THE SUM OF POWERS OF SUBTREE SIZES FOR CONDITIONED GALTON–WATSON TREES

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Abstract. We study the additive functional $X_n(\alpha)$ on conditioned Galton–Watson trees given, for arbitrary complex α, by summing the αth power of all subtree sizes. Allowing complex α is advantageous, even for the study of real α, since it allows us to use powerful results from the theory of analytic functions in the proofs.

For $\operatorname{Re}\alpha < 0$, we prove that $X_n(\alpha)$, suitably normalized, has a complex normal limiting distribution; moreover, as processes in α, the weak convergence holds in the space of analytic functions in the left half-plane. We establish, and prove similar process-convergence extensions of, limiting distribution results for α in various regions of the complex plane. We focus mainly on the case where $\operatorname{Re}\alpha > 0$, for which $X_n(\alpha)$, suitably normalized, has a limiting distribution that is *not* normal but does not depend on the offspring distribution ξ of the conditioned Galton–Watson tree, assuming only that $\mathbb{E}\,\xi = 1$ and $0 < \operatorname{Var}\xi < \infty$. Under a weak extra moment assumption on ξ, we prove that the convergence extends to moments, ordinary and absolute and mixed, of all orders.

At least when $\operatorname{Re}\alpha > \frac{1}{2}$, the limit random variable $Y(\alpha)$ can be expressed as a function of a normalized Brownian excursion.

1. Introduction and main results

In the study of random trees, one important part is the study of *additive functionals.* These are functionals of rooted trees of the type

$$F(T) := \sum_{v \in T} f(T_v), \tag{1.1}$$

where v ranges over all nodes of the tree T, T_v is the subtree consisting of v and all its descendants, and f is a given functional of trees, often called the *toll function.* Equivalently, additive functionals may be defined by the recursion

$$F(T) := f(T) + \sum_{i=1}^{d} F(T_{v(i)}), \tag{1.2}$$

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where d is the degree of the root o of T and $v(1), \dots, v(d)$ are the children of o. (All trees in this paper are rooted.)

We are mainly interested in the case when $T = \mathcal{T}_n$ is some random tree of order $|\mathcal{T}_n| = n$, and we study asymptotics of $F(\mathcal{T}_n)$ as $n \to \infty$. Such problems have been studied by many authors, for different classes of functionals f and different classes of random trees $\mathcal{T}_n$; some examples are [29; 21; 17; 22; 28; 61; 38; 11; 51; 39; 1; 9].

In the present paper we consider the case where the toll function is $f_\alpha(T) := |T|^\alpha$ for some constant α, and $\mathcal{T}_n$ is a conditioned Galton–Watson tree, defined by some offspring distribution ξ with $\mathbb{E}\,\xi = 1$ and $0 < \operatorname{Var} \xi < \infty$; see Section 2.1 for definitions and note that this includes for example uniformly random labelled trees, ordered trees, and binary trees. (We use these standing assumptions on $\mathcal{T}_n$ and ξ throughout the paper, whether said explictly or not.) Some previous papers dealing with this situation, in varying generality, are [21; 17; 11; 1; 9]. We denote the corresponding additive functional (1.1) by F_α; thus $F_\alpha(T)$ is the sum of the αth power of all subtree sizes for T. We also introduce the following notation:

$$X_n(\alpha) := F_\alpha(\mathcal{T}_n) := \sum_{v \in \mathcal{T}_n} |\mathcal{T}_{n,v}|^\alpha, \tag{1.3}$$

$$\widetilde{X}_n(\alpha) := X_n(\alpha) - \mathbb{E}\,X_n(\alpha). \tag{1.4}$$

Note that for $\alpha = 0$, we trivially have $X_n(0) = F_0(\mathcal{T}_n) = n$. The case $\alpha = 1$ yields, as is well known, the *total pathlength*, see Example 1.25.

Previous papers have studied the case when α is real, but we consider these variables for arbitrary complex α. This is advantageous, even for the study of real α, since it allows us to use powerful results from the theory of analytic functions in the proofs. We also find new phenomena for non-real α (for example Theorem 1.20). Note that $X_n(\alpha)$ and $\widetilde{X}_n(\alpha)$ are random entire functions of α, for any given n. [The expectation in (1.4) exists because, for a given n, the variable $X_n(\alpha)$ takes only a finite number of different values.]

We begin with the case $\operatorname{Re}\alpha < 0$, where $X_n(\alpha)$ is asymptotically normal as an easy consequence of [38, Theorem 1.5 and Remark 1.6]. More precisely, the following holds. (Proofs of this and other theorems stated here are given later.) We say that a complex random variable ζ is *normal* if $(\operatorname{Re}\zeta, \operatorname{Im}\zeta)$ has a two-dimensional normal distribution. (See [32, Section 1.4], and note that a real normal variable is a special case.)

Theorem 1.1. *Let $\mathcal{T}_n$ be a conditioned Galton–Watson tree defined by an offspring distribution ξ with $\mathbb{E}\,\xi = 1$ and $0 < \sigma^2 := \operatorname{Var}\xi < \infty$. Then there exists a family of centered complex normal random variables $\widehat{X}(\alpha)$, $\operatorname{Re}\alpha < 0$, such that, as $n \to \infty$,*

$$n^{-1/2}\widetilde{X}_n(\alpha) = \frac{X_n(\alpha) - \mathbb{E}\,X_n(\alpha)}{\sqrt{n}} \xrightarrow{\text{d}} \widehat{X}(\alpha), \qquad \operatorname{Re}\alpha < 0. \tag{1.5}$$

Moreover, $\widehat{X}(\alpha)$ is a (random) analytic function of α, and the convergence (1.5) *holds in the space $\mathcal{H}(H_-)$ of analytic functions in the left half-plane $H_- := \{\alpha : \operatorname{Re}\alpha < 0\}$. Furthermore,*

$$\overline{\widehat{X}(\alpha)} = \widehat{X}(\overline{\alpha}), \qquad \alpha \in H_-. \tag{1.6}$$

The covariance function $\mathbb{E}\big(\widehat{X}(\alpha)\widehat{X}(\beta)\big)$ *is an analytic function of two variables* $\alpha, \beta \in H_-$*, and, as* $n \to \infty$*,*

$$n^{-1}\operatorname{Cov}\big(X_n(\alpha), X_n(\beta)\big) \to \mathbb{E}\big[\widehat{X}(\alpha)\widehat{X}(\beta)\big], \qquad \alpha, \beta \in H_-. \tag{1.7}$$

The convergence in $\mathcal{H}(H_-)$ means uniform convergence on compact sets and implies joint convergence for different α in (1.5); see Section 2.2.

The distribution of the limit $\widehat{X}(\alpha)$ depends on the offspring distribution ξ in a rather complicated way. Since the variables $\widehat{X}(\alpha)$ are complex normal, and (1.6) holds, the joint distribution of all $\widehat{X}(\alpha)$ is determined by the covariance function $\mathbb{E}\big(\widehat{X}(\alpha)\widehat{X}(\beta)\big)$, $\alpha, \beta \in H_-$. We give a formula for this in (5.1), but we do not know any simple way to evaluate it.

In most parts of the paper we assume $\operatorname{Re}\alpha > 0$. We introduce a normalization that will turn out to be correct for $\operatorname{Re}\alpha > 0$ and define

$$Y_n(\alpha) := n^{-\alpha-\frac{1}{2}} X_n(\alpha), \tag{1.8}$$

$$\widetilde{Y}_n(\alpha) := n^{-\alpha-\frac{1}{2}} \widetilde{X}_n(\alpha) = Y_n(\alpha) - \mathbb{E}\, Y_n(\alpha). \tag{1.9}$$

Then the following holds.

Theorem 1.2. *There exists a family of complex random variables* $\widetilde{Y}(\alpha)$*,* $\operatorname{Re}\alpha > 0$*, such that if* $\mathcal{T}_n$ *is a conditioned Galton–Watson tree defined by an offspring distribution* ξ *with* $\mathbb{E}\,\xi = 1$ *and* $0 < \sigma^2 := \operatorname{Var}\xi < \infty$*, then, as* $n \to \infty$*,*

$$\sigma n^{-\alpha-\frac{1}{2}} \widetilde{X}_n(\alpha) = \sigma \widetilde{Y}_n(\alpha) \xrightarrow{\mathrm{d}} \widetilde{Y}(\alpha), \qquad \operatorname{Re}\alpha > 0. \tag{1.10}$$

Moreover, $\widetilde{Y}(\alpha)$ *is a (random) analytic function of* α*, and the convergence* (1.10) *holds in the space* $\mathcal{H}(H_+)$ *of analytic functions in the right half-plane* $H_+ := \{\alpha : \operatorname{Re}\alpha > 0\}$*.*

Here $\widetilde{Y}(\alpha)$ is *not* normal. [In fact, it follows from (1.20) and (1.21) below that if $\alpha > \frac{1}{2}$, then $\widetilde{Y}(\alpha)$ is bounded below.] On the other hand, note that the family $\widetilde{Y}(\alpha)$ does *not* depend on the offspring distribution ξ; it is the same for all conditioned Galton–Watson trees satisfying our conditions $\mathbb{E}\,\xi = 1$ and $0 < \sigma^2 < \infty$, and thus the asymptotics of $\widetilde{X}_n$ depends on ξ only through the scaling factor σ. Hence, we have universality of the limit when $\operatorname{Re}\alpha > 0$, but not when $\operatorname{Re}\alpha < 0$.

We can add moment convergence to Theorem 1.2, at least provided we add a weak extra moment assumption.

Theorem 1.3. *Assume, in addition to the conditions on* ξ *in Theorem 1.2, that* $\mathbb{E}\,\xi^{2+\delta} < \infty$ *for some* $\delta > 0$*. Then, the limit* (1.10) *holds with all moments, ordinary and absolute. In other words, if* $\operatorname{Re}\alpha > 0$*, then* $\mathbb{E}\,|\widetilde{Y}(\alpha)|^r < \infty$ *for every* $r < \infty$*; furthermore, for any integer* $\ell \geqslant 1$*,*

$$n^{-\ell(\alpha+\frac{1}{2})}\, \mathbb{E}\big[\widetilde{X}_n(\alpha)^\ell\big] = \mathbb{E}\big[\widetilde{Y}_n(\alpha)^\ell\big] \to \sigma^{-\ell}\, \mathbb{E}\big[\widetilde{Y}(\alpha)^\ell\big], \qquad \operatorname{Re}\alpha > 0, \tag{1.11}$$

and similarly for absolute moments and mixed moments of $\widetilde{X}_n(\alpha)$ *and* $\overline{\widetilde{X}_n(\alpha)}$*.*

Moreover, for each fixed ℓ*,* (1.11) *and its analogues for absolute moments and mixed moments hold uniformly for* α *in any fixed compact subset of* H_+*; the limit*

$\mathbb{E}\,\widetilde{Y}(\alpha)^\ell$ is an analytic function of $\alpha \in H_+$ while absolute moments and mixed moments of $\widetilde{Y}(\alpha)$ and $\overline{\widetilde{Y}(\alpha)}$ are continuous functions of $\alpha \in H_+$.

The result extends to joint moments for several $\alpha \in H_+$. The moments of $\widetilde{Y}(\alpha)$ may be computed by (1.20) and the recursion formula (1.25)–(1.26) below. Note that $\widetilde{Y}(\alpha)$ is centered: $\mathbb{E}\,\widetilde{Y}(\alpha) = 0$; this follows, e.g., by the case $\ell = 1$ of (1.11). See also Remark 1.15 and Example 1.16.

Remark 1.4. We conjecture that Theorem 1.3 holds also without the extra moment condition. Note that even without that condition, (1.11) holds for $\alpha \neq \frac{1}{2}$ as a simple consequence of Theorem 1.12 below. The case $\alpha = \frac{1}{2}$ is more complicated, but has been treated directly in the special case $\xi \sim \mathrm{Bi}(2, \frac{1}{2})$ (binary trees) by [21]; that special case satisfies $\mathbb{E}\,\xi^r < \infty$ for every r, but it seems likely that the proof in [21] can be adapted to the general case by arguments similar to those in Section 12. However, we have not pursued this and leave it as an open problem. See also [9]. □

Theorems 1.1 and 1.2 are stated for the centered variables $\widetilde{X}_n(\alpha)$. We obtain results for $X_n(\alpha)$ by combining Theorems 1.1–1.2 with the asymptotics for the expectation $\mathbb{E}\,X_n(\alpha)$ given in the next theorem, but we first need more notation.

Let $\mathcal{T}$ be the Galton–Watson tree (without conditioning) defined by the offspring distribution ξ; see Section 2.1. It follows from (2.6) that $f_\alpha(\mathcal{T}) = |\mathcal{T}|^\alpha$ has a finite expectation if and only if $\operatorname{Re}\alpha < \frac{1}{2}$, and we define

$$\mu(\alpha) := \mathbb{E}\, f_\alpha(\mathcal{T}) = \mathbb{E}\,|\mathcal{T}|^\alpha = \sum_{n=1}^{\infty} n^\alpha\, \mathbb{P}(|\mathcal{T}| = n), \qquad \operatorname{Re}\alpha < \tfrac{1}{2}. \tag{1.12}$$

This is an analytic function in the half-plane $\operatorname{Re}\alpha < \frac{1}{2}$. Note that $\mu(\alpha)$ depends on the offspring distribution ξ, although we do not show this in the notation. Note also that $\mu(\alpha)$ has a singularity at $\alpha = \frac{1}{2}$; in fact, it is easily seen from (2.6) that

$$\mu(\alpha) \sim \frac{(2\pi\sigma^2)^{-1/2}}{\frac{1}{2} - \alpha}, \qquad \text{as } \alpha \nearrow \tfrac{1}{2}. \tag{1.13}$$

Remark 1.5. In Section 10 (Theorem 10.7), we show by a rather complicated argument that although $\mu(\alpha) \to \infty$ as $\alpha \nearrow \frac{1}{2}$ (se (1.13)), $\mu(\alpha)$ has a continuous extension to all other points on the line $\operatorname{Re}\alpha = \frac{1}{2}$. □

It is shown by Aldous [2] that if we construct a random fringe tree $\mathcal{T}_{n,V}$ by first choosing a random conditioned Galton–Watson tree $\mathcal{T}_n$ as above, and then a random node V in the tree, then $\mathcal{T}_{n,V}$ converges in distribution as $n \to \infty$ to the random Galton–Watson tree $\mathcal{T}$. This was sharpened in [37, Theorem 7.12] to the corresponding 'quenched' result: the conditional distribution of $\mathcal{T}_{n,V}$ given $\mathcal{T}_n$ converges in probability to the distribution of $\mathcal{T}$. As a consequence (see Section 3), we obtain the following results, which show the central role of $\mu(\alpha)$ in the study of $X_n(\alpha)$.

Theorem 1.6. (i) *If $\operatorname{Re}\alpha \leqslant 0$, then as $n \to \infty$,*

$$\mathbb{E}\,X_n(\alpha) = \mu(\alpha)n + o(n). \tag{1.14}$$

(ii) *If $\operatorname{Re}\alpha \leqslant 0$, then $X_n(\alpha)/n \xrightarrow{\text{p}} \mu(\alpha)$.*

The following theorem improves and extends the estimate (1.14); in particular, note that [in parts (i) and (ii)] the error term in (1.14) is improved to $o\big(n^{1/2}\big)$ for $\operatorname{Re}\alpha < 0$ and $O\big(n^{1/2}\big)$ for $\operatorname{Re}\alpha = 0$.

Theorem 1.7. *The following estimates hold as $n \to \infty$, in all cases uniformly for α in compact subsets of the indicated domains.*

(i) *If* $\operatorname{Re}\alpha < 0$, *then*

$$\mathbb{E}\, X_n(\alpha) = \mu(\alpha)n + o\big(n^{1/2}\big). \tag{1.15}$$

(ii) *If* $-\frac12 < \operatorname{Re}\alpha < \frac12$, *then*

$$\mathbb{E}\, X_n(\alpha) = \mu(\alpha)n + \frac{1}{\sqrt{2}\sigma}\frac{\Gamma(\alpha-\frac12)}{\Gamma(\alpha)}\, n^{\alpha+\frac12} + o\big(n^{(\operatorname{Re}\alpha)_+ + \frac12}\big). \tag{1.16}$$

(iii) *If* $\operatorname{Re}\alpha > \frac12$, *then*

$$\mathbb{E}\, X_n(\alpha) = \frac{1}{\sqrt{2}\sigma}\frac{\Gamma(\alpha-\frac12)}{\Gamma(\alpha)} n^{\alpha+\frac12} + o\big(n^{\alpha+\frac12}\big). \tag{1.17}$$

(iv) *If* $\alpha = \frac12$, *then*

$$\mathbb{E}\, X_n(1/2) = \frac{1}{\sqrt{2\pi\sigma^2}} n\log n + o\big(n\log n\big). \tag{1.18}$$

Remark 1.8. As shown in Theorem 10.8(i), the estimate (1.16) holds also for $\alpha = \frac12 + \mathrm{i}y$, $y \neq 0$, where $\mu(\alpha)$ is the continuous extension described in Remark 1.5. □

Theorems 1.1 and 1.7(i) together yield the following variant of (1.5).

Theorem 1.9. *If* $\operatorname{Re}\alpha < 0$, *then, as $n \to \infty$,*

$$\frac{X_n(\alpha) - n\mu(\alpha)}{\sqrt{n}} \overset{\mathrm{d}}{\longrightarrow} \widehat{X}(\alpha). \tag{1.19}$$

Moreover, this holds in the space $\mathcal{H}(H_-)$.

Similarly, Theorems 1.2 and 1.7 [parts (iii) and (ii)] yield the following. We define, for $\operatorname{Re}\alpha > 0$ and $\alpha \neq \frac12$, the complex random variable

$$Y(\alpha) := \widetilde{Y}(\alpha) + \frac{1}{\sqrt{2}}\frac{\Gamma(\alpha-\frac12)}{\Gamma(\alpha)}. \tag{1.20}$$

Theorem 1.10. (i) *If* $\operatorname{Re}\alpha > \frac12$, *then, as $n \to \infty$,*

$$Y_n(\alpha) := n^{-\alpha-\frac12} X_n(\alpha) \overset{\mathrm{d}}{\longrightarrow} \sigma^{-1} Y(\alpha). \tag{1.21}$$

(ii) *If* $0 < \operatorname{Re}\alpha < \frac12$, *then, as $n \to \infty$,*

$$n^{-\alpha-\frac12}\big[X_n(\alpha) - n\mu(\alpha)\big] \overset{\mathrm{d}}{\longrightarrow} \sigma^{-1} Y(\alpha). \tag{1.22}$$

Moreover, in both cases, this holds in the space $\mathcal{H}(D)$ for the indicated domain D.

Remark 1.11. As shown in Theorem 10.8(ii), the limit result (1.22) holds also for $\alpha = \frac12 + \mathrm{i}y$, $y \neq 0$, where $\mu(\alpha)$ is the continuous extension of Remark 1.5. □

We can add moment convergence to Theorem 1.10, too.

Theorem 1.12. *The limits* (1.21) *and* (1.22) *hold with all moments, for* $\operatorname{Re}\alpha > \frac{1}{2}$, *and* $0 < \operatorname{Re}\alpha < \frac{1}{2}$, *respectively. In other words, for any integer* $\ell \geqslant 1$, *if* $\operatorname{Re}\alpha > \frac{1}{2}$, *then*

$$\mathbb{E}\, X_n(\alpha)^\ell = \sigma^{-\ell}\,\mathbb{E}\, Y(\alpha)^\ell n^{\ell(\alpha+\frac{1}{2})} + o\bigl(n^{\ell(\alpha+\frac{1}{2})}\bigr), \tag{1.23}$$

and if $0 < \operatorname{Re}\alpha < \frac{1}{2}$, *then*

$$\mathbb{E}\bigl[X_n(\alpha) - n\mu(\alpha)\bigr]^\ell = \sigma^{-\ell}\,\mathbb{E}\, Y(\alpha)^\ell n^{\ell(\alpha+\frac{1}{2})} + o\bigl(n^{\ell(\alpha+\frac{1}{2})}\bigr). \tag{1.24}$$

Moreover, in both cases, the moments $\kappa_\ell = \kappa_\ell(\alpha) := \mathbb{E}\, Y(\alpha)^\ell$ *are given by the recursion formula*

$$\kappa_1 = \frac{\Gamma(\alpha - \frac{1}{2})}{\sqrt{2}\,\Gamma(\alpha)}, \tag{1.25}$$

and, for $\ell \geqslant 2$, *with* $\alpha' := \alpha + \frac{1}{2}$,

$$\kappa_\ell = \frac{\ell\Gamma(\ell\alpha' - 1)}{\sqrt{2}\,\Gamma(\ell\alpha' - \frac{1}{2})}\kappa_{\ell-1} + \frac{1}{4\sqrt{\pi}}\sum_{j=1}^{\ell-1}\binom{\ell}{j}\frac{\Gamma(j\alpha' - \frac{1}{2})\Gamma((\ell-j)\alpha' - \frac{1}{2})}{\Gamma(\ell\alpha' - \frac{1}{2})}\kappa_j\kappa_{\ell-j}. \tag{1.26}$$

The result extends to joint moments; see Section 12.6.

Remark 1.13. For the case of random binary trees [the case $\xi \sim \operatorname{Bi}(2, \frac{1}{2})$] and real α, Theorems 1.10 and 1.12 were shown already by Fill and Kapur [21], by the method used here in Section 12 to show Theorem 1.12 (namely, singularity analysis of generating functions and the method of moments). Recently (and independently), the case of uniformly random ordered trees [$\xi \sim \operatorname{Ge}(\frac{1}{2})$, in connection with a study of Dyck paths] has been shown (also by such methods) by Caracciolo, Erba and Sportiello [9], and they have extended their result to general ξ, at least when ξ has a finite exponential moment [personal communication]. □

Remark 1.14. Theorem 1.10(i) has also been shown by Delmas, Dhersin and Sciauveau [11] (for $\alpha > 1$, or for full binary trees) and Abraham, Delmas and Nassif [1] (in general). (They consider only real α, but their results extend immediately to complex α.) The results in these papers are more general and allow more general toll functions, and they show how the result can be formulated in an interesting way as convergence of random measures defined by the trees; moreover, they consider also more general conditioned Galton–Watson trees, where $\operatorname{Var}(\xi)$ may be infinite provided ξ belongs to the domain of attraction of a stable distribution. We do not consider such extensions here. □

Remark 1.15. Centered moments $\mathbb{E}\,\widetilde{Y}(\alpha)^k$ can as always be found from the ordinary moments given by the recursion above. Alternatively, [21, Proposition 3.9] gives a (more complicated) recursion formula for the centered moments that yields them directly. [The formula there is given for real α, but it extends to complex α with $\operatorname{Re}\alpha > 0$ by the same proof or by analytic continuation. Note also the different normalizations: Y there is our $\sqrt{2}Y(\alpha)$.] Another formula for centered moments is given by [9, Proposition 7] [again with a different normalization: x_p there is our $2^{-1/2}Y(p)$]. □

Example 1.16. Consider for simplicity real $\alpha > 0$. It follows from (1.25)–(1.26) that

$$\begin{aligned}\mathbb{E}\,\widetilde{Y}(\alpha)^2 &= \operatorname{Var} Y(\alpha) = \kappa_2 - \kappa_1^2 \\ &= \frac{\Gamma(2\alpha)\Gamma(\alpha-\frac12)}{\Gamma(2\alpha+\frac12)\Gamma(\alpha)} + \frac{\Gamma(\alpha-\frac12)^2}{4\sqrt{\pi}\Gamma(2\alpha+\frac12)} - \frac{\Gamma(\alpha-\frac12)^2}{2\Gamma(\alpha)^2}, \qquad \alpha \neq \tfrac12. \end{aligned} \tag{1.27}$$

Moreover, the moments of $\widetilde{Y}(\alpha)$ (which do not depend on ξ) are continuous functions of α by Theorem 1.3, and thus we can obtain the variance $\operatorname{Var}\widetilde{Y}(\frac12)$ by taking the limit of (1.27) as $\alpha \to \frac12$. A simple calculation using Taylor and Laurent expansions of $\Gamma(z)$ yields, cf. [21, Remark 3.6(c)(iv)],

$$\mathbb{E}\,\widetilde{Y}(\tfrac12)^2 = \operatorname{Var}\widetilde{Y}(\tfrac12) = \frac{4\log 2}{\pi} - \frac{\pi}{4}. \tag{1.28}$$

Higher moments of $\widetilde{Y}(\frac12)$ can be calculated in the same way. The moments of $\widetilde{Y}(\frac12)$ were originally found in [21, Proposition 3.8 and Theorem 3.10(b)], and given by a recursion there. [Note again that Y there is our $\sqrt{2}Y(\frac12)$.] See [9, Proposition 7 and Table 3] for another formula and explicit expressions up to order 5 (again with a different normalization). □

Theorems 1.1 and 1.2, or 1.9 and 1.10, show that the asymptotic distribution exhibits a phase transition at $\operatorname{Re}\alpha = 0$.

Remark 1.17. We do not know how to bridge the gap between the two cases $\operatorname{Re}\alpha < 0$ and $\operatorname{Re}\alpha > 0$. Moreover, we do not know the asymptotic distribution, if any, when $\operatorname{Re}\alpha = 0$ (excepting the trivial case $\alpha = 0$ when $X_n(0) = n$ is deterministic), although we note that Theorem 1.6(ii) yields a weaker result on convergence in probability. However, we conjecture that $(n\log n)^{-1/2}\widetilde{X}_n(\mathrm{i}t)$ converges in distribution to a symmetric complex normal distribution, for any $t \neq 0$. □

Problem 1.18. Does $X_n(\mathrm{i}t)$ have an asymptotic distribution, after suitable normalization, for (fixed and real) $t \neq 0$? If so, what is it?

Remark 1.19. For real $\alpha \searrow 0$, (1.25)–(1.26) show that $\mathbb{E}\,Y(\alpha)^2 \to 0$, and thus $Y(\alpha) \xrightarrow{\mathrm{p}} 0$. [See also (1.27).] As remarked in [21, Remark 3.6(e)], one can use (1.25)–(1.26) and the method of moments to show that

$$\alpha^{-1/2}Y(\alpha) \xrightarrow{\mathrm{d}} N(0, 2-2\log 2), \qquad \alpha \searrow 0. \tag{1.29}$$

If we consider complex α with $\operatorname{Re}\alpha > 0$, and let $\alpha \to 0$ from various different directions, then $\alpha^{-1/2}Y(\alpha)$ converges in distribution to various different limits, each of which has a certain complex normal distribution; see Appendix C.

