \in E_-(\sigma)$ and $j \in E_+(\sigma)$. Finally, in the remaining case $i \in E_+(\sigma)$ and $j \in E_-(\sigma)$, (iii) applies. Hence, (4.1) holds for every pair (i, j) with $1 \leqslant i < j \leqslant m$, and thus $\mathbf{k}$ is an occurrence of σ. $\square$

Let

$$\Delta_i = \Delta_i(\pi) := \pi_i - i, \qquad i \in [n], \tag{4.2}$$

and note that $\Delta_i > 0$ if $i \in E_+(\pi)$ and $\Delta_i \leqslant 0$ if $i \in E_-(\pi)$.

Lemma 4.2. *Lemma 4.1 holds also if* (4.1) *in* (iii) *is replaced by*

$$\sigma_i > \sigma_j \iff k_j - k_i < |\Delta_{k_i}| + |\Delta_{k_j}|. \tag{4.3}$$

Proof. Suppose that (i)–(ii) hold, and that $i < j$ with $i \in E_+(\sigma)$ and $j \in E_-(\sigma)$. Then $k_i \in E_+(\pi)$ and $k_j \in E_-(\pi)$, and thus

$$\pi_{k_i} - \pi_{k_j} = k_i - k_j + \Delta_{k_i} - \Delta_{k_j} = k_i - k_j + |\Delta_{k_i}| + |\Delta_{k_j}|. \tag{4.4}$$

Consequently, (4.1) holds if and only if (4.3) does. $\square$

Before proceeding, we use Lemma 4.2 to give a useful upper bound for $n_\sigma(\pi)$. Let

$$\bar\Delta = \bar\Delta(\pi) := \max_{1\leqslant i\leqslant n} |\Delta_i|. \tag{4.5}$$

Then, (4.3) implies that $0 \leqslant k_j - k_i \leqslant 2\bar\Delta$ when (i,j) is an inversion in σ. Since the inversion graph Γ_σ is connected, this implies

$$0 < k_i - k_1 \leqslant 2m\bar\Delta, \qquad i = 2,\dots,m. \tag{4.6}$$

Hence, the number of occurrences $\mathbf{k}$ of σ with a given choice of k_1 is at most $(2m\bar\Delta)^{m-1}$, and thus

$$n_\sigma(\pi) \leqslant (2m)^{m-1} n\bar\Delta^{m-1} = O(n\bar\Delta^{m-1}). \tag{4.7}$$

Now let $\pi = \boldsymbol{\pi}_{\mathbf{321},n}$ be random. By the BJS bijection, the uniformly random $\boldsymbol{\pi}_{\mathbf{321},n}$ corresponds to a uniformly random Dyck path $\gamma \in \mathfrak{D}_{2n}$ by $\boldsymbol{\pi}_{\mathbf{321},n} = \pi_\gamma$. We use the notation in Subsection 2.3; we sometimes write γ, $\pi = \pi_\gamma$, or σ as arguments of various sets or quantities for clarity, but often we omit them.

It is well-known that a random Dyck path converges in distribution to a Brownian excursion after suitable normalization as $n \to \infty$. To be precise,

$$\frac{\gamma(2nt)}{\sqrt{2n}} \overset{\mathrm{d}}{\longrightarrow} \mathbf{e}(t) \tag{4.8}$$

as random elements of $C[0,1]$, see [33]. We use the Skorohod coupling theorem [34, Theorem 4.30], and may thus assume in the remainder of the proof that the Dyck paths, and thus the permutations $\boldsymbol{\pi}_{\mathbf{321},n}$, are coupled for different n such that (4.8) holds a.s. In other words,

$$\gamma(i) = \sqrt{2n}\Bigl(\mathbf{e}\Bigl(\frac{i}{2n}\Bigr) + o(1)\Bigr), \tag{4.9}$$

where, as throughout this proof, $o(1) \to 0$ as $n \to \infty$, uniformly in $i \in [n]$ (and in other similar variables later). However, the $o(1)$ may depend on the random $\boldsymbol{\pi}_{\mathbf{321},n}$, γ and $\mathbf{e}$. $O(\dots)$ below is interpreted similarly.

Hoffman, Rizzolo and Slivken [24, Section 2] show that a random Dyck path with very high probability satisfies some regularity properties there called 'Petrov conditions', moreover, they show some deterministic consequences of these properties (at least for large n). By the Borel–Cantelli lemma, the 'Petrov conditions' thus a.s. hold for all large n, so we may assume that these conditions and their consequences hold for γ.

In particular, by [24, Lemma 2.7], if $j \in \mathcal{D}$, then $|\pi_\gamma(j) - j - \gamma(2j)| < 10n^{0.4}$, while if $j \notin \mathcal{D}$, then $|\pi_\gamma(j) - j + \gamma(2j)| < 10n^{0.4}$. Hence, recalling the notation (4.2) (with $\pi = \pi_\gamma$) and using (4.9) and (2.4),

$$\Delta_j = \begin{cases} \gamma(2j) + O(n^{0.4}) = \sqrt{2n}\,\mathbf{e}(j/n) + o\bigl(n^{1/2}\bigr), & j \in \mathcal{D} = E_+(\pi_\gamma). \\ -\gamma(2j) + O(n^{0.4}) = -\sqrt{2n}\,\mathbf{e}(j/n) + o\bigl(n^{1/2}\bigr), & j \in \mathcal{D}^{\mathsf{c}} = E_-(\pi_\gamma), \end{cases} \tag{4.10}$$

and consequently, for all $j \in [n]$,

$$|\Delta_j| = \sqrt{2n}\,\mathbf{e}(j/n) + o\bigl(n^{1/2}\bigr). \tag{4.11}$$

Note that (4.11) implies, by the definition (4.5),

$$\bar{\Delta} = O\bigl(n^{1/2}\bigr). \tag{4.12}$$

Let, for $k \in [n]$, $\mathcal{A}_k$ be the set of all occurrences $\mathbf{k} = (k_1, \dots, k_m)$ of σ in π_γ such that $k_1 = k$. Thus $n_\sigma(\pi_\gamma) = \sum_{k=1}^{n} |\mathcal{A}_k|$.

