A GRAPHON COUNTER EXAMPLE

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Abstract. We give an example of a graphon such that there is no equivalent graphon with a degree function that is (weakly) increasing.

1. Introduction

A central fact in the theory of dense graph limits (see e.g. the book by Lovász [7]) is that each graph limit can be represented by a graphon, but this representation is not unique. We say that two graphons are *equivalent* (also called *weakly isomorphic*) if they define the same graph limit; thus there is a bijection between graph limits and equivalence classes of graphons. (Recall that equivalence of graphons can be described by the homomorphism densities being the same; furthermore, it is equivalent to the cut distance being 0; see [7] for details.)

Recall that graphons are symmetric measurable functions $W : \Omega \times \Omega \to [0,1]$, where $\Omega = (\Omega, \mathcal{F}, \mu)$ is a probability space. We may always choose Ω to be $[0,1]$ with Lebesgue measure, in the sense that any graphon is equivalent to a graphon defined on $[0,1]$, but it is often advantageous to use graphons defined on other probability spaces Ω too.

The characterization of equivalence between graphons is known to be complicated. Any two graphons on the same space Ω that are equal a.e. are equivalent, and every graphon is equivalent to any the pull-back of it by a measure preserving map (see below for definitions), but equivalence is not limited to this. See e.g. [8], [1], [5], [2] and [6].

Given a graph limit, it would be desirable to somehow define a canonical graphon representing it (at least up to equality a.e.); in other words, to define a canonical choice of a graphon in the corresponding equivalence class. In some special cases, this can be done in a natural way. For example, see [4], a graph limit that is the limit of a sequence of threshold graphs can always be represented by a graphon $W(x,y)$ on $[0,1]$ that only takes values in $\{0,1\}$, and furthermore is increasing in each coordinate separately (we say that a function $f(x)$ is increasing if $f(x) \leqslant f(y)$ when $x \leqslant y$); moreover, two such graphons are equivalent if and only if they are a.e. equal. There is thus a canonical graphon representing each threshold graph limit.

Similarly, if a graphon $W(x,y)$ defined on $[0,1]$ has a degree function

$$\mathfrak{D}(x) = \mathfrak{D}_W(x) := \int_0^1 W(x,y)\,\mathrm{d}y \tag{1.1}$$

Date: 6 September, 2019; revised 17 January, 2020.

Partly supported by the Knut and Alice Wallenberg Foundation.

that is a strictly increasing function $[0,1] \to [0,1]$, then it is not difficult to show that any equivalent graphon that also has an increasing degree function is a.e. equal to W; see Section 3 for details. Hence, a graphon with a strictly increasing degree function can be regarded as a canonical choice in its equivalence class.

Of course, not every graphon is equivalent to such a graphon; for example not a graphon with a constant degree function. Nevertheless, this leads to the following interesting question. We repeat that we use 'increasing' in the weak sense (also known as 'weakly increasing'): f is increasing if $f(x) \leqslant f(y)$ when $x \leqslant y$;

Problem. *Given any graphon W, does there exist an equivalent graphon on $[0,1]$ with an increasing degree function* (1.1)*?*

The purpose of this note is to show that this is *not* the case.

Theorem 1. *There exists a graphon on $[0,1]$ such that there is no equivalent graphon on $[0,1]$ with a (weakly) increasing degree function.*

We prove this theorem by giving a simple explicit example in (2.1). The example is similar to, and inspired by, standard examples such as [7, Example 7.11] showing that two equivalent graphons are not necessarily pull-backs of each other.

Remark 2. The analogue for finite graphs of the problem above for graphons is the trivial fact that the vertices of a graph can be ordered with (weakly) increasing vertex degrees. Note that there will always be ties, so even for a finite graph, this does not define a unique canonical labelling.

1.1. **Some notation.** $[0,1]$ will, as above, be regarded as a probability space equipped with the Lebesgue measure and the Lebesque σ-field. (We might also use the Borel σ-field. For the present paper, this makes no difference; for other purposes, the choice of σ-field may have some importance.)

Let $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$ be two probability spaces. A function $\varphi : \Omega_1 \to \Omega_2$ is *measure preserving* if $\mu_1(\varphi^{-1}(A)) = \mu_2(A)$ for any measurable $A \subseteq \Omega_2$. If W is a graphon on Ω_2 and $\varphi : \Omega_1 \to \Omega_2$ is measure preserving, then the *pull-back* W^φ is the graphon $W^\varphi(x,y) := W\bigl(\varphi(x), \varphi(y)\bigr)$ defined on Ω_1. As mentioned above, a pull-back W^φ is always equivalent to W.