If we instead let $\alpha \to \mathrm{i}t$ with $t \neq 0$ real, then (1.25)–(1.26) imply that the (complex) moments $\mathbb{E}\,Y(\alpha)^\ell$ converge. However, the absolute moment $\mathbb{E}\,|Y(\alpha)|^2 \to \infty$ by a similar calculation; see (12.80). It can be shown, again by the method of moments, that in this case, $(\operatorname{Re}\alpha)^{1/2}Y(\alpha)$ converges in distribution to a symmetric complex normal distribution; see Appendix D. As a consequence, the imaginary axis is a.s. a natural boundary for the random analytic functions $Y(\cdot)$ and $\widetilde{Y}(\cdot)$ i.e., they have no analytic extension to any larger domain; see again Appendix D for details. □

Theorems 1.7 and 1.10 show another phase transition at $\operatorname{Re}\alpha = \frac{1}{2}$; this phase transition comes from the behavior of the mean $\mathbb{E}\, X_n(\alpha)$, while the fluctuations $\widetilde{X}_n(\alpha)$ vary analytically by Theorem 1.2. To be precise, there is a singularity at $\alpha = \frac{1}{2}$, as shown by (1.13) together with (1.16) or (1.22). For non-real α on the line $\operatorname{Re}\alpha = \frac{1}{2}$, the situation is more complicated. As said in Remarks 1.5, 1.8, and 1.11, the results for $\operatorname{Re}\alpha < \frac{1}{2}$ extend continuously to $\operatorname{Re}\alpha = \frac{1}{2}$, $\alpha \neq \frac{1}{2}$. Moreover, the next theorem (Theorem 1.20) shows that if we add a weak moment assumption on ξ, then we can extend Theorems 1.7 and 1.10 *analytically* across the line $\operatorname{Re}\alpha = \frac{1}{2}$, and also refine the result at the exceptional case $\alpha = \frac{1}{2}$. [The results now depend on ξ through more than just σ^2, see (6.39).] Hence, assuming a higher moment, there is a singularity at $\alpha = \frac{1}{2}$ but no other singularities at the line $\operatorname{Re}\alpha = \frac{1}{2}$. However, in general (without higher moments), $\mu(\alpha)$ *cannot* be extended analytically across the line $\operatorname{Re}\alpha = \frac{1}{2}$, see Theorem 11.1; hence, in general the entire line $\operatorname{Re}\alpha = \frac{1}{2}$ is a singularity—in other words, a phase transition.

Theorem 1.20. *Suppose that* $\mathbb{E}\,\xi^{2+\delta} < \infty$ *for some* $\delta \in (0, 1]$. *Then:*

(i) $\mu(\alpha)$ *can be analytically continued to a meromorphic function in* $\operatorname{Re}\alpha < \frac{1}{2} + \frac{\delta}{2}$, *with a single pole at* $\alpha = \frac{1}{2}$ *with residue* $-1/\sqrt{2\pi\sigma^2}$.

(ii) *Using this extension of* $\mu(\alpha)$, (1.16) *holds, uniformly on compact sets, for* $-\frac{1}{2} < \operatorname{Re}\alpha < \frac{1}{2} + \frac{\delta}{2}$ *with* $\alpha \neq \frac{1}{2}$.

(iii) *For some constant c (depending on the offspring distribution),*

$$\mathbb{E}\, X_n(\tfrac{1}{2}) = \frac{1}{\sqrt{2\pi\sigma^2}} n \log n + cn + o(n). \tag{1.30}$$

Remark 1.21. If ξ has higher moments, then $\mu(\alpha)$ can be continued even further: see Theorem 6.5. In particular, if ξ has finite moments of all orders, then $\mu(\alpha)$ can be continued to a meromorphic function in the entire complex plane $\mathbb{C}$, with poles at $j + \frac{1}{2}$, $j = 0, 1, 2, \dots$ (or possibly a subset thereof). □

Theorem 1.22. *Suppose that* $\mathbb{E}\,\xi^{2+\delta} < \infty$ *for some* $\delta \in (0, 1]$. *Then:*

(i) *The limit in distribution* (1.22) *holds for all* $\alpha \in D := \{\alpha \neq \frac{1}{2} : 0 < \operatorname{Re}\alpha < \frac{1}{2} + \frac{\delta}{2}\}$; *moreover* (1.22) *holds in* $\mathcal{H}(D)$.

(ii) *For some constant c (depending on the offspring distribution),*

$$n^{-1}\Big[X_n(\tfrac{1}{2}) - \frac{1}{\sqrt{2\pi\sigma^2}} n \log n\Big] \xrightarrow{\mathrm{d}} \sigma^{-1}\widetilde{Y}(\tfrac{1}{2}) + c. \tag{1.31}$$

The constants c in (1.30) and (1.31) are equal. The proof yields the formula (6.39).

Remark 1.23. The phase transitions at $\operatorname{Re}\alpha = 0$ and $\operatorname{Re}\alpha = \frac{1}{2}$ can be explained as follows. Consider for simplicity real α, when all terms in (1.1) are positive. The expected number of subtrees $\mathcal{T}_{n,v}$ of order k is roughly $n\,\mathbb{P}(|\mathcal{T}| = k) = \Theta(nk^{-3/2})$, by [37, Theorem 7.12] (see Section 3) and (2.6). Hence, if $\alpha > \frac{1}{2}$, $\mathbb{E}\, X_n(\alpha)$ is dominated by the rather few large $\mathcal{T}_{n,v}$ of size $\Theta(n)$; there are roughly $\Theta(n^{1/2})$ such trees, which explains the order $n^{\alpha+\frac{1}{2}}$ of $\mathbb{E}\, X_n(\alpha)$. For $\alpha < \frac{1}{2}$, $\mathbb{E}\, X_n(\alpha)$ is dominated by the small subtrees $\mathcal{T}_{n,v}$, of size $O(1)$, and this yields the linear behavior of $\mathbb{E}\, X_n(\alpha)$ in Theorem 1.7.

For $\alpha < 0$, the fluctuations, too, are dominated by the small subtrees (as shown in the proof of [38, Theorem 1.5]); there are $\approx n$ of these, and they are only weakly dependent on each other, and as a result $X_n(\alpha)$ has an asymptotic normal distribution with the usual scaling.

For $0 < \alpha < \frac{1}{2}$, on the other hand, the mean $\mathbb{E}\, X_n(\alpha)$ is dominated by the small subtrees as just said, but fluctuations are dominated by the large subtrees of order $\Theta(n)$. (To see this, note that for $\alpha > 0$ and $\varepsilon > 0$, the contribution to $X_n(\alpha)$ from subtrees of order $\leqslant \varepsilon n$ has variance $O\bigl(\varepsilon^{2\alpha} n^{2\alpha+1}\bigr)$ by [38, Theorem 6.7].) Hence, we have the same asymptotic behavior of $\widetilde{X}_n(\alpha)$ as for larger α. The large subtrees are more strongly dependent on each other, and lead to a non-normal limit; on the other hand, asymptotically they do not depend on details in the offspring distribution. $\square$

At least when $\operatorname{Re}\alpha > \frac{1}{2}$, the limit random variable $Y(\alpha)$ can be expressed as a function of a normalized Brownian excursion $(\mathbf{e}(t))$. [Recall that $(\mathbf{e}(t))$ is a random continuous function on $[0, 1]$; see, e.g., [52] for a definition.] For a function f defined on an interval, define

$$m(f; s, t) := \inf_{u \in [s,t]} f(u). \tag{1.32}$$

The general representation formula for $\operatorname{Re}\alpha > \frac{1}{2}$ is a little bit complicated, and we give three closely related versions (1.33)–(1.35), where the first two are related by mirror symmetry and the third, symmetric, formula is the average of the two preceding. (See further the proof, which also gives a fourth formula (7.27). The representations (1.35) and (1.36) were stated in [18, (4.2)–(4.3), see also Examples 4.6 and 4.7]; the present paper gives, after a long delay, the proof promised there.) Note that the integrals in (1.33)–(1.35) converge (absolutely) a.s. when $\operatorname{Re}\alpha > \frac{1}{2}$, since $\mathbf{e}(t)$ is a.s. Hölder(γ)-continuous for every $\gamma < \frac{1}{2}$, and thus, e.g., $|\mathbf{e}(t) - m(\mathbf{e}; s, t)| \leqslant C(t-s)^{\gamma}$ for some random constant C. (This well-known fact follows e.g. from the corresponding fact for Brownian motion together with the construction of $\mathbf{e}$ from the excursions of the Brownian motion, see [52, Theorem I.(2.2) and Chapter XII.2–3].)

We also give a simpler expression (1.36) valid for $\operatorname{Re}\alpha > 1$. [The integral in (1.36) diverges for $\operatorname{Re}\alpha \leqslant 1$.]

Theorem 1.24. (i) *If* $\operatorname{Re}\alpha > \frac{1}{2}$, *then, jointly for all such* α,

$$\begin{aligned} Y(\alpha) &\overset{\mathrm{d}}{=} 2\alpha \int_0^1 t^{\alpha-1}\mathbf{e}(t)\,\mathrm{d}t \\ &\qquad - 2\alpha(\alpha-1) \iint\limits_{0<s<t<1} (t-s)^{\alpha-2}\bigl[\mathbf{e}(t) - m(\mathbf{e}; s, t)\bigr]\,\mathrm{d}s\,\mathrm{d}t \qquad (1.33) \\ &= 2\alpha \int_0^1 (1-t)^{\alpha-1}\mathbf{e}(t)\,\mathrm{d}t \\ &\qquad - 2\alpha(\alpha-1) \iint\limits_{0<s<t<1} (t-s)^{\alpha-2}\bigl[\mathbf{e}(s) - m(\mathbf{e}; s, t)\bigr]\,\mathrm{d}s\,\mathrm{d}t \qquad (1.34) \end{aligned}$$

$$= \alpha \int_0^1 \bigl[t^{\alpha-1} + (1-t)^{\alpha-1}\bigr] \mathbf{e}(t)\,\mathrm{d}t$$
$$- \alpha(\alpha-1) \iint\limits_{0<s<t<1} (t-s)^{\alpha-2} \left[\mathbf{e}(s) + \mathbf{e}(t) - 2m(\mathbf{e}; s,t)\right] \mathrm{d}s\,\mathrm{d}t. \tag{1.35}$$

(ii) *If* $\operatorname{Re}\alpha > 1$*, we have also the simpler representation*

$$Y(\alpha) \overset{\mathrm{d}}{=} 2\alpha(\alpha-1) \iint\limits_{0<s<t<1} (t-s)^{\alpha-2} m(\mathbf{e}; s,t)\,\mathrm{d}s\,\mathrm{d}t. \tag{1.36}$$

Example 1.25. For $\alpha = 1$, (1.33)–(1.35) reduce to

$$Y(1) = 2\int_0^1 \mathbf{e}(t)\,\mathrm{d}t, \tag{1.37}$$

twice the *Brownian excursion area.* In fact, with $d(v)$ denoting the depth of a given node v, it is easy to see that

$$X_n(1) = \sum_{v\in\mathcal{T}_n} |\mathcal{T}_{n,v}| = \sum_{v\in\mathcal{T}_n} (d(v)+1) = n + \sum_{v\in\mathcal{T}_n} d(v), \tag{1.38}$$

i.e., n plus the *total pathlength.* The convergence of the total pathlength, suitably rescaled, to the Brownian excursion area was shown by Aldous [3, 4], see also [34]. The Brownian excursion area has been studied by many authors in various contexts, for example [43; 44; 57; 58; 59; 55; 25; 24; 33], see also [36] and the further references there.

Furthermore, for $\alpha = 2$, (1.36) reduces to

$$Y(2) \overset{\mathrm{d}}{=} 4 \iint\limits_{0<s<t<1} m(\mathbf{e}; s,t)\,\mathrm{d}s\,\mathrm{d}t. \tag{1.39}$$

This too was studied in [34], where $Y(2)$ was denoted η. Moreover, the random variable $P(\mathcal{T}_n)$ there equals $X_n(1) - n$, $Q(\mathcal{T}_n)$ equals $X_n(2) - n^2$, and the Wiener index $W(\mathcal{T}_n) = nP(\mathcal{T}_n) - Q(\mathcal{T}_n)$ equals $nX_n(1) - X_n(2)$. Hence, the limit theorem [34, Theorem 3.1] follows from Theorems 1.10 and 1.24.

Moreover, as noted by [21], Theorem 1.12 yields for $\alpha = 1$ a recursion formula for the moments of the Brownian excursion area, which is equivalent to the formulas given by [57; 58; 59; 25; 24], see also [36, Section 2]. Similarly, also noted by [21], Theorem 1.12 yields for $\alpha = 2$ the recursion formula for moments of $Y(2)$ given in [34]. More generally, the recursion in [34] for mixed moments of $Y(1)$ and $Y(2)$ follows from Theorem 12.9 below. □

Remark 1.26. For $\alpha > \frac{1}{2}$, a different (but equivalent) representation of the limit $Y(\alpha)$ as a function of a Brownian excursion $\mathbf{e}$ is given by Delmas, Dhersin and Sciauveau [11, (1.10) and (2.6)]. That representation can also be written as a functional of the Brownian continuum random tree; see Abraham, Delmas and Nassif [1, Theorem 1.1]. □

Remark 1.27. As demonstrated in Section 8, it follows from the proof of Theorem 1.2 given in that section that there exists a representation of $Y(\alpha)$ as a (measurable) functional of $\mathbf{e}$ also for $0 < \operatorname{Re}\alpha \leqslant \frac{1}{2}$. However, this is only an existence statement, and we do not know any explicit representation. More precisely, there exists a measurable function $\Psi : H_+ \times C[0,1] \to \mathbb{C}$ such that

$$Y(\alpha) = \Psi(\alpha, \mathbf{e}), \qquad \operatorname{Re}\alpha > 0, \tag{1.40}$$

where $\mathbf{e}$ as above is a Brownian excursion. Moreover, $\Psi(\alpha, f)$ is an analytic function of $\alpha \in H_+$ for every $f \in C[0,1]$. For $\operatorname{Re}\alpha > \frac{1}{2}$, $\Psi(\alpha, \mathbf{e})$ is a.s. given by the formulas (1.33)–(1.35), and for $\operatorname{Re}\alpha > 1$ also by (1.36). Hence, in principle, $\Psi(\alpha, \mathbf{e})$ is given by an analytic extension of (1.36) to all $\alpha \in H_+$, and such an extension (necessarily unique) exists a.s. (Note that for $\operatorname{Re}\alpha < 1$, the double integrals in (1.33)–(1.35) do not converge for every function $\mathbf{e} \in C[0,1]$, so we can only claim existence of the extension a.s.)

We concede that the existence of an analytic extension $\Psi(\alpha, \mathbf{e})$ gives a "representation" of $Y(\alpha)$ only in a rather abstract sense. □

Problem 1.28. Find an explicit representation for $Y(\alpha)$ as a function of $\mathbf{e}$ for $0 < \operatorname{Re}\alpha < \frac{1}{2}$, or even for $0 < \alpha < \frac{1}{2}$.

Finally, we consider real α and let $\alpha \to \infty$. We show the following asymptotic result yielding a limit of the limit in Theorem 1.10; this improves a result in [21] which shows the existence of such a limit together with (1.43). Let $B(t)$, $t \geqslant 0$, be a standard Brownian motion, and let

$$S(t) := \sup_{s \in [0,t]} B(s) \tag{1.41}$$

be the corresponding supremum process.

Theorem 1.29. *As $\alpha \to +\infty$ along the real axis, we have $\alpha^{1/2} Y(\alpha) \xrightarrow{\mathrm{d}} Y_\infty$, where Y_∞ is a random variable with the representation*

$$Y_\infty = \int_0^\infty e^{-t} S(t)\,\mathrm{d}t. \tag{1.42}$$

and moments

$$\mathbb{E}\, Y_\infty^k = 2^{-k/2}\sqrt{k!}, \qquad k \geqslant 0, \tag{1.43}$$

and more generally, for real or complex r,

$$\mathbb{E}\, Y_\infty^r = 2^{-r/2}\sqrt{\Gamma(r+1)}, \qquad \operatorname{Re} r > -1. \tag{1.44}$$

Further representations of Y_∞ are given in (9.24) and (9.26).

Remark 1.30. Since convergence in the space $\mathcal{H}(D)$ (for a domain $D \subseteq \mathbb{C}$) of a sequence of analytic functions implies convergence of their derivatives, the results above imply corresponding results for $X_n'(\alpha)$ and $Y_n'(\alpha)$ (and also for higher derivatives). Note that $X_n'(\alpha)$ is the additive functional given by the toll function $\frac{\mathrm{d}}{\mathrm{d}\alpha} f_\alpha(T) = |T|^\alpha \log|T|$. In particular, we have

$$X_n'(0) = \sum_{v \in \mathcal{T}_n} \log|\mathcal{T}_{n,v}| = \log \prod_{v \in \mathcal{T}_n} |\mathcal{T}_{n,v}|, \tag{1.45}$$

which is known as the *shape functional*, see e.g. [16; 46]. Unfortunately, because of the phase transition at $\operatorname{Re}\alpha = 0$, most of our results do not include 0 in their domains. The exception is Theorem 1.7(ii), which implies

$$\mathbb{E}\, X_n'(0) = \mu'(0)n + o\bigl(n^{1/2}\log n\bigr), \tag{1.46}$$

where the error term is obtained from (1.16) and Cauchy's estimates using the circle $|z| = 1/\log n$. More precise estimates of $\mathbb{E}\, X_n'(0)$ have been proved by [16; 21; 17] [random binary trees, the case $\xi \sim \mathrm{Bi}(2, \frac{1}{2})$], and [46] (general ξ with an exponential moment); furthermore, these papers also give results for the variance (which is of order $n \log n$). Moreover, asymptotic normality of $X_n'(0)$ has been shown in special cases by Pittel [50] [random labelled trees, the case $\xi \sim \mathrm{Po}(1)$], Fill and Kapur [21] [random binary trees, the case $\xi \sim \mathrm{Bi}(2, \frac{1}{2})$], and Caracciolo, Erba and Sportiello [9] [random ordered trees, the case $\xi \sim \mathrm{Ge}(\frac{1}{2})$]. We have been able to extend this to general ξ, assuming $\mathbb{E}\,\xi^{2+\delta} < \infty$ for some $\delta > 0$, by suitable modifications of the arguments in Section 12 (we might provide details in future work). It seems to be an open problem to show asymptotic normality of $X_n'(0)$ for arbitrary ξ with $0 < \operatorname{Var}\xi < \infty$ (and $\mathbb{E}\,\xi = 1$, as always).

Note that although the asymptotic normality of $X_n'(0)$ does not follow from the results in the present paper, it fits well together with Theorem 1.1 which shows that $X_n(\alpha)$ is asymptotically normal for every $\alpha < 0$. □

The contents of the paper are as follows. Section 2 contains some preliminaries. Section 3 gives the simple proof of Theorem 1.6. Section 4 shows two lemmas on tightness, and Section 5 then gives a short proof of Theorem 1.1. Section 6 is a detailed study of the expectation $\mathbb{E}\, X_n(\alpha)$. Section 7 treats convergence to Brownian excursion and functions thereof. Section 8 gives some remaining proofs. Section 9 discusses the limit as real $\alpha \to +\infty$. Sections 10 and 11 give proofs and a counterexample, respectively, for the case $\operatorname{Re}\alpha = \frac{1}{2}$. Section 12 studies moments and gives proofs of Theorems 1.3 and 1.12. This section uses a method different from that of the previous sections; the two methods complement each other and combine in the proof of Theorem 1.3. Finally, Appendix A discusses calculation of $\mu(\alpha)$ and gives some examples of it; Appendix B gives a proof of a technical lemma in Section 12, together with some background on polylogarithms used in the proof; Appendices C and D give proofs of the additional results claimed in Remark 1.19.

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2. Preliminaries and notation

2.1. **Conditioned Galton–Watson trees.** Given a non-negative integer-valued random variable ξ, with distribution $\mathcal{L}(\xi)$, the *Galton–Watson tree* $\mathcal{T}$ with offspring distribution $\mathcal{L}(\xi)$ is constructed recursively by starting with a root and giving each node a number of children that is a new copy of ξ, independent of the numbers of children of the other nodes. Obviously, only the distribution $\mathcal{L}(\xi)$ of ξ matters; we abuse language and say also that $\mathcal{T}$ has offspring distribution ξ. Furthermore, let $\mathcal{T}_n$ be $\mathcal{T}$ conditioned on having exactly n nodes; this is called a *conditioned Galton–Watson tree*. (We consider only n such that $\mathbb{P}(|\mathcal{T}| = n) > 0$.)

We assume that $\mathbb{P}(\xi = 0) > 0$, since otherwise the tree $\mathcal{T}$ is a.s. infinite. In fact, we consider here only the *critical* case $\mathbb{E}\,\xi = 1$; in this case $\mathcal{T}$ is a.s. finite (provided $\mathbb{P}(\xi \neq 1) > 0$). It is well known that in most cases, but not all, a conditioned Galton–Watson tree with an offspring distribution ξ' with an expectation $\mathbb{E}\,\xi' \neq 1$ is equivalent to a conditioned Galton–Watson tree with another offspring distribution ξ satisfying $\mathbb{E}\,\xi = 1$, so this is only a minor restriction. See e.g. [37, Section 4] for details.

We also assume $0 < \operatorname{Var}\xi < \infty$ (but usually no higher moment assumptions).

Remark 2.1. More generally, a *simply generated random tree* T_n defined by a given sequence of non-negative weights $(\phi_k)_0^\infty$ is a random ordered tree with n nodes such that for every ordered tree T with $|T| = n$, the probability $\mathbb{P}(\mathcal{T}_n = T)$ is proportional to $\prod_{v\in T} \phi_{\delta^+(v)}$, where $\delta^+(v)$ denotes the outdegree of v, see e.g. [45] or [12, Section 1.2.7]. Every conditioned Galton–Watson tree is a simply generated random tree, and the converse holds under a weak condition. In particular, if the generating function $\Phi(z) := \sum_{k=0}^\infty \phi_k z^k$ has a positive radius of convergence R and there exists τ with $0 < \tau < R$ and $\tau\Phi'(\tau) = \Phi(\tau)$ (which is a common assumption in studies of simply generated random trees), then the simply generated random tree T_n equals a conditioned Galton–Watson tree defined by a suitable ξ with $\mathbb{E}\,\xi = 1$; furthermore, this ξ has finite moment generating function $\mathbb{E}\,e^{t\xi} < \infty$ at some $t > 0$, and thus finite moments of all orders. Again, see e.g. [37, Section 4] for details. □

Let $\xi_1, \xi_2, \dots$ be independent copies of ξ and define

$$S_n := \sum_{i=1}^n \xi_i. \tag{2.1}$$

It is well known (see Otter [48], or [37, Theorem 15.5] and the further references given there) that for any $n \geqslant 1$,

$$\mathbb{P}(|\mathcal{T}| = n) = \frac{1}{n}\,\mathbb{P}(S_n = n-1). \tag{2.2}$$

In particular, (1.12) can be written

$$\mu(\alpha) = \sum_{n=1}^\infty n^{\alpha-1}\,\mathbb{P}(S_n = n-1), \qquad \operatorname{Re}\alpha < \tfrac12. \tag{2.3}$$

For some examples where exact (and in one case rational) values of $\mu(\alpha)$ can be computed when α is a negative integer, see Appendix A.

Recall that the *span* of an integer-valued random variable ξ, denoted $\operatorname{span}(\xi)$, is the largest integer h such that $\xi \in a + h\mathbb{Z}$ a.s. for some $a \in \mathbb{Z}$; we consider only ξ with $\mathbb{P}(\xi = 0) > 0$ and then the span is the largest integer h such that $\xi/h \in \mathbb{Z}$ a.s., i.e., the greatest common divisor of $\{n : \mathbb{P}(\xi = n) > 0\}$. (Typically, $h = 1$, but we have for example $h = 2$ in the case of full binary trees, when $\xi \in \{0, 2\}$.) The local limit theorem for discrete random variables can in our setting can be stated as follows; see, e.g., [41, Theorem 1.4.2] or [49, Theorem VII.1].

Lemma 2.2 (Local limit theorem)**.** *Suppose that ξ is an integer-valued random variable with $\mathbb{P}(\xi = 0) > 0$, $\mathbb{E}\,\xi = 1$, $0 < \sigma^2 := \operatorname{Var}\xi < \infty$, and span h. Then, as $n \to \infty$, uniformly in all $m \in h\mathbb{Z}$,*

$$\mathbb{P}(S_n = m) = \frac{h}{\sqrt{2\pi\sigma^2 n}}\Big[e^{-(m-n)^2/(2n\sigma^2)} + o(1)\Big]. \tag{2.4}$$

□

In particular, for any fixed $\ell \in \mathbb{Z}$, as $n \to \infty$ with $n \equiv \ell \pmod h$,

$$\mathbb{P}(S_n = n - \ell) \sim \frac{h}{\sqrt{2\pi\sigma^2}}\, n^{-1/2}. \tag{2.5}$$

Combining (2.2) and (2.5) we see that

$$\mathbb{P}(|\mathcal{T}| = n) \sim \frac{h}{\sqrt{2\pi\sigma^2}}\, n^{-3/2} \tag{2.6}$$

as $n \to \infty$ with $n \equiv 1 \pmod h$. [The probability is 0 when $n \not\equiv 1 \pmod h$.]

We will for simplicity assume in some proofs below that the span of ξ equals 1; then (2.6) is valid as $n \to \infty$ without restriction. However, this is just for convenience, and the results hold also for $h > 1$, using standard modifications of the arguments. (We leave these to the reader, but give sometimes a hint.)

2.2. **Random analytic functions.** For a domain (non-empty open connected set) $D \subseteq \mathbb{C}$, let $\mathcal{H}(D)$ denote the space of all analytic functions on D, equipped with the usual topology of uniform convergence on compact sets; this is a topological vector space with the topology given by the seminorms $p_K(f) := \sup_{z\in K} |f(z)|$, with K ranging over all compact subsets of D. The space $\mathcal{H}(D)$ is a Fréchet space, i.e., a locally convex space with a topology that can be defined by a complete translation-invariant metric, and it has (by Montel's theorem on normal families) the property that every closed bounded subset is compact, see e.g. [53, §1.45] or [60, Example 10.II and Theorem 14.6]. Furthermore, $\mathcal{H}(D)$ is separable. $\mathcal{H}(D)$ is thus a Polish space (i.e., a complete separable metric space). We equip $\mathcal{H}(D)$ with its Borel σ-field, and note that this is generated by the point evaluations $f \mapsto f(z)$, $z \in D$. [This can be seen by choosing an increasing sequence (K_i) of compact sets with $D = \bigcup_i K_i$, and a countable dense subset (f_j) of $\mathcal{H}(D)$, and noting that then the sets $U_{i,j,n} := \{f : p_{K_i}(f - f_j) < 1/n\}$ form a countable basis of the topology of $\mathcal{H}(D)$; furthermore, each $U_{i,j,n}$ belongs to the σ-field generated by the point evaluations. We omit the standard details.] It follows from this and the monotone class theorem that the distribution of a random function f in $\mathcal{H}(D)$ is determined by its finite-dimensional distributions (i.e., the distributions of finite sets of point evaluations).

