Patterns in random permutations avoiding some other patterns

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Abstract

Consider a random permutation drawn from the set of permutations of length n that avoid a given set of one or several patterns of length 3. We show that the number of occurrences of another pattern has a limit distribution, after suitable scaling. In several cases, the limit is normal, as it is in the case of unrestricted random permutations; in other cases the limit is a non-normal distribution, depending on the studied pattern. In the case when a single pattern of length 3 is forbidden, the limit distributions can be expressed in terms of a Brownian excursion.

The analysis is made case by case; unfortunately, no general method is known, and no general pattern emerges from the results.

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1 Introduction

Let $\mathfrak{S}_n$ be the set of permutations of $[n] := \{1,\dots,n\}$, and $\mathfrak{S}_* := \bigcup_{n\geq 1}\mathfrak{S}_n$. If $\sigma = \sigma_1\cdots\sigma_m \in \mathfrak{S}_m$ and $\pi = \pi_1\cdots\pi_n \in \mathfrak{S}_n$, then an *occurrence* of σ in π is a subsequence $\pi_{i_1}\cdots\pi_{i_m}$, with $1 \leq i_1 < \cdots < i_m \leq n$, that has the same order as σ, i.e., $\pi_{i_j} < \pi_{i_k} \iff \sigma_j < \sigma_k$ for all $j,k \in [m]$. We let $n_\sigma(\pi)$ be the number of occurrences of σ in π, and note that

$$\sum_{\sigma\in\mathfrak{S}_m} n_\sigma(\pi) = \binom{n}{m}, \tag{1}$$

for every $\pi \in \mathfrak{S}_n$. For example, an inversion is an occurrence of 21, and thus $n_{21}(\pi)$ is the number of inversions in π.

We say that π *avoids* another permutation τ if $n_\tau(\pi) = 0$. Let

$$\mathfrak{S}_n(\tau) := \{\pi \in \mathfrak{S}_n : n_\tau(\pi) = 0\}, \tag{2}$$

the set of permutations of length n that avoid τ. More generally, for any set $T = \{\tau_1,\dots,\tau_k\}$ of permutations, let

$$\mathfrak{S}_n(T) = \mathfrak{S}_n(\tau_1,\dots,\tau_k) := \bigcap_{i=1}^{k}\mathfrak{S}_n(\tau_i), \tag{3}$$

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the set of permutations of length n that avoid all $\tau_i \in T$. We also let $\mathfrak{S}_*(T) := \bigcup_{n=1}^{\infty} \mathfrak{S}_n(T)$ be the set of T-avoiding permutations of arbitrary length.

The classes $\mathfrak{S}_*(\tau)$ and, more generally, $\mathfrak{S}_*(T)$ have been studied for a long time. For examples relevant to analysis of algorithms, see e.g. [13, Exercise 2.2.1-5] (π can be obtained by a stack if and only if $\pi \in \mathfrak{S}_n(312)$; equivalently: π is stack-sortable if and only if $\pi \in \mathfrak{S}_n(312)$); [13, Exercise 2.2.1-10,11] and [17] (π is deque-sortable if and only if π $\pi \in \mathfrak{S}_n(2431, 4231)$; [16] ($\pi$ can be sorted by 2 parallel queues if and only if $\pi \in \mathfrak{S}_n(321)$. Further examples are given in [15], Exercises 6.19 x (321), y (312), ee (321), ff (312), ii (231), oo (132), xx (321); 6.25 g (321); 6.39 k, l ($\{2413, 3142\}$), m ($\{1342, 1324\}$); 6.47 a ($\{4231, 3412\}$); 6.48 (1342). See also [3].

In particular, one classical problem is to enumerate the sets $\mathfrak{S}_n(T)$, either exactly or asymptotically, see e.g. [3, Chapters 4–5] and [14].

The general problem that concerns us is to take a fixed set T of one or several permutations and let $\boldsymbol{\pi}_{T;n}$ be a uniformly random T-avoiding permutation, i.e., a uniformly random element of $\mathfrak{S}_n(T)$, and then study the asymptotic distribution of the random variable $n_\sigma(\boldsymbol{\pi}_{T;n})$ (as $n \to \infty$) for some other fixed permutation σ. (Only σ that are themselves T-avoiding are interesting, since otherwise $n_\sigma(\boldsymbol{\pi}_{T;n}) = 0$.)

Here we study the cases when T is a set of permutations of length 3. The cases when T contains a permutation of length ≤ 2 are trivial, since then there is at most one permutation in $\mathfrak{S}_n(T)$ for any n. The case of forbidding one or several permutations of length ≥ 4 seems much more complicated, but there are recent impressive results for $\mathfrak{S}_n(2413, 3142)$ (separable permutations) by Bassino, Bouvel, Féray, Gerin, and Pierrot [2], with generalizations to some other classes in [1].

There are $2^6 = 64$ sets T of permutations of length 3. Of these, every T that contains $\{123, 321\}$, and every T with $|T| \geq 4$ is trivial, in the sense that $\mathfrak{S}_n(T)$ contains at most 2 elements for any $n \geq 5$ (see [14]). Ignoring these cases, there are $1 + 6 + 14 + 16 = 37$ remaining cases (with $|T| = 0, 1, 2, 3$, respectively), and by symmetries, see Appendix A, these reduce to $1 + 2 + 4 + 4 = 11$ non-equivalent cases, which are treated in Sections 2–12. For further details, see [12], [8], [9], [10]; these papers also contain further references to related work, and to some of the many papers by various authors that study other properties of random τ-avoiding permutations.