2. The example

Our example is the graphon

$$W(x,y) := \begin{cases} 4xy, & x,y \in (0,\frac12), \\ 1/2, & x+y > 3/2, \\ 0, & \text{otherwise}. \end{cases} \tag{2.1}$$

Note that the degree function is given by

$$\mathfrak{D}(x) := \int_0^1 W(x,y)\,\mathrm{d}y = \begin{cases} \frac12 x, & x \in (0,\frac12), \\ \frac12(x-\frac12), & x \in (\frac12, 1). \end{cases} \tag{2.2}$$

Suppose that W is equivalent to a graphon W_1 on $[0,1]$ that has an increasing degree function $\mathfrak{D}_1(x) := \int_0^1 W_1(x,y)\,\mathrm{d}y$; we will show that this leads to a contradiction.

The equivalence $W \cong W_1$ implies by [1, Corollary 2.7], see also [7, Corollary 10.35] and [6, Theorem 8.6], that there exist a probability space (Ω, μ) and two measure preserving maps $\varphi, \psi : \Omega \to [0,1]$ such that $W^\varphi = W_1^\psi$ a.e., i.e.,

$$W\bigl(\varphi(x), \varphi(y)\bigr) = W_1\bigl(\psi(x), \psi(y)\bigr), \qquad \text{a.e. on } \Omega^2. \tag{2.3}$$

(The probability space (Ω, μ) can be taken as $[0,1]$ with Lebesgue measure, but we have no need for this. Instead, we prefer to use the notation Ω and μ to distinguish between this space and $[0,1]$, which hopefully will make the proof easier to follow.)

Since φ and ψ are measure preserving, we have for every Borel measurable $f \geqslant 0$ on $[0,1]$,

$$\int_0^1 f(x)\,\mathrm{d}x = \int_\Omega f(\varphi(x))\,\mathrm{d}\mu(x) = \int_\Omega f(\psi(x))\,\mathrm{d}\mu(x). \tag{2.4}$$

We use this repeatedly below.

In particular, (2.3) and (2.4) imply that for a.e. $x \in \Omega$

$$\begin{aligned}\mathfrak{D}\bigl(\varphi(x)\bigr) &= \int_0^1 W\bigl(\varphi(x), y\bigr)\,\mathrm{d}y = \int_\Omega W\bigl(\varphi(x), \varphi(y)\bigr)\,\mathrm{d}\mu(y) \\ &= \int_\Omega W_1\bigl(\psi(x), \psi(y)\bigr)\,\mathrm{d}\mu(y) = \int_0^1 W_1\bigl(\psi(x), y\bigr)\,\mathrm{d}y = \mathfrak{D}_1\bigl(\psi(x)\bigr).\end{aligned} \tag{2.5}$$

Hence, for every real $r \in (0, \frac14]$, using (2.2),

$$\begin{aligned}&\lambda\bigl\{x \in [0,1] : \mathfrak{D}_1(x) \leqslant r\bigr\} = \mu\bigl\{x \in \Omega : \mathfrak{D}_1(\psi(x)) \leqslant r\bigr\} \\ &\quad = \mu\bigl\{x \in \Omega : \mathfrak{D}(\varphi(x)) \leqslant r\bigr\} = \lambda\bigl\{x \in [0,1] : \mathfrak{D}(x) \leqslant r\bigr\} = 4r.\end{aligned} \tag{2.6}$$

Since we have assumed that $\mathfrak{D}_1$ is increasing, this implies

$$\mathfrak{D}_1(x) = x/4, \qquad x \in (0,1). \tag{2.7}$$

Define

$$h(x) := \lambda\bigl\{y : W(x,y) \notin \{0, \tfrac12\}\bigr\} = \begin{cases} \frac12, & x \in (0, \frac12), \\ 0, & x \in (\frac12, 1), \end{cases} \tag{2.8}$$

and, similarly,

$$h_1(x) := \lambda\bigl\{y : W_1(x,y) \notin \{0, \tfrac12\}\bigr\}. \tag{2.9}$$

Then (2.3) implies, similarly to (2.5), for a.e. $x \in \Omega$,

$$\begin{aligned}h(\varphi(x)) &= \lambda\bigl\{y : W(\varphi(x), y) \notin \{0, \tfrac12\}\bigr\} \\ &= \mu\bigl\{y : W(\varphi(x), \varphi(y)) \notin \{0, \tfrac12\}\bigr\} \\ &= \mu\bigl\{y : W_1(\psi(x), \psi(y)) \notin \{0, \tfrac12\}\bigr\} \\ &= \lambda\bigl\{y : W_1(\psi(x), y) \notin \{0, \tfrac12\}\bigr\} = h_1(\psi(x)).\end{aligned} \tag{2.10}$$

This will yield our contradiction. We first calculate h_1.