We can use the general theory in e.g. Billingsley [5] or Kallenberg [40] for convergence in distribution of random functions in $\mathcal{H}(D)$. In particular, recall that a sequence (X_n) of random variables in a metric space $\mathcal{S}$ is *tight* if for every $\varepsilon > 0$, there exists a compact subset $K \subseteq \mathcal{S}$ such that $\mathbb{P}(X_n \in K) > 1 - \varepsilon$ for every n. Prohorov's theorem [5, Theorems 6.1–6.2], [40, Theorem 16.3] says that in a Polish space, a sequence X_n is tight if and only if the corresponding sequence of distributions $\mathcal{L}(X_n)$ is relatively compact, i.e., each subsequence has a subsubsequence that converges in distribution.

It is easy to characterize tightness in $\mathcal{H}(D)$ in terms of tightness of real-valued random variables.

Lemma 2.3. *Let D be a domain in $\mathbb{C}$, and let $(X_n(z))$ be a sequence of random analytic functions on D. Then the following are equivalent.*

(i) *The sequence $(X_n(z))$ is tight in $\mathcal{H}(D)$.*
(ii) *The sequence $(\sup_{z\in K}|X_n(z)|)$ is tight for every compact $K \subset D$.*
(iii) *The sequence $(\sup_{z\in B}|X_n(z)|)$ is tight for every closed disc $B \subset D$.*

Proof. This proof is an easy exercise that we include for completeness.

(i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) is trivial.

(iii) $\Longrightarrow$ (i). Assume that (iii) holds and choose a sequence of closed discs $B_j \subset D$, $j \geqslant 1$, such that the interiors B_j° cover D. Let $\varepsilon > 0$. Then, by (iii), for each j there exists $M_j < \infty$ such that $\mathbb{P}(\sup_{z\in B_j}|X_n(z)| > M_j) < 2^{-j}\varepsilon$. Let $L := \{f \in \mathcal{H}(D) : \sup_{z\in B_j}|f(z)| \leqslant M_j \text{ for all } j\}$. Each compact subset K of D is covered by a finite collection of open discs B_j°, and it follows that there exists $M_K < \infty$ such that if $f \in L$, then $p_K(f) := \sup_{z\in K}|f(z)| \leqslant M_K$. In other words, $\sup_{f\in L} p_K(f) < \infty$ for each compact $K \subset D$, which says that L is bounded in $\mathcal{H}(D)$, because the topology is defined by the seminorms p_K [60, Proposition 14.5]. Moreover, L is a closed set in $\mathcal{H}(D)$, and thus L is compact in $\mathcal{H}(D)$ by the Montel property mentioned above. Furthermore, $\mathbb{P}(X_n \notin L) < \sum_{j=1}^{\infty} 2^{-j}\varepsilon = \varepsilon$. ☐

This leads to the following simple sufficient condition.

Lemma 2.4. *Let D be a domain in $\mathbb{C}$ and let $(X_n(z))$ be a sequence of random analytic functions in $\mathcal{H}(D)$. Suppose that there exists a function $\gamma : D \to (0,\infty)$, bounded on each compact subset of D, such that $\mathbb{E}|X_n(z)| \leqslant \gamma(z)$ for every $z \in D$. Then the sequence (X_n) is tight in $\mathcal{H}(D)$.*

Proof. Let $B \subset D$ be a closed disc. There exists a circle $\Gamma \subset D$ such that B lies in the interior of Γ. If $f \in \mathcal{H}(D)$, then the value $f(z)$ at a point inside Γ can be expressed by a Poisson integral $\int_\Gamma P(z,w)f(w)|\mathrm{d}w|$ over the circle Γ, where P is the Poisson kernel. (This is because analytic functions are harmonic. See e.g. [54, 11.4, 11.12, and 11.13].) Furthermore, the Poisson kernel is continuous, and thus bounded by some constant C_1 for all $z \in B$ and $w \in \Gamma$. Consequently, for every $f \in \mathcal{H}(D)$ we have

$$\sup_{z\in B}|f(z)| \leqslant C_1 \int_\Gamma |f(w)|\,|\mathrm{d}w|. \tag{2.7}$$

Applying this to $X_n(z)$ and taking the expectation, we obtain

$$\begin{aligned}\mathbb{E}\sup_{z\in B}|X_n(z)| &\leqslant C_1\,\mathbb{E}\int_\Gamma |X_n(w)|\,|\mathrm{d}w| = C_1\int_\Gamma \mathbb{E}\,|X_n(w)|\,|\mathrm{d}w| \\ &\leqslant C_1\int_\Gamma \gamma(w)\,|\mathrm{d}w| < \infty. \end{aligned}\tag{2.8}$$

Hence the sequence (X_n) satisfies Lemma 2.3(iii) (by Markov's inequality), and the conclusion follows by Lemma 2.3. □

We shall also use the following, which again uses properties of analytic functions.

Lemma 2.5. *Let D be a domain in $\mathbb{C}$ and let E be a subset of D that has a limit point in D. (I.e., there exists a sequence $z_n \in E$ of distinct points and $z_\infty \in D$ such that $z_n \to z_\infty$.) Suppose that (X_n) is a tight sequence of random elements of $\mathcal{H}(D)$ and that there exists a family of random variables $\{Y_z : z \in E\}$ such that for each $z \in E$, $X_n(z) \overset{\mathrm{d}}{\longrightarrow} Y_z$ and, moreover, this holds jointly for any finite set of $z \in E$. Then $X_n \overset{\mathrm{d}}{\longrightarrow} Y$ in $\mathcal{H}(D)$, for some random function $Y(z) \in \mathcal{H}(D)$. Furthermore, $Y(z) \overset{\mathrm{d}}{=} Y_z$, jointly for any finite set of $z \in E$. That is, Y restricted to E and (Y_z) have the same finite-dimensional distributions, and thus have the same distribution as random elements of $\mathbb{C}^E$.*

Proof. It suffices to consider the case when $E = \{z_1, z_2, \dots\}$ with $z_n \to z_\infty \in D$. The result then is a special case of Bousquet-Mélou and Janson [8, Lemma 7.1]; in the notation there we take $\mathcal{S}_1 = \mathcal{H}(D)$, $\mathcal{S}_2 = \mathbb{C}^E$ and let ϕ be the obvious restriction map $f(z) \mapsto (f(z_i))_{i=1}^\infty$; note that ϕ is injective by the standard uniqueness for analytic functions. The assumption of joint convergence $X_n(z) \overset{\mathrm{d}}{\longrightarrow} Y_z$ for any finite subset of E is equivalent to the convergence $\phi(X_n) \overset{\mathrm{d}}{\longrightarrow} (Y_{z_i})$ in $\mathbb{C}^E$, since this space has the product topology [5, p. 19]. The conclusion follows from [8, Lemma 7.1]. □

Remark 2.6. Lemma 2.5 may fail if we do not assume joint convergence; i.e., if only $X_n(z) \overset{\mathrm{d}}{\longrightarrow} Y_z$ for each $z \in E$ separately. For a counterexample, let $D = \mathbb{C}$ and $E = \{z : |z| = 1\}$; further, let U be uniformly distributed on the unit circle $\{z : |z| = 1\}$, let $X_{2n}(z) := U$ (a constant function) and $X_{2n+1}(z) := Uz$. Then $X_n(z) \overset{\mathrm{d}}{\longrightarrow} U$ for each fixed $z \in E$, and (X_n) is tight in $\mathcal{H}(D)$ by Lemma 2.4 with $\gamma(z) := \max\{1, |z|\}$, but X_n does not converge in $\mathcal{H}(\mathbb{C})$; for example, $X_n(0)$ does not converge in distribution.

We do not know whether it would be sufficient to assume $X_n(z) \overset{\mathrm{d}}{\longrightarrow} Y_z$ for each $z \in E$ separately in the case when E contains a non-empty open set. □

2.3. **Dominated convergence.** To show uniformity in α of various estimates, we use the following simple, but perhaps not so well known, version of Lebesgue's dominated convergence theorem.

Lemma 2.7. *Let $\mathcal{A}$ be an arbitrary index set. Suppose that, for $\alpha \in \mathcal{A}$ and $n \geqslant 1$, $f_{\alpha,n}(x)$ are measurable functions on a measure space $(\mathcal{S}, \mathcal{F}, \mu)$, and that for a.e. fixed $x \in \mathcal{S}$, we have $f_{\alpha,n}(x) \to g_\alpha(x)$ as $n \to \infty$, uniformly in $\alpha \in \mathcal{A}$. Suppose furthermore that $h(x)$ is an integrable function on $\mathcal{S}$, such that $|f_{\alpha,n}(x)| \leqslant h(x)$ a.e. for each α and n. Then $\int_{\mathcal{S}} f_{\alpha,n}(x)\,\mathrm{d}\mu(x) \to \int_{\mathcal{S}} g_\alpha(x)\,\mathrm{d}\mu(x)$ as $n \to \infty$, uniformly in $\alpha \in \mathcal{A}$.*

Proof. Note first that the assumptions imply $|g_\alpha(x)| \leqslant h(x)$ a.e. for each α; hence, $\bigl|f_{\alpha,n}(x) - g_\alpha(x)\bigr| \leqslant 2h(x)$ a.e. Let α_n be an arbitrary sequence of elements of $\mathcal{A}$. Then $\int_{\mathcal{S}} \bigl(f_{\alpha_n,n}(x) - g_{\alpha_n}(x)\bigr)\,\mathrm{d}\mu(x) \to 0$ as $n \to \infty$ by the standard dominated convergence theorem. The result follows. □

Remark 2.8. Suppose that the assumptions of Lemma 2.7 hold, and furthermore that $\mathcal{A}$ is an open set in the complex plane and that $g_\alpha(x)$ is an analytic function of α for every $x \in \mathcal{S}$, and jointly measurable in α and x. Then the limit $G(\alpha) := \int_{\mathcal{S}} g_\alpha(x)\,\mathrm{d}\mu(x)$ is an analytic function of $\alpha \in \mathcal{A}$. To see this, note again that the assumptions imply $|g_\alpha(x)| \leqslant h(x)$ a.e. for each α. It follows by dominated convergence that $G(\alpha)$ is a continuous function of α, and by Fubini's theorem that the line integral of $G(\alpha)$ around the boundary of any closed triangle inside $\mathcal{A}$ is 0; hence $G(\alpha)$ is analytic by Morera's theorem. □

2.4. **Further notation.** We denote the distance between two nodes v and w in a tree by $d(v,w)$. Furthermore, we let $d(v) := d(v,o)$ denote the distance from v to the root o; this is usually called the *depth* of v.

For two nodes v, w of a rooted tree T, $v \prec w$ means that w is a descendant of v. Thus, $w \in T_v \iff w \succeq v$. Furthermore, $v \wedge w$ denotes the last common ancestor of v and w. Thus,

$$u \preceq v \wedge w \iff (u \preceq v) \wedge (u \preceq w). \tag{2.9}$$

For real numbers x and y, $x \wedge y$ is another notation for $\min(x,y)$. Furthermore, $x_+ := \max(x,0)$ and $x_- := -\min(x,0)$.

Unspecified limits are as $n \to \infty$.

$C, C_1, \dots$ and $c, c_1, \dots$ denote positive constants (typically with large and small values, respectively), not necessarily the same at different places. The constants may depend on the offspring distribution ξ; they may also depend on other parameters that are indicated as arguments.

3. The case $\operatorname{Re}\alpha \leqslant 0$, convergence in probability

Proof of Theorem 1.6. By (1.3), recalling that V is a random node in $\mathcal{T}_n$,

$$\mathbb{E}\big(f_\alpha(\mathcal{T}_{n,V}) \mid \mathcal{T}_n\big) = \frac{1}{n}\sum_{v\in\mathcal{T}_n} |\mathcal{T}_{n,v}|^\alpha = \frac{1}{n}X_n(\alpha), \tag{3.1}$$

and consequently

$$\mathbb{E}\, f_\alpha(\mathcal{T}_{n,V}) = \frac{1}{n}\,\mathbb{E}\, X_n(\alpha). \tag{3.2}$$

The random trees defined in Section 1 may be regarded as random elements of the countable discrete set $\mathfrak{T}$ of finite ordered rooted trees. As noted just before the statement of Theorem 1.6 in Section 1, Aldous [2] shows that $\mathcal{T}_{n,V} \overset{\mathrm{d}}{\longrightarrow} \mathcal{T}$, as random elements of $\mathfrak{T}$. If $\operatorname{Re}\alpha \leqslant 0$, then f_α is a bounded function on $\mathfrak{T}$, trivially continuous since $\mathfrak{T}$ is discrete. Hence, it follows from (3.2) that

$$\frac{1}{n}X_n(\alpha) = \mathbb{E}\, f_\alpha(\mathcal{T}_{n,V}) \to \mathbb{E}\, f_\alpha(\mathcal{T}) = \mu(\alpha), \tag{3.3}$$

showing (1.14).

Similarly, by [37, Theorem 7.12], the conditional distribution of $\mathcal{T}_{n,V}$ given $\mathcal{T}_n$ converges (as a random element of the space of probability distributions on $\mathfrak{T}$) in probability to the distribution of $\mathcal{T}$, which by (3.1) yields part (ii) of Theorem 1.6. □

4. Tightness

Recall the notation at (1.3)–(1.4) and (1.8)–(1.9).

Lemma 4.1. (i) *For* $\operatorname{Re}\alpha < 0$ *and all* $n \geqslant 1$, $\mathbb{E}\,|\widetilde{X}_n(\alpha)|^2 \leqslant C(\alpha)n$, *for some constant* $C(\alpha) = O\big(1 + |\operatorname{Re}\alpha|^{-2}\big)$*; thus* $C(\alpha)$ *is bounded on each proper half-space* $\{\alpha : \operatorname{Re}\alpha < -\varepsilon < 0\}$.

(ii) *For* $\operatorname{Re}\alpha > 0$ *and all* $n \geqslant 1$, $\mathbb{E}\,|\widetilde{X}_n(\alpha)|^2 \leqslant C(\alpha)n^{2\operatorname{Re}\alpha+1}$ *and thus* $\mathbb{E}\,|\widetilde{Y}_n(\alpha)|^2 \leqslant C(\alpha)$, *for some constant* $C(\alpha) = O\big(1 + (\operatorname{Re}\alpha)^{-2}\big)$*; thus* $C(\alpha)$ *is bounded on each proper half-space* $\{\alpha : \operatorname{Re}\alpha > \varepsilon > 0\}$.

Proof. Recall the notation $f_\alpha(T) := |T|^\alpha$. We apply [38, Theorem 6.7] to (the real and imaginary parts of) the functional $f(T) := f_\alpha(T)\cdot \mathbf{1}_{|T|\leqslant n}$. Since $|f(\mathcal{T}_k)| = |f_\alpha(\mathcal{T}_k)| = |k^\alpha| = k^{\operatorname{Re}\alpha}$ for $k \leqslant n$, and $f(\mathcal{T}_k) = 0$ for $k > n$, this yields

$$\big(\mathbb{E}\,|\widetilde{X}_n(\alpha)|^2\big)^{1/2} \leqslant C_1 n^{1/2}\Big(\sup_{k\leqslant n} k^{\operatorname{Re}\alpha} + \sum_{k=1}^n k^{\operatorname{Re}\alpha-1}\Big) \leqslant \begin{cases} C_2(\alpha)n^{1/2}, & \operatorname{Re}\alpha < 0,\\ C_3(\alpha)n^{\operatorname{Re}\alpha+\frac12}, & \operatorname{Re}\alpha > 0,\end{cases}$$

with $C_2(\alpha) = O\big(1 + |\operatorname{Re}\alpha|^{-1}\big)$ and $C_3(\alpha) = O\big(1 + |\operatorname{Re}\alpha|^{-1}\big)$. ◻

Lemma 4.2. (i) *The family of random functions* $n^{-1/2}\widetilde{X}_n(\alpha)$ *is tight in the space* $\mathcal{H}(H_-)$.

(ii) *The family of random functions* $\widetilde{Y}_n(\alpha) := n^{-\alpha-\frac12}\widetilde{X}_n(\alpha)$ *is tight in the space* $\mathcal{H}(H_+)$.

Proof. This is an immediate consequence of Lemmas 2.4 and 4.1 (and the Cauchy–Schwarz inequality). ◻

5. The case $\operatorname{Re}\alpha < 0$

Proof of Theorem 1.1. For a fixed real $\alpha < 0$, [38, Theorem 1.5] yields (1.5) with $\widehat{X}(\alpha) \sim N\big(0, \gamma^2(\alpha)\big)$ for some $\gamma^2(\alpha) \geqslant 0$. Furthermore, as remarked in [38], [38, Theorem 1.5] extends, by the Cramér–Wold device, to joint convergence for several functionals. By considering $\operatorname{Re} f_\alpha$ and $\operatorname{Im} f_\alpha$, we thus obtain (1.5) for complex $\alpha \in H_-$; furthermore, we obtain joint convergence for any finite set of (real or complex) such α. The convergence in $\mathcal{H}(H_-)$ now follows from Lemmas 2.5 and 4.2(i).

The symmetry (1.6) is now obvious, since the corresponding formula for $X_n(\alpha)$ follows trivially from the definition (1.3). Finally, (1.7) follows from [38, (1.16)] and polarization (i.e., considering linear combinations). ◻

Remark 5.1. Furthermore, [38, (1.17)] and polarization yields a formula for the covariance function, for $\operatorname{Re}\alpha, \operatorname{Re}\beta < 0$:

$$\mathbb{E}\big(\widehat{X}(\alpha)\widehat{X}(\beta)\big) = \mathbb{E}\big(f_\alpha(\mathcal{T})\big(F_\beta(\mathcal{T}) - |\mathcal{T}|\mu(\beta)\big)\big) + \mathbb{E}\big(f_\beta(\mathcal{T})\big(F_\alpha(\mathcal{T}) - |\mathcal{T}|\mu(\alpha)\big)\big) - \mu(\alpha+\beta) + \big(1 - \sigma^{-2}\big)\mu(\alpha)\mu(\beta). \quad (5.1)$$

◻

6. The mean

Lemma 6.1. *For any complex α,*

$$\mathbb{E}\,X_n(\alpha) = n\sum_{k=1}^{n}\frac{\mathbb{P}(S_{n-k}=n-k)\,\mathbb{P}(S_k=k-1)}{\mathbb{P}(S_n=n-1)}k^{\alpha-1}. \tag{6.1}$$

Proof. By [38, Lemma 5.1], summing over k,

$$\mathbb{E}\,X_n(\alpha) = \mathbb{E}\,F_\alpha(\mathcal{T}_n) = \sum_{k=1}^{n} n\frac{\mathbb{P}(S_{n-k}=n-k)}{\mathbb{P}(S_n=n-1)}\,\mathbb{E}\,f_{\alpha,k}(\mathcal{T}) \tag{6.2}$$

where $f_{\alpha,k}(T) := f_\alpha(T)\mathbf{1}_{|T|=k} = k^\alpha\mathbf{1}_{|T|=k}$ and thus, using (2.2),

$$\mathbb{E}\,f_{\alpha,k}(\mathcal{T}) = k^\alpha\,\mathbb{P}(|\mathcal{T}|=k) = k^{\alpha-1}\,\mathbb{P}(S_k=k-1). \tag{6.3}$$

The result follows. ◻

We now prove Theorem 1.7. We begin with part (i), which follows from [38], and part (iii), which is rather easy.

Proof of Theorem 1.7(i). The estimate (1.15) is an instance of [38, (1.13)], and the proof in [38] shows that the estimate holds uniformly in each half-space $\operatorname{Re}\alpha < -\varepsilon < 0$. ◻

Proof of Theorem 1.7(iii). We write (6.1) as $\mathbb{E}\,X_n(\alpha) = n^{\alpha-\frac12}\sum_{k=1}^{n} g_{n,\alpha}(k)$ where

$$g_{n,\alpha}(k) := \frac{\mathbb{P}(S_{n-k}=n-k)\,\mathbb{P}(S_k=k-1)}{\mathbb{P}(S_n=n-1)}k^{\alpha-1}n^{-\alpha+\frac32}. \tag{6.4}$$

Thus, converting the sum in (6.1) to an integral by letting $k := \lceil xn\rceil$,

$$n^{-\alpha-\frac12}\,\mathbb{E}\,X_n(\alpha) = n^{-1}\sum_{k=1}^{n} g_{n,\alpha}(k) = \int_0^1 g_{n,\alpha}(\lceil xn\rceil)\,\mathrm{d}x. \tag{6.5}$$

Assume for simplicity $\operatorname{span}(\xi)=1$. [Otherwise, replace $\lceil xn\rceil$ by xn rounded upwards to the nearest integer $k\equiv 1 \pmod{\operatorname{span}(\xi)}$, and make minor modifications.] For any fixed $x\in(0,1)$, it then follows from (2.5) that as $n\to\infty$, for any fixed α and uniformly for α in a compact set,

$$g_{n,\alpha}(\lceil xn\rceil) \sim \frac{(n-nx)^{-1/2}(nx)^{-1/2}}{\sqrt{2\pi\sigma^2}\,n^{-1/2}}(nx)^{\alpha-1}n^{-\alpha+\frac32} = \frac{1}{\sqrt{2\pi\sigma^2}}(1-x)^{-1/2}x^{\alpha-\frac32}. \tag{6.6}$$

Furthermore, (2.5) similarly also implies that, for n so large that $\mathbb{P}(S_n=n-1)>0$,

$$|g_{n,\alpha}(\lceil xn\rceil)| \leqslant C(1-x)^{-1/2}x^{-(\operatorname{Re}\alpha-\frac32)_-} \tag{6.7}$$

for some constant C (depending on the offspring distribution, but not on α). Since we assume $\operatorname{Re}\alpha > \frac12$, the right-hand side of (6.7) is integrable, and thus dominated convergence and (6.6) yield, evaluating a beta integral,

$$\int_0^1 g_{n,\alpha}(\lceil xn\rceil)\,\mathrm{d}x \to \frac{1}{\sqrt{2\pi\sigma^2}}\int_0^1 x^{\alpha-3/2}(1-x)^{-1/2}\,\mathrm{d}x$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} B\bigl(\alpha - 1/2, 1/2\bigr) = \frac{1}{\sqrt{2\pi\sigma^2}} \frac{\Gamma(\alpha-1/2)\Gamma(1/2)}{\Gamma(\alpha)}$$
$$= \frac{1}{\sqrt{2}\sigma} \frac{\Gamma(\alpha-1/2)}{\Gamma(\alpha)}. \tag{6.8}$$

Moreover, using Lemma 2.7, this holds uniformly for α in each compact subset of $\{\alpha : \operatorname{Re}\alpha > \frac{1}{2}\}$. The result follows by (6.5). ☐

Before completing the proof of Theorem 1.7, we give another lemma with a related estimate for $\mathbb{E}\, X_n(\alpha)$. We define, compare (1.12) and (2.3), for any complex α,

$$\mu_n(\alpha) := \mathbb{E}\bigl(|\mathcal{T}|^\alpha \mathbf{1}_{|\mathcal{T}|\leqslant n}\bigr) = \sum_{k=1}^n k^\alpha\, \mathbb{P}(|\mathcal{T}| = k) = \sum_{k=1}^n k^{\alpha-1}\, \mathbb{P}(S_k = k-1). \tag{6.9}$$

Lemma 6.2. *If* $\operatorname{Re}\alpha > -\frac{1}{2}$, *then, as* $n \to \infty$,

$$\mathbb{E}\, X_n(\alpha) = n\mu_n(\alpha) + \frac{1}{\sqrt{2}\sigma}\left[\frac{\Gamma(\alpha-\frac{1}{2})}{\Gamma(\alpha)} - \frac{\pi^{-1/2}}{\alpha-\frac{1}{2}}\right] n^{\alpha+\frac{1}{2}} + o\bigl(n^{(\operatorname{Re}\alpha)_+ + \frac{1}{2}}\bigr). \tag{6.10}$$

Moreover, this holds uniformly for any compact set of α *with* $\operatorname{Re}\alpha > -\frac{1}{2}$.

Remark 6.3. For $\alpha = \frac{1}{2}$, the square bracket in (6.10) is interpreted by continuity. With $\psi(x) := \Gamma'(x)/\Gamma(x)$, the value at $\frac{1}{2}$ is easily found to be $\pi^{-1/2}\bigl(\psi(1) - \psi(\frac{1}{2})\bigr) = (2\log 2)\pi^{-1/2}$, using [47, 5.4.12–13]. ☐

Proof. This time we use (6.1) and (6.9) to obtain, with $g_{n,\alpha}(k)$ as in (6.4), cf. (6.5),

$$\mathbb{E}\, X_n(\alpha) - n\mu_n(\alpha) = n^{\alpha-\frac{1}{2}} \sum_{k=1}^n \bigl[g_{n,\alpha}(k) - n^{\frac{3}{2}-\alpha} k^{\alpha-1}\, \mathbb{P}(S_k = k-1)\bigr],$$
$$= n^{\alpha-\frac{1}{2}} \sum_{k=1}^n h_{n,\alpha}(k), \tag{6.11}$$

where, see (6.4),

$$h_{n,\alpha}(k) := g_{n,\alpha}(k) - n^{\frac{3}{2}-\alpha} k^{\alpha-1}\, \mathbb{P}(S_k = k-1)$$
$$= n^{\frac{3}{2}-\alpha} k^{\alpha-1}\, \mathbb{P}(S_k = k-1) \left[\frac{\mathbb{P}(S_{n-k} = n-k)}{\mathbb{P}(S_n = n-1)} - 1\right]. \tag{6.12}$$

We use once more (2.5) and see that, assuming for simplicity that ξ has span 1, for any fixed $x \in (0,1)$, for any fixed α and uniformly for α in a compact set,

$$h_{n,\alpha}(\lceil xn \rceil) \to \frac{1}{\sqrt{2\pi\sigma^2}} x^{\alpha-\frac{3}{2}} \Bigl[(1-x)^{-1/2} - 1\Bigr]. \tag{6.13}$$

Furthermore, by (2.5), for all n, k, and α,

$$n^{\frac{3}{2}-\alpha} k^{\alpha-1}\, \mathbb{P}(S_k = k-1) = O\Bigl(\Bigl(\frac{k}{n}\Bigr)^{\operatorname{Re}\alpha - \frac{3}{2}}\Bigr). \tag{6.14}$$

If $1 \leqslant k \leqslant n/2$, then by [38, Lemma 5.2(i)],

$$\frac{\mathbb{P}(S_{n-k} = n-k)}{\mathbb{P}(S_n = n-1)} - 1 = O\left(\frac{k}{n}\right) + o\bigl(n^{-1/2}\bigr), \tag{6.15}$$

and if $n/2 < k \leqslant n$, then by [38, Lemma 5.2(ii)],

$$\frac{\mathbb{P}(S_{n-k} = n-k)}{\mathbb{P}(S_n = n-1)} - 1 = O\left(\frac{n^{1/2}}{(n-k+1)^{1/2}}\right). \tag{6.16}$$

For $k \geqslant n^{1/2}$, the bound in (6.15) is $O(k/n)$. Let $h^*_{n,\alpha}(k) := h_{n,\alpha}(k)\mathbf{1}_{k \geqslant n^{1/2}}$, and fix α with $\operatorname{Re}\alpha > -\frac{1}{2}$. Then, combining (6.12) and (6.14)–(6.16), for all n and $x \in (0,1)$,

$$h^*_{n,\alpha}(\lceil xn \rceil) = O\bigl(x^{-(\operatorname{Re}\alpha - \frac{1}{2})_-} + (1-x)^{-1/2}\bigr). \tag{6.17}$$

This bound is integrable, and thus dominated convergence and (6.13) yield

$$\begin{aligned} n^{-1} \sum_{n^{1/2} \leqslant k \leqslant n} h_{n,\alpha}(k) &= \int_0^1 h^*_{n,\alpha}(\lceil xn \rceil)\,\mathrm{d}x \\ &\to \frac{1}{\sqrt{2\pi\sigma^2}} \int_0^1 x^{\alpha - \frac{3}{2}} \Bigl[(1-x)^{-1/2} - 1\Bigr]\,\mathrm{d}x. \end{aligned} \tag{6.18}$$

The integral on the right-hand side of (6.18) converges for any α with $\operatorname{Re}\alpha > -\frac{1}{2}$, and defines an analytic function in that region. If $\operatorname{Re}\alpha > \frac{1}{2}$, we have

$$\begin{aligned} \int_0^1 x^{\alpha - \frac{3}{2}} \Bigl[(1-x)^{-1/2} - 1\Bigr]\,\mathrm{d}x &= \int_0^1 x^{\alpha - \frac{3}{2}} (1-x)^{-1/2}\,\mathrm{d}x - \int_0^1 x^{\alpha - \frac{3}{2}}\,\mathrm{d}x \\ &= B\bigl(\alpha - \tfrac{1}{2}, \tfrac{1}{2}\bigr) - \frac{1}{\alpha - \frac{1}{2}} \\ &= \frac{\Gamma(\alpha - \frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\alpha)} - \frac{1}{\alpha - \frac{1}{2}}. \end{aligned} \tag{6.19}$$

The right-hand side in (6.19) is analytic for $\operatorname{Re}\alpha > -\frac{1}{2}$ (with a removable singularity at $\alpha = \frac{1}{2}$), and thus by analytic continuation, (6.19) holds as soon as $\operatorname{Re}\alpha > -\frac{1}{2}$.