We have shown above that if $\mathbf{k} \in \mathcal{A}_k$, then (4.6) holds, and thus, using (4.12),

$$|k_i - k| = O(\bar{\Delta}) = O\bigl(n^{1/2}\bigr) = o(n). \tag{4.13}$$

Since $\mathbf{e}(t)$ is continuous, it thus follows from (4.11) that

$$|\Delta_{k_i}| = \sqrt{2n}\,\mathbf{e}(k_i/n) + o\bigl(n^{1/2}\bigr) = \sqrt{2n}\,\mathbf{e}(k/n) + o\bigl(n^{1/2}\bigr). \tag{4.14}$$

Hence, in (4.3), we have

$$|\Delta_{k_i}| + |\Delta_{k_j}| = 2^{3/2} n^{1/2} \mathbf{e}(k/n) + o\bigl(n^{1/2}\bigr). \tag{4.15}$$

Motivated by (4.15), let $\mathcal{A}'_k$ be the set of m-tuples $\mathbf{k} = (k_1, \dots, k_m)$ with $k = k_1 < \dots < k_m$ such that Lemma 4.1(i)–(ii) hold, and, furthermore, for every $i \in E_+(\sigma)$ and $j \in E_-(\sigma)$ with $i < j$,

$$\sigma_i > \sigma_j \iff k_j - k_i < 2^{3/2} n^{1/2} \mathbf{e}(k/n). \tag{4.16}$$

Note that this agrees with the characterization of $\mathcal{A}_k$ implied by Lemma 4.2 except that the bound $|\Delta_{k_i}| + |\Delta_{k_j}|$ in (4.3) is replaced by $2^{3/2} n^{1/2} \mathbf{e}(k/n)$. Consequently, if $\mathbf{k} \in \mathcal{A}_k \Delta \mathcal{A}'_k$, then for some pair (i, j) either

$$|\Delta_{k_i}| + |\Delta_{k_j}| \leqslant k_j - k_i \leqslant 2^{3/2} n^{1/2} \mathbf{e}(k/n) \tag{4.17}$$

or conversely.

Furthermore, if $\mathbf{k} \in \mathcal{A}'_k$, then (4.16) shows that

$$0 \leqslant k_j - k_i \leqslant 2^{3/2} n^{1/2} \max_t \mathbf{e}(t) = O\bigl(n^{1/2}\bigr) \tag{4.18}$$

for every inversion (i, j) of σ, and thus, by the argument used above for (4.6), $k_i - k = O\bigl(n^{1/2}\bigr)$ for every $i \leqslant m$; as a consequence, (4.14) holds for $\mathbf{k} \in \mathcal{A}'_k$ too.

It follows that if $\mathbf{k} \in \mathcal{A}_k \Delta \mathcal{A}'_k$, then $|k_i - k| = O\bigl(n^{1/2}\bigr)$ for $i = 2, \dots, m$, and furthermore, for some pair (i, j) with $1 \leqslant i < j \leqslant m$,

$$\bigl|k_j - k_i - 2^{3/2} n^{1/2} \mathbf{e}(k/n)\bigr| = o\bigl(n^{1/2}\bigr). \tag{4.19}$$

It follows that

$$\bigl|\mathcal{A}_k \Delta \mathcal{A}'_k\bigr| = o\bigl(n^{(m-1)/2}\bigr). \tag{4.20}$$

Hence, we may in the sequel consider $\mathcal{A}'_k$ instead of $\mathcal{A}_k$.

Next, let $\mathcal{A}^0$ be the set of m-tuples $\mathbf{k} = (k_1, \dots, k_m) \in [1, n]^m$ such that Lemma 4.1(i)–(ii) hold (with $\pi = \pi_\gamma$); i.e.,

$$\mathcal{A}^0 = \prod_{i=1}^{m} E_{\varepsilon_i}(\pi_\gamma), \tag{4.21}$$

where $\varepsilon_i \in \{+, -\}$ is such that $i \in E_{\varepsilon_i}(\sigma)$. Note also that $\sigma_1 > 1$ since σ is a block of length > 1, and thus $1 \in E_+(\sigma)$, i.e., $\varepsilon_1 = +$.

Furthermore, let $\mathcal{B}_k$ be the set of m-tuples $\mathbf{k} = (k_1, \dots, k_m) \in [1, n]^m$ such that $k_1 = k$ and

$$(k_2 - k, \dots, k_m - k) \in \mathcal{P}_k := 2^{3/2} n^{1/2} \mathbf{e}(k/n) \Pi_\sigma, \tag{4.22}$$

where Π_σ is the polytope defined in Section 3; note that this means that $k = k_1 \leqslant k_2 \dots \leqslant k_m$ and that the equivalences (4.16) hold (for pairs (i, j) as above), except in some cases of equality. Consequently, $\mathcal{A}'_k$ equals $\mathcal{A}^0 \cap \mathcal{B}_k$, except possibly for some points on the boundary, and thus, recalling (4.20).

$$|\mathcal{A}_k| = |\mathcal{A}'_k| + o\big(n^{(m-1)/2}\big) = |\mathcal{A}^0 \cap \mathcal{B}_k| + o\big(n^{(m-1)/2}\big). \tag{4.23}$$

Furthermore, (4.22) implies that, recalling $\lambda_{m-1}(\Pi_\sigma) = v_\sigma$,

$$\begin{aligned} |\mathcal{B}_k| &= \lambda_{m-1}\big(\mathcal{P}_k\big) + O\big(n^{(m-2)/2}\big) \\ &= \big(2^{3/2} n^{1/2} \mathbf{e}(k/n)\big)^{m-1} v_\sigma + O\big(n^{(m-2)/2}\big). \end{aligned} \tag{4.24}$$

The idea is now that roughly each second point belongs to $E_+(\pi_\gamma)$ and each second to $E_-(\pi_\gamma)$, and thus $|\mathcal{A}^0 \cap \mathcal{B}_k| \approx 2^{-(m-1)} |\mathcal{B}_k|$. We make this precise in the following lemma.