The cases studied here, i.e., the non-trivial cases with $T \subset \mathfrak{S}_3$, all have asymptotic distributions of one of the following two types.

I. Normal limits: For every $\sigma \in \mathfrak{S}_*(T)$, there exists constants α, β, γ such that, as $n \to \infty$,

$$\frac{n_\sigma(\boldsymbol{\pi}_{T;n}) - \beta n^\alpha}{n^{\alpha-1/2}} \xrightarrow{\mathrm{d}} N\bigl(0, \gamma^2\bigr), \tag{4}$$

with convergence of all moments. Furthermore, assuming $|\sigma| \geq 2$, $\gamma^2 > 0$, so the limit is not deterministic, except possibly for one $\sigma \in \mathfrak{S}_m(T)$ for each length $m \geq 2$.

In particular, $\mathbb{E}\, n_\sigma(\boldsymbol{\pi}_{T;n}) \sim \beta n^\alpha$. Note that (4) implies concentration, in the sense

$$\frac{n_\sigma(\boldsymbol{\pi}_{T;n})}{\mathbb{E}\, n_\sigma(\boldsymbol{\pi}_{T;n})} \xrightarrow{\mathrm{p}} 1. \tag{5}$$

II. Non-normal limits without concentration: For every $\sigma \in \mathfrak{S}_*(T)$, there exists a constant α such that

$$\frac{n_\sigma(\boldsymbol{\pi}_{T;n})}{n^\alpha} \xrightarrow{\mathrm{d}} W_\sigma, \tag{6}$$

T	$\lvert\mathfrak{S}_n(T)\rvert$	type I	type II	as. variance $= 0$
$\emptyset$	$n!$	$\lvert\sigma\rvert$		
$\{132\}$	C_n		$(\lvert\sigma\rvert + D(\sigma))/2$	$m \cdots 1$
$\{321\}$	C_n		$(\lvert\sigma\rvert + B(\sigma))/2$	$1 \cdots m$
$\{132, 312\}$	2^{n-1}	$\lvert\sigma\rvert$		
$\{231, 312\}$	2^{n-1}	$B(\sigma)$		$1 \cdots m$
$\{231, 321\}$	2^{n-1}	$B(\sigma)$		$1 \cdots m$
$\{132, 321\}$	$\binom{n}{2} + 1$		$\lvert\sigma\rvert$	
$\{231, 312, 321\}$	F_{n+1}	$B(\sigma)$		$1 \cdots m$
$\{132, 231, 312\}$	n		$\lvert\sigma\rvert$	
$\{132, 231, 321\}$	n		$\lvert\sigma\rvert - 1$ or $\lvert\sigma\rvert$	$1 \cdots m$
$\{132, 213, 321\}$	n		$\lvert\sigma\rvert$	
$\{2413, 3142\}$	s_{n-1}		$\lvert\sigma\rvert$	

Table 1 The table shows whether $n_\sigma(\boldsymbol{\pi}_{T;n})$ has limits of type I or II; furthermore, the exponent $\alpha = \alpha(\sigma)$ is given in the column for the type. The last column shows the exceptional cases, if any, where the asymptotic variance vanishes. $C_n := \frac{1}{n+1}\binom{2n}{n}$ is a Catalan number; F_{n+1} is a Fibonacci number ($F_0 = 0$, $F_1 = 1$); s_{n-1} is a Schröder number; $D(\sigma)$ is the number of descents and $B(\sigma)$ is the number of blocks in σ.

with convergence of all moments, for some random variable $W_\sigma > 0$. Hence, also

$$\frac{n_\sigma(\boldsymbol{\pi}_{T;n})}{\mathbb{E}\, n_\sigma(\boldsymbol{\pi}_{T;n})} \xrightarrow{\mathrm{d}} W'_\sigma, \tag{7}$$

with convergence of all moments, for some random variable $W'_\sigma > 0$ (necessarily with $\mathbb{E}\, W'_\sigma = 1$). Furthermore, assuming $|\sigma| \geq 2$, $\operatorname{Var} W_\sigma > 0$, so W_σ and W'_σ are not deterministic, except possibly for one $\sigma \in \mathfrak{S}_m(T)$ for each length $m \geq 2$.

▶ Remark. In all cases studied here, if there are any exceptional $\sigma \in \mathfrak{S}_*(T)$ with $\sigma \geq 2$ such that the limit in (4) or (6) is deterministic, i.e., the asymptotic variance is 0, then the exceptional σ are either all identity permutations $1 \cdots m$, or all decreasing permutations $m \cdots 1$. Furthermore, these exceptional cases arise because almost all of the $\binom{n}{|\sigma|}$ patterns in $\boldsymbol{\pi}_{T;n}$ of length $|\sigma|$ are occurrences of σ; more precisely, $\mathbb{E}\big(\binom{n}{|\sigma|} - n_\sigma(\boldsymbol{\pi}_{T;n})\big) = O\big(n^{|\sigma|-1}\big)$ for the exceptional cases of type I and $O\big(n^{|\sigma|-1/2}\big)$ for the cases of type II. (It follows that (5) holds also for the latter.)

We summarize the results for T consisting of permutations of length 3 in Table 1; for reference, we include the number $|\mathfrak{S}_n(T)|$ of T-avoiding permutations of length n, see e.g. [13, Exercises 2.2.1-4,5], [15, Exercise 6.19ee,ff], [3, Corollary 4.7], and [14]. We include also the case $T = \{2413, 3142\}$ from [2]; see [17] for the enumeration.