By combining (6.11), (6.18), and (6.19), we obtain the main terms in (6.10). However, it remains to show that the terms with $k < n^{1/2}$ in (6.11) are negligible. For this we use again (6.14) and (6.15) and obtain

$$\begin{aligned} \sum_{k < \sqrt{n}} h_{n,\alpha}(k) &\leqslant C \sum_{k < \sqrt{n}} \left(\frac{k}{n}\right)^{\operatorname{Re}\alpha - \frac{1}{2}} + o\bigl(n^{-1/2}\bigr) \sum_{k < \sqrt{n}} \left(\frac{k}{n}\right)^{-(\operatorname{Re}\alpha)_- - \frac{3}{2}} \\ &\leqslant C_1(\alpha) n^{\frac{1}{2}(\operatorname{Re}\alpha + \frac{1}{2}) + \frac{1}{2} - \operatorname{Re}\alpha} + o\bigl(n^{1 + (\operatorname{Re}\alpha)_-}\bigr) \\ &= o\bigl(n^{1 + (\operatorname{Re}\alpha)_-}\bigr). \end{aligned} \tag{6.20}$$

This shows that the contribution to (6.11) for $k < n^{1/2}$ is $o\bigl(n^{\operatorname{Re}\alpha + \frac{1}{2} + (\operatorname{Re}\alpha)_-}\bigr) = o\bigl(n^{(\operatorname{Re}\alpha)_+ + \frac{1}{2}}\bigr)$, which completes the proof of (6.10).

Moreover, the estimates (6.13), (6.17), and (6.20) hold uniformly in any compact subset of $\{\alpha : \operatorname{Re}\alpha > -\frac{1}{2}\}$, which using Lemma 2.7 gives the uniformity in (6.10). ☐

Proof of Theorem 1.7(ii). Assume again for simplicity that ξ has span 1. Then (2.5) yields

$$\mathbb{P}(S_k = k-1) = \frac{1}{\sqrt{2\pi\sigma^2}} k^{-\frac{1}{2}}(1+\varepsilon_k), \tag{6.21}$$

with $\varepsilon_k \to 0$ as $k \to \infty$, and thus, using dominated convergence, for $\operatorname{Re}\alpha < \frac{1}{2}$,

$$\begin{aligned} n^{\frac{1}{2}-\alpha}\big[\mu(\alpha) - \mu_n(\alpha)\big] &= n^{\frac{1}{2}-\alpha} \sum_{k=n+1}^{\infty} k^{\alpha-1}\,\mathbb{P}(S_k = k-1) \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_1^\infty \left(\frac{\lceil xn\rceil}{n}\right)^{\alpha-\frac{3}{2}} (1+\varepsilon_{\lceil xn\rceil})\,\mathrm{d}x \\ &\to \frac{1}{\sqrt{2\pi\sigma^2}} \int_1^\infty x^{\alpha-\frac{3}{2}}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi\sigma^2}(\frac{1}{2}-\alpha)}. \end{aligned} \tag{6.22}$$

Moreover, by Lemma 2.7, (6.22) holds uniformly in every half-plane $\operatorname{Re}\alpha < b < \frac{1}{2}$. The result follows by combining (6.10) and (6.22). ☐

Proof of Theorem 1.7(iv). By (6.9) and (6.21), again assuming $\operatorname{span}(\xi) = 1$,

$$\mu_n(\tfrac{1}{2}) = \sum_{k=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} k^{-1}(1+\varepsilon_k) = \frac{1}{\sqrt{2\pi\sigma^2}} \log n + o(\log n), \tag{6.23}$$

and the result follows from Lemma 6.2. ☐

Proof of Theorem 1.9. This follows, as said in the introduction, immediately from Theorems 1.1 and 1.7(i). ☐

6.1. **Extensions assuming higher moments.** We first prove Theorem 1.20 where we assume $\mathbb{E}\,\xi^{2+\delta} < \infty$ for some $\delta \in (0,1]$. For an example (without higher moments) where $\mu(\alpha)$ *cannot* be extended analytically across the line $\alpha = \frac{1}{2}$, see Theorem 11.1 in Section 11.

Proof of Theorem 1.20. Assume again for simplicity that $\operatorname{span}(\xi) = 1$. Then the assumption $\mathbb{E}\,\xi^{2+\delta} < \infty$ implies that (2.5) can be improved to

$$\mathbb{P}(S_n = n-1) = \frac{1}{\sqrt{2\pi\sigma^2}} n^{-1/2} + r(n), \tag{6.24}$$

with

$$r(n) = O\big(n^{-\frac{1}{2}-\frac{\delta}{2}}\big), \tag{6.25}$$

see [30, Theorem 6.1], [31, Theorem 4.5.3 and 4.5.4].

(i): Consequently, with $\zeta(\cdot)$ denoting the Riemann zeta function, (2.3) yields

$$\mu(\alpha) = \frac{1}{\sqrt{2\pi\sigma^2}} \sum_{n=1}^{\infty} n^{\alpha-\frac{3}{2}} + \sum_{n=1}^{\infty} n^{\alpha-1} r(n) = \frac{\zeta(\frac{3}{2}-\alpha)}{\sqrt{2\pi\sigma^2}} + \sum_{n=1}^{\infty} n^{\alpha-1} r(n), \tag{6.26}$$

where the final sum by (6.25) converges and is analytic in α for $\operatorname{Re}\alpha < \frac{1}{2} + \frac{\delta}{2}$. It is well known that the Riemann zeta function can be extended to a meromorphic function in the complex plane, with a single pole at 1 with residue 1. The result follows. [If $\operatorname{span}(\xi) > 1$, we use the Hurwitz zeta function [47, §25.11] instead of the Riemann zeta function.]

(ii): Let $D^\delta := \{\alpha \neq \frac{1}{2} : -\frac{1}{2} < \operatorname{Re}\alpha < \frac{1}{2} + \frac{\delta}{2}\}$, $D^\delta_- := \{\alpha \in D^\delta : \operatorname{Re}\alpha < \frac{1}{2}\}$, and $D^\delta_+ := \{\alpha \in D^\delta : \operatorname{Re}\alpha > \frac{1}{2}\}$. Furthermore, fix a compact subset K of D^δ. Define, for $k \geqslant 2$ and for $k = 1$ and $\operatorname{Re}\alpha > \frac{1}{2}$,

$$a(k,\alpha) := k^{\alpha-\frac{3}{2}} - (\alpha - \tfrac{1}{2})^{-1}\big[k^{\alpha-\frac{1}{2}} - (k-1)^{\alpha-\frac{1}{2}}\big], \tag{6.27}$$

$$b(k,\alpha) := \frac{1}{\sqrt{2\pi\sigma^2}} a(k,\alpha) + k^{\alpha-1} r(k). \tag{6.28}$$

Note that for $k \geqslant 2$ and $\alpha \in K$, by a Taylor expansion,

$$a(k,\alpha) = O\big(k^{\operatorname{Re}\alpha - \frac{5}{2}}\big), \tag{6.29}$$

where the implied constant depends only on K, and thus, using also (6.25),

$$b(k,\alpha) = O\big(k^{\operatorname{Re}\alpha - \frac{3}{2} - \frac{\delta}{2}}\big). \tag{6.30}$$

By (6.24), (6.27), and (6.28),

$$k^{\alpha-1}\,\mathbb{P}(S_k = k-1) = \frac{(\alpha-\frac{1}{2})^{-1}}{\sqrt{2\pi\sigma^2}}\big[k^{\alpha-\frac{1}{2}} - (k-1)^{\alpha-\frac{1}{2}}\big] + b(k,\alpha), \tag{6.31}$$

where either $k \geqslant 2$ or $k \geqslant 1$ and $\alpha \in D^\delta_+$.

It follows from (6.29) that $\sum_{k=2}^\infty a(k,\alpha)$ converges for $\alpha \in D^\delta$ and defines an analytic function there. Furthermore, if $\alpha \in D^\delta_-$, then, summing the telescoping sum,

$$\sum_{k=2}^\infty a(k,\alpha) = \zeta\big(\tfrac{3}{2} - \alpha\big) - 1 + \big(\alpha - \tfrac{1}{2}\big)^{-1}, \tag{6.32}$$

and consequently, by (6.26),

$$\mu(\alpha) = \frac{1}{\sqrt{2\pi\sigma^2}}\left(\sum_{k=2}^\infty a(k,\alpha) + 1 - \big(\alpha - \tfrac{1}{2}\big)^{-1}\right) + \sum_{k=1}^\infty k^{\alpha-1} r(k). \tag{6.33}$$

Both sides of (6.33) are analytic in D^δ, so by analytic continuation, (6.33) holds for all $\alpha \in D^\delta$ (and also for $\operatorname{Re}\alpha \leqslant -\frac{1}{2}$). In particular, for $\alpha \in D^\delta_+$, where $a(1,\alpha)$ is defined,

$$\mu(\alpha) = \frac{1}{\sqrt{2\pi\sigma^2}} \sum_{k=1}^\infty a(k,\alpha) + \sum_{k=1}^\infty k^{\alpha-1} r(k) = \sum_{k=1}^\infty b(k,\alpha). \tag{6.34}$$

We now analyze $\mu_n(\alpha)$ further. First, for $\alpha \in K_- := K \cap D^\delta_-$, using (6.31) and (6.30),

$$\begin{aligned}\mu(\alpha) - \mu_n(\alpha) &= \sum_{k=n+1}^{\infty} k^{\alpha-1}\,\mathbb{P}(S_k = k-1)\\ &= \sum_{k=n+1}^{\infty}\Bigl(\frac{(\alpha-\frac12)^{-1}}{\sqrt{2\pi\sigma^2}}\bigl[k^{\alpha-\frac12} - (k-1)^{\alpha-\frac12}\bigr] + b(k,\alpha)\Bigr)\\ &= -\frac{(\alpha-\frac12)^{-1}}{\sqrt{2\pi\sigma^2}} n^{\alpha-\frac12} + O\bigl(n^{\operatorname{Re}\alpha-\frac12-\frac{\delta}{2}}\bigr). \end{aligned} \tag{6.35}$$

Next, consider $\alpha \in D^\delta_+$. By (6.31) and (6.34), for $\alpha \in K_+ := K \cap D^\delta_+$,

$$\begin{aligned}\mu_n(\alpha) - \mu(\alpha) &= \sum_{k=1}^{n}\Bigl(\frac{(\alpha-\frac12)^{-1}}{\sqrt{2\pi\sigma^2}}\bigl[k^{\alpha-\frac12} - (k-1)^{\alpha-\frac12}\bigr] + b(k,\alpha)\Bigr) - \sum_{k=1}^{\infty} b(k,\alpha)\\ &= \frac{(\alpha-\frac12)^{-1}}{\sqrt{2\pi\sigma^2}} n^{\alpha-\frac12} - \sum_{k=n+1}^{\infty} b(k,\alpha)\\ &= \frac{(\alpha-\frac12)^{-1}}{\sqrt{2\pi\sigma^2}} n^{\alpha-\frac12} + O\bigl(n^{\operatorname{Re}\alpha-\frac12-\frac{\delta}{2}}\bigr). \end{aligned} \tag{6.36}$$

We have obtained the same estimate for the two ranges in (6.35) and (6.36), and can combine them to obtain, for $\alpha \in K_- \cup K_+$,

$$\mu_n(\alpha) - \mu(\alpha) = \frac{(\alpha-\frac12)^{-1}}{\sqrt{2\pi\sigma^2}} n^{\alpha-\frac12} + O\bigl(n^{\operatorname{Re}\alpha-\frac12-\frac{\delta}{2}}\bigr). \tag{6.37}$$

Furthermore, for each n, $\mu_n(\alpha) - \mu(\alpha)$ is a continuous function in D^δ, and thus (6.37) holds for $\alpha \in \overline{K_- \cup K_+}$ by continuity. If K is a closed disc, then $K = \overline{K_- \cup K_+}$, and thus (6.37) hold for $\alpha \in K$. In general, any compact $K \subset D^\delta$ can be covered by a finite union of closed discs $K_i \subset D^\delta$, and it follows that (6.37) holds uniformly in $\alpha \in K$ for each compact $K \subset D^\delta$.

Combining (6.10) and (6.37), we obtain (1.16), uniformly on each compact $K \subset D^\delta$.

(iii): By (6.24)–(6.25), (6.21) holds with $\varepsilon_k = O(k^{-\delta/2})$. Consequently, (6.23) is improved to

$$\mu_n(\tfrac12) = \sum_{k=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} k^{-1}(1+\varepsilon_k) = \frac{1}{\sqrt{2\pi\sigma^2}}\log n + c_1 + o(1), \tag{6.38}$$

with $c_1 = (2\pi\sigma^2)^{-1/2}\bigl(\gamma + \sum_{k=1}^\infty \varepsilon_k/k\bigr)$. The result (1.30) follows from (6.10). ☐

Remark 6.4. The proof shows, using Remark 6.3, that the constant c in Theorem 1.20(iii) is given by

$$c = \sum_{k=1}^{\infty} \frac1k\Bigl[k^{1/2}\,\mathbb{P}(S_k = k-1) - \frac{1}{\sqrt{2\pi\sigma^2}}\Bigr] - \frac{1}{\sqrt{2\pi\sigma^2}}\psi\Bigl(\frac12\Bigr), \tag{6.39}$$

where $\psi(\frac12) = -(2\log 2 + \gamma)$ [47, 5.4.13]. ☐

We next show that Theorem 1.20(i) extends to the case $\delta > 1$, at least if δ is an integer.

Theorem 6.5. *If* $\mathbb{E}\,\xi^k < \infty$ *for an integer* $k \geqslant 3$*, then* $\mu(\alpha)$ *can be continued as a meromorphic function in* $\operatorname{Re}\alpha < \frac{k-1}{2}$ *with simple poles at* $\{\frac12, \frac32, \frac52, \dots\}$ *(or possibly a subset of these points) and no other poles.*

Typically, all these points $\ell - \frac12$ (with $1 \leqslant \ell < k/2$) are poles; however in special cases, $\mu(\alpha)$ might be regular at some of these points, see Example 6.8.

Proof. Assume for simplicity that $\operatorname{span}(\xi) = 1$. In this case, see [49, Theorem VII.13] (with slightly different notation), (6.24) can be refined to

$$\mathbb{P}(S_n = n-1) = \frac{e^{-x^2/2}}{\sqrt{2\pi\sigma^2 n}}\Bigl[1 + \sum_{\nu=1}^{k-2} \tilde q_\nu(x) n^{-\nu/2}\Bigr] + o\bigl(n^{-(k-1)/2}\bigr) \tag{6.40}$$

where $x = -1/(\sigma\sqrt{n})$ and $\tilde q_\nu$ is a polynomial (independent of n) whose coefficients depend on the cumulants of ξ of order up to $\nu+2$, see [49, VI.(1.14)] for details. The polynomial $\tilde q_\nu$ is odd if ν is odd, and is even if ν is even; hence the term $\tilde q_\nu(x)n^{-\nu/2}$ is a polynomial in n^{-1} for every ν, and expanding $e^{-x^2/2} = e^{-1/(2\sigma^2 n)}$ into its Taylor series and rearranging, we obtain from (6.40)

$$\mathbb{P}(S_n = n-1) = \sum_{j=0}^{\lfloor k/2\rfloor - 1} a_j n^{-j-\frac12} + r(n) \tag{6.41}$$

with $r(n) = o\bigl(n^{-(k-1)/2}\bigr)$, for some coefficients a_j. Consequently (2.3) yields, cf. (6.26),

$$\mu(\alpha) = \sum_{j=0}^{\lfloor k/2\rfloor - 1} a_j \zeta\bigl(\tfrac32 + j - \alpha\bigr) + \sum_{n=1}^{\infty} n^{\alpha-1} r(n), \tag{6.42}$$

where the final sum is analytic in $\operatorname{Re}\alpha < (k-1)/2$, which proves the result. □

Remark 6.6. The proof of Theorem 6.5 shows that the residue of $\mu(\alpha)$ at $\alpha = j + \frac12$ (assumed to be less than $\frac{k-1}{2}$) is $-a_j$, where a_j is the coefficient in the expansion (6.41) and can be calculated from the cumulants $\varkappa_2 = \sigma^2, \varkappa_3 \dots, \varkappa_{2j+2}$ of ξ. For example, see Theorem 1.20(i), the residue at $\frac12$ is $-a_0 = -1/\sqrt{2\pi\sigma^2}$. As another example, a calculation (which we omit) shows that if $k > 4$, the residue at $\frac32$ is

$$-a_1 = \frac{1}{\sqrt{2\pi\sigma^2}}\Bigl(\frac{1}{2\sigma^2} - \frac{\varkappa_3}{2\sigma^4} - \frac{\varkappa_4}{8\sigma^4} + \frac{5\varkappa_3^2}{24\sigma^6}\Bigr). \tag{6.43}$$

□

Example 6.7. Consider the case of uniformly random labelled trees, which is given by $\xi \sim \operatorname{Po}(1)$. In this case,

$$\mathbb{P}(S_n = n-1) = \mathbb{P}(\operatorname{Po}(n) = n-1) = \frac{n^{n-1}}{(n-1)!}e^{-n} \tag{6.44}$$

which by Stirling's formula, see e.g. [47, 5.11.1], has a (divergent) asymptotic expansion that can be written

$$\sim (2\pi n)^{-1/2} \exp\Bigl(-\sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k-1)n^{2k-1}}\Bigr) \tag{6.45}$$

where B_{2k} are the Bernoulli numbers. Expanding the exponential in (6.45) (as a formal power series), we obtain coefficients a_k such that for any integer J we have

$$\mathbb{P}(S_n = n-1) = \sum_{j=0}^{J} a_j n^{-j-\frac12} + o\bigl(n^{-J-\frac12}\bigr), \tag{6.46}$$

which is the same as (6.41), and it follows by the argument above that $\mu(\alpha)$ has residue $-a_j$ at $j+\frac12$.

For example, $a_0 = (2\pi)^{-1/2}$, see (6.24) and (6.26), and $a_1 = -\frac{1}{12}(2\pi)^{-1/2}$, showing that $\mu(\alpha)$ has a pole with residue $\frac{1}{12}(2\pi)^{-1/2}$ at $\frac32$. (This agrees with (6.43) since $\varkappa_k = 1$ for every $k \geqslant 1$.) ☐

Example 6.8. We construct an example where ξ is bounded, so Theorem 6.5 applies for every k and $\mu(\alpha)$ is meromorphic in the entire complex plane, and furthermore $\mu(\alpha)$ is regular at $\alpha = \frac32$.

We use three parameters m, s, and A, where $m \geqslant 10$ is a fixed integer (we may take $m = 10$), $s \in [0, m)$, and A is a large integer. Let $\xi = \xi_{m,s,A}$ take the values $0, 1, A, mA$ with the probabilities

$$\mathbb{P}(\xi = mA) = \frac{s}{2m^2 A}, \tag{6.47}$$

$$\mathbb{P}(\xi = A) = \frac{1}{2A}, \tag{6.48}$$

$$\mathbb{P}(\xi = 1) = \frac12 - \frac{s}{2m}, \tag{6.49}$$

$$\mathbb{P}(\xi = 0) = 1 - \mathbb{P}(\xi = 1) - \mathbb{P}(\xi = A) - \mathbb{P}(\xi = mA). \tag{6.50}$$

Then $\mathbb{E}\,\xi = 1$ and $\operatorname{span}(\xi) = 1$. Keep m and s fixed, and let $A \to \infty$; then

$$\sigma^2 \sim \mathbb{E}\,\xi^2 \sim \frac{sA}{2} + \frac{A}{2} = \frac{1+s}{2}A, \tag{6.51}$$

$$\varkappa_3 \sim \mathbb{E}\,\xi^3 \sim \frac{smA^2}{2} + \frac{A^2}{2} = \frac{1+sm}{2}A^2, \tag{6.52}$$

$$\varkappa_4 \sim \mathbb{E}\,\xi^4 \sim \frac{sm^2A^3}{2} + \frac{A^3}{2} = \frac{1+sm^2}{2}A^3. \tag{6.53}$$

Denote the parenthesized factor in (6.43) by $f(m,\alpha,A)$. It follows from (6.51)–(6.53) that as $A \to \infty$ with fixed m and s,

$$f(m,s,A) = -\frac{1+sm^2}{4(1+s)^2}A + \frac{5(1+sm)^2}{12(1+s)^3}A + o(A) = \bigl(g(m,s) + o(1)\bigr)A, \tag{6.54}$$

where

$$g(m,s) := -\frac{1+sm^2}{4(1+s)^2} + \frac{5(1+sm)^2}{12(1+s)^3} = \frac{5(1+sm)^2 - 3(1+s)(1+sm^2)}{12(1+s)^3}. \tag{6.55}$$

For $s = 0$, the final numerator in (6.55) is $2 > 0$, and thus $g(m, 0) > 0$. For $s = 1$, the final numerator is $5(1 + m)^2 - 6 - 6m^2 < 0$, and thus $g(m, 1) < 0$. Hence, by (6.54), we may choose a large A such that $f(m, 0, A) > 0$ and $f(m, 1, A) < 0$. Then, by continuity, these exists $s \in (0, 1)$ such that $f(m, s, A) = 0$, and (6.43) shows that for the corresponding ξ, we have the residue 0 at $\frac{3}{2}$, i.e., there is no pole there and $\mu(\alpha)$ is regular at $\frac{3}{2}$. □

7. Brownian representations

We use the well-known result by Aldous [3, 4] that represents a conditioned Galton–Watson tree asymptotically by a Brownian excursion $(\mathbf{e}(t))$ in the following way (under the conditions $\mathbb{E}\,\xi = 1$ and $\sigma^2 := \operatorname{Var}\xi < \infty$ that also we assume). (See also Le Gall [42] and Drmota [12, Chapter 4.1].)

Consider the *depth-first walk* on the tree $\mathcal{T}_n$; this is a walk $v(1), \dots, v(2n-1)$ on the nodes of $\mathcal{T}_n$, where $v(1) = v(2n-1)$ is the root o, and each time we come to a node, we proceed to the first unvisited child of the node, if there is any, and otherwise to the parent. For convenience, we also define $v(0) = v(2n) = o$. We define $W_n(i) := d(v(i))$, and extend W_n to the interval $[0, 2n]$ by linear interpolation between the integers. Furthermore, we scale W_n to a function on $[0, 1]$ by

$$\widehat{W}_n(t) := \sigma n^{-1/2} W_n(2nt). \tag{7.1}$$

Then $\widehat{W}_n$ is a random continuous funtion on [0,1], and is thus a random element of the Banach space $C[0, 1]$. One of the main results of Aldous [4, Theorem 23 with Remark 2] is that, as random elements of $C[0, 1]$,

$$(\widehat{W}_n(t)) \xrightarrow{\mathrm{d}} (2\mathbf{e}(t)). \tag{7.2}$$

We can think of $W_n(t)$ as the position of a worm that crawls on the edges of the tree, visiting each edge twice (once in each direction).

We define $v(x)$ also for non-integer $x \in [0, 2n]$ as either $v(\lfloor x \rfloor)$ or $v(\lceil x \rceil)$, choosing between these two the node more distant from the root. Thus,

$$d(v(x)) = \lceil W_n(x) \rceil. \tag{7.3}$$

For a node v, let $i'_v := \min\{i \geqslant 1 : v(i) = v\}$ and $i''_v := \max\{i \leqslant 2n-1 : v(i) = v\}$, i.e., the first and last times that v is visited (with $i'_o = 1$ and $i''_o = 2n-1$). Then the subtree $\mathcal{T}_{n,v}$ is visited during the interval $[i'_v, i''_v]$, and $i''_v - i'_v = 2(|\mathcal{T}_{n,v}| - 1)$. Let

$$J_v := \{x \in (0, 2n) : v(x) \succeq v\}. \tag{7.4}$$

Then $J_v = (i'_v - 1, i''_v + 1)$, and thus J_v is an interval of length

$$|J_v| = i''_v - i'_v + 2 = 2|\mathcal{T}_{n,v}|. \tag{7.5}$$

We can now prove Theorem 1.24. When $\operatorname{Re}\alpha > 1$, all four expressions (1.33)–(1.36) are equivalent by elementary calculus, so part (ii) follows from part (i). Nevertheless, we begin with a straightforward proof of the simpler part (ii), and then show how part (i) can be proved by a similar, but more complicated, argument. Since we have not yet proved convergence of $Y_n(\alpha)$, we state the result as the following two lemmas.