Lemma 4.3. *If the 'Petrov conditions' hold for γ, and $1 \leqslant a \leqslant b \leqslant n$, then*

$$|[a, b] \cap \mathcal{D}| = \tfrac{1}{2}(b - a) + O\big((b - a)^{0.6} + n^{0.18}\big). \tag{4.25}$$

Here, the $O(\dots)$ is uniform in all such γ, a and b.

Proof. Since (4.25) is trivial for small n, we may assume that n is large enough when needed below.

The 'Petrov conditions' [24, Definition 2.3] include that if $|j - i| \geqslant n^{0.3}$, then

$$|D_j - D_i - 2(j - i)| < 0.1 |i - j|^{0.6}. \tag{4.26}$$

If $|j - i| < n^{0.3}$, let $\ell = \min(i, j) - \lceil n^{0.3} \rceil$ or $\ell = \max(i, j) + \lceil n^{0.3} \rceil$, chosen such that $\ell \in [1, n]$. Then, by (4.26) for the pairs (i, ℓ) and (j, ℓ) and the triangle inequality,

$$|D_j - D_i - 2(j - i)| < 0.2\big(|j - i| + \lceil n^{0.3} \rceil\big)^{0.6} = O\big(|j - i|^{0.6} + n^{0.18}\big). \tag{4.27}$$

Now, let i and j be such that $D_{i-1} < a \leqslant D_i$ and $D_{j-1} < b \leqslant D_j$. Then $[a, b) \cap \mathcal{D} = \{D_i, \dots, D_{j-1}\}$ and thus $|[a, b) \cap \mathcal{D}| = j - i$. Furthermore, by [24, Lemma 2.5],

$$|a - D_i| \leqslant |D_i - D_{i-1}| \leqslant n^{0.18}, \qquad |b - D_j| \leqslant |D_j - D_{j-1}| \leqslant n^{0.18}. \tag{4.28}$$

Consequently, using (4.28) together with (4.26) or (4.27),

$$b - a = D_j - D_i + O\bigl(n^{0.18}\bigr) = 2(j - i) + O\bigl(|j - i|^{0.6} + n^{0.18}\bigr). \tag{4.29}$$

This yields (4.25), since (for the error term) either $j - i = 0$ or $j - i \leqslant 1 + D_{j-1} - D_i \leqslant b - a$. □

Let $N := \lfloor n^{0.6} \rfloor$ and $i_\nu := \lfloor in/N \rfloor$, $0 \leqslant \nu \leqslant N$. Partition $(0, n]$ into N intervals $I_\nu = (i_{\nu-1}, i_\nu]$, $1 \leqslant \nu \leqslant N$, of lengths $|I_\nu| = n^{0.4} + O(1)$.

For $\nu_2 \dots \nu_m \in [N]$, let $\mathcal{Q}_{k;\nu_2,\dots,\nu_m} := \{k\} \times \prod_{j=2}^m I_{\nu_j}$, and let

$$\mathcal{N}_k := \{(\nu_2, \dots, \nu_m) : \nu_2 < \dots < \nu_m \text{ and } \mathcal{Q}_{k;\nu_2,\dots,\nu_m} \subseteq \mathcal{B}_k\} \tag{4.30}$$

$$\mathcal{B}''_k := \bigcup_{(\nu_2,\dots,\nu_m)\in\mathcal{N}_k} \mathcal{Q}_{k;\nu_2,\dots,\nu_m}. \tag{4.31}$$

Thus, $\mathcal{B}''_k \subseteq \mathcal{B}_k$. Furthermore, if $\mathbf{k} = (k_1, \dots, k_m) \in \mathcal{B}_k \setminus \mathcal{B}''_k$, then $\mathbf{k} \in Q_{k:\nu_2,\dots,\nu_m}$ for some $\nu_2, \dots, \nu_m$ such that either $\nu_j = \nu_{j+1}$ for some j, or $Q_{k:\nu_2,\dots,\nu_m} \not\subseteq \mathcal{B}_k$; in both cases, the point in $\mathcal{P}_k$ corresponding to $\mathbf{k}$ by (4.22) has distance $O\bigl(n^{0.4}\bigr)$ to the boundary of $\mathcal{P}_k$, and it follows that

$$|\mathcal{B}_k \setminus \mathcal{B}''_k| = O\bigl(n^{(m-2)/2+0.4}\bigr) = o\bigl(n^{(m-1)/2}\bigr). \tag{4.32}$$

Let

$$\mathcal{A}''_k := \mathcal{A}^0 \cap \mathcal{B}''_k = \bigcup_{(\nu_2,\dots,\nu_m)\in\mathcal{N}_k} \mathcal{A}^0 \cap \mathcal{Q}_{k;\nu_2,\dots,\nu_m}. \tag{4.33}$$

Then $\mathcal{A}''_k \subseteq \mathcal{A}^0 \cap \mathcal{B}_k = \mathcal{A}'_k$, and

$$|\mathcal{A}'_k \setminus \mathcal{A}''_k| \leqslant |\mathcal{B}_k \setminus \mathcal{B}''_k| = o\bigl(n^{(m-1)/2}\bigr). \tag{4.34}$$

Furthermore, for each $(i_2, \dots, i_m) \in \mathcal{N}_k$, (4.21) shows that if $k \in E_+(\pi_\gamma) = E_{\varepsilon_1}(\pi_\gamma)$, then

$$\mathcal{A}^0 \cap \mathcal{Q}_{k;\nu_2,\dots,\nu_m} = \{k\} \times \prod_{j=2}^m \bigl(I_{\nu_j} \cap E_{\varepsilon_j}(\pi_\gamma)\bigr). \tag{4.35}$$