We see no obvious pattern in the existence of limits of type I or II in Table 1. Moreover, the proofs, sketched below, are done case by case; we have not succeeded to prove any general results, treating all (or at least some) forbidden sets T at the same time.

▶ Remark. We do not know whether a general set of forbidden permutations T has limits in distribution of $n_\sigma(\boldsymbol{\pi}_{T;n})$ (after normalization) at all, and even if limits exist, there is no known reason implying that they have to be of type I or II above; other types of limits are conceivable.

▶ Remark. The non-normal limits in the cases $\{132\}$, $\{321\}$ and $\{2413, 3142\}$ can all be expressed as functionals of a Brownian excursion $\mathbf{e}$, see [8, 9, 2]. However, the expressions in these three cases are, in general, quite different (and obtained by quite different arguments), so there is no obvious hope for a unification. (The other cases of non-normal limits in Table 1 are different, and of a more elementary kind.)

1.1 Some notation

Let $\iota = \iota_n$ be the identity permutation of length n.

If $\sigma \in \mathfrak{S}_m$ and $\tau \in \mathfrak{S}_n$, their *composition* $\sigma * \tau \in \mathfrak{S}_{m+n}$ is defined by letting τ act on $[m+1, m+n]$ in the natural way; more formally, $\sigma * \tau = \pi \in \mathfrak{S}_{m+n}$ where $\pi_i = \sigma_i$ for $1 \leq i \leq m$, and $\pi_{j+m} = \tau_j + m$ for $1 \leq j \leq n$. We say that a permutation $\pi \in \mathfrak{S}_*$ is *decomposable* if $\pi = \sigma * \tau$ for some $\sigma, \tau \in \mathfrak{S}_*$, and *indecomposable* otherwise; we also call an indecomposable permutation a *block*.

It is easy to see that any permutation $\pi \in \mathfrak{S}_*$ has a unique decomposition $\pi = \pi_1 * \cdots * \pi_\ell$ into indecomposable permutations (blocks) $\pi_1, \ldots, \pi_\ell$; we call these the *blocks of* π. (These are useful to characterize the permutations in some of the classes below.)

2 No restriction, $T = \emptyset$

As a background, consider first the case $T = \emptyset$, so $\mathfrak{S}_n(T) = \mathfrak{S}_n$; the set of all $n!$ permutations of length n. It is well-known, see Bóna [4, 5] and [12, Theorem 4.1], that if $\boldsymbol{\pi}_n$ is a uniformly random permutation in $\mathfrak{S}_n$, then $n_\sigma(\boldsymbol{\pi}_n)$ has an asymptotic normal distribution as $n \to \infty$ for every fixed permutation σ:

▶ **Theorem 1** (Bóna [4, 5]). *If $|\sigma| = m \geq 2$ then, as $n \to \infty$, for some $\gamma^2 > 0$,*

$$\frac{n_\sigma(\boldsymbol{\pi}_n) - \frac{1}{m!}\binom{n}{m}}{n^{m-1/2}} \xrightarrow{\mathrm{d}} N\big(0, \gamma^2\big). \tag{8}$$

Sketch of proof. A random permutation $\boldsymbol{\pi}_n$ can be obtained by taking i.i.d. random variables $X_1, \ldots, X_n \sim U(0,1)$ and considering their ranks. Then

$$n_\sigma(\boldsymbol{\pi}_n) = \sum_{i_1 < \cdots < i_m} f\big(X_{i_1}, \ldots, X_{i_m}\big) \tag{9}$$

for a suitable (indicator) function f. This sum is an asymmetric U-statistic, and the result follows by general results on U-statistics, see [6] and [11]. ◀

▶ Remark. The asymptotic variance γ^2 depends on σ. It can be calculated explicitly, and the same holds for all parameters γ^2 (or μ) in the limit theorems below. Moreover, the convergence (8) holds with convergence of all moments, and it holds jointly for any set of σ; also this holds for all later limit theorems too.

3 Avoiding 132

Consider next the cases when T consists of a single permutation of length 3. The symmetries in Appendix A leave two non-equivalent cases. In this section we avoid $T = \{132\}$; equivalent cases are $\{213\}$, $\{231\}$, $\{312\}$. Recall that the standard Brownian excursion $\mathbf{e}(x)$ is a random non-negative function on $[0,1]$. Let

$$\lambda(\sigma) := |\sigma| + D(\sigma) \tag{10}$$

where $D(\sigma)$ is the number of *descents* in σ, i.e., indices i such that $\sigma_i > \sigma_{i+1}$ or (as a convenient convention) $i = |\sigma|$. Note that $1 \le D(\sigma) \le |\sigma|$, and thus

$$|\sigma| + 1 \le \lambda(\sigma) \le 2|\sigma|, \tag{11}$$

with the extreme values $\lambda(\sigma) = |\sigma| + 1$ if and only if $\sigma = 1 \cdots k$, and $\lambda(\sigma) = 2|\sigma|$ if and only if $\sigma = k \cdots 1$, for some $k = |\sigma|$.

▶ **Theorem 2** ([8]). *There exist strictly positive random variables Λ_σ such that as $n \to \infty$,*

$$n_\sigma(\boldsymbol{\pi}_{132;n})/n^{\lambda(\sigma)/2} \xrightarrow{\mathrm{d}} \Lambda_\sigma. \tag{12}$$

Sketch of proof. The analysis is based on a well-known bijection with binary trees and Dyck paths, and the, also well-known, convergence in distribution of random Dyck paths to a Brownian excursion. For (not so simple) details, see [8]. ◀

The limit variables Λ_σ in Theorem 2 can be expressed as functionals of a Brownian excursion $\mathbf{e}(x)$, see [8]; the description is, in general, rather complicated, but some cases are simple. Moments of the variables Λ_σ can be calculated by a recursion formula given in [8].