Lemma 7.1. *If* $\operatorname{Re}\alpha > 1$*, then* $Y_n(\alpha) \xrightarrow{\mathrm{d}} \sigma^{-1}Y(\alpha)$ *as* $n \to \infty$*, with* $Y(\alpha)$ *given by* (1.36)*. Moreover, this holds jointly for any finite set of such* α*.*

Proof. We assume $\operatorname{Re}\alpha > 1$, and then (7.5) implies

$$(2|\mathcal{T}_{n,v}|)^\alpha = \iint\limits_{\substack{x,y\in J_v\\ x<y}} \alpha(\alpha-1)(y-x)^{\alpha-2}\,\mathrm{d}x\,\mathrm{d}y. \tag{7.6}$$

Hence,

$$2^\alpha X_n(\alpha) = \sum_{v\in\mathcal{T}_n}(2|\mathcal{T}_{n,v}|)^\alpha = \iint\limits_{0<x<y<2n} \alpha(\alpha-1)(y-x)^{\alpha-2}\sum_{v\in\mathcal{T}_n}\mathbf{1}_{x,y\in J_v}\,\mathrm{d}x\,\mathrm{d}y. \tag{7.7}$$

Now, by (7.4) and (2.9), $x, y \in J_v \iff v \preceq v(x) \wedge v(y)$, and thus

$$\sum_{v\in\mathcal{T}_n}\mathbf{1}_{x,y\in J_v} = \#\{v : v \preceq v(x)\wedge v(y)\} = d(v(x)\wedge v(y)) + 1. \tag{7.8}$$

Furthermore, from the construction of the depth-first walk,

$$d(v(x)\wedge v(y)) = \lceil m(W_n; x, y)\rceil. \tag{7.9}$$

recalling the notation (1.32). [Actually, $m(W_n; x, y)$ is an integer except when $v(x)$ is an ancestor of $v(y)$ or conversely.] Combining (7.7)–(7.9) and (7.1) yield

$$\begin{aligned}
2^\alpha X_n(\alpha) &= \iint\limits_{0<x<y<2n} \alpha(\alpha-1)(y-x)^{\alpha-2}\bigl[d(v(x)\wedge v(y)) + 1\bigr]\,\mathrm{d}x\,\mathrm{d}y\\
&= \iint\limits_{0<x<y<2n} \alpha(\alpha-1)(y-x)^{\alpha-2}\bigl[m(W_n; x, y) + O(1)\bigr]\,\mathrm{d}x\,\mathrm{d}y\\
&= \alpha(\alpha-1)(2n)^\alpha \iint\limits_{0<s<t<1} (t-s)^{\alpha-2}\bigl[m(n^{1/2}\sigma^{-1}\widehat{W}_n; s, t) + O(1)\bigr]\,\mathrm{d}s\,\mathrm{d}t.
\end{aligned} \tag{7.10}$$

Since $\widehat{W}_n \xrightarrow{\mathrm{d}} 2\mathbf{e}$ in $C[0,1]$ by (7.2), and the integral below defines a continuous functional on $C[0,1]$ because $\iint(t-s)^{\alpha-2}\,\mathrm{d}s\,\mathrm{d}t$ converges (absolutely), it follows that

$$\begin{aligned}
\sigma n^{-\alpha-\frac12}X_n(\alpha) &= \alpha(\alpha-1)\iint\limits_{0<s<t<1}(t-s)^{\alpha-2}m(\widehat{W}_n; s, t)\,\mathrm{d}s\,\mathrm{d}t + O\bigl(n^{-1/2}\bigr)\\
&\xrightarrow{\mathrm{d}} \alpha(\alpha-1)\iint\limits_{0<s<t<1}(t-s)^{\alpha-2}m(2\mathbf{e}; s, t)\,\mathrm{d}s\,\mathrm{d}t = Y(\alpha).
\end{aligned} \tag{7.11}$$

In other words, recalling (1.8), $\sigma Y_n(\alpha) \xrightarrow{\mathrm{d}} Y(\alpha)$.

Joint convergence for several α follows by the same argument. ∎

Lemma 7.2. *If* $\operatorname{Re}\alpha > \frac12$*, then* $Y_n(\alpha) \xrightarrow{\mathrm{d}} \sigma^{-1}Y(\alpha)$ *as* $n \to \infty$*, with* $Y(\alpha)$ *given by* (1.33)–(1.35)*. Moreover, this holds jointly for any finite set of such* α*.*

Proof. Fix α with $\operatorname{Re}\alpha > \frac{1}{2}$. We begin with a calculus fact (assuming only that $\operatorname{Re}\alpha > 0$). For any $0 < a < b < \infty$,

$$(b-a)^\alpha = \alpha \int_a^b x^{\alpha-1}\,\mathrm{d}x - \alpha(\alpha-1) \iint\limits_{0<x<a<y<b} (y-x)^{\alpha-2}\,\mathrm{d}x\,\mathrm{d}y. \tag{7.12}$$

We apply this to the interval $(a,b) = J_v$ in (7.4) and obtain, using (7.5),

$$(2|\mathcal{T}_{n,v}|)^\alpha = \alpha \int_{x\in J_v} x^{\alpha-1}\,\mathrm{d}x - \alpha(\alpha-1) \iint\limits_{0<x<y,\, x\notin J_v,\, y\in J_v} (y-x)^{\alpha-2}\,\mathrm{d}x\,\mathrm{d}y$$

and thus, summing over all nodes v of $\mathcal{T}_n$,

$$\begin{aligned} 2^\alpha \sum_v |\mathcal{T}_{n,v}|^\alpha &= \alpha \int_0^{2n} x^{\alpha-1} \sum_{v\in\mathcal{T}_n} \mathbf{1}_{x\in J_v}\,\mathrm{d}x \\ &\qquad - \alpha(\alpha-1) \iint\limits_{0<x<y<2n} (y-x)^{\alpha-2} \sum_{v\in\mathcal{T}_n} \mathbf{1}_{x\notin J_v,\, y\in J_v}\,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{7.13}$$

Now, using (7.4) and (7.3),

$$\sum_{v\in\mathcal{T}_n} \mathbf{1}_{x\in J_v} = \#\{v : v(x) \succeq v\} = d(v(x)) + 1 = \lceil W_n(x) \rceil + 1 \tag{7.14}$$

and similarly, using also (2.9) and (7.9),

$$\begin{aligned} \sum_{v\in\mathcal{T}_n} \mathbf{1}_{x\notin J_v,\, y\in J_v} &= \#\{v : v(x) \not\succeq v \text{ and } v(y) \succeq v\} \\ &= \#\{v : v(y) \succeq v\} - \#\{v : v(x) \wedge v(y) \succeq v\} \\ &= d(v(y)) - d(v(x) \wedge v(y)) \\ &= \lceil W_n(y) \rceil - \lceil m(W_n; x, y) \rceil. \end{aligned} \tag{7.15}$$

Consequently, recalling the definitions (1.3) and (1.8) of $X_n(\alpha)$ and $Y_n(\alpha)$,

$$\begin{aligned} 2^\alpha X_n(\alpha) &= \alpha \int_0^{2n} x^{\alpha-1} \bigl(\lceil W_n(x) \rceil + 1\bigr)\,\mathrm{d}x \\ &\quad - \alpha(\alpha-1) \iint\limits_{0<x<y<2n} (y-x)^{\alpha-2} \bigl(\lceil W_n(y) \rceil - \lceil m(W_n; x, y) \rceil\bigr)\,\mathrm{d}x\,\mathrm{d}y \end{aligned} \tag{7.16}$$

and thus

$$\begin{aligned} Y_n(\alpha) &= \alpha \int_0^1 t^{\alpha-1} n^{-1/2} \bigl(\lceil W_n(2nt) \rceil + 1\bigr)\,\mathrm{d}t \\ &- \alpha(\alpha-1) \iint\limits_{0<s<t<1} (t-s)^{\alpha-2} n^{-1/2} \bigl(\lceil W_n(2nt) \rceil - \lceil m(W_n; 2ns, 2nt) \rceil\bigr)\,\mathrm{d}s\,\mathrm{d}t. \end{aligned} \tag{7.17}$$

The first integral in (7.17) is no problem; it converges (in distribution) by (7.1) and (7.2), just as the integral at the end of the proof of Lemma 7.1, because $\int t^{\alpha-1}\,\mathrm{d}t$ converges (absolutely).

The second integral, however, is more difficult, since $\iint (t-s)^{\alpha-2}\,\mathrm{d}s\,\mathrm{d}t$ diverges if $\operatorname{Re}\alpha \leqslant 1$. We therefore use a truncation argument. For $0<\varepsilon<1$ we split $Y_n(\alpha) = Z_{n,\varepsilon}(\alpha) + Z'_{n,\varepsilon}(\alpha)$, where

$$\begin{aligned} Z_{n,\varepsilon}(\alpha) := \alpha \int_0^1 t^{\alpha-1} n^{-1/2} \bigl(\lceil W_n(2nt)\rceil + 1\bigr)\,\mathrm{d}t \\ - \alpha(\alpha-1) \iint\limits_{t-s>\varepsilon} (t-s)^{\alpha-2} n^{-1/2} \bigl(\lceil W_n(2nt)\rceil - \lceil m(W_n; 2ns, 2nt)\rceil\bigr)\,\mathrm{d}s\,\mathrm{d}t \end{aligned} \tag{7.18}$$

and

$$\begin{aligned} Z'_{n,\varepsilon}(\alpha) := \\ - \alpha(\alpha-1) \iint\limits_{0<t-s<\varepsilon} (t-s)^{\alpha-2} n^{-1/2} \bigl(\lceil W_n(2nt)\rceil - \lceil m(W_n; 2ns, 2nt)\rceil\bigr)\,\mathrm{d}s\,\mathrm{d}t. \end{aligned} \tag{7.19}$$

For each fixed α with $\operatorname{Re}\alpha > 0$ and each fixed $0<\varepsilon<1$,

$$\begin{aligned} \sigma Z_{n,\varepsilon}(\alpha) = \alpha \int_0^1 t^{\alpha-1} \widehat{W}_n(t)\,\mathrm{d}t \\ - \alpha(\alpha-1) \iint\limits_{t-s>\varepsilon} (t-s)^{\alpha-2} \bigl(\widehat{W}_n(t) - m(\widehat{W}_n; s,t)\bigr)\,\mathrm{d}s\,\mathrm{d}t + O\bigl(n^{-1/2}\bigr) \end{aligned} \tag{7.20}$$

and thus, by (7.2) and the continuous mapping theorem,

$$\begin{aligned} \sigma Z_{n,\varepsilon}(\alpha) \xrightarrow{\mathrm{d}} Z_\varepsilon(\alpha) := 2\alpha \int_0^1 t^{\alpha-1} \mathbf{e}(t)\,\mathrm{d}t \\ - 2\alpha(\alpha-1) \iint\limits_{t-s>\varepsilon} (t-s)^{\alpha-2} \bigl(\mathbf{e}(t) - m(\mathbf{e}; s,t)\bigr)\,\mathrm{d}s\,\mathrm{d}t. \end{aligned} \tag{7.21}$$

We now use the assumption $\operatorname{Re}\alpha > \frac12$. We define $Y(\alpha)$ by (1.33), noting that the integrals converge, as said in Section 1, because $\mathbf{e}(t)$ is Hölder(γ)-continuous for every $\gamma < \frac12$. This shows that as $\varepsilon \to 0$,

$$Z_\varepsilon(\alpha) \to Y(\alpha) \tag{7.22}$$

a.s. (and thus in distribution). Furthermore, let β be real with $\frac12 < \beta < (\operatorname{Re}\alpha \wedge 1)$. It follows from (7.19) that

$$|Z'_{n,\varepsilon}(\alpha)| \leqslant \frac{|\alpha(\alpha-1)|}{|\beta(\beta-1)|}\, \varepsilon^{\operatorname{Re}\alpha-\beta} Z'_{n,\varepsilon}(\beta). \tag{7.23}$$

Furthermore, by (7.17), $Z'_{n,\varepsilon}(\beta) \leqslant Y_n(\beta)$, and by Theorem 1.7(iii) we have $\mathbb{E}\, Y_n(\beta) = O(1)$. Consequently, (7.23) implies

$$\mathbb{E}\,|Y_n(\alpha) - Z_{n,\varepsilon}(\alpha)| = \mathbb{E}\,|Z'_{n,\varepsilon}(\alpha)| = O\bigl(\varepsilon^{\operatorname{Re}\alpha-\beta}\bigr). \tag{7.24}$$

Consequently, $Z_{n,\varepsilon}(\alpha) \xrightarrow{\mathrm{p}} Y_n(\alpha)$ as $\varepsilon \to 0$ uniformly in n, i.e., for any $\delta > 0$, $\sup_n \mathbb{P}(|Y_n(\alpha) - Z_{n,\varepsilon}(\alpha)| > \delta) \to 0$. This together with the facts (7.21) and (7.22) imply the result $\sigma Y_n(\alpha) \xrightarrow{\mathrm{d}} Y(\alpha)$, see e.g. [5, Theorem 4.2] or [40, Theorem 4.28]. Joint convergence for several α follows by the same argument.

It remains to show that (1.34)–(1.35) are equal to $Y(\alpha)$. Let us temporarily denote these expressions by $Y^{(1)}(\alpha)$ and $Y^{(2)}(\alpha)$.

Note that, a.s.,

$$\begin{aligned}(\alpha-1)\iint_{t-s>\varepsilon}(t-s)^{\alpha-2}\mathbf{e}(t)\,\mathrm{d}s\,\mathrm{d}t &= (\alpha-1)\int_\varepsilon^1 \mathbf{e}(t)\int_0^{t-\varepsilon}(t-s)^{\alpha-2}\,\mathrm{d}s\,\mathrm{d}t \\ &= \int_\varepsilon^1 \mathbf{e}(t)\bigl(t^{\alpha-1}-\varepsilon^{\alpha-1}\bigr)\,\mathrm{d}t \\ &= \int_0^1 \mathbf{e}(t)\bigl(t^{\alpha-1}-\varepsilon^{\alpha-1}\bigr)\,\mathrm{d}t + O\bigl(\varepsilon^{\operatorname{Re}\alpha}\bigr) \qquad (7.25)\end{aligned}$$

and hence

$$Z_\varepsilon(\alpha) = 2\alpha\varepsilon^{\alpha-1}\int_0^1 \mathbf{e}(t)\,\mathrm{d}t + 2\alpha(\alpha-1)\iint_{t-s>\varepsilon}(t-s)^{\alpha-2}m(\mathbf{e};s,t)\,\mathrm{d}s\,\mathrm{d}t + O\bigl(\varepsilon^{\operatorname{Re}\alpha}\bigr). \qquad (7.26)$$

Consequently, (7.22) yields the formula

$$Y(\alpha) = \lim_{\varepsilon\to 0}\left(2\alpha\varepsilon^{\alpha-1}\int_0^1 \mathbf{e}(t)\,\mathrm{d}t + 2\alpha(\alpha-1)\iint_{t-s>\varepsilon}(t-s)^{\alpha-2}m(\mathbf{e};s,t)\,\mathrm{d}s\,\mathrm{d}t\right). \qquad (7.27)$$

If we replace $\mathbf{e}(t)$ by the reflected $\mathbf{e}(1-t)$, the right-hand side of (7.27) is unchanged, while $Y(\alpha)$ defined by (1.33) becomes $Y^{(1)}(\alpha)$ defined by (1.34). Consequently, $Y^{(1)}(\alpha) = Y(\alpha)$ a.s. Furthermore, $Y^{(2)}(\alpha) = [Y(\alpha) + Y^{(1)}(\alpha)]/2$, and thus also $Y^{(2)}(\alpha) = Y(\alpha)$ a.s. □

Remark 7.3. For $\alpha = k \geqslant 2$ integer, an alternative argument uses the following identity, obtained by extending (7.8) to several nodes:

$$\begin{aligned}\sum_{v\in\mathcal{T}_n}|\mathcal{T}_{n,v}|^k &= \sum_{v,v_1,\dots,v_k\in\mathcal{T}_n}\mathbf{1}_{v_1,\dots,v_k\geq v} \\ &= 2^{-k}\int_0^{2n}\cdots\int_0^{2n}\bigl(d(v(x_1)\wedge\cdots\wedge v(x_k))+1\bigr)\,\mathrm{d}x_1\cdots\mathrm{d}x_k \\ &= 2^{-k}k!\int\cdots\int_{0<x_1<\cdots<x_k<2n}\lceil m(W_n;x_1,x_k)\rceil\,\mathrm{d}x_1\cdots\mathrm{d}x_k + n^k \\ &= n^k k(k-1)\iint_{0<t_1<t_k<1}\lceil m(W_n;2nt_1,2nt_k)\rceil(t_k-t_1)^{k-2}\,\mathrm{d}t_1\,\mathrm{d}t_k + n^k. \qquad (7.28)\end{aligned}$$

This easily shows Lemma 7.1 with (1.36) in this case. A similar, but simpler, argument yields (1.37) for $k = 1$, see (1.38). □

8. Proofs of Theorem 1.2 and remaining limit theorems

Proof of Theorem 1.2. Theorem 1.2 now follows from Lemma 2.5, with $D = H_+$ and $E = \{\alpha : \operatorname{Re}\alpha > 1\}$, using Lemmas 4.2(ii) and 7.1. ☐

Proof of Remark 1.27. This is implicit in the proof above, but we add some details. Let D and E be as in the proof of Theorem 1.2. (Alternatively, take $E := [2, 3]$.) Let $\varphi : \mathcal{H}(D) \to C(E)$ be the restriction mapping $f \mapsto f|_E$, and let $\psi : C[0, 1] \to C(E)$ be the mapping taking $\mathbf{e} \in C[0, 1]$ to the element of $C(E)$ that maps $\alpha \in E$ to the right-hand side of (1.36); both φ and ψ are continuous and thus measurable. Let also Y denote the random function $Y(\alpha) \in \mathcal{H}(D)$. The proof above (in particular, Lemma 2.5) shows that $\varphi(Y) \overset{\mathrm{d}}{=} \psi(\mathbf{e})$, and thus we may assume

$$\varphi(Y) = \psi(\mathbf{e}) \qquad \text{a.s.} \tag{8.1}$$

(The skeptical reader might apply [40, Corollary 6.11] for the last step.) Furthermore, φ is injective, and both $\mathcal{H}(D)$ and $C(E)$ are Polish spaces; thus the range $R := \varphi(\mathcal{H}(D))$ is a Borel set in $C(E)$, and the inverse function $\varphi^{-1} : R \to \mathcal{H}(D)$ is measurable, see e.g. [10, Theorem 8.3.7 and Proposition 8.3.5]. By (8.1), we have $Y = \varphi^{-1}\bigl(\psi(\mathbf{e})\bigr)$ a.s. Consequently, (1.40) holds with

$$\Psi(\alpha, f) := \begin{cases} \varphi^{-1}\bigl(\psi(f)\bigr)(\alpha), & \psi(f) \in R, \\ 0, & \text{otherwise.} \end{cases} \tag{8.2}$$

☐

Proofs of Theorems 1.10 and 1.22. These results follow immediately from Theorem 1.2 and the estimates of $\mathbb{E}\, X_n(\alpha)$ in Theorems 1.7 and 1.20. ☐

Proof of Theorem 1.24. Theorem 1.24 follows from Theorem 1.10(i) and Lemmas 7.1–7.2, comparing the limits. More precisely, this yields equality in distribution jointly for any finite number of α, which implies equality jointly for all α since the distribution of $Y(\alpha)$ in $\mathcal{H}(H^+)$ is determined by the finite-dimensional distributions, see Section 2.2. ☐

9. The limit as $\alpha \to \infty$

We introduce more notation. As above, $\mathbf{e}(t)$, $t \in [0, 1]$, is a normalized Brownian excursion, and $m(\mathbf{e}; s, t)$ is defined by (1.32). We further define

$$m(s) := m\bigl(\mathbf{e}; s; \tfrac{1}{2}\bigr), \qquad\qquad m'(s) := m\bigl(\mathbf{e}; \tfrac{1}{2}, 1 - s\bigr) \tag{9.1}$$

for $0 \leqslant s \leqslant \frac{1}{2}$; for convenience we extend m and m' to continuous functions on $[0, \infty)$ by defining $m(s) = m'(s) := m(\frac{1}{2}) = \mathbf{e}(\frac{1}{2})$ for $s > \frac{1}{2}$. Furthermore,

- $(B(t))$ is a standard Brownian motion on $[0, \infty)$.
- $(S(t) := \sup_{s \in [0,t]} B(s))$ is the corresponding supremum process.
- $(\tau(a) := \min\{t : B(t) = a\}, a \geqslant 0)$ is the corresponding family of hitting times.

- $(R(t))$ is a three-dimensional Bessel process on $[0,\infty)$, i.e., $(R(t)) \overset{\mathrm{d}}{=} (|B^{(3)}(t)|)$, where $(B^{(3)}(t) = \big(B_1(t), B_2(t), B_3(t)\big))$ is a three-dimensional Brownian motion (so B_1, B_2, B_3 are three independent copies of B). It is well known that a.s. $R(0)=0$, $R(s)>0$ for all $s>0$ and $R(s)\to\infty$ as $s\to\infty$ [52, §VI.3].
- $J(t) := \inf_{s\geqslant t} R(s)$, $t \geqslant 0$, is the future minimum of R. By Pitman's theorem [52, VI.(3.5)], as stochastic processes in $C[0,\infty)$ we have

$$(J(t)) \overset{\mathrm{d}}{=} (S(t)). \tag{9.2}$$

- $(J'(t), t \geqslant 0)$ is an independent copy of the stochastic process $(J(t))$. Similarly, $(S'(t))$ is an independent copy of $(S(t))$ and $(\tau'(a))$ is an independent copy of $(\tau(a))$.

For notational convenience, we also define, using (1.36), for $r > -1$,

$$W_r := \iint\limits_{0<s<t<1} (t-s)^r m(\mathbf{e}; s,t)\,\mathrm{d}s\,\mathrm{d}t = \frac{1}{2(r+1)(r+2)} Y(r+2). \tag{9.3}$$

The assertion $\alpha^{1/2}Y(\alpha) \overset{\mathrm{d}}{\longrightarrow} Y_\infty$ in Theorem 1.29 is thus equivalent to $r^{5/2}W_r \overset{\mathrm{d}}{\longrightarrow} \frac{1}{2}Y_\infty$ as $r\to\infty$.

Lemma 9.1. *As $r\to\infty$ we have jointly (i.e., bivariately for sequences of processes) $r^{1/2}m(x/r) \overset{\mathrm{d}}{\longrightarrow} J(x)$ and $r^{1/2}m'(x/r) \overset{\mathrm{d}}{\longrightarrow} J'(x)$ in $C[0,T]$, for any $T<\infty$.*

Remark 9.2. Convergence in $C[0,T]$ for every fixed T is equivalent to convergence in $C[0,\infty)$, see e.g. [40, Proposition 16.6], so the conclusion may as well be stated as joint convergence in distribution in $C[0,\infty)$. □

Proof. Let us first consider m. We use the representation, see e.g. [6, II.(1.5)],

$$\mathbf{e}(t) \overset{\mathrm{d}}{=} (1-t)R\Big(\frac{t}{1-t}\Big) \tag{9.4}$$

as processes on $[0,1)$. Hence, using Brownian scaling, for $x\in[0,r)$ we have, as processes,

$$r^{1/2}\mathbf{e}(x/r) \overset{\mathrm{d}}{=} \Big(1-\frac{x}{r}\Big)r^{1/2}R\Big(\frac{x/r}{1-(x/r)}\Big) \overset{\mathrm{d}}{=} \Big(1-\frac{x}{r}\Big)R\Big(\frac{x}{1-(x/r)}\Big) \tag{9.5}$$

and thus, for $x\in[0,r/2]$,

$$r^{1/2}m(x/r) \overset{\mathrm{d}}{=} \min_{x\leqslant t\leqslant r/2}\Big(1-\frac{t}{r}\Big)R\Big(\frac{t}{1-(t/r)}\Big). \tag{9.6}$$

Recall that a.s. $R(t)\to\infty$ as $t\to\infty$. Hence, given T, we can choose a (random) $T_1\geqslant T$ such that $R(t)\geqslant 2\sup_{u\in[T,2T]}R(u)$ for all $t\geqslant T_1$. It follows that if $T_1\leqslant t\leqslant r/2$, then

$$\Big(1-\frac{t}{r}\Big)R\Big(\frac{t}{1-(t/r)}\Big) \geqslant \tfrac{1}{2}\cdot 2\sup_{u\in[T,2T]}R(u) \geqslant R\Big(\frac{T}{1-(T/r)}\Big). \tag{9.7}$$

Hence, if $x \leqslant T$ and $r \geqslant 2T_1$, the minimum in (9.6) equals the minimum over $x \leqslant t \leqslant T_1$. Furthermore, as $r \to \infty$, since R is continuous,

$$\min_{x\leqslant t\leqslant T_1}\Bigl(1-\frac{t}{r}\Bigr)R\Bigl(\frac{t}{1-(t/r)}\Bigr) \to \min_{x\leqslant t\leqslant T_1} R(t) = \min_{x\leqslant t<\infty} R(t) = J(x) \tag{9.8}$$

uniformly for $x \in [0,T]$, i.e. in $C[0,T]$. Consequently,

$$\min_{x\leqslant t\leqslant r/2}\Bigl(1-\frac{t}{r}\Bigr)R\Bigl(\frac{t}{1-(t/r)}\Bigr) \xrightarrow{\text{a.s.}} J(x) \tag{9.9}$$

in $C[0,T]$, and (9.6) implies

$$r^{1/2}m(x/r) \xrightarrow{\text{d}} J(x) \qquad \text{in } C[0,T], \tag{9.10}$$

which proves the assertion about m. By symmetry also

$$r^{1/2}m'(x/r) \xrightarrow{\text{d}} J(x) \overset{\text{d}}{=} J'(x) \qquad \text{in } C[0,T], \tag{9.11}$$

since $(\mathbf{e}(1-t)) \overset{\text{d}}{=} (\mathbf{e}(t))$ and thus $(m'(t)) \overset{\text{d}}{=} (m(t))$ (as random functions in $C[0,\infty)$).