Furthermore, $E_+(\pi_\gamma) = \mathcal{D}$ and $E_-(\pi_\gamma) = [n] \setminus \mathcal{D}$ by (2.4), and thus Lemma 4.3 shows that $|I_{\nu_j} \cap E_{\varepsilon_j}| = \frac{1}{2}|I_{\nu_j}|\bigl(1 + o(1)\bigr)$ for every j, regardless of the value of ε_j. Hence we obtain, using (4.33), (4.35), and the fact that $|\mathcal{B}''_k| \leqslant |\mathcal{B}_k| = O(n^{(m-1)/2})$ by (4.24), provided $k \in E_+(\pi_\gamma)$,

$$\begin{aligned}
|\mathcal{A}''_k| &= \sum_{(\nu_2,\dots,\nu_m)\in\mathcal{N}_k} |\mathcal{A}^0 \cap \mathcal{Q}_{k;\nu_2,\dots,\nu_m}| \\
&= \sum_{(\nu_2,\dots,\nu_m)\in\mathcal{N}_k} \bigl(2^{-(m-1)} + o(1)\bigr)|\mathcal{Q}_{k;\nu_2,\dots,\nu_m}| \\
&= \bigl(2^{-(m-1)} + o(1)\bigr)|\mathcal{B}''_k| = 2^{-(m-1)}|\mathcal{B}''_k| + o\bigl(n^{(m-1)/2}\bigr).
\end{aligned} \tag{4.36}$$

Consequently, by (4.23), (4.34), (4.36), (4.32), (4.24),

$$\begin{aligned}|\mathcal{A}_k| &= |\mathcal{A}'_k| + o\bigl(n^{(m-1)/2}\bigr) = |\mathcal{A}''_k| + o\bigl(n^{(m-1)/2}\bigr)\\ &= 2^{-(m-1)}|\mathcal{B}''_k| + o\bigl(n^{(m-1)/2}\bigr) = 2^{-(m-1)}|\mathcal{B}_k| + o\bigl(n^{(m-1)/2}\bigr)\\ &= 2^{(m-1)/2}n^{(m-1)/2}\mathbf{e}(k/n)^{m-1}v_\sigma + o\bigl(n^{(m-1)/2}\bigr), \end{aligned} \tag{4.37}$$

provided $k \in \mathbb{E}_+(\pi_\gamma)$; otherwise $\mathcal{A}_k = \emptyset$.

Finally,

$$\begin{aligned} n_\sigma(\pi_\gamma) &= \sum_{k=1}^{n} |\mathcal{A}_k|\\ &= 2^{(m-1)/2}v_\sigma n^{(m-1)/2} \sum_{k\in E_+(\pi_\gamma)} \mathbf{e}(k/n)^{m-1} + o\bigl(n^{(m+1)/2}\bigr). \end{aligned} \tag{4.38}$$

For each interval I_ν, using the continuity of $\mathbf{e}$ and Lemma 4.3,

$$\begin{aligned} \sum_{k\in E_+(\pi_\gamma)\cap I_\nu} \mathbf{e}(k/n)^{m-1} &= \sum_{k\in E_+(\pi_\gamma)\cap I_\nu} \bigl(\mathbf{e}(i_\nu/n)^{m-1} + o(1)\bigr)\\ &= |E_+(\pi_\gamma)\cap I_\nu|\bigl(\mathbf{e}(i_\nu/n)^{m-1} + o(1)\bigr)\\ &= \tfrac12 |I_\nu| \mathbf{e}(i_\nu/n)^{m-1} + o\bigl(|I_\nu|\bigr)\\ &= \tfrac12 \int_{I_\nu} \mathbf{e}(x/n)^{m-1}\,\mathrm{d}x + o\bigl(|I_\nu|\bigr). \end{aligned} \tag{4.39}$$

Summing over all I_ν, we thus obtain by (4.38), recalling (and justifying) (3.2),

$$\begin{aligned} n_\sigma(\pi_\gamma) &= 2^{(m-3)/2}v_\sigma n^{(m-1)/2}\int_0^n \mathbf{e}(x/n)^{m-1}\,\mathrm{d}x + o\bigl(n^{(m+1)/2}\bigr)\\ &= w_\sigma n^{(m+1)/2}\int_0^1 \mathbf{e}(t)^{m-1}\,\mathrm{d}t + o\bigl(n^{(m+1)/2}\bigr). \end{aligned} \tag{4.40}$$

This proves (1.9)–(1.10) in the case $\ell = 1$, i.e., σ is a block, and $m > 1$. (The case $m = 1$ is trivial.)

Consider now the general case when σ has $\ell \geqslant 1$ blocks $\sigma^1, \dots, \sigma^\ell$. We continue with the assumptions above; in particular $\pi = \pi_\gamma$, (4.8) holds a.s., and the 'Petrov conditions' hold for γ.

Let $j_1, \dots, j_\ell$ be the positions in σ where the blocks start; thus $j_1 = 1$ and $j_{p+1} = j_p + m_p$, $1 \leqslant p < \ell$. Then $\mathbf{k} = (k_1, \dots, k_m)$ is an occurrence of σ in π if and only if each $(k_{j_p}, \dots, k_{j_p+m_p-1})$ is an occurrence of σ^p in π, and furthermore $k_i < k_j$ whenever $i < k_p \leqslant j$ for some p. In particular, this implies, with the obvious definition of $\mathcal{A}_k(\sigma^p)$,

$$k_{j_1} < k_{j_2} < \dots < k_{j_\ell} \quad \text{and} \quad (k_{j_p}, \dots, k_{j_p+m_p-1}) \in \mathcal{A}_{k_{j_p}}(\sigma^p) \text{ for all } p. \tag{4.41}$$