▶ **Example 3.** In the special case $\sigma = 12$, $\Lambda_{12} = \sqrt{2}\int_0^1 \mathbf{e}(x)\,\mathrm{d}x$, see [8, Example 7.6]; this is (apart from the factor $\sqrt{2}$) the well-known *Brownian excursion area*, see e.g. [7] and the references there.

For the number n_{21} of inversions, we thus have

$$\frac{\binom{n}{2} - n_{21}(\boldsymbol{\pi}_{132;n})}{n^{3/2}} = \frac{n_{12}(\boldsymbol{\pi}_{132;n})}{n^{3/2}} \xrightarrow{\mathrm{d}} \Lambda_{12} = \sqrt{2}\int_0^1 \mathbf{e}(x)\,\mathrm{d}x. \tag{13}$$

By symmetries, see Appendix A, the left-hand side can also be seen as the number of inversions $n_{21}(\boldsymbol{\pi}_{231;n})$ or $n_{21}(\boldsymbol{\pi}_{312;n})$, normalized by $n^{3/2}$, where we instead avoid 231 or 312.

4 Avoiding 321

In this section we avoid $T = \{321\}$. The case $T = \{123\}$ is equivalent.

$\mathfrak{S}_n(321)$ is treated in detail in [9]. As for $\mathfrak{S}_n(132)$ in Section 3, the analysis is based on a well-known bijection with Dyck paths, but the details are very different, and so are in general the resulting limit distributions.

▶ **Theorem 4** ([9]). *Let $\sigma \in \mathfrak{S}_*(321)$. Let $m := |\sigma|$, and suppose that σ has ℓ blocks of lengths $m_1, \dots, m_\ell$. Then, as $n \to \infty$,*

$$n_\sigma(\boldsymbol{\pi}_{321;n})/n^{(m+\ell)/2} \xrightarrow{\mathrm{d}} W_\sigma \tag{14}$$

for a positive random variable W_σ that can be represented as

$$W_\sigma = w_\sigma \int_{0<t_1<\cdots<t_\ell<1} \mathbf{e}(t_1)^{m_1-1} \cdots \mathbf{e}(t_\ell)^{m_\ell-1}\,\mathrm{d}t_1 \cdots \mathrm{d}t_\ell, \tag{15}$$

where w_σ is positive constant.

Sketch of proof. As for Theorem 2, the analysis is based on a bijection with Dyck paths, and the convergence in distribution of random Dyck paths to a Brownian excursion. For details, see [8]. ◀

In this case, we have an explicit general formula (15) for the limit variables. On the other hand, we do not know how to compute even the mean $\mathbb{E}\, W_\sigma$ in general; see [9] for calculations in various special cases.

▶ **Example 5.** Let $\sigma = 21$. Then $w_{21} = 2^{-1/2}$, see [9], and thus (14)–(15), with $\ell = 1$ and $m_1 = m = 2$, yield for the number of inversions,

$$\frac{n_{21}(\boldsymbol{\pi}_{321;n})}{n^{3/2}} \xrightarrow{\mathrm{d}} 2^{-1/2} \int_0^1 \mathbf{e}(x)\,\mathrm{d}x. \tag{16}$$

Note that the limit in (16) differs from the one in (13) by a factor 2.

5 Avoiding $\{132, 312\}$

In this section we avoid $T = \{132, 312\}$. Equivalent sets are $\{132, 231\}$, $\{213, 231\}$, $\{213, 312\}$.

▶ **Theorem 6.** *For any $m \geq 2$ and $\sigma \in \mathfrak{S}_m(132, 312)$, as $n \to \infty$,*

$$\frac{n_\sigma(\boldsymbol{\pi}_{132,312;n}) - 2^{1-m} n^m/m!}{n^{m-1/2}} \xrightarrow{\mathrm{d}} N(0, \gamma^2). \tag{17}$$

Sketch of proof. It was shown by [14, Proposition 12] (in an equivalent formulation) that *a permutation π belongs to the class $\mathfrak{S}_*(132, 312)$ if and only if every entry π_i is either a maximum or a minimum.* We encode a permutation $\pi \in \mathfrak{S}_n(132, 312)$ by a sequence $\xi_2, \dots, \xi_n \in \{\pm 1\}^{n-1}$, where $\xi_j = 1$ if π_j is a maximum in π, and $\xi_j = -1$ if π_j is a minimum. This is a bijection, and hence the code for a uniformly random $\boldsymbol{\pi}_{132,312;n}$ has $\xi_2, \dots, \xi_n$ i.i.d. with the symmetric Bernoulli distribution $\mathbb{P}(\xi_j = 1) = \mathbb{P}(\xi_j = -1) = \frac{1}{2}$.

Let $\sigma \in \mathfrak{S}_m(132, 312)$ have the code $\eta_2, \dots, \eta_m$. Then $\pi_{i_1} \cdots \pi_{i_m}$ is an occurrence of σ in π if and only if $\xi_{i_j} = \eta_j$ for $2 \leq j \leq m$. Consequently, $n_\sigma(\boldsymbol{\pi}_{132,312;n})$ is a U-statistic

$$n_\sigma(\boldsymbol{\pi}_{132,312;n}) = \sum_{i_1 < \dots < i_m} f(\xi_{i_1}, \dots, \xi_{i_m}), \tag{18}$$

where

$$f(\xi_1, \dots, \xi_m) := \prod_{j=2}^{m} \mathbf{1}\{\xi_j = \eta_j\}. \tag{19}$$

Note that f does not depend on the first argument.