If $0 < a < b < 1$, then, using (2.7), (2.4), (2.10), (2.5), and (2.4) again,

$$\int_a^b h_1(x)\,\mathrm{d}x = \int_0^1 h_1(x)\mathbf{1}\Bigl\{\frac{a}{4} < \mathfrak{D}_1(x) < \frac{b}{4}\Bigr\}\,\mathrm{d}x$$

$$= \int_\Omega h_1(\psi(x)) \mathbf{1}\Big\{\frac{a}{4} < \mathfrak{D}_1(\psi(x)) < \frac{b}{4}\Big\} \,\mathrm{d}\mu(x)$$
$$= \int_\Omega h(\varphi(x)) \mathbf{1}\Big\{\frac{a}{4} < \mathfrak{D}(\varphi(x)) < \frac{b}{4}\Big\} \,\mathrm{d}\mu(x)$$
$$= \int_0^1 h(x) \mathbf{1}\Big\{\frac{a}{4} < \mathfrak{D}(x) < \frac{b}{4}\Big\} \,\mathrm{d}x. \tag{2.11}$$

However, by (2.8) and (2.2),

$$\int_0^1 h(x) \mathbf{1}\Big\{\frac{a}{4} < \mathfrak{D}(x) < \frac{b}{4}\Big\} \,\mathrm{d}x = \frac{1}{2}\int_0^{1/2} \mathbf{1}\Big\{\frac{a}{4} < \mathfrak{D}(x) < \frac{b}{4}\Big\} \,\mathrm{d}x$$
$$= \frac{1}{2}\lambda\Big(\frac{a}{2}, \frac{b}{2}\Big) = \frac{b-a}{4}. \tag{2.12}$$

Consequently, (2.11) and (2.12) show that for every $a \in (0,1)$ and $\varepsilon \in (0, 1-a)$,

$$\frac{1}{\varepsilon}\int_a^{a+\varepsilon} h_1(x) \,\mathrm{d}x = \frac{1}{\varepsilon} \cdot \frac{\varepsilon}{4} = \frac{1}{4}. \tag{2.13}$$

However, by the Lebesgue differentiation theorem, as $\varepsilon \to 0$, this converges a.e. to $h_1(x)$. Hence,

$$h_1(x) = \frac{1}{4} \qquad \text{a.e. } x \in [0,1]. \tag{2.14}$$

We may now complete the proof. It follows from (2.14) that $h_1(\psi(x)) = \frac{1}{4}$ a.e. on Ω, while (2.8) implies that $h(x) \neq \frac{1}{4}$ a.e. on $[0,1]$, and thus $h(\varphi(x)) \neq \frac{1}{4}$ a.e. on Ω. Thus (2.10) yields a contradiction.

Consequently, there is no graphon W_1 equivalent to W with increasing degree function. □

3. Strictly increasing degree functions

In this section, we give a proof of the following result, mentioned in the introduction. This result is not new; it is mentioned in Delmas, Dhersin and Sciauveau [3] (without proof), and it may also have been observed earlier. We do not know any published proof, so we give one for completeness.

Theorem 3. *If $W(x,y)$ is a graphon defined on $[0,1]$ such that its degree function $\mathfrak{D}(x)$ is a strictly increasing function $[0,1] \to [0,1]$, then any equivalent graphon that also has a strictly increasing degree function is a.e. equal to W.*

Proof. Suppose that W_1 is an equivalent graphon on $[0,1]$ that has a strictly increasing degree function $\mathfrak{D}_1$. As in Section 2, there exists a probability space (Ω, μ) and measure preserving maps $\varphi, \psi : \Omega \to [0,1]$ such that (2.3)–(2.5) hold. By (2.5), for a.e. $x, y \in \Omega$,

$$\varphi(x) < \varphi(y) \implies \mathfrak{D}(\varphi(x)) < \mathfrak{D}(\varphi(y)) \implies \mathfrak{D}_1(\psi(x)) < \mathfrak{D}_1(\psi(y))$$
$$\implies \psi(x) < \psi(y). \tag{3.1}$$

We may interchange W and W_1 and thus, for a.e. x, y,

$$\varphi(x) < \varphi(y) \iff \psi(x) < \psi(y). \tag{3.2}$$

Consequently, for a.e. $x \in \Omega$,

$$\varphi(x) = \lambda\{t \in [0,1] : t < \varphi(x)\} = \mu\{y \in \Omega : \varphi(y) < \varphi(x)\}$$
$$= \mu\{y \in \Omega : \psi(y) < \psi(x)\} = \lambda\{t \in [0,1] : t < \psi(x)\} = \psi(x). \quad (3.3)$$

This together with (2.3) shows that $W\bigl(\varphi(x), \varphi(y)\bigr) = W_1\bigl(\varphi(x), \varphi(y)\bigr)$ a.e. on Ω^2, and a final use of the fact that φ is measure preserving shows that $W(s,t) = W_1(s,t)$ for a.e. $s, t \in [0,1]$. $\square$

Remark 4. Theorem 3 can easily be slightly extended to show that also there is no equivalent graphon with a weakly but not strictly increasing degree funtion. We omit the proof.

Acknowledgement

I thank two anonymous referees for helpful comments.

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