On the other hand, if (4.41) holds and furthermore $k_{j_{p+1}} > k_{j_p} + n^{0.6}$, say, for each $p < \ell$, then (4.13) implies that, assuming n is large enough, $\mathbf{k}$ is an occurrence of σ in π. Consequently, with $q_p := k_{j_p}$,

$$n_\sigma(\pi_\gamma) = \sum_{1\leqslant q_1<\dots<q_\ell\leqslant n} \prod_{p=1}^{\ell} |\mathcal{A}_{q_p}(\sigma^p)| + o\big(n^{(m+\ell)/2}\big). \tag{4.42}$$

We use again the intervals I_ν above, and obtain

$$n_\sigma(\pi_\gamma) = \sum_{\nu_1<\dots<\nu_\ell} \prod_{p=1}^{\ell} \Big(\sum_{q_p\in I_{\nu_p}} |\mathcal{A}_{q_p}(\sigma^p)| \Big) + o\big(n^{(m+\ell)/2}\big). \tag{4.43}$$

For each p with $m_p > 1$, we argue as in (4.38)–(4.39) and obtain

$$\sum_{q_p\in I_\nu} |\mathcal{A}_{q_p}(\sigma^p)| = w_{\sigma^p} n^{(m_p-1)/2} \int_{I_\nu} \mathbf{e}(x/n)^{m_p-1}\,\mathrm{d}x + o\big(n^{(m_p-1)/2}|I_\nu|\big). \tag{4.44}$$

Furthermore, if $m_p = 1$, then $\mathcal{A}_k(\sigma^p) = \{k\}$ and $|\mathcal{A}_k(\sigma^p)| = 1$ for every $k \in [n]$, and thus (4.44) holds trivially, with $w_\sigma = 1$ as given by (3.2).

Finally, (4.43)–(4.44) together with (3.3) yield, with W_σ given by (1.10),

$$\begin{aligned} n_\sigma(\pi_\gamma) &= w_\sigma n^{(m-\ell)/2} \sum_{\nu_1<\dots<\nu_\ell} \prod_{p=1}^{\ell} \int_{I_{\nu_p}} \mathbf{e}(x_p/n)^{m_p-1}\,\mathrm{d}x_p + o\big(n^{(m+\ell)/2}\big) \\ &= w_\sigma n^{(m-\ell)/2} \int_{0<x_1<\dots<x_\ell<n} \mathbf{e}(x_1/n)^{m_1-1}\cdots \mathbf{e}(x_\ell/n)^{m_\ell-1}\,\mathrm{d}x_1\cdots\mathrm{d}x_\ell \\ &\qquad + o\big(n^{(m+\ell)/2}\big) \\ &= n^{(m+\ell)/2} W_\sigma + o\big(n^{(m+\ell)/2}\big). \end{aligned}$$

This completes the proof of (1.9)–(1.10).

Since the proof shows a.s. convergence (under the coupling assumption in the proof), joint convergence for several σ follows immediately.

In order to show moment convergence, we first prove another lemma.

For a Dyck path $\gamma \in \mathfrak{D}_{2n}$, let

$$M(\gamma) := \max_{0\leqslant i\leqslant 2n} \gamma(i). \tag{4.45}$$

Lemma 4.4. (i) *Let $M_n := M(\gamma)$, where γ is a uniformly random Dyck path of length $2n$. Then, for every fixed $r < \infty$, the random variables $(M_n/n^{1/2})^r$, $n \geqslant 1$, are uniformly integrable.*

(ii) *Let $\bar\Delta_n := \bar\Delta(\boldsymbol{\pi}_{\mathbf{321},n})$. Then, for every fixed $r < \infty$, the random variables $(\bar\Delta_n/n^{1/2})^r$, $n \geqslant 1$, are uniformly integrable.*

Proof. (i): We use the well-known bijection between Dyck paths $\gamma \in \mathfrak{D}_{2n}$ and ordered rooted trees T_γ with $n+1$ vertices, where γ encodes the depth-first walk on T_γ, see e.g. [4; 20]. Then $M_n = \max\gamma = H(T_\gamma)$, the height of the tree T_γ. Furthermore, T_γ is a uniformly random ordered rooted tree with $n+1$ vertices, and can thus be represented as a conditioned Galton–Watson tree with a Geometric offspring distribution, see e.g. [2; 20]; hence we can apply [1, Theorem 1.2], and conclude that for all $n \geqslant 1$ and $x \geqslant 0$,

$$\mathbb{P}\big(M_n/\sqrt{n} \geqslant x\big) = \mathbb{P}\big(H(T_\gamma) \geqslant x\sqrt{n}\big) \leqslant Ce^{-c(x\sqrt{n})^2/(n+1)} \leqslant Ce^{-c_1x^2}. \tag{4.46}$$

Consequently, for any fixed $r > 0$,

$$\mathbb{E}\big(M_n/\sqrt{n}\big)^{r+1} = (r+1)\int_0^\infty x^r\,\mathbb{P}\big(M_n/\sqrt{n} \geqslant x\big) \leqslant C \tag{4.47}$$

and the conclusion follows, see [23, Theorem 5.4.2].

(ii): By the BJS bijection, the uniformly random $\boldsymbol{\pi}_{\mathbf{321},n}$ corresponds to a uniformly random Dyck path $\gamma \in \mathfrak{D}_{2n}$ by $\boldsymbol{\pi}_{\mathbf{321},n} = \pi_\gamma$. We use the notation in Subsection 2.3.