The result now follows from the theory of U-statistics [6], [11]. ◀

▶ **Example 7.** For the number of inversions, we have $\sigma = 21$ and $m = 2$, $\eta_2 = -1$. A calculation yields $\mu = \frac{1}{2}$ and $\gamma^2 = \frac{1}{12}$, and thus Theorem 6 yields

$$\frac{n_{21}(\boldsymbol{\pi}_{132,312;n}) - n^2/4}{n^{3/2}} \xrightarrow{\mathrm{d}} N\big(0, \tfrac{1}{12}\big), \tag{20}$$

6 Avoiding $\{231, 312\}$

In this section we avoid $T = \{231, 312\}$. The only equivalent set is $\{132, 213\}$.

▶ **Theorem 8.** *Let $\sigma \in \mathfrak{S}_m(231, 312)$ have block lengths $\ell_1, \dots, \ell_b$. Then, as $n \to \infty$,*

$$\frac{n_\sigma(\boldsymbol{\pi}_{231,312;n}) - n^b/b!}{n^{b-1/2}} \xrightarrow{\mathrm{d}} N(0, \gamma^2). \tag{21}$$

Sketch of proof. It was shown by [14, Proposition 12] (in an equivalent form) that *a permutation π belongs to the class $\mathfrak{S}_*(231, 312)$ if and only if every block in π is decreasing, i.e., of the type $\ell(\ell-1)\cdots 21$ for some ℓ.* Hence there exists exactly one block of each length $\ell \geq 1$, and a permutation $\pi \in \mathfrak{S}_*(231, 312)$ can be encoded by its sequence of block lengths. In this section, let $\pi_{\ell_1,\dots,\ell_b}$ denote the permutation in $\mathfrak{S}_*(231, 312)$ with block lengths $\ell_1, \dots, \ell_b$.

A uniformly random permutation $\boldsymbol{\pi}_{231,312;n}$ can be generated as $\pi_{L_1,\dots,L_B}$, where the block lengths $L_1, \dots, L_B$ are obtained from an infinite i.i.d. sequence $L_1, L_2, \dots \sim \mathrm{Ge}(\frac{1}{2})$, stopped at B such that $L_1+\dots+L_B \geq n$, and then adjusting L_B such that $L_1+\dots+L_B = n$.

Let $\sigma \in \mathfrak{S}_*(231, 312)$ have block lengths $\ell_1, \dots, \ell_b$, so that $\sigma = \pi_{\ell_1,\dots,\ell_b}$. Then,

$$n_\sigma\bigl(\pi_{L_1,\dots,L_B}\bigr) = \sum_{1\leq i_1<\dots<i_b\leq B} \prod_{j=1}^{b} \binom{L_{i_j}}{\ell_i}. \tag{22}$$

This is again a kind of U-statistic, but it is based on the sequence $L_1, \dots, L_B$ of random length B, obtained by stopping the infinite sequence L_i. Nevertheless, general results for U-statistics cover this modification and yield the result, see [11]. ◀

▶ **Example 9.** For the number of inversions, we have $\sigma = 21$ and $b = 1$, $\ell_1 = 2$. A calculation yields $\gamma^2 = 6$, and Theorem 8 yields

$$\frac{n_{21}(\boldsymbol{\pi}_{231,312;n}) - n}{n^{1/2}} \xrightarrow{\mathrm{d}} N(0, 6). \tag{23}$$

7 Avoiding $\{231, 321\}$

In this section we avoid $T = \{231, 321\}$. Equivalent sets are $\{123, 132\}$, $\{123, 213\}$, $\{312, 321\}$.

▶ **Theorem 10.** *Let $\sigma \in \mathfrak{S}_m(231, 321)$ have block lengths $\ell_1, \dots, \ell_b$, and let b_1 be the number of blocks of length $\ell_i = 1$. Then, as $n \to \infty$,*

$$\frac{n_\sigma(\boldsymbol{\pi}_{231,321;n}) - 2^{b_1-b} n^b / b!}{n^{b-1/2}} \xrightarrow{\mathrm{d}} N\bigl(0, \gamma^2\bigr). \tag{24}$$

Sketch of proof. It was shown by [14, Proposition 12] (in an equivalent form) that *a permutation π belongs to the class $\mathfrak{S}_*(231, 321)$ if and only if every block in π is of the type $\ell 12\cdots(\ell-1)$ for some ℓ.* Thus, as in Section 6, a permutation in $\mathfrak{S}_*(231, 321)$ is determined by its block lengths, and these can be arbitrary. Hence, a uniformly random $\boldsymbol{\pi}_{231,321;n}$ has block lengths $L_1, \dots, L_B$ with the same distribution as in Section 6. Letting now σ be the permutation in $\mathfrak{S}_*(231, 321)$ with block lengths $\ell_1, \dots, \ell_b$, $n_\sigma(\boldsymbol{\pi}_{231,321;n})$ is a function of the block lengths $L_1, \dots, L_B$ that is similar (but not identical) to (22). This time some lower order terms appear, but they may be neglected, and the remainder is a U-statistic similar to the one in the proof of Theorem 8, and the result follows in the same way. ◀