It remains to prove joint convergence to independent limits. Let

$$m_1(s) := m(\mathbf{e}; s, r^{-2/3}), \qquad m_1'(s) := m(\mathbf{e}; 1-r^{-2/3}, 1-s) \tag{9.12}$$

(for r with $s \leqslant r^{-2/3} \leqslant \frac{1}{2}$). We may assume that the left and right sides of (9.5) are equal, and then $m(x/r) = m_1(x/r)$ whenever the minimum in (9.6) equals the minimum over $t \in [x, r^{1/3}]$; in particular, this holds if $x \leqslant T$ and $r^{1/3} \geqslant T_1$ defined above. (This implies $r \geqslant 2r^{1/3} \geqslant 2T_1$.) Consequently,

$$\mathbb{P}\bigl(m(x/r) = m_1(x/r) \text{ for all } x \in [0,T]\bigr) \geqslant \mathbb{P}\bigl(T_1 \leqslant r^{1/3}\bigr) \to 1 \tag{9.13}$$

as $r \to \infty$. By symmetry, also

$$\mathbb{P}\bigl(m'(x/r) = m_1'(x/r) \text{ for all } x \in [0,T]\bigr) \to 1. \tag{9.14}$$

Next, we may assume $R(t) = |B^{(3)}(t)|$ and that equality holds in (9.4). Define the modification $\tilde{R}(t) := |B^{(3)}(t) - B^{(3)}(1)|$ and the corresponding $\tilde{\mathbf{e}}(t) := (1-t)\tilde{R}\bigl(t/(1-t)\bigr)$ and $\tilde{m}_1'(s) := m(\tilde{\mathbf{e}}; 1-r^{-2/3}, 1-s)$. Then $|\tilde{R}(t) - R(t)| \leqslant |B^{(3)}(1)|$ for all t, and thus $|\tilde{\mathbf{e}}(t) - \mathbf{e}(t)| \leqslant (1-t)|B^{(3)}(1)|$ and $|\tilde{m}_1'(s) - m_1'(s)| \leqslant r^{-2/3}|B^{(3)}(1)|$. Consequently, assuming $r^{1/3} \geqslant T$,

$$\sup_{x\leqslant T} |r^{1/2}\tilde{m}_1'(x/r) - r^{1/2}m_1'(x/r)| \leqslant r^{-1/6}|B^{(3)}(1)| \xrightarrow{\text{p}} 0. \tag{9.15}$$

Let ρ denote the metric in $C[0,T]$. By (9.14) and (9.15),

$$\rho\bigl(r^{1/2}\tilde{m}_1'(x/r), r^{1/2}m'(x/r)\bigr) \xrightarrow{\text{p}} 0 \tag{9.16}$$

as $r \to \infty$. Thus by (9.11),

$$r^{1/2}\tilde{m}_1'(x/r) \xrightarrow{\text{d}} J(x) \qquad \text{in } C[0,T], \tag{9.17}$$

Now, for $x \leqslant T$ and large r, $\tilde{m}_1'(x/r)$ depends only on $\tilde{\mathbf{e}}(t)$ for $t \geqslant \frac{1}{2}$, and thus on $\tilde{R}(t)$ for $t \geqslant 1$. However, $\bigl(\tilde{R}(t) = |B^{(3)}(t) - B^{(3)}(1)|,\ t \geqslant 1\bigr)$ is independent of $\bigl(R(t) =$

$|B^{(3)}(t)|,\ t \leqslant 1\big)$, and thus of $\big(\mathbf{e}(t),\ t \leqslant \frac12\big)$ and of $\big(m(s),\ s \leqslant \frac12\big)$. Consequently, we can combine (9.10) and (9.17) to

$$\big(r^{1/2}m(x/r), r^{1/2}\tilde{m}_1'(x/r)\big) \xrightarrow{\mathrm{d}} \big(J(x), J'(x)\big) \qquad \text{in } C[0,T]\times C[0,T],$$

with independent limits $(J(x))$ and $(J'(x))$. Finally, the result follows by using (9.16) again. ☐

Lemma 9.3. *As $r \to \infty$,*

$$r^{5/2}W_r \xrightarrow{\mathrm{d}} W_\infty := \iint\limits_{x,y>0} e^{-x-y}\big(J(x)\wedge J'(y)\big)\,\mathrm{d}x\,\mathrm{d}y. \tag{9.18}$$

Proof. Note first that for some constant c (in fact, $c = \mathbb{E}\,|B^{(3)}(1)| = \sqrt{8/\pi}$), $\mathbb{E}\,R(t) = ct^{1/2}$. Hence, $\mathbb{E}\,J(x) \leqslant \mathbb{E}\,R(x) = cx^{1/2}$ and

$$\mathbb{E}\iint\limits_{x,y>0} e^{-x-y}\big(J(x)\wedge J'(y)\big)\,\mathrm{d}x\,\mathrm{d}y \leqslant \iint\limits_{x,y>0} e^{-x-y}cx^{1/2}\,\mathrm{d}x\,\mathrm{d}y < \infty.$$

Consequently, the double integral in (9.18) converges a.s.

If $s \leqslant \frac12 \leqslant t$, then by (9.1), $m(\mathbf{e}; s,t) = m(s)\wedge m'(1-t)$. Noting this, we define a truncated version of W_r by, for $r \geqslant 2T$ and substituting $s = x/r$ and $t = 1-y/r$,

$$\begin{aligned} W_r^T &:= \iint\limits_{\substack{0<s<T/r\\ 1-T/r<t<1}} (t-s)^r m(\mathbf{e}; s,t)\,\mathrm{d}s\,\mathrm{d}t \\ &= r^{-2}\int_0^T\!\!\int_0^T \Big(1-\frac{y}{r}-\frac{x}{r}\Big)^r \big(m(x/r)\wedge m'(y/r)\big)\,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{9.19}$$

Since $\big(1-\frac{y}{r}-\frac{x}{r}\big)^r \to e^{-y-x}$ uniformly for $x,y \in [0,T]$ as $r\to\infty$, it follows from Lemma 9.1 and the continuous mapping theorem that for each fixed $T<\infty$, as $r\to\infty$ we have

$$r^{5/2}W_r^T \xrightarrow{\mathrm{d}} W_\infty^T := \int_0^T\!\!\int_0^T e^{-x-y}\big(J(x)\wedge J'(y)\big)\,\mathrm{d}x\,\mathrm{d}y. \tag{9.20}$$

Furthermore, $W_\infty^T \to W_\infty$ a.s. as $T\to\infty$.

Moreover, by (9.4), $\mathbb{E}\,\mathbf{e}(t) = ct^{1/2}(1-t)^{1/2}$ and thus $\mathbb{E}\,m(\mathbf{e};s,t) \leqslant \mathbb{E}\,\mathbf{e}(s) \leqslant cs^{1/2}$. Hence, for $r \geqslant 2T > 0$, and again with the substitutions $s = x/r$ and $t = 1-(y/r)$, we have

$$\begin{aligned} \mathbb{E}(W_r - W_r^T) &= \iint\limits_{\{T/r<s<t<1\}\cup\{0<s<t<1-T/r\}} (t-s)^r\,\mathbb{E}\,m(\mathbf{e};s,t)\,\mathrm{d}s\,\mathrm{d}t \\ &\leqslant r^{-2}\iint\limits_{[0,r)^2\setminus[0,T]^2} \Big(1-\frac{y}{r}-\frac{x}{r}\Big)_+^r c\Big(\frac{x}{r}\Big)^{1/2}\,\mathrm{d}x\,\mathrm{d}y \\ &\leqslant cr^{-5/2}\iint\limits_{[0,\infty)^2\setminus[0,T]^2} e^{-x-y}x^{1/2}\,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{9.21}$$

Hence,

$$\limsup_{r\to\infty} \mathbb{E}\,|r^{5/2}W_r - r^{5/2}W_r^T| \to 0 \tag{9.22}$$

as $T \to \infty$. This shows, by [5, Theorem 4.2] or [40, Theorem 4.28] again, that we can let $T \to \infty$ inside (9.20) and obtain the conclusion (9.18). ☐

Proof of Theorem 1.29. By (9.3), Lemma 9.3 can be written

$$\alpha^{1/2}Y(\alpha) \xrightarrow{\mathrm{d}} Y_\infty := 2W_\infty \tag{9.23}$$

as $\alpha \to \infty$. We now give some equivalent expressions for the limit. First, by (9.2),

$$Y_\infty \overset{\mathrm{d}}{=} 2\int_0^\infty\int_0^\infty e^{-x-y}\bigl(S(x)\wedge S'(y)\bigr)\,\mathrm{d}x\,\mathrm{d}y. \tag{9.24}$$

Secondly, note that $\tau(a) \leqslant x \iff S(x) \geqslant a$; thus τ and S are inverses of each other. Similarly, we may assume that τ' is the inverse of S'. By Fubini's theorem,

$$\begin{aligned}
Y_\infty &\overset{\mathrm{d}}{=} 2\int_0^\infty\int_0^\infty e^{-x-y}\bigl(S(x)\wedge S'(y)\bigr)\,\mathrm{d}x\,\mathrm{d}y \\
&= 2\iiint\limits_{0\leqslant s\leqslant S(x)\wedge S'(y)} e^{-x-y}\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}y \\
&= 2\iiint\limits_{\tau(s)\leqslant x,\,\tau'(s)\leqslant y} e^{-x-y}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s \\
&= 2\int_0^\infty e^{-\tau(s)-\tau'(s)}\,\mathrm{d}s.
\end{aligned} \tag{9.25}$$

However, $(\tau(s))$ and $(\tau'(s))$ are independent processes with independent increments, and thus $(\tau(s)+\tau'(s))$ has independent increments. Furthermore, for each fixed s, $\tau(2s)-\tau(s) \overset{\mathrm{d}}{=} \tau(s)$ and is independent of $\tau(s)$, and hence $\tau(s)+\tau'(s) \overset{\mathrm{d}}{=} \tau(2s)$. It follows that the stochastic process $(\tau(s)+\tau'(s))$ equals in distribution $(\tau(2s))$. Hence, we also have the representation

$$Y_\infty \overset{\mathrm{d}}{=} 2\int_0^\infty e^{-\tau(2s)}\,\mathrm{d}s = \int_0^\infty e^{-\tau(s)}\,\mathrm{d}s. \tag{9.26}$$

The same Fubini argument in the opposite direction now gives

$$\begin{aligned}
Y_\infty \overset{\mathrm{d}}{=} \int_0^\infty e^{-\tau(s)}\,\mathrm{d}s &= \iint\limits_{x\geqslant\tau(s)} e^{-x}\,\mathrm{d}x\,\mathrm{d}s \\
&= \iint\limits_{0\leqslant s\leqslant S(x)} e^{-x}\,\mathrm{d}s\,\mathrm{d}x = \int_0^\infty e^{-x}S(x)\,\mathrm{d}x.
\end{aligned} \tag{9.27}$$

This shows (1.42).

It remains to calculate the moments of Y_∞. For integer moments we use (9.26). Recall, see e.g. [52, Proposition II.3.7 and Sections III.3–4], that $\tau(s)$ is a stable process with stationary independent increments and

$$\mathbb{E}\, e^{-s\tau(t)} = e^{-t\sqrt{2s}}, \qquad s,t \geqslant 0. \tag{9.28}$$

Define $\Delta\tau(s,s') := \tau(s') - \tau(s)$. Then, by symmetry and the change of variables $t_1 = s_1$, $t_2 = s_2 - s_1$, …, $t_k = s_k - s_{k-1}$, noting that the increments $\Delta\tau(s_{i-1}, s_i)$ are independent and $\Delta\tau(s_{i-1}, s_i) \overset{\mathrm{d}}{=} \tau(t_i)$ (with $s_0 = 0$), we have

$$\begin{aligned}
\mathbb{E}\, Y_\infty^k &= k! \int_{0<s_1<s_2<\dots<s_k} \mathbb{E}\, e^{-\tau(s_1)-\dots-\tau(s_k)} \,\mathrm{d}s_1 \cdots \mathrm{d}s_k \\
&= k! \int_{0<s_1<s_2<\dots<s_k} \mathbb{E}\, e^{-k\Delta\tau(0,s_1)-(k-1)\Delta\tau(s_1,s_2)-\dots-\Delta\tau(s_{k-1},s_k)} \,\mathrm{d}s_1 \cdots \mathrm{d}s_k \\
&= k! \int_{t_1,\dots,t_k>0} \mathbb{E}\, e^{-k\tau(t_1)}\, \mathbb{E}\, e^{-(k-1)\tau(t_2)} \dots \mathbb{E}\, e^{-\tau(t_k)} \,\mathrm{d}t_1 \cdots \mathrm{d}t_k \\
&= k! \int_{t_1,\dots,t_k>0} e^{-t_1\sqrt{2k}-t_2\sqrt{2(k-1)}-\dots-t_k\sqrt{2}} \,\mathrm{d}t_1 \cdots \mathrm{d}t_k \\
&= k! \prod_1^k \frac{1}{\sqrt{2j}} = 2^{-k/2} k!^{1/2},
\end{aligned} \tag{9.29}$$

which is (1.43).

In order to extend this to non-integer moments, let

$$Z := \log Y_\infty + \frac{1}{2}\log 2, \tag{9.30}$$

and let Z' be an independent copy of Z. Then, for integer $k \geqslant 1$,

$$\mathbb{E}\big(\big(e^{Z+Z'}\big)^k\big) = \mathbb{E}\, e^{k(Z+Z')} = \big(\mathbb{E}\, e^{kZ}\big)^2 = \big(2^{k/2}\,\mathbb{E}\, Y_\infty^k\big)^2 = k!, \tag{9.31}$$

and thus $V := e^{Z+Z'} \sim \mathrm{Exp}(1)$, since an exponential distribution is determined by its moments. Hence, for any real $r > -1$,

$$\big(\mathbb{E}\, e^{rZ}\big)^2 = \mathbb{E}\, e^{rZ+rZ'} = \mathbb{E}\, V^r = \int_0^\infty x^r e^{-x}\,\mathrm{d}x = \Gamma(r+1), \tag{9.32}$$

and thus $\mathbb{E}\, e^{rZ} = \sqrt{\Gamma(r+1)}$. Since $e^{rZ} = 2^{r/2} Y_\infty^r$, (1.44) follows, for real r. Finally, (1.44) is extended to complex r by analytic continuation, or by (9.32) again, now knowing that the expectations exist. □

Remark 9.4. The characteristic function $\varphi_Z(t)$ of the random variable Z in (9.30) is thus $\Gamma(1+\mathrm{i}t)^{1/2}$, which decreases exponentially as $t \to \pm\infty$; hence Z has by Fourier inversion a continuous density

$$f_Z(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\mathrm{i}tx}\varphi_Z(t)\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\mathrm{i}tx}\Gamma(1+\mathrm{i}t)^{1/2}\,\mathrm{d}t, \tag{9.33}$$

see e.g. [15, Theorem XV.3.3]; furthermore, by a standard argument, we may differentiate repeatedly under the integral sign, and thus the density function $f_Z(x)$ is infinitely differentiable. (In fact, it follows from Stirling's formula that $\varphi_Z(t) = \Gamma(1+\mathrm{i}t)^{1/2}$

belongs to the Schwartz class $\mathcal{S}(\mathbb{R})$ of infinitely differentiable functions such that every derivative decreases faster than $|x|^{-k}$ for any $k < \infty$; hence $f_Z \in \mathcal{S}(\mathbb{R})$, see [60, Theorem 25.1].)

Consequently, also Y_∞ is absolutely continuous, with a density $f_Y(x)$ that is infinitely differentiable on $(0,\infty)$. Results on the asymptotics of the density function $f_Y(x)$ of Y_∞ as $x \to 0$ and $x \to \infty$ are given in [21]. ☐

Remark 9.5. $2^{1/2}Y_\infty$ has moments $\sqrt{k!}$, and it follows that if Y'_∞ is an independent copy of Y_∞, then $2Y_\infty Y'_\infty$ has moments $k!$ and $2Y_\infty Y'_\infty \sim \mathrm{Exp}(1)$. Hence, the distribution of $2^{1/2}Y_\infty$ is a "square root" of Exp(1), in the sense of taking products of independent variables.

Moreover, if we let $(\tau(s))$ be another stable subordinator, with $\mathbb{E}\, e^{-s\tau(t)} = e^{-ts^\gamma}$ $(0 < \gamma < 1)$ instead of (9.28), then (9.26) defines by the same calculations a random variable $Y_{(\gamma)}$ with

$$\mathbb{E}\, Y_{(\gamma)}^k = (k!)^{1-\gamma}. \tag{9.34}$$

In particular, choosing $\gamma = 1 - (1/m)$, we obtain an m^{th} root of the exponential distribution Exp(1).

Recalling that $V \sim \mathrm{Exp}(1)$ and taking logarithms, this shows that $\log V$ is infinitely divisible, and thus the same holds for $-\log V$, which has a Gumbel distribution. This has been known for a long time, and a calculation shows that $-\log V$ has a Lévy measure with a density $\sum_{j=1}^\infty e^{-jx}/x = x^{-1}(e^x-1)^{-1}$, $x > 0$; see, e.g., [56, Examples 11.1 and 11.10]. See also [7, Example 7.2.3]. ☐

10. Extensions to $\operatorname{Re}\alpha = \frac12$

In this section, we show the extensions to $\operatorname{Re}\alpha = \frac12$ claimed in Remarks 1.5, 1.8, and 1.11. These require different methods from the ones used above.

Let $\varphi(t) := \mathbb{E}\, e^{\mathrm{i}t\xi}$ be the characteristic function of the offspring distribution ξ. Furthermore, let $\tilde\xi := \xi - \mathbb{E}\,\xi = \xi - 1$, and denote its characteristic function by

$$\tilde\varphi(t) := \mathbb{E}\, e^{\mathrm{i}t\tilde\xi} = e^{-\mathrm{i}t}\varphi(t), \qquad t \in \mathbb{R}. \tag{10.1}$$

Since $\mathbb{E}\,\tilde\xi = 0$ and $\mathbb{E}\,\tilde\xi^2 = \sigma^2$, we have $\tilde\varphi(t) = 1 - \frac{\sigma^2}{2}t^2 + o(t^2)$; hence

$$\tilde\varphi(t) = 1 - \frac{\sigma^2}{2}t^2\big[1+\gamma(t)\big], \qquad t \in \mathbb{R}, \tag{10.2}$$

for some continuous function $\gamma(t)$ on $\mathbb{R}$ such that $\gamma(0) = 0$.

We also let

$$\rho(t) := 1 - \tilde\varphi(t) = \frac{\sigma^2}{2}t^2\big[1+\gamma(t)\big]. \tag{10.3}$$

Since ξ is integer-valued, φ and $\tilde\varphi$ are 2π-periodic. Note that $\mathbb{P}(\tilde\xi = -1) > 0$ and thus $\tilde\varphi(t) \neq 1$ if $0 < |t| \leqslant \pi$ [also when $\operatorname{span}(\xi) > 1$]; hence (10.2) and continuity imply

$$\operatorname{Re}\rho(t) = 1 - \operatorname{Re}\tilde\varphi(t) \geqslant c_1 t^2, \qquad 0 \leqslant |t| \leqslant \pi, \tag{10.4}$$

for some $c_1 > 0$. Furthermore, if $\operatorname{span}(\xi) = h \geqslant 1$, then $\varphi(\pm 2\pi/h) = 1$ but $|\varphi(t)| < 1$ for $0 < |t| < 2\pi/h$, and it follows similarly from (10.2) and continuity that

$$|\widetilde{\varphi}(t)| = |\varphi(t)| \leqslant 1 - c_2 t^2 \leqslant e^{-c_2 t^2}, \qquad 0 \leqslant |t| \leqslant \pi/h. \tag{10.5}$$

Lemma 10.1. *If* $\operatorname{Re}\alpha < \frac{1}{2}$, *then*

$$\mu(\alpha) = \frac{1}{2\pi\Gamma(1-\alpha)} \int_{-\pi}^{\pi} \int_0^{\infty} x^{-\alpha} \frac{\varphi(t)}{e^x - \widetilde{\varphi}(t)} \, \mathrm{d}x \, \mathrm{d}t, \tag{10.6}$$

where the double integral is absolutely convergent.

Proof. Let $\operatorname{Re}\alpha < \frac{1}{2}$. Fourier inversion and (2.3) yield

$$\mu(\alpha) = \sum_{n=1}^{\infty} n^{\alpha-1} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-\mathrm{i}(n-1)t} \varphi(t)^n \, \mathrm{d}t = \sum_{n=1}^{\infty} n^{\alpha-1} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\mathrm{i}t} \widetilde{\varphi}(t)^n \, \mathrm{d}t. \tag{10.7}$$

Let $\operatorname{span}(\xi) = h \geqslant 1$. It follows from the estimate (10.5) that

$$\int_{-\pi}^{\pi} |\widetilde{\varphi}(t)|^n \, \mathrm{d}t = h \int_{-\pi/h}^{\pi/h} |\widetilde{\varphi}(t)|^n \, \mathrm{d}t \leqslant h \int_{-\infty}^{\infty} e^{-c_2 n t^2} \, \mathrm{d}t = C_1 n^{-1/2}. \tag{10.8}$$

Hence,

$$\sum_{n=1}^{\infty} \int_{-\pi}^{\pi} \left| n^{\alpha-1} e^{\mathrm{i}t} \widetilde{\varphi}(t)^n \right| \mathrm{d}t \leqslant C_1 \sum_{n=1}^{\infty} n^{\operatorname{Re}\alpha - \frac{3}{2}} < \infty. \tag{10.9}$$

Thus we may interchange the order of summation and integration in (10.7) and obtain

$$\mu(\alpha) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\mathrm{i}t} \sum_{n=1}^{\infty} n^{\alpha-1} \widetilde{\varphi}(t)^n \, \mathrm{d}t. \tag{10.10}$$

The sum $\sum_{n=1}^{\infty} n^{\alpha-1} \widetilde{\varphi}(t)^n$ is known as the polylogarithm $\operatorname{Li}_{1-\alpha}(\widetilde{\varphi}(t))$ [47, §25.12(ii)]. It can be expressed as an integral [47, 25.12.11] by a standard argument, which we adapt as follows: Since $\operatorname{Re}\alpha < \frac{1}{2} < 1$, we have $n^{\alpha-1}\Gamma(1-\alpha) = \int_0^\infty x^{-\alpha} e^{-nx} \, \mathrm{d}x$ and thus (10.10) yields

$$\Gamma(1-\alpha)\mu(\alpha) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} \int_0^{\infty} x^{-\alpha} e^{-nx} \widetilde{\varphi}(t)^n e^{\mathrm{i}t} \, \mathrm{d}x \, \mathrm{d}t. \tag{10.11}$$

Again, this expression is absolutely convergent as a consequence of (10.8) and (10.9), and thus we may again interchange the order of summation and integration and obtain

$$\begin{aligned} \Gamma(1-\alpha)\mu(\alpha) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_0^{\infty} x^{-\alpha} \sum_{n=1}^{\infty} e^{-nx} \widetilde{\varphi}(t)^n e^{\mathrm{i}t} \, \mathrm{d}x \, \mathrm{d}t \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_0^{\infty} x^{-\alpha} \frac{e^{-x}\widetilde{\varphi}(t)}{1 - e^{-x}\widetilde{\varphi}(t)} e^{\mathrm{i}t} \, \mathrm{d}x \, \mathrm{d}t. \end{aligned} \tag{10.12}$$

This yields (10.6), with absolute convergence. □

We next modify (10.6) by ignoring terms that are analytic at $\operatorname{Re}\alpha = \frac{1}{2}$; more precisely, we ignore terms that are analytic in $D_1 := \{\alpha : 0 < \operatorname{Re}\alpha < 1\}$.

Lemma 10.2. *There exists a function* $h(\alpha) \in \mathcal{H}(D_1)$ *such that if* $0 < \operatorname{Re}\alpha < \frac{1}{2}$, *then*

$$\mu(\alpha) = \frac{\Gamma(\alpha)}{2\pi}\int_{-\pi}^{\pi} \rho(t)^{-\alpha}\,\mathrm{d}t + h(\alpha). \tag{10.13}$$

Remark 10.3. Since $\rho(t) \neq 0$ for $0 < |t| \leqslant \pi$, the integral $\int_{t_0 \leqslant |t| \leqslant \pi} \rho(t)^{-\alpha}\,\mathrm{d}t$ is an entire function of α for any $t_0 \in (0, \pi]$, and thus the integral in (10.13) can be replaced by the integral over $|t| \leqslant t_0$ for any such t_0. □

Proof of Lemma 10.2. First, for $x \geqslant 1$ and $\operatorname{Re}\alpha > 0$, the integrand in (10.6) is $O(e^{-x})$ so the double integral over $\{x \geqslant 1,\, t \in (-\pi, \pi)\}$ converges and defines an analytic function $h_1 \in \mathcal{H}(D_1)$. We may thus consider the integral for $0 < x < 1$ only.

Next, using (10.3) and (10.4), for $x > 0$ we have

$$\bigl|e^x - \widetilde{\varphi}(t)\bigr| \geqslant \operatorname{Re}\bigl(e^x - \widetilde{\varphi}(t)\bigr) = e^x - 1 + \operatorname{Re}\rho(t) \geqslant x + c_1 t^2. \tag{10.14}$$

Hence, using $|\varphi(t) - 1| \leqslant C_2 t$ (since $\mathbb{E}\,\xi < \infty$),

$$\int_{-\pi}^{\pi}\int_0^1 \Bigl|x^{-\alpha}\frac{\varphi(t) - 1}{e^x - \widetilde{\varphi}(t)}\Bigr|\,\mathrm{d}x\,\mathrm{d}t \leqslant C_3 \int_{-\pi}^{\pi}\int_0^1 x^{-\operatorname{Re}\alpha}\frac{|t|}{x + t^2}\,\mathrm{d}x\,\mathrm{d}t. \tag{10.15}$$

Now, for $0 < x < 1$,

$$\int_0^{\pi} \frac{t}{x + t^2}\,\mathrm{d}t \leqslant \int_0^{\sqrt{x}} \frac{t}{x}\,\mathrm{d}t + \int_{\sqrt{x}}^{\pi} \frac{t}{t^2}\,\mathrm{d}t = \frac{1}{2} + \log\pi - \log\sqrt{x} = O(1 + |\log x|) \tag{10.16}$$

and thus (10.15) converges for $\operatorname{Re}\alpha < 1$. It follows that if we replace the numerator $\varphi(t)$ by 1 in (10.6) (with $x < 1$ only), then the difference is in $\mathcal{H}(D_1)$.

Similarly, for $0 < x < 1$ and $|t| \leqslant \pi$,

$$\Bigl|\frac{1}{e^x - \widetilde{\varphi}(t)} - \frac{1}{x + 1 - \widetilde{\varphi}(t)}\Bigr| \leqslant \frac{e^x - 1 - x}{(x + c_1 t^2)^2} \leqslant 1, \tag{10.17}$$

and we may thus also replace the denominator $e^x - \widetilde{\varphi}(t)$ by $x + 1 - \widetilde{\varphi}(t) = x + \rho(t)$.