If $j \in \mathcal{D} = \mathcal{D}(\gamma)$, then $j = D_i$ for some i, and thus, using (2.5),

$$0 \leqslant \pi_\gamma(j) - j = (A_i + 1) - D_i = 1 + \gamma(A_i + D_i) \leqslant 1 + M(\gamma). \tag{4.48}$$

On the other hand, if $j \notin \mathcal{D}$, then $D_i < j < D_{i+1}$ for some i, and by [24, Lemmas 2.4 and 2.6], with very high probability $1 - O(n^{-r-1})$, $|\pi_\gamma(j) - j + y_i| < 7n^{0.4}$ and thus (for large n)

$$|\pi_\gamma(j) - j| \leqslant 7n^{0.4} + y_i \leqslant n^{0.5} + M(\gamma). \tag{4.49}$$

It follows from (4.48) and (4.49) that with very high probability,

$$\bar{\Delta}_n = \bar{\Delta}(\pi_\gamma) = \max_j |\pi_\gamma(j) - j| \leqslant n^{0.5} + M_n. \tag{4.50}$$

Let $\mathcal{E}_n$ be the event that (4.50) holds. Then the exceptional event $\mathcal{E}_n^{\mathsf{c}}$ has probability $O(n^{-r-1})$, say. Consequently, using (4.50) on $\mathcal{E}_n$ and the trivial bound $\bar{\Delta}_n \leqslant n$ on $\mathcal{E}_n^{\mathsf{c}}$, and applying (i),

$$\mathbb{E}\big(\bar{\Delta}_n/\sqrt{n}\big)^{r+1} \leqslant \mathbb{E}\big(1 + M_n/\sqrt{n}\big)^{r+1} + O\big(n^{(r+1)/2}\cdot n^{-r-1}\big) = O(1). \tag{4.51}$$

The conclusion follows, see again [23, Theorem 5.4.2]. □

Competion of the proof of Theorem 1.2. We have proved (1.9)–(1.10) above. Furthermore, (4.7) applied to each block σ^p shows that

$$n_\sigma(\pi_\gamma) \leqslant \prod_{p=1}^{\ell} n_{\sigma^p}(\pi_\gamma) = O\big(n^\ell \bar{\Delta}^{m-\ell}\big). \tag{4.52}$$

Hence, for any fixed $r > 0$,

$$\big(n_\sigma(\pi_\gamma)/n^{(m+\ell)/2}\big)^r \leqslant Cn^{r\ell - r(m+\ell)/2}\bar{\Delta}^{r(m-\ell)} = C\big(\bar{\Delta}/n^{1/2}\big)^{r(m-\ell)}, \tag{4.53}$$

which is uniformly integrable by Lemma 4.4. Consequently, the left-hand side of (4.53) is uniformly integrable, for any fixed $r > 0$, and the convergence in distribution (1.9) implies convergence of moments too. Convergence of mixed moments follows by the same argument. □

5. Moment calculations

Moments of the limiting random variable W_σ in (1.10), and thus asymptotics of the moments of $n_\sigma(\boldsymbol{\pi}_{\mathbf{321},n})$, can often be calculated explicitly. We do not know a single method that covers all cases, so we present here some different methods, with overlapping applicability. We give some example which illustrate the methods, and leave further cases to the reader.

5.1. **Using known results.** In the special cases in Examples 1.4 and 1.5, W_σ is (up to the constant w_σ) given by an integral $\int_0^1 \mathbf{e}(t)^k$, or by a power of this integral. Hence moments of W_σ are given by moments of this integral, and these moments can be found by recursion formulas, see [29] and the references there for $k = 1$, [40] for $k = 2$ and [43] for the general case.

Example 5.1. For $\sigma = \mathbf{231}$ and $\mathbf{312}$ we have by Theorem 1.2 and (3.10) the same limit in distribution

$$W_{\mathbf{231}} = W_{\mathbf{312}} = \frac{1}{2}\int_0^1 \mathbf{e}(t)^2 \,\mathrm{d}t. \tag{5.1}$$

(In fact, $n_{\mathbf{231}}(\boldsymbol{\pi}_{\mathbf{321},n})$ and $n_{\mathbf{312}}(\boldsymbol{\pi}_{\mathbf{321},n})$ have the same distribution for any n, as is easily seen because, in general, $n_{\sigma^{-1}}(\pi^{-1}) = n_\sigma(\pi)$.) By [40, Table 2], (5.1) yields e.g. $\mathbb{E}\, W_{\mathbf{231}} = 1/4$, $\mathbb{E}\, W_{\mathbf{231}}^2 = 19/240$ and $\mathbb{E}\, W_{\mathbf{231}}^3 = 631/20160$.

Example 5.2. Let $\sigma = \mathbf{214365}$. Thus σ consists of $\ell = 3$ blocks, which all are $\mathbf{21}$. Hence, (3.9) yields $w_\sigma = w_{\mathbf{21}}^3 = 2^{-3/2}$, and (1.14) yields

$$W_{\mathbf{214365}} = \frac{2^{-3/2}}{6}\Bigl(\int_0^1 \mathbf{e}(t)\,\mathrm{d}t\Bigr)^3. \tag{5.2}$$

Hence, using e.g. [29, Table 1], $\mathbb{E}\, W_{\mathbf{214365}} = 5\sqrt{\pi}/512$.

5.2. **The joint density function.** First, for any $0 < t_1 < \cdots < t_\ell$, the joint distribution of $\bigl(\mathbf{e}(t_1), \dots, \mathbf{e}(t_\ell)\bigr)$ has an explicit density, see [42, Section 11.3, page 464] (using the characterization of $\mathbf{e}(t)$ as a three-dimensional Bessel bridge). Thus, using (1.15), $\mathbb{E}\, W_\sigma$ can always be expressed as a 2ℓ-dimensional multiple integral; furthermore, higher moments can similarly be expressed using multiple integrals of higher dimensions. However, we do not know how to calculate these integrals, except in the simplest cases.