▶ **Example 11.** For the number of inversions, we have $\sigma = 21$ and $b = 1$, $\ell_1 = 2$, $b_1 = 0$. A calculation yields $\gamma^2 = 1/4$, and Theorem 10 yields

$$\frac{n_{21}(\boldsymbol{\pi}_{231,321;n}) - n/2}{n^{1/2}} \xrightarrow{\mathrm{d}} N(0, \tfrac{1}{4}). \tag{25}$$

In fact, in this special case it can be seen that we have the exact distribution

$$n_{21}(\boldsymbol{\pi}_{231,321;n}) \sim \mathrm{Bi}\bigl(n-1, \tfrac{1}{2}\bigr). \tag{26}$$

8 Avoiding $\{132, 321\}$

In this section we avoid $T = \{132, 321\}$. Equivalent sets are $\{123, 231\}$, $\{123, 312\}$, $\{213, 321\}$.

It was shown in [14, Proposition 13] that *a permutation π belongs to $\mathfrak{S}_*(132, 321)$ if and only if either $\pi = \iota_n$ for some n, or $\pi = \pi_{k,\ell,m}$ for some $k, \ell \geq 1$ and $m \geq 0$, where, in this section,*

$$\pi_{k,\ell,m} := (\ell + 1, \dots, \ell + k, 1, \dots, \ell, k + \ell + 1, \dots, k + \ell + m) \in \mathfrak{S}_{k+\ell+m}. \tag{27}$$

Recall that the Dirichlet distribution $\mathrm{Dir}(1, 1, 1)$ is the uniform distribution on the simplex $\{(x, y, z) \in \mathbb{R}^3_+ : x + y + z = 1\}$.

▶ **Theorem 12.** *Let $\sigma \in \mathfrak{S}_*(132, 321)$. Then the following hold as $n \to \infty$.*

(i) *If $\sigma = \pi_{i,j,p}$ for some i, j, p, then*

$$n^{-(i+j+p)} n_\sigma(\boldsymbol{\pi}_{132,321;n}) \xrightarrow{\mathrm{d}} W_{i,j,p} := \frac{1}{i!\, j!\, p!} X^i Y^j Z^p, \tag{28}$$

where $(X, Y, Z) \sim \mathrm{Dir}(1, 1, 1)$.

(ii) *If $\sigma = \iota_i$, then*

$$n^{-i} n_\sigma(\boldsymbol{\pi}_{132,321;n}) \xrightarrow{\mathrm{d}} W_i := \frac{1}{i!}\big((X + Z)^i + (Y + Z)^i - Z^i\big), \tag{29}$$

with $(X, Y, Z) \sim \mathrm{Dir}(1, 1, 1)$ as in (i).

Sketch of proof. For asymptotic results, we may ignore the case when $\boldsymbol{\pi}_{132,321;n} = \iota_n$. Conditioning on $\boldsymbol{\pi}_{132,321;n} \neq \iota_n$, we have $\boldsymbol{\pi}_{132,321;n} = \pi_{K,L,n-K-L}$, where K and L are random with (K, L) uniformly distributed over the set $\{K, L \geq 1 : K + L \leq n\}$. As $n \to \infty$, we thus have

$$\Big(\frac{K}{n}, \frac{L}{n}, \frac{n - K - L}{n}\Big) \xrightarrow{\mathrm{d}} (X, Y, Z) \sim \mathrm{Dir}(1, 1, 1). \tag{30}$$

If $\sigma = \pi_{i,j,p}$ for some i, j, p, then it is easily seen that

$$n_\sigma(\pi_{k,\ell,m}) = \binom{k}{i}\binom{\ell}{j}\binom{m}{p}. \tag{31}$$

Similarly, if $\sigma = \iota_i$, then, by inclusion-exclusion,

$$n_\sigma(\pi_{k,\ell,m}) = \binom{k+m}{i} + \binom{\ell+m}{i} - \binom{m}{i}. \tag{32}$$

These exact formulas and (30) yield the results. ◀

▶ **Corollary 13.** *The number of inversions has the asymptotic distribution*

$$n^{-2} n_{21}(\boldsymbol{\pi}_{132,321;n}) \xrightarrow{\mathrm{d}} W := XY, \tag{33}$$

with (X, Y) as above; the limit variable W has density function

$$2\log\big(1 + \sqrt{1 - 4x}\big) - 2\log\big(1 - \sqrt{1 - 4x}\big), \qquad 0 < x < 1/4, \tag{34}$$

and moments

$$\mathbb{E}\, W^r = 2\frac{r!^2}{(2r + 2)!}, \qquad r > 0. \tag{35}$$

9 Avoiding $\{231, 312, 321\}$

We proceed to sets of three forbidden patterns. In this section we avoid $T = \{231, 312, 321\}$. An equivalent set is $\{123, 132, 213\}$.

▶ **Theorem 14.** *Let $\sigma \in \mathfrak{S}_m(231, 312, 321)$ have block lengths $\ell_1, \dots, \ell_b$. Then, as $n \to \infty$,*

$$\frac{n_\sigma(\boldsymbol{\pi}_{231,312,321;n}) - \mu n^b/b!}{n^{b-1/2}} \xrightarrow{\mathrm{d}} N\bigl(0, \gamma^2\bigr), \tag{36}$$

for some constants μ and γ^2.

Sketch of proof. It was shown in [14, Proposition 15*] (in an equivalent form) that *a permutation π belongs to the class $\mathfrak{S}_*(231, 312, 321)$ if and only if every block in π is decreasing and has length ≤ 2, i.e., every block is* 1 *or* 21*.* Hence, a permutation $\pi \in \mathfrak{S}_n(231, 312, 321)$ is uniquely determined by its sequence of block lengths $L_1, \dots, L_B$, where each $L_i \in \{1, 2\}$ and $L_1 + \dots + L_B = n$.