This yields

$$\mu(\alpha) = \frac{1}{2\pi\Gamma(1 - \alpha)}\int_{-\pi}^{\pi}\int_0^1 x^{-\alpha}\frac{1}{x + \rho(t)}\,\mathrm{d}x\,\mathrm{d}t + h_2(\alpha). \tag{10.18}$$

with $h_2 \in \mathcal{H}(D_1)$. We now reintroduce $x \geqslant 1$, noting that $\operatorname{Re}\rho(t) \geqslant 0$ and thus, for $\operatorname{Re}\alpha > 0$,

$$\int_{-\pi}^{\pi}\int_1^{\infty} \Bigl|\frac{x^{-\alpha}}{x + \rho(t)}\Bigr|\,\mathrm{d}x\,\mathrm{d}t \leqslant 2\pi\int_1^{\infty} x^{-\operatorname{Re}\alpha - 1}\,\mathrm{d}x < \infty. \tag{10.19}$$

Hence, for $\alpha \in D_1$,

$$\mu(\alpha) = \frac{1}{2\pi\Gamma(1 - \alpha)}\int_{-\pi}^{\pi}\int_0^{\infty} \frac{x^{-\alpha}}{x + \rho(t)}\,\mathrm{d}x\,\mathrm{d}t + h(\alpha), \tag{10.20}$$

with $h \in \mathcal{H}(D_1)$, and (10.13) follows by a standard beta integral: for $0 < \operatorname{Re}\alpha < 1$ and $\rho \notin (-\infty, 0]$ we have

$$\begin{aligned}\int_0^{\infty} \frac{x^{-\alpha}}{x + \rho}\,\mathrm{d}x &= \rho^{-\alpha}\int_0^{\infty} \frac{x^{-\alpha}}{x + 1}\,\mathrm{d}x = \rho^{-\alpha}B(1 - \alpha, \alpha) \\ &= \rho^{-\alpha}\Gamma(1 - \alpha)\Gamma(\alpha),\end{aligned} \tag{10.21}$$

where the first equality holds for all $\rho > 0$ by a change of variables and therefore for all $\rho \notin (-\infty, 0]$ by analytic continuation. $\square$

Recall the function $\gamma(\alpha)$ defined by (10.2).

Lemma 10.4. *For any $a > 0$ we have*

$$\int_0^\infty \frac{|\gamma(at) - \gamma(t)|}{t} \, \mathrm{d}t < \infty. \tag{10.22}$$

Proof. By (10.2), recalling $\mathbb{E}\,\tilde{\xi} = 0$ and $\mathbb{E}\,\tilde{\xi}^2 = \sigma^2$, we have

$$-\frac{\sigma^2 t^2}{2}\gamma(t) = \widetilde{\varphi}(t) - 1 + \frac{\sigma^2 t^2}{2} = \mathbb{E}\, e^{\mathrm{i}t\tilde{\xi}} - 1 - \mathbb{E}(\mathrm{i}t\tilde{\xi}) - \frac{1}{2}\,\mathbb{E}(\mathrm{i}t\tilde{\xi})^2. \tag{10.23}$$

Define

$$\psi_1(x) := e^{\mathrm{i}x} - 1 - \mathrm{i}x, \tag{10.24}$$

$$\psi_2(x) := e^{\mathrm{i}x} - 1 - \mathrm{i}x - \tfrac{1}{2}(\mathrm{i}x)^2. \tag{10.25}$$

Then (10.23) implies

$$\gamma(t) = -\frac{2}{\sigma^2 t^2}\,\mathbb{E}\,\psi_2(t\tilde{\xi}) \tag{10.26}$$

and thus

$$\gamma(at) - \gamma(t) = \frac{2}{\sigma^2 t^2}\,\mathbb{E}\Big[\psi_2(t\tilde{\xi}) - \frac{1}{a^2}\psi_2(at\tilde{\xi})\Big]. \tag{10.27}$$

Fix $a > 0$. Taylor's formula yields the standard estimate $|\psi_2(x)| \leqslant |x|^3$, and thus

$$|\psi_2(x) - a^{-2}\psi_2(ax)| \leqslant C|x|^3. \tag{10.28}$$

Furthermore, $\psi_2(x) - a^{-2}\psi_2(ax) = \psi_1(x) - a^{-2}\psi_1(ax)$ by cancellation, and $|\psi_1(x)| \leqslant 2|x|$ and thus

$$|\psi_1(x) - a^{-2}\psi_1(ax)| \leqslant C|x|. \tag{10.29}$$

Consequently,

$$|\psi_2(x) - a^{-2}\psi_2(ax)| \leqslant C\big(|x| \wedge |x|^3\big). \tag{10.30}$$

Combining (10.27) and (10.30) we obtain, for $t \neq 0$,

$$|\gamma(at) - \gamma(t)| \leqslant Ct^{-2}\,\mathbb{E}\big(|t\tilde{\xi}| \wedge |t\tilde{\xi}|^3\big). \tag{10.31}$$

Hence,

$$\begin{aligned}
\int_0^\infty \frac{|\gamma(at) - \gamma(t)|}{t} \, \mathrm{d}t &\leqslant C \int_0^\infty \mathbb{E}\big(|t^{-2}\tilde{\xi}| \wedge |\tilde{\xi}|^3\big) \, \mathrm{d}t \\
&= C\bigg(\int_0^{|\tilde{\xi}|^{-1}} |\tilde{\xi}|^3 \, \mathrm{d}t + \int_{|\tilde{\xi}|^{-1}}^\infty t^{-2}|\tilde{\xi}| \, \mathrm{d}t\bigg) \\
&= C\,\mathbb{E}\big(|\tilde{\xi}|^2 + |\tilde{\xi}|^2\big) = 2C\sigma^2 < \infty. \qquad \square
\end{aligned}$$

Remark 10.5. Lemma 10.4 and its proof hold with $\tilde{\xi}$ replaced by any random variable X with $\mathbb{E}\,X = 0$ and $\mathbb{E}\,X^2 < \infty$. $\square$

Remark 10.6. Note, in contrast, that the integral $\int_0^1 |\gamma(t)|t^{-1}\,\mathrm{d}t$ may diverge; hence some cancellation is essential in Lemma 10.4. In fact, it is not difficult to show, using similar arguments, that $\int_0^1 |\gamma(t)|t^{-1}\,\mathrm{d}t < \infty$ if and only if $\mathbb{E}\,\tilde{\xi}^2\log|\tilde{\xi}| < \infty$. (Since $\gamma(t) \to -1$ as $t \to \infty$, we cannot here integrate to ∞.) □

The function $\mu(\alpha)$ is defined by (1.12) for $\operatorname{Re}\alpha < \frac{1}{2}$. As noted at (1.13), $\mu(\alpha) \to \infty$ as $\alpha \nearrow \frac{1}{2}$. However, $\mu(\alpha)$ has a continuous extension to all other points on the line $\operatorname{Re}\alpha = \frac{1}{2}$.

Theorem 10.7. *The function $\mu(\alpha)$ has a continuous extension to the set $\{\alpha : \operatorname{Re}\alpha \leqslant \frac{1}{2}\}\backslash\{\frac{1}{2}\}$.*

Proof. For $0 < s \leqslant \pi$ and $\operatorname{Re}\alpha < \frac{1}{2}$, let

$$f_s(\alpha) := \Bigl(\frac{\sigma^2}{2}\Bigr)^{\alpha}\int_{-s}^{s}\rho(t)^{-\alpha}\,\mathrm{d}t = \int_{-s}^{s} t^{-2\alpha}\bigl[1+\gamma(t)\bigr]^{-\alpha}\,\mathrm{d}t. \tag{10.32}$$

Let $a > 0$ and let $s_0 := \pi/(1 \vee a)$. Then, for $0 < s \leqslant s_0$, we have

$$f_{as}(\alpha) = \int_{-as}^{as} t^{-2\alpha}\bigl[1+\gamma(t)\bigr]^{-\alpha}\,\mathrm{d}t = a^{1-2\alpha}\int_{-s}^{s} t^{-2\alpha}\bigl[1+\gamma(at)\bigr]^{-\alpha}\,\mathrm{d}t. \tag{10.33}$$

Fix $B < \infty$ and let $D^B := \{\alpha : 0 \leqslant \operatorname{Re}\alpha < \frac{1}{2},\ |\operatorname{Im}\alpha| \leqslant B\}$. By (10.32) and (10.33), uniformly for $\alpha \in D^B$, noting that $1+\gamma(t) \neq 0$ for $0 < |t| \leqslant \pi$ by (10.2), we have

$$\begin{aligned}\bigl|a^{2\alpha-1}f_{as}(\alpha) - f_s(\alpha)\bigr| &\leqslant \int_{-s}^{s}\bigl|(1+\gamma(at))^{-\alpha} - (1+\gamma(t))^{-\alpha}\bigr|\bigl|t^{-2\alpha}\bigr|\,\mathrm{d}t\\ &\leqslant C\int_{-s}^{s}|\gamma(at)-\gamma(t)|t^{-2\operatorname{Re}\alpha}\,\mathrm{d}t\\ &\leqslant C\int_{0}^{s}|\gamma(at)-\gamma(t)|t^{-1}\,\mathrm{d}t,\end{aligned} \tag{10.34}$$

which tends to 0 as $s \to 0$ by Lemma 10.4.

Let

$$F_s(\alpha) := a^{2\alpha-1}\bigl(f_\pi(\alpha) - f_{as}(\alpha)\bigr) - \bigl(f_\pi(\alpha) - f_s(\alpha)\bigr). \tag{10.35}$$

We have just shown in (10.34) that as $s \to 0$ we have

$$F_s(\alpha) \to \bigl(a^{2\alpha-1} - 1\bigr)f_\pi(\alpha) \tag{10.36}$$

uniformly in D^B. For $s \in (0, s_0]$, $F_s(\alpha)$ is an entire function, see Remark 10.3, and in particular continuous on $\overline{D^B}$. Hence, the sequence $F_{1/n}(\alpha)$, which is uniformly convergent on D^B by (10.36), is a Cauchy sequence in $C(\overline{D^B})$, and thus converges uniformly on $\overline{D^B}$ to some continuous limit. Together with (10.36) again, this shows that $(a^{2\alpha-1}-1)f_\pi(\alpha)$ has a continuous extension to $\overline{D^B}$.

This holds for any $a > 0$. We now choose $a = e^{1/B}$; then $a^{2\alpha-1} \neq 1$ in $\overline{D^B}\backslash\{\frac{1}{2}\}$, and thus $f_\pi(\alpha)$ has a continuous extension to $\overline{D^B}\backslash\{\frac{1}{2}\}$. Since B is arbitrary, this shows that $f_\pi(\alpha)$ has a continuous extension to $\{\alpha : 0 \leqslant \operatorname{Re}\alpha \leqslant \frac{1}{2}\}\backslash\{\frac{1}{2}\}$.

Finally, the definition (10.32) shows that the same holds for $\int_{-\pi}^{\pi}\rho(t)^{-\alpha}\,\mathrm{d}t$, and the result follows by Lemma 10.2. □

In the sequel, $\mu(\alpha)$ is defined for $\operatorname{Re}\alpha = \frac{1}{2}$, $\alpha \neq \frac{1}{2}$, as this continuous extension.

Theorem 10.8. (i) *The estimate* (1.16) *in Theorem 1.7*(ii) *holds also for* $\alpha = \frac{1}{2}+\mathrm{i}y$, $y \neq 0$. *Moreover,* (1.16) *holds uniformly on compact subsets of* $\{\alpha : -\frac{1}{2} < \operatorname{Re}\alpha \leqslant \frac{1}{2}\}\backslash\{\frac{1}{2}\}$.

(ii) *The limit result* (1.22) *in Theorem 1.10*(ii) *holds also for* $\alpha = \frac{1}{2}+\mathrm{i}y$, $y \neq 0$. *Moreover,* (1.22) *holds in the space* $C(\widehat{D})$ *of continuous functions on the set* $\widehat{D} := \{\alpha : 0 < \operatorname{Re}\alpha \leqslant \frac{1}{2}\}\backslash\{\frac{1}{2}\}$.

The topology in $C(\widehat{D})$ is defined by uniform convergence on compact subsets of $\widehat{D}$.

Proof. Part (ii) follows by Theorem 1.2 and (i), so it suffices to prove (i).

In this proof, let $D := \{\alpha : 0 < \operatorname{Re}\alpha < \frac{3}{4}\}$, $D_- := \{\alpha : 0 < \operatorname{Re}\alpha < \frac{1}{2}\}$, and, for $B > 0$, $D_-^B := \{\alpha \in D_- : |\operatorname{Im}\alpha| \leqslant B\}$, $\widehat{D}^B := \{\alpha \in \widehat{D} : |\operatorname{Im}\alpha| \leqslant B\}$.

By (2.3) and (6.9), for $\operatorname{Re}\alpha < \frac{1}{2}$ we have

$$\mu(\alpha) - \mu_n(\alpha) = \sum_{k=n+1}^{\infty} k^{\alpha-1}\,\mathbb{P}(S_k = k-1). \tag{10.37}$$

Imitating the proof of Lemma 10.1 we obtain, cf. (10.12), for $\alpha \in D_-$,

$$\begin{aligned}
\Gamma(1-\alpha)\bigl(\mu(\alpha)-\mu_n(\alpha)\bigr) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^{\infty} x^{-\alpha}\sum_{k=n+1}^{\infty} e^{-kx}\widetilde{\varphi}(t)^k e^{\mathrm{i}t}\,\mathrm{d}x\,\mathrm{d}t \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^{\infty} x^{-\alpha}\frac{e^{-(n+1)x}\widetilde{\varphi}(t)^{n+1}}{1-e^{-x}\widetilde{\varphi}(t)} e^{\mathrm{i}t}\,\mathrm{d}x\,\mathrm{d}t \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^{\infty} x^{-\alpha}\frac{e^{-nx}\widetilde{\varphi}(t)^{n}}{e^{x}-\widetilde{\varphi}(t)} \varphi(t)\,\mathrm{d}x\,\mathrm{d}t
\end{aligned} \tag{10.38}$$

and thus, by the change of variables $x \mapsto x/n$, $t \mapsto t/\sqrt{n}$, we have

$$f_n(\alpha) := n^{\frac{1}{2}-\alpha}\Gamma(1-\alpha)\bigl(\mu(\alpha)-\mu_n(\alpha)\bigr) \tag{10.39}$$

$$= \frac{1}{2\pi}\int_{-\pi\sqrt{n}}^{\pi\sqrt{n}}\int_0^{\infty} x^{-\alpha}\frac{e^{-x}\widetilde{\varphi}(t/\sqrt{n})^{n}}{n[e^{x/n}-\widetilde{\varphi}(t/\sqrt{n})]} \varphi(t/\sqrt{n})\,\mathrm{d}x\,\mathrm{d}t \tag{10.40}$$

Denote the integrand in (10.40) by $g_n(\alpha,x,t)$, and let this define $g_n(\alpha,x,t)$ for any $\alpha \in D$. Note that for any fixed $\alpha \in D$, $x > 0$, and $t \in \mathbb{R}$, by (10.2),

$$g_n(\alpha,x,t) \to x^{-\alpha}\frac{e^{-x-\frac{\sigma^2}{2}t^2}}{x+\frac{\sigma^2}{2}t^2} =: g(\alpha,x,t). \tag{10.41}$$

Furthermore, (10.41) trivially holds uniformly for $\alpha \in D$. Note also that, by (10.4),

$$\begin{aligned}
\bigl|n\bigl(e^{x/n}-\widetilde{\varphi}(t/\sqrt{n})\bigr)\bigr| &\geqslant \operatorname{Re}\bigl(n\bigl(e^{x/n}-\widetilde{\varphi}(t/\sqrt{n})\bigr)\bigr) \geqslant x + n\operatorname{Re}\bigl(1-\widetilde{\varphi}(t/\sqrt{n})\bigr) \\
&\geqslant x + c_1 t^2.
\end{aligned} \tag{10.42}$$

Let $h := \operatorname{span}(\xi)$. If $h > 1$, consider first t with $\pi\sqrt{n}/h < |t| \leqslant \pi\sqrt{n}$. For such t, (10.42) implies $\bigl|e^{x/n} - \widetilde{\varphi}(t/\sqrt{n})\bigr| \geqslant c$, and thus $|g_n(\alpha, x, t)| \leqslant Cn^{-1}x^{-\operatorname{Re}\alpha}e^{-x}$. Hence, the integral (10.40) restricted to $|t| > \pi\sqrt{n}/h$ is $O\bigl(n^{-1/2}\bigr)$, uniformly in D.

Next (for any h), for $\alpha \in D$ and $|t| \leqslant \pi\sqrt{n}/h$, (10.5) and (10.42) yield

$$|g_n(\alpha, x, t)| \leqslant x^{-\operatorname{Re}\alpha}\frac{e^{-x-ct^2}}{x + ct^2} \leqslant \bigl(1 + x^{-3/4}\bigr)\frac{e^{-x-ct^2}}{x + ct^2}. \tag{10.43}$$

The right-hand side is integrable over $(x, t) \in (0, \infty) \times ((-\infty, -1) \cup (1, \infty))$; hence the integral (10.40) restricted to $1 < |t| \leqslant \pi\sqrt{n}/h$ converges by Lemma 2.7 uniformly on D to the corresponding integral of $g(\alpha, x, t)$, which is an analytic function $h_1(\alpha) \in \mathcal{H}(D)$ by Remark 2.8.

Similarly, for $x \geqslant 1$, using (10.42) again,

$$|g_n(\alpha, x, t)| \leqslant x^{-\operatorname{Re}\alpha}\frac{e^{-x}}{x + c_1t^2} \leqslant e^{-x} \tag{10.44}$$

and it follows by Lemma 2.7 and Remark 2.8 that the integral (10.40) restricted to $(x, t) \in (1, \infty) \times (-1, 1)$ converges uniformly to an analytic function $h_2(\alpha) \in \mathcal{H}(D)$.

It remains to consider the integral in (10.40) over $(x, t) \in Q := (0, 1) \times (-1, 1)$. We modify this integral in several steps.

We first replace e^{-x} by 1 in the numerator of $g_n(\alpha, x, t)$; the absolute value of the difference is bounded, using (10.42) again, by

$$x^{-\operatorname{Re}\alpha}\frac{1 - e^{-x}}{x + c_1t^2} \leqslant x^{-\operatorname{Re}\alpha} \leqslant 1 + x^{-3/4} \tag{10.45}$$

and thus Lemma 2.7 and Remark 2.8 show that the integral of the difference over $(x, t) \in Q$ converges uniformly to an analytic function $h_3(\alpha) \in \mathcal{H}(D)$.

Similarly, we then replace $\widetilde{\varphi}(t/\sqrt{n})^n$ by 1 in the resulting integral; the difference is by (10.42) and (10.2), using $|1 - \widetilde{\varphi}(t/\sqrt{n})^n| \leqslant n|1 - \widetilde{\varphi}(t/\sqrt{n})|$, bounded by

$$x^{-\operatorname{Re}\alpha}\frac{Ct^2}{x + c_1t^2} \leqslant Cx^{-\operatorname{Re}\alpha} \leqslant C(1 + x^{-3/4}) \tag{10.46}$$

and again the integral of the difference over Q converges uniformly to an analytic function $h_4(\alpha) \in \mathcal{H}(D)$.

Next, we replace in the denominator $e^{x/n} - \widetilde{\varphi}(t/\sqrt{n})$ by $(x/n) + \rho(t/\sqrt{n})$. The resulting error is by (10.17) bounded by $x^{-\operatorname{Re}\alpha}\frac{1}{n}$ so the error in the integral over Q is $O(n^{-1})$, uniformly in $\alpha \in D$.

Similarly, $\varphi(t/\sqrt{n}) = 1 + O(t/\sqrt{n})$, so replacing the factor $\varphi(t/\sqrt{n})$ by 1 yields an error in the integral over Q that is bounded, for $\alpha \in D$, by

$$\frac{C}{\sqrt{n}}\int_{-1}^{1}\int_{0}^{1} x^{-3/4}\frac{|t|}{x + t^2}\,\mathrm{d}x\,\mathrm{d}t = O\bigl(n^{-1/2}\bigr), \tag{10.47}$$

since the integral converges by (10.16).

Summarizing the development so far, we have shown that

$$f_n(\alpha) = \frac{1}{2\pi}\int_{-1}^{1}\int_{0}^{1} x^{-\alpha}\frac{1}{x + n\rho(t/\sqrt{n})}\,\mathrm{d}x\,\mathrm{d}t + h_5(\alpha) + o(1), \tag{10.48}$$

uniformly in D_-, for some $h_5(\alpha) \in \mathcal{H}(D)$.

Define, for $a > 0$ and $\alpha \in D_-$,

$$F_{n,a}(\alpha) := \int_{-1}^{1}\int_{0}^{1} \frac{x^{-\alpha}}{x + na^{-2}\rho(at/\sqrt{n})}\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_{-1}^{1}\int_{0}^{1} \frac{x^{-\alpha}}{x + \frac{\sigma^2}{2}t^2\bigl[1+\gamma(at/\sqrt{n})\bigr]}\,\mathrm{d}x\,\mathrm{d}t, \tag{10.49}$$

noting that the integrals converge by (10.4) and the fact that

$$\int_{-1}^{1}\int_{0}^{1} \frac{|x^{-\alpha}|}{x+t^2}\,\mathrm{d}x\,\mathrm{d}t \leqslant \pi \int_{0}^{1} x^{-\operatorname{Re}\alpha - \frac{1}{2}}\,\mathrm{d}x < \infty. \tag{10.50}$$

Thus, (10.48) can be written, uniformly in D_-,

$$f_n(\alpha) = \frac{1}{2\pi}F_{n,1}(\alpha) + h_5(\alpha) + o(1). \tag{10.51}$$

Fix $a > 1$. Then, for $\alpha \in D_-$ (and $n \geqslant a^2$, say), using Lemma 10.4 we have

$$\bigl|F_{n,a}(\alpha) - F_{n,1}(\alpha)\bigr| \leqslant \int_{-1}^{1}\int_{0}^{1} |x^{-\alpha}| \frac{Ct^2\bigl|\gamma(at/\sqrt{n}) - \gamma(t/\sqrt{n})\bigr|}{(x+ct^2)^2}\mathrm{d}x\,\mathrm{d}t$$
$$\leqslant C\int_{-1}^{1}\bigl|\gamma(at/\sqrt{n}) - \gamma(t/\sqrt{n})\bigr| \int_{0}^{1} x^{-1/2}\frac{\mathrm{d}x}{x+t^2}\,\mathrm{d}t$$
$$\leqslant C\int_{-1}^{1}\frac{\bigl|\gamma(at/\sqrt{n}) - \gamma(t/\sqrt{n})\bigr|}{|t|}\,\mathrm{d}t$$
$$= C\int_{-1/\sqrt{n}}^{1/\sqrt{n}}\frac{\bigl|\gamma(at) - \gamma(t)\bigr|}{|t|}\,\mathrm{d}t \to 0, \tag{10.52}$$

as $n \to \infty$. Moreover, by the change of variables $x \mapsto a^{-2}x$, $t \mapsto a^{-1}t$,

$$F_{n,a}(\alpha) = a^{2\alpha-1}\int_{-a}^{a}\int_{0}^{a^2} \frac{x^{-\alpha}}{x + n\rho(t/\sqrt{n})}\,\mathrm{d}x\,\mathrm{d}t, \tag{10.53}$$

which differs from $a^{2\alpha-1}F_{n,1}(\alpha)$ by an integral which, using Lemma 2.7 and Remark 2.8 again, converges uniformly to some function $h_6(\alpha) \in \mathcal{H}(D)$.

It follows that, uniformly for $\alpha \in D_-$,

$$\bigl(a^{2\alpha-1} - 1\bigr)F_{n,1}(\alpha) = F_{n,a}(\alpha) - F_{n,1}(\alpha) - \bigl(F_{n,a}(\alpha) - a^{2\alpha-1}F_{n,1}(\alpha)\bigr) \to -h_6(\alpha). \tag{10.54}$$

Consequently, (10.51) shows that $\bigl(a^{2\alpha-1} - 1\bigr)f_n(\alpha)$ converges uniformly in D_- to some function $h_7(\alpha) \in \mathcal{H}(D)$, which, recalling the definition (10.39) of $f_n(\alpha)$, shows that

$$\bigl(a^{2\alpha-1} - 1\bigr)n^{\frac{1}{2}-\alpha}\bigl(\mu(\alpha) - \mu_n(\alpha)\bigr) = h_8(\alpha) + o(1), \tag{10.55}$$

uniformly in D_-^B, for some function $h_8(\alpha) \in \mathcal{H}(D)$ and every $B > 0$. By (6.22),

$$h_8(\alpha) = \bigl(a^{2\alpha-1} - 1\bigr)\frac{1}{\sqrt{2\pi\sigma^2}(\frac{1}{2} - \alpha)} \tag{10.56}$$

for $\alpha \in D_-$, and thus by analytic continuation for $\alpha \in D\backslash\{\frac{1}{2}\}$.

By Theorem 10.7, $\mu(\alpha)$ is continuous on $\widehat{D}$, and so are $\mu_n(\alpha)$ (which is an entire function) and $h_8(\alpha)$. Hence, by continuity, (10.55) holds uniformly in every $\widehat{D}^B$.

Finally, for any compact set $K \subset \widehat{D}$, we can choose $a > 1$ such that $a^{2\alpha-1} \neq 1$ on K, and then (10.55) and (10.56) show that, uniformly for $\alpha \in K$,

$$n^{\frac{1}{2}-\alpha}\big[\mu(\alpha) - \mu_n(\alpha)\big] = \frac{1}{\sqrt{2\pi\sigma^2}(\frac{1}{2}-\alpha)} + o(1) \tag{10.57}$$

as $n \to \infty$. The result (1.16), uniformly on K, follows from (10.57) and Lemma 6.2.

This shows that (1.16) holds uniformly on any compact subset of $\widehat{D}$, and in particular on any compact subset of $\{\alpha : \frac{1}{4} \leqslant \operatorname{Re}\alpha \leqslant \frac{1}{2}\}\backslash\{\frac{1}{2}\}$. Since Theorem 1.7(ii) implies that (1.16) holds uniformly on any compact subset of $\{\alpha : -\frac{1}{2} < \operatorname{Re}\alpha \leqslant \frac{1}{4}\}$, it follows that it holds uniformly on any compact subset of $\{\alpha : -\frac{1}{2} < \operatorname{Re}\alpha \leqslant \frac{1}{2}\}\backslash\{\frac{1}{2}\}$. □

11. An example where $\mu(\alpha)$ has no analytic extension

Theorem 10.7 shows that $\mu(\alpha)$ has a continuous extension to the line $\operatorname{Re}\alpha = \frac{1}{2}$, except at $\alpha = \frac{1}{2}$. However, in general, μ cannot be extended analytically across this line; in fact the derivative $\mu'(\alpha)$ may diverge as α approaches this line. In particular, Theorem 1.20(i) does not hold (in general) without the extra moment assumption there.