In particular, this method works well for the expectation in the special case when there is only one non-trivial block (i.e., a block of length > 1). A special case of the joint density given in [42, Section XI.3] is that for any fixed $t \in (0, 1)$, $\mathbf{e}(t)$ is a positive random variable with the density

$$\frac{\sqrt{2}}{\sqrt{\pi t^3(1-t)^3}} x^2 e^{-x^2/(2t(1-t))}, \qquad x > 0. \tag{5.3}$$

(This also follows easily from (2.6).) Furthermore, (5.3) implies by a standard calculation which we omit that if $t \in (0, 1)$ and $r > -3$, then

$$\mathbb{E}[\mathbf{e}(t)^r] = 2^{r/2+1}\pi^{-1/2}\bigl(t(1-t)\bigr)^{r/2}\Gamma\Bigl(\frac{r+3}{2}\Bigr). \tag{5.4}$$

We can now calculate $\mathbb{E}\, W_\sigma$ for any σ that only has one non-trivial block.

Example 5.3. Let $\sigma = \mathbf{1243}$, with blocks $\mathbf{1}, \mathbf{1}, \mathbf{21}$. Thus $w_{\mathbf{1243}} = w_{\mathbf{1}}^2 w_{\mathbf{21}} = w_{\mathbf{21}} = 2^{-1/2}$ by (3.9). Furthermore, by (1.10),

$$W_{\mathbf{1243}} = w_{\mathbf{1243}} \int_{0<t_1<t_2<t_3<1} \mathbf{e}(t_3)\,\mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}t_3 = 2^{-3/2} \int_0^1 t^2 \mathbf{e}(t)\,\mathrm{d}t. \tag{5.5}$$

By (5.4), this yields

$$\begin{aligned} \mathbb{E}\, W_{\mathbf{1243}} &= 2^{-3/2} \int_0^1 t^2\, \mathbb{E}\,\mathbf{e}(t)\,\mathrm{d}t = \pi^{-1/2} \int_0^1 t^{5/2}(1-t)^{1/2}\,\mathrm{d}t \\ &= \pi^{-1/2} \frac{\Gamma(7/2)\Gamma(3/2)}{\Gamma(5)} = \frac{5}{128}\sqrt{\pi}. \end{aligned} \tag{5.6}$$

5.3. **Continuum random tree.** Our next method uses a (minor) part of Aldous's theory of the Brownian continuum random tree [2; 3; 4], in particular [4, Corollary 22 and Lemma 21], which among other things yield a simple description (in terms of binary trees with random edge lengths) of the distribution of the random vector $\bigl(\mathbf{e}(U_1), \dots, \mathbf{e}(U_\ell)\bigr)$, where $\ell \geqslant 1$ and $U_1, \dots, U_\ell \sim U(0,1)$ are i.i.d. and independent of $\mathbf{e}$.

In particular, this leads to the following. (One can obtain (5.7) also by integrating (5.4), but the proof below requires less computations.)

Lemma 5.4. (i) *If $r > -2$, then*

$$\mathbb{E} \int_0^1 \mathbf{e}(t)^r\,\mathrm{d}t = 2^{-r/2}\Gamma\Bigl(\frac{r}{2}+1\Bigr). \tag{5.7}$$

(ii) *If $r, s > -1$, then*

$$\begin{aligned} &\mathbb{E} \int_0^1 \int_0^1 \mathbf{e}(t)^r \mathbf{e}(u)^s\,\mathrm{d}t\,\mathrm{d}u \\ &\quad = 2^{-(r+s)/2}\Bigl(\frac{r+s+2}{(r+1)(s+1)} - \frac{\Gamma(r+1)\Gamma(s+1)}{\Gamma(r+s+2)}\Bigr)\Gamma\Bigl(\frac{r+s}{2}+1\Bigr). \end{aligned} \tag{5.8}$$

Proof. (i): For $\ell = 1$, the description of Aldous [4] simply says that $2\mathbf{e}(U_1)$ has a Rayleigh distribution with density $xe^{-x^2/2}$, $x > 0$. Hence, for any $r > 0$,

$$2^r\, \mathbb{E} \int_0^1 \mathbf{e}(t)^r\,\mathrm{d}t = 2^r\, \mathbb{E}\bigl[\mathbf{e}(U_1)^r\bigr] = \int_0^\infty x^{r+1} e^{-x^2/2}\,\mathrm{d}x = 2^{r/2}\Gamma\Bigl(\frac{r}{2}+1\Bigr),$$

where the final integral is evaluated using a standard change of variables, see e.g. [41, (5.9.1)]. This yields (5.7).

(ii): For $\ell = 2$, the description of Aldous [4] says that

$$\bigl(2\mathbf{e}(U_1), 2\mathbf{e}(U_2)\bigr) \overset{\mathrm{d}}{=} (L_1 + L_2, L_1 + L_3), \tag{5.9}$$

where (L_1, L_2, L_3) has the density $(x_1+x_2+x_3)e^{-(x_1+x_2+x_3)^2/2}$, $x_1, x_2, x_3 > 0$. Consequently, for any $r, s > -1$, using the change of variables $z =$

$x_1 + x_2 + x_3$, $x = x_2/z$, $y = (x_1 + x_2)/z$,

$$
\begin{aligned}
2^{r+s}\,\mathbb{E}\int_0^1\int_0^1 \mathbf{e}(x)^r\mathbf{e}(y)^s\,\mathrm{d}x\,\mathrm{d}y &= 2^{r+s}\,\mathbb{E}\big[\mathbf{e}(U_1)^r\mathbf{e}(U_2)^s\big]\\
&= \mathbb{E}\big((L_1+L_2)^r(L_1+L_3)^s\big)\\
&= \int_{x_1,x_2,x_3>0}(x_1+x_2)^r(x_1+x_3)^s(x_1+x_2+x_3)e^{-(x_1+x_2+x_3)^2/2}\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\\
&= \int_{z=0}^\infty\iint_{0<x<y<1} y^r(1-x)^s z^{r+s+3}e^{-z^2/2}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\\
&= \frac{1}{r+1}\int_0^1\big(1-x^{r+1}\big)(1-x)^s\,\mathrm{d}x\cdot\int_0^\infty z^{r+s+3}e^{-z^2/2}\,\mathrm{d}z\\
&= \Big(\frac{1}{(r+1)(s+1)} - \frac{\Gamma(r+2)\Gamma(s+1)}{(r+1)\Gamma(r+s+3)}\Big)2^{(r+s+2)/2}\Gamma\Big(\frac{r+s}{2}+2\Big).
\end{aligned}
$$