Let $p := (\sqrt{5} - 1)/2$, the golden ratio, so that $p + p^2 = 1$. Let X be a random variable with the distribution

$$\mathbb{P}(X = 1) = p, \qquad \mathbb{P}(X = 2) = p^2. \tag{37}$$

Consider an i.i.d. sequence $X_1, X_2, \dots$ of copies of X, and let $S_k := \sum_{i=1}^k X_i$. Let further $B(n) := \min\{k : S_k \ge n\}$. Then, conditioned on $S_{B(n)} = n$, the sequence $X_1, \dots, X_{B(n)}$ has the same distribution as the sequence $L_1, \dots, L_B$ of block lengths of a uniformly random permutation $\boldsymbol{\pi}_{231,312,321;n}$.

Consequently, $n_\sigma(\boldsymbol{\pi}_{231,312,321;n})$ can be expressed as a U-statistic based on $X_1, \dots, X_B$, conditioned as above. This conditioning does not affect the asymptotic distribution, see [11], and the result follows again by general results for U-statistics. ◀

▶ **Example 15.** For the number of inversions, $\sigma = 21$ we have $b = 1$. A calculation yields $\mu = 1 - p = (3 - \sqrt{5})/2$ and $\gamma^2 = 5^{-3/2}$. Consequently,

$$\frac{n_{21}(\boldsymbol{\pi}_{231,312,321;n}) - \frac{3-\sqrt{5}}{2} n}{n^{1/2}} \xrightarrow{\mathrm{d}} N\bigl(0, 5^{-3/2}\bigr). \tag{38}$$

10 Avoiding $\{132, 231, 312\}$

In this section we avoid $\{132, 231, 312\}$. Equivalent sets are $\{132, 213, 231\}$, $\{132, 213, 312\}$, $\{213, 231, 312\}$.

It was shown in [14, Proposition 16*] (in an equivalent form) that $\mathfrak{S}_n(132, 231, 312) = \{\pi_{k,n-k} : 1 \le k \le n\}$, *where, in this section,*

$$\pi_{k,\ell} := (k, \dots, 1, k + 1, \dots, k + \ell) \in \mathfrak{S}_{k+\ell}, \qquad k \ge 1,\ \ell \ge 0. \tag{39}$$

▶ **Theorem 16.** *Let $\sigma \in \mathfrak{S}_*(132, 231, 312)$. Then the following hold as $n \to \infty$, with $U \sim$* $\mathsf{U}(0, 1)$.

(i) *If $\sigma = \pi_{k,m-k}$ with $2 \le k \le m$, then*

$$n^{-m} n_\sigma(\boldsymbol{\pi}_{132,231,312;n}) \xrightarrow{\mathrm{d}} W_{k,m-k} := \frac{1}{k!\,(m-k)!} U^k (1 - U)^{m-k}. \tag{40}$$

(ii) *If $\sigma = \pi_{1,m-1} = \iota_m$, then*

$$n^{-m} n_\sigma(\boldsymbol{\pi}_{132,231,312;n}) \xrightarrow{\mathrm{d}} W_{1,m-1} := \frac{1}{(m-1)!} U(1-U)^{m-1} + \frac{1}{m!}(1-U)^m$$
$$= \frac{1}{m!}\bigl(1+(m-1)U\bigr)(1-U)^{m-1}. \tag{41}$$

Sketch of proof. The random $\boldsymbol{\pi}_{132,231,312;n} = \pi_{K,n-K}$, where $K \in [n]$ is uniformly random. Obviously, as $n \to \infty$,

$$K/n \xrightarrow{\mathrm{d}} U \sim \mathsf{U}(0,1). \tag{42}$$

Furthermore, if $\sigma = \pi_{k,\ell}$, then it is easy to see that

$$n_\sigma\bigl(\pi_{K,n-K}\bigr) = \begin{cases} \binom{K}{k}\binom{n-K}{\ell}, & k \geq 2, \\ K\binom{n-K}{\ell} + \binom{n-K}{\ell+1}, & k = 1. \end{cases} \tag{43}$$

The results follow. ◀

▶ **Corollary 17.** *The number of inversions has the asymptotic distribution*

$$n^{-2} n_{21}(\boldsymbol{\pi}_{132,231,312;n}) \xrightarrow{\mathrm{d}} W := U^2/2 \tag{44}$$

with $U \sim \mathsf{U}(0,1)$. Thus, $2W \sim B(\frac{1}{2},1)$, and W has moments

$$\mathbb{E}\, W^r = \frac{1}{2^r(2r+1)}, \qquad r > 0. \tag{45}$$

11 Avoiding $\{132, 231, 321\}$

In this section we avoid $\{132, 231, 321\}$. Equivalent sets are $\{123, 132, 231\}$, $\{123, 213, 312\}$, $\{213, 312, 321\}$, $\{123, 132, 312\}$, $\{123, 213, 231\}$, $\{132, 312, 321\}$, $\{213, 231, 321\}$.

It was shown in [14, Proposition 16*] (in an equivalent form) that $\mathfrak{S}_n(132, 231, 321) = \{\pi_{k,n-k} : 1 \leq k \leq n\}$, *where, in this section,*

$$\pi_{k,\ell} := (k, 1, \dots, k-1, k+1, \dots, k+\ell) \in \mathfrak{S}_{k+\ell}, \qquad k \geq 1,\ \ell \geq 0. \tag{46}$$

▶ **Theorem 18.** *Let $\sigma \in \mathfrak{S}_*(132, 231, 321)$. Then the following hold as $n \to \infty$, with $U \sim$* $\mathsf{U}(0,1)$.

(i) *If $\sigma = \pi_{k,m-k}$ with $2 \leq k \leq m$, then*

$$n^{-(m-1)} n_\sigma(\boldsymbol{\pi}_{132,231,321;n}) \xrightarrow{\mathrm{d}} W_{k,m-k} := \frac{1}{(k-1)!\,(m-k)!} U^{k-1}(1-U)^{m-k}. \tag{47}$$

(ii) *If $\sigma = \pi_{1,m-1} = \iota_m$, then*

$$n^{-m} n_\sigma(\boldsymbol{\pi}_{132,231,321;n}) = \frac{1}{m!} + O\bigl(n^{-1}\bigr) \xrightarrow{\mathrm{p}} \frac{1}{m!}. \tag{48}$$

Sketch of proof. The random permutation $\boldsymbol{\pi}_{132,231,321;n} = \pi_{K,n-K}$, where $K \in [n]$ is uniformly random. The results follow similarly to the proof of Theorem 16. ◀

▶ **Corollary 19.** *The number of inversions $n_{21}(\boldsymbol{\pi}_{132,231,321;n})$ has a uniform distribution on $\{0, \dots, n-1\}$, and thus the asymptotic distribution*

$$n^{-1} n_{21}(\boldsymbol{\pi}_{132,231,321;n}) \xrightarrow{\mathrm{d}} U \sim \mathsf{U}(0,1). \tag{49}$$

12 Avoiding $\{132, 213, 321\}$

In this section we avoid $\{132, 213, 321\}$. An equivalent sets is $\{123, 231, 312\}$.

It was shown in [14, Proposition 16*] (in an equivalent form) that $\mathfrak{S}_n(132, 213, 321) = \{\pi_{k,n-k} : 1 \leq k \leq n\}$, *where, in this section,*

$$\pi_{k,\ell} := (\ell + 1, \dots, \ell + k, 1, \dots, \ell) \in \mathfrak{S}_{k+\ell}, \qquad k \geq 1,\ \ell \geq 0. \tag{50}$$

▶ **Theorem 20.** *Let $\sigma \in \mathfrak{S}_*(132, 213, 321)$. Then the following hold as $n \to \infty$, with $U \sim$ $\mathsf{U}(0, 1)$.*

(i) *If $\sigma = \pi_{k,m-k}$ with $1 \leq k \leq m - 1$, then*

$$n^{-m} n_\sigma(\boldsymbol{\pi}_{132,213,321;n}) \xrightarrow{\mathrm{d}} W_{k,m-k} := \frac{1}{k!\,(m-k)!} U^k (1 - U)^{m-k}. \tag{51}$$

(ii) *If $\sigma = \pi_{m,0} = \iota_m$, then*

$$n^{-m} n_\sigma(\boldsymbol{\pi}_{132,213,321;n}) \xrightarrow{\mathrm{d}} W_{m,0} := \frac{1}{m!} \bigl(U^m + (1 - U)^m\bigr). \tag{52}$$

Sketch of proof. Similarly to the proof of Theorem 16. ◀

▶ **Corollary 21.** *The number of inversions has the asymptotic distribution*

$$n^{-2} n_{21}(\boldsymbol{\pi}_{132,213,321;n}) \xrightarrow{\mathrm{d}} W := U(1 - U), \tag{53}$$

with $U \sim \mathsf{U}(0, 1)$. Thus, $4W \sim B(1, \frac{1}{2})$, and W has moments

$$\mathbb{E}\, W^r = \frac{\Gamma(r+1)^2}{\Gamma(2r+2)}, \qquad r > 0. \tag{54}$$

A Symmetries

For any permutation $\pi = \pi_1 \cdots \pi_n$, define its *inverse* π^{-1} in the usual way, and its *reversal* and *complement* by

$$\pi^{\mathsf{r}} := \pi_n \cdots \pi_1, \tag{55}$$

$$\pi^{\mathsf{c}} := (n + 1 - \pi_1) \cdots (n + 1 - \pi_n). \tag{56}$$

These three operations generate a group $\mathfrak{G}$ of 8 symmetries (isomorphic to the dihedral group D_4). It is easy to see that for any symmetry $\mathsf{s} \in \mathfrak{G}$,

$$n_{\sigma^{\mathsf{s}}}(\pi^{\mathsf{s}}) = n_\sigma(\pi). \tag{57}$$

Thus, if we define $T^{\mathsf{s}} := \{\tau^{\mathsf{s}} : \tau \in T\}$, then

$$\mathfrak{S}_n(T^{\mathsf{s}}) = \{\pi^{\mathsf{s}} : \pi \in \mathfrak{S}_n(T)\}, \tag{58}$$

and, for any permutation σ,

$$n_{\sigma^{\mathsf{s}}}(\boldsymbol{\pi}_{T^{\mathsf{s}};n}) \overset{\mathrm{d}}{=} n_\sigma(\boldsymbol{\pi}_{T;n}). \tag{59}$$

We say that the sets of forbidden permutations T and T^{s} are *equivalent*, and note that (59) implies that it suffices to consider one set T in each equivalence class $\{T^{\mathsf{s}} : \mathsf{s} \in \mathfrak{G}\}$.

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