Simple manipulations of the Gamma functions yield (5.8). □

Proof of Corollary 1.7. Since σ is assumed to be indecomposable, (1.15) holds with $\ell = 1$ and $m_1 = m$, and thus, using Lemma 5.4,

$$\mathbb{E}\,W_\sigma = w_\sigma\int_0^1\mathbb{E}[\mathbf{e}(t)^{m-1}]\,\mathrm{d}t = w_\sigma 2^{-(m-1)/2}\Gamma\big(\tfrac{m+1}{2}\big). \tag{5.10}$$

Thus (1.16) follows from (1.15).

Similarly,

$$\mathbb{E}[W_\sigma^2] = w_\sigma^2\,\mathbb{E}\int_0^1\int_0^1\mathbf{e}(t)^{m-1}\mathbf{e}(u)^{m-1}\,\mathrm{d}t\,\mathrm{d}u \tag{5.11}$$

is given by (5.8), and (1.17) follows. □

Note that in (1.10), we integrate only over $t_1 < \cdots < t_\ell$, while the method based on [4] used here yields the integral over $[0,1]^\ell$, without restriction on the order of the variables. This was not a problem in Corollary 1.7, when σ is indecomposable so $\ell = 1$. The method also applies when $\ell > 1$ in the special case when all blocks have the same lengths $m_1 = \cdots = m_\ell$; see Example 1.5. In these cases, higher moments of W_σ can be calculated by the same method, although the calculations become more and more involved; the method in Subsection 5.1 seems simpler in these cases.

The method applies when $\ell = 2$ for the expectation (but not for the variance or higher moments) also when $m_1 \neq m_2$, as consequence of the following lemma.

Lemma 5.5. *If $r, s > -1$, then*

$$\mathbb{E}\int_{0<t_1<t_2<1}\mathbf{e}(t_1)^r\mathbf{e}(t_2)^s\,\mathrm{d}t_1\,\mathrm{d}t_2 = \frac{1}{2}\,\mathbb{E}\int_0^1\int_0^1\mathbf{e}(t_1)^r\mathbf{e}(t_2)^s\,\mathrm{d}t_1\,\mathrm{d}t_2. \tag{5.12}$$

Proof. Since the distribution of $\mathbf{e}$ is invariant under reflection, $\mathbf{e}(t) \overset{\mathrm{d}}{=} \mathbf{e}(1-t)$ (as random functions),

$$\mathbb{E}\int_{0<t_1<t_2<1} \mathbf{e}(t_1)^r\mathbf{e}(t_2)^s \,\mathrm{d}t_1\,\mathrm{d}t_2 = \mathbb{E}\int_{1>t_1>t_2>0} \mathbf{e}(t_1)^r\mathbf{e}(t_2)^s \,\mathrm{d}t_1\,\mathrm{d}t_2 \tag{5.13}$$

and (5.12) follows. □

Proof of Corollary 1.8. Lemmas 5.5 and Lemma 5.4(i) yield (1.18). □

As mentioned in Section 1, Corollaries 1.7 and 1.8 proved here imply (1.4)–(1.6).

5.4. **Brownian bridge.** If all blocks of σ have odd length, then the exponents in (1.10) are even, and thus we can use the representation (2.6) and write $\mathbb{E}\bigl[\mathbf{e}(t_1)^{m_1-1}\cdots\mathbf{e}(t_\ell)^{m_\ell-1}\bigr]$ as the expectation of a polynomial in the jointly Gaussian variables $\mathbf{b}_k(t_i)$. This expectation is a polynomial in the covariances $\operatorname{Cov}\bigl(\mathbf{b}_k(t_i),\mathbf{b}_\kappa(t_j)\bigr) = \delta_{k\kappa}t_i(1-t_j)$ (for $t_i < t_j$), see e.g. [28, Thorem 1.28], so (1.15) reduces to the integral of a polynomial over the given simplex, which is calculated by elementary calculus. Higher moments can be calculated similarly.

Example 5.6. Let $\sigma = \mathbf{2314675}$, with the blocks $\mathbf{231}$, $\mathbf{1}$, $\mathbf{231}$. Then $w_\sigma = w_{\mathbf{231}}^2 = 1/4$ by (3.10), and

$$W_{\mathbf{2314675}} = \frac{1}{4}\int_{0<t_1<t_2<t_3<1} \mathbf{e}(t_1)^2\mathbf{e}(t_3)^2 \,\mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}t_3. \tag{5.14}$$

Furthermore, using (2.6) and symmetry,

$$\begin{aligned}\mathbb{E}\bigl[\mathbf{e}(t_1)^2\mathbf{e}(t_3)^2\bigr] &= 3\,\mathbb{E}\bigl[\mathbf{b}_1(t_1)^2\mathbf{b}_1(t_3)^2\bigr] + 6\,\mathbb{E}\bigl[\mathbf{b}_1(t_1)^2\mathbf{b}_2(t_3)^2\bigr]\\ &= 3\bigl(t_1(1-t_1)t_3(1-t_3) + 2t_1^2(1-t_3)^2\bigr) + 6t_1(1-t_1)t_3(1-t_3). \end{aligned}\tag{5.15}$$

Hence,

$$\begin{aligned}\mathbb{E}\,W_{\mathbf{2314675}} &= \frac{1}{4}\int_{0<t_1<t_3<1}\bigl(9t_1(1-t_1)t_3(1-t_3) + 6t_1^2(1-t_3)^2\bigr)(t_3-t_1)\,\mathrm{d}t_1\,\mathrm{d}t_3\\ &= \frac{31}{3360}.\end{aligned}\tag{5.16}$$

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