Theorem 11.1. *There exists ξ with $\mathbb{E}\,\xi = 1$ and $0 < \operatorname{Var}\xi < \infty$ such that for any α_0 with $\operatorname{Re}\alpha_0 = \frac{1}{2}$, $\limsup_{\alpha\to\alpha_0,\,\operatorname{Re}\alpha<\frac{1}{2}} |\mu'(\alpha)| = \infty$. In particular, $\mu(\alpha)$ has no analytic extension in a neighborhood of any such α_0. In other words, the line $\operatorname{Re}\alpha = \frac{1}{2}$ is a natural boundary for $\mu(\alpha)$.*

We shall first prove three lemmas. Instead of working with $\mu(\alpha)$ directly, we shall use Lemma 10.2 (and, for convenience, Remark 10.3). We define, for any function $\rho(t)$ and a complex α,

$$F(\rho;\alpha) := \int_{-1}^{1} \rho(t)^{-\alpha}\,\mathrm{d}t. \tag{11.1}$$

Note that if $\rho(t) \geqslant ct^2$ (as will be the case below), then this integral is finite for $\operatorname{Re}\alpha < \frac{1}{2}$, at least, and defines an analytic function there. If $F(\rho;\alpha)$ extends analytically to a larger domain, we will use the same notation for the extension (even if the integral (11.1) diverges).

If $\rho(t) = 1 - \mathbb{E}\,e^{\mathrm{i}t\tilde{\xi}}$ as in (10.3), we also write $F(\xi;\alpha)$.

By Lemma 10.2 and Remark 10.3, Theorem 11.1 follows if we prove the statement with $\mu(\alpha)$ replaced by $F(\xi;\alpha)$.

We define in this section the domains $D_0 := \{\alpha : \frac{1}{4} < \operatorname{Re}\alpha < \frac{3}{4}\}$, $D_- := \{\alpha : \frac{1}{4} < \operatorname{Re}\alpha < \frac{1}{2}\}$ and $D^* := D_0\backslash\{\frac{1}{2}\}$. (These choices are partly for convenience; we could take D_0 larger.)

If (g_N) is a sequence of functions in a domain D, we write $O_{\mathcal{H}(D)}(g_N(\alpha))$ for any sequence of functions $f_N \in \mathcal{H}(D)$ such that $f_N(\alpha)/g_N(\alpha)$ is bounded on each compact $K \subset D$, uniformly in N. (Often, $g_N(\alpha)$ will not depend on α.) We extend the definition to functions $g_{N,t}(\alpha)$ and $f_{N,t}(\alpha)$ depending also on an additional parameter t, requiring uniformity also in t.

It will be convenient to work with a restricted set of offspring distributions ξ. Let $\mathcal{P}_1$ be the set of all probability distributions $(p_k)_0^\infty$ on $\{0,1,2,\dots\}$ such that $p_0, p_1, p_2 > 0.1$, and if ξ has the distribution $(p_k)_0^\infty$, then $\mathbb{E}\,\xi = \sum_k kp_k = 1$, $\operatorname{Var}\xi = \sum_k (k-1)^2 p_k = 2$ and $\mathbb{E}\,\xi^3 = \sum_k k^3 p_k < \infty$. (The set $\mathcal{P}_1$ is clearly non-empty. A concrete example is $(0.52, 0.2, 0.2, 0, 0, 0.08)$.) We write $\xi \in \mathcal{P}_1$ for $\mathcal{L}(\xi) \in \mathcal{P}_1$.

If $\xi \in \mathcal{P}_1$, then $\sigma^2 = 2$ and $\mathbb{E}\,\xi^3 < \infty$, and thus $\tilde\varphi(t) = 1 - t^2 + O(t^3)$; hence $\rho(t) = t^2 + O(t^3)$. Moreover, since $\mathbb{P}(\tilde\xi = j) > 0.1$ for $j = \pm 1$, we have

$$\operatorname{Re}\rho(t) = \operatorname{Re}\bigl(1 - \mathbb{E}\,e^{\mathrm{i}t\tilde\xi}\bigr) = \mathbb{E}\bigl(1 - \cos t\tilde\xi\bigr) \geqslant 0.2(1-\cos t) \geqslant ct^2, \tag{11.2}$$

for $|t| \leqslant \pi$, uniformly for all $\xi \in \mathcal{P}_1$.

Lemma 11.2. *If $\xi \in \mathcal{P}_1$, then $F(\xi;\alpha)$ extends to a function in $\mathcal{H}(D^*)$.*

Proof. $\mu(\alpha) \in \mathcal{H}(D^*)$ by Theorem 1.20(i) (or Theorem 6.5), and the result follows by Lemma 10.2 and Remark 10.3. ☐

Lemma 11.3. *If $\xi_N \in \mathcal{P}_1$ for $N \geqslant 1$ and $\xi_N \xrightarrow{\mathrm{d}} \xi$, then $F(\xi_N;\alpha) \to F(\xi;\alpha)$ in $\mathcal{H}(D_-)$.*

Note that we do not assume $\xi \in \mathcal{P}_1$. (In fact, it is easy to see that the lemma extends to arbitrary ξ_N and ξ with expectation 1 and finite, non-zero variance.)

Proof. Let $\rho(t) := 1 - \mathbb{E}\,e^{\mathrm{i}t\tilde\xi}$ and $\rho_N(t) := 1 - \mathbb{E}\,e^{\mathrm{i}t\tilde\xi_N}$, where as usual $\tilde\xi := \xi - 1$ and $\tilde\xi_N := \xi_N - 1$. Since $\xi_N \xrightarrow{\mathrm{d}} \xi$, $\rho_N(t) \to \rho(t)$ for every t. Lemma 2.7 together with the estimate (11.2) show that $F(\xi_N;\alpha) = F(\rho_N;\alpha) \to F(\xi;\alpha)$ uniformly on every compact subset of D_-. ☐

Lemma 11.4. *If $\xi \in \mathcal{P}_1$ and $y \in \mathbb{R}\setminus\{0\}$, then there exists a sequence $\xi_N \in \mathcal{P}_1$, $N \geqslant 1$, such that, as $N \to \infty$, $\xi_N \xrightarrow{\mathrm{d}} \xi$ and $\bigl|\frac{\mathrm{d}}{\mathrm{d}\alpha}F(\xi_N;\alpha)\bigr|_{\alpha=\frac12+\mathrm{i}y} \to \infty$ for any fixed real $y \neq 0$.*

Proof. Let $a_N := (\log N)^{-1/2}$ and let ξ_N have the distribution

$$\mathcal{L}(\xi_N) = \mathcal{L}(\xi) + a_N\Bigl[\frac{2}{N^2}\bigl(\delta_N - N\delta_1 + (N-1)\delta_0\bigr) - \frac{N-1}{N}\bigl(\delta_2 - 2\delta_1 + \delta_0\bigr)\Bigr] \tag{11.3}$$

where δ_j is unit mass at j. Since $a_N \to 0$, and $\mathbb{P}(\xi = j) > 0.1 > 0$ for $j = 0,1,2$, this is clearly a probability distribution if N is large enough. Furthermore, $\xi_N \xrightarrow{\mathrm{d}} \xi$ as $N \to \infty$, and $\mathbb{E}\,\xi_N = \mathbb{E}\,\xi$, $\mathbb{E}\,\xi_N^2 = \mathbb{E}\,\xi^2$ and $\mathbb{E}\,\xi_N^3 < \infty$, and thus $\xi_N \in \mathcal{P}_1$, provided N is large enough. (We assume in the rest of this proof that N is large enough whenever necessary, without further mention. We can define ξ_N arbitrarily for small N.)

Let $\varphi_N(t) := \mathbb{E}\, e^{\mathrm{i}t\xi_N}$ and, recalling (10.24)–(10.25),

$$\begin{aligned}\Delta_N(t) &:= \varphi_N(t) - \varphi(t) = \mathbb{E}\, e^{\mathrm{i}t\xi_N} - \mathbb{E}\, e^{\mathrm{i}t\xi}\\ &= a_N\Big[\frac{2}{N^2}\big(e^{\mathrm{i}Nt} - Ne^{\mathrm{i}t} + N - 1\big) - \frac{N-1}{N}\big(e^{2\mathrm{i}t} - 2e^{\mathrm{i}t} + 1\big)\Big]\\ &= a_N\Big[\frac{2}{N^2}\big(\psi_1(Nt) - N\psi_1(t)\big) - \frac{N-1}{N}\big((\mathrm{i}t)^2 + O(t^3)\big)\Big]\\ &= 2a_N N^{-2}\big(\psi_2(Nt) - N\psi_2(t)\big) + O(a_N t^3)\\ &= 2a_N N^{-2}\psi_2(Nt) + O(a_N t^3), \end{aligned} \tag{11.4}$$

since $\psi_2(x) = O(x^3)$. We further define

$$\widetilde{\psi}(t) := 2\frac{\psi_2(t)}{t^2} = \frac{2e^{\mathrm{i}t} - 2 - 2\mathrm{i}t + t^2}{t^2} = 2\frac{\psi_1(t)}{t^2} + 1. \tag{11.5}$$

Then $\widetilde{\psi}$ is bounded and continuous on $\mathbb{R}$, $\widetilde{\psi}(t) = O(t)$ and $\widetilde{\psi}(t) = 1 + O(t^{-1})$. Furthermore, (11.4) yields

$$\Delta_N(t) = a_N t^2 \widetilde{\psi}(Nt) + O(a_N t^3). \tag{11.6}$$

In particular, $\Delta_N(t) = O(a_N t^2)$ for $|t| \leqslant \pi$.

We further let $\widetilde{\varphi}_N(t) := \mathbb{E}\, e^{\mathrm{i}t\tilde{\xi}_N}$, $\rho_N(t) := 1 - \widetilde{\varphi}_N(t)$ and, using (11.6),

$$\tilde{\Delta}_N(t) := \rho_N(t) - \rho(t) = -e^{-\mathrm{i}t}\Delta_N(t) = -a_N t^2\widetilde{\psi}(Nt) + O(a_N t^3). \tag{11.7}$$

In particular,

$$\tilde{\Delta}_N(t) = O(a_N t^2), \qquad |t| \leqslant \pi. \tag{11.8}$$

Let $\rho_0(t) := t^2$, and let $\delta(t) := \rho(t) - \rho_0(t)$. Then $\delta(t) = O(t^3)$, since $\operatorname{Var}\xi = 2$ and $\mathbb{E}\,\xi^3 < \infty$. The general formula, for any twice continuously differentiable function f,

$$f(x+y+z) - f(x+y) - f(x+z) + f(x) = yz\int_0^1\int_0^1 f''(x + sy + tz)\,\mathrm{d}s\,\mathrm{d}t$$

implies together with (11.2) and (11.8), for $\alpha \in D_0$,

$$\begin{aligned}\big|(\rho(t) + \tilde{\Delta}_N(t))^{-\alpha} - \rho(t)^{-\alpha} - \big((\rho_0(t) + \tilde{\Delta}_N(t))^{-\alpha} - \rho_0(t)^{-\alpha}\big)\big|&\\ \leqslant C|\tilde{\Delta}_N(t)|\,|\delta(t)|\,|\alpha|\,|\alpha+1|\,|t|^{-2(\operatorname{Re}\alpha+2)}&\\ = O\big(a_N|\alpha|^2|t|^{1-2\operatorname{Re}\alpha}\big).& \end{aligned} \tag{11.9}$$

Hence, integrating over t and recalling (11.1),

$$F(\rho + \tilde{\Delta}_N;\alpha) - F(\rho;\alpha) - \big(F(\rho_0 + \tilde{\Delta}_N;\alpha) - F(\rho_0;\alpha)\big) = O_{\mathcal{H}(D_0)}(a_N). \tag{11.10}$$

Next, let $\Delta^*_N(t) := -a_N t^2\widetilde{\psi}(Nt)$. Then $\tilde{\Delta}_N(t) - \Delta^*_N(t) = O(a_N t^3)$ by (11.7), and thus, by the mean value theorem and (11.8), for $|t| \leqslant \pi$,

$$\begin{aligned}\big|(\rho_0(t) + \tilde{\Delta}_N(t))^{-\alpha} - (\rho_0(t) + \Delta^*_N(t))^{-\alpha}\big| &\leqslant C|\tilde{\Delta}_N(t) - \Delta^*_N(t)|\,|\alpha|\,|t|^{-2(\operatorname{Re}\alpha+1)}\\ &= O\big(a_N|\alpha||t|^{1-2\operatorname{Re}\alpha}\big).\end{aligned}$$

Hence, by an integration,

$$F(\rho_0 + \tilde{\Delta}_N;\alpha) - F(\rho_0 + \Delta^*_N;\alpha) = O_{\mathcal{H}(D_0)}(a_N). \tag{11.11}$$

Now consider $F(\rho_0+\Delta_N^*;\alpha)-F(\rho_0;\alpha)$. Let $\chi(t) := \mathbf{1}_{|t|>1}$. Then, considering first $t>0$, for $\alpha \in D_-$,

$$\begin{aligned}
\int_0^1 \bigl[(\rho_0(t)+\Delta_N^*(t))^{-\alpha}-\rho_0(t)^{-\alpha}\bigr]\,\mathrm{d}t &= \int_0^1 t^{-2\alpha}\bigl[(1-a_N\widetilde{\psi}(Nt))^{-\alpha}-1\bigr]\,\mathrm{d}t \\
&= \int_0^1 t^{-2\alpha}\bigl[(1-a_N\widetilde{\psi}(Nt))^{-\alpha}-(1-a_N\chi(Nt))^{-\alpha}\bigr]\,\mathrm{d}t \\
&\qquad + \bigl[(1-a_N)^{-\alpha}-1\bigr]\int_{1/N}^1 t^{-2\alpha}\,\mathrm{d}t \\
&= N^{2\alpha-1}\int_0^N t^{-2\alpha}\bigl[(1-a_N\widetilde{\psi}(t))^{-\alpha}-(1-a_N\chi(t))^{-\alpha}\bigr]\,\mathrm{d}t \\
&\qquad + \bigl[(1-a_N)^{-\alpha}-1\bigr]\frac{1}{1-2\alpha}\bigl(1-N^{2\alpha-1}\bigr). \qquad (11.12)
\end{aligned}$$

Since $\widetilde{\psi}(t)-\chi(t) = O\bigl(|t|\wedge|t^{-1}|\bigr)$, and $a_N\chi(t)=O(a_N)=o(1)$, with $\chi(t)=0$ for $0\leqslant t<1$, a Taylor expansion yields, uniformly for $t\in\mathbb{R}$,

$$\begin{aligned}
&(1-a_N\widetilde{\psi}(t))^{-\alpha}-(1-a_N\chi(t))^{-\alpha} \\
&\qquad = \alpha a_N\bigl(\widetilde{\psi}(t)-\chi(t)\bigr)+O_{\mathcal{H}(D_0)}\bigl(a_N|\widetilde{\psi}(t)-\chi(t)|(a_N|\widetilde{\psi}(t)|+a_N\chi(t))\bigr) \\
&\qquad = \alpha a_N\bigl(\widetilde{\psi}(t)-\chi(t)\bigr)+O_{\mathcal{H}(D_0)}\bigl(a_N^2(|t|^2\wedge|t|^{-1})\bigr). \qquad (11.13)
\end{aligned}$$

Using (11.13) and a Taylor expansion of $(1-a_N)^{-\alpha}$ in (11.12), we obtain for $\alpha\in D_-$,

$$\begin{aligned}
&\int_0^1 \bigl[(\rho_0(t)+\Delta_N^*(t))^{-\alpha}-\rho_0(t)^{-\alpha}\bigr]\,\mathrm{d}t \\
&\qquad = N^{2\alpha-1}\int_0^N t^{-2\alpha}\alpha a_N\bigl(\widetilde{\psi}(t)-\chi(t)\bigr)\,\mathrm{d}t-\frac{\alpha a_N}{1-2\alpha}N^{2\alpha-1} \\
&\qquad\qquad + O_{\mathcal{H}(D*)}\bigl(a_N^2N^{2\alpha-1}\bigr)+O_{\mathcal{H}(D*)}(a_N). \qquad (11.14)
\end{aligned}$$

Furthermore, using again $\widetilde{\psi}(t)-\chi(t)=O\bigl(|t^{-1}|\bigr)$,

$$\int_N^\infty t^{-2\alpha}\bigl(\widetilde{\psi}(t)-\chi(t)\bigr)\,\mathrm{d}t = O\bigl(N^{-2\operatorname{Re}\alpha}\bigr), \qquad (11.15)$$

so we may as well integrate to ∞ on the right-hand side of (11.14).

For $\alpha\in D_-$, recalling (11.5),

$$\int_0^\infty t^{-2\alpha}\bigl(\widetilde{\psi}(t)-\chi(t)\bigr)\,\mathrm{d}t = 2\int_0^\infty \psi_1(t)t^{-2\alpha-2}\,\mathrm{d}t+\int_0^1 t^{-2\alpha}\,\mathrm{d}t \qquad (11.16)$$

Furthermore, if $\alpha\in D_-$ and $\operatorname{Re}\zeta\geqslant 0$, then

$$\int_0^\infty \bigl(e^{-\zeta t}-1+\zeta t\bigr)t^{-2\alpha-2}\,\mathrm{d}t = \zeta^{2\alpha+1}\Gamma(-2\alpha-1); \qquad (11.17)$$

the case $\zeta=1$ is well known [47, 5.9.5], the case $\zeta>0$ follows by a change of variables, the case $\operatorname{Re}\zeta>0$ follows by analytic continuation, and the case $\operatorname{Re}\zeta\geqslant 0$

follows by continuity. Recalling (10.24), we take $\zeta = -\mathrm{i}$ in (11.17), and obtain from (11.16), for $\alpha \in D_-$.

$$\int_0^\infty t^{-2\alpha}\big(\widetilde{\psi}(t) - \chi(t)\big)\,\mathrm{d}t = 2(-\mathrm{i})^{2\alpha+1}\Gamma(-2\alpha-1) + \frac{1}{1-2\alpha}. \tag{11.18}$$

Combining (11.14)–(11.15) and (11.18), we obtain (for $\alpha \in D_-$)

$$\begin{aligned}\int_0^1 \big[(\rho_0(t) + \Delta_N^*(t))^{-\alpha} - \rho_0(t)^{-\alpha}\big]\,\mathrm{d}t &= 2\alpha(-\mathrm{i})^{2\alpha+1}\Gamma(-2\alpha-1)a_N N^{2\alpha-1} \\ &\quad + O_{\mathcal{H}(D^*)}\big(a_N^2 N^{2\alpha-1}\big) + O_{\mathcal{H}(D^*)}(a_N).\end{aligned} \tag{11.19}$$

The integral over $(-1,0)$ yields the same result with $(-\mathrm{i})^{2\alpha+1}$ replaced by $\mathrm{i}^{2\alpha+1}$, e.g. by conjugating (11.19) and α. Consequently,

$$\begin{aligned}F(\rho_0 + \Delta_N^*;\alpha) - F(\rho_0;\alpha) &= 2\alpha\big(\mathrm{i}^{2\alpha+1} + (-\mathrm{i})^{2\alpha+1}\big)\Gamma(-2\alpha-1)a_N N^{2\alpha-1} \\ &\quad + O_{\mathcal{H}(D^*)}\big(a_N^2 N^{2\alpha-1}\big) + O_{\mathcal{H}(D^*)}(a_N).\end{aligned} \tag{11.20}$$

For convenience, we write

$$G(\alpha) := 2\alpha\big(\mathrm{i}^{2\alpha+1} + (-\mathrm{i})^{2\alpha+1}\big)\Gamma(-2\alpha-1) = 2\alpha\big(\mathrm{i}e^{\mathrm{i}\pi\alpha} - \mathrm{i}e^{-\mathrm{i}\pi\alpha}\big)\Gamma(-2\alpha-1). \tag{11.21}$$

Combining (11.10), (11.11), and (11.20) yield, for $\alpha \in D_-$,

$$\begin{aligned}F(\rho_N;\alpha) = F(\rho + \tilde{\Delta}_N;\alpha) &= F(\rho;\alpha) + a_N G(\alpha) N^{2\alpha-1} \\ &\quad + O_{\mathcal{H}(D^*)}\big(a_N^2 N^{2\alpha-1}\big) + O_{\mathcal{H}(D^*)}(a_N).\end{aligned} \tag{11.22}$$

By Lemma 11.2, all terms in (11.22) are analytic in D^*, and thus (11.22) holds for $\alpha \in D^*$.

Note that if f_N and g_N are functions such that $f_N(\alpha) = O_{\mathcal{H}(D^*)}(g_N(\alpha))$, then $f_N(\alpha) = g_N(\alpha)h_N(\alpha)$ with $h_N(\alpha) = O_{\mathcal{H}(D^*)}(1)$. By Cauchy's estimate, $h_N'(\alpha) = O_{\mathcal{H}(D^*)}(1)$, and it follows that $f_N'(\alpha) = O_{\mathcal{H}(D^*)}(g_N(\alpha)) + O_{\mathcal{H}(D^*)}(g_N'(\alpha))$. Hence, taking derivatives in (11.22) and then putting $\alpha = \frac{1}{2} + \mathrm{i}y$ for a fixed $y \neq 0$ yields

$$\begin{aligned}F'(\rho_N;\alpha) &= F'(\rho;\alpha) + 2(\log N)a_N G(\alpha)N^{2\alpha-1} + O\big(a_N^2 \log N\big) + O(a_N) \\ &= 2G(\alpha)(\log N)a_N N^{2\mathrm{i}y} + O(1).\end{aligned} \tag{11.23}$$

Since $G(\alpha) = -4\alpha(\cosh \pi y)\Gamma(-2-2y\mathrm{i}) \neq 0$, $|N^{2\mathrm{i}y}| = 1$ and $a_N \log N = (\log N)^{1/2} \to \infty$, (11.23) shows that $|F'(\xi_N; \frac{1}{2} + \mathrm{i}y)| = |F'(\rho_N; \frac{1}{2} + \mathrm{i}y)| \to \infty$ as $N \to \infty$. ☐

Proof of Theorem 11.1. Let $(y_n)_1^\infty$ be an enumeration of all non-zero rational numbers. We shall construct sequences $x_n \in (\frac{1}{4}, \frac{1}{2})$ and $\xi_n \in \mathcal{P}_1$, $n = 1, 2, \dots$, such that, with $z_n := x_n + \mathrm{i}y_n \in D_-$,

$$|F'(\xi_n, z_k)| > k, \qquad k = 1, \dots, n, \tag{11.24}$$

and, furthermore, the total variation distance

$$d_{\mathrm{TV}}(\xi_n, \xi_{n-1}) < 2^{-n}. \tag{11.25}$$

We construct the sequences inductively. Suppose that ξ_{n-1} is constructed. (For $n = 1$, we let ξ_0 be any element of $\mathcal{P}_1$.) By Lemma 11.4, there exists a sequence $\xi_{n-1,N} \in \mathcal{P}_1$ such that, as $N \to \infty$, $\xi_{n-1,N} \xrightarrow{\mathrm{d}} \xi_{n-1}$ and $|F'(\xi_{n-1,N}; \frac{1}{2} + \mathrm{i}y_n)| \to \infty$. By Lemma 11.3, then $F(\xi_{n-1,N}; \alpha) \to F(\xi_{n-1}; \alpha)$ in $\mathcal{H}(D_-)$. This implies $F'(\xi_{n-1,N}; \alpha) \to F'(\xi_{n-1}; \alpha)$ in $\mathcal{H}(D_-)$, and in particular, $F'(\xi_{n-1,N}; z_k) \to F'(\xi_{n-1}; z_k)$ for $1 \leqslant k \leqslant n-1$. Since (11.24) holds for $n-1$ by the induction hypothesis, it follows that $|F'(\xi_{n-1,N}; z_k)| > k$ for $1 \leqslant k \leqslant n-1$ for all large N. Furthermore, if we choose N large enough, $|F'(\xi_{n-1,N}; \frac{1}{2} + \mathrm{i}y_n)| > n$ and $d_{\mathrm{TV}}(\xi_{n-1,N}, \xi_{n-1}) < 2^{-n}$.

We choose a large N such that these properties hold and let $\xi_n := \xi_{n-1,N}$. Then (11.24) holds for $k = 1, \dots, n-1$. Furthermore, since $\xi_n \in \mathcal{P}_1$, $F(\xi_n; \alpha) \in \mathcal{H}(D^*)$, and thus $F'(\xi_n; \alpha)$ is continuous in D^*. Hence $|F'(\xi_n; x + \mathrm{i}y_n)| \to |F'(\xi_n; \frac{1}{2} + \mathrm{i}y_n)|$ as $x \to \frac{1}{2}$, and we can choose $x_n \in (\frac{1}{4}, \frac{1}{2})$ with $\frac{1}{2} - x_n < \frac{1}{n}$ such that $|F'(\xi_n; x + \mathrm{i}y_n)| > n$.

This completes the construction of x_n and ξ_n. By (11.25), the distributions $\mathcal{L}(\xi_n)$ form a Cauchy sequence in total variation distance, so there exists a random variable ξ with $\xi_n \xrightarrow{\mathrm{d}} \xi$. Clearly, ξ is non-negative and integer-valued. Moreover, since $\xi_n \in \mathcal{P}_1$ we have $\mathbb{E}\,\xi_n^2 = \operatorname{Var}\xi_n + (\mathbb{E}\,\xi_n)^2 = 3$, for every n, and thus the sequence ξ_n is uniformly integrable, so $\mathbb{E}\,\xi = \lim_{n\to\infty} \mathbb{E}\,\xi_n = 1$. Furthermore, by Fatou's lemma, $\mathbb{E}\,\xi^2 \leqslant 3 < \infty$. Note that ξ does not necessarily belong to $\mathcal{P}_1$; in fact, it is easily seen from (11.26) below that $\xi \notin \mathcal{P}_1$. Nevertheless (11.2) holds for every ξ_n (with the same c) and thus (11.2) holds for ξ too. In particular $\mathbb{P}(\xi \neq 1) > 0$ so $\operatorname{Var}\xi > 0$.

Lemma 11.3 shows that $F(\xi_n; \alpha) \to F(\xi; \alpha)$ in $\mathcal{H}(D_-)$, and thus $F'(\xi_n; \alpha) \to F'(\xi; \alpha)$ for every $\alpha \in D_-$. Hence, (11.24) implies

$$|F'(\xi; z_k)| \geqslant k \tag{11.26}$$

for every k. Thus, $|F'(\xi; z_n)| \to \infty$ as $n \to \infty$.

Now take any $y \in \mathbb{R}$ and let $\alpha_0 := \frac{1}{2} + \mathrm{i}y$. There is an infinite number of points y_n in each neighborhood of y, so we can find a subsequence converging to y. Since $x_n \to \frac{1}{2}$, it follows that there is a subsequence of $z_n = x_n + \mathrm{i}y_n$ that converges to α_0. Suppose first that $y \neq 0$, so $\alpha_0 \neq \frac{1}{2}$. Then it follows from Lemma 10.2 (with Remark 10.3) and Theorem 10.7 that, as $n \to \infty$ along the subsequence,

$$\mu'(z_n) = \frac{1}{2\pi}\Gamma(z_n)F'(\xi; z_n) + O(1) \tag{11.27}$$

and thus, by (11.26), $|\mu'(z_n)| \to \infty$.

This proves the claim in Theorem 11.1 for every α_0 with $\operatorname{Re}\alpha_0 = \frac{1}{2}$ and $\alpha_0 \neq \frac{1}{2}$. The case $\alpha_0 = \frac{1}{2}$ follows easily, either by noting that the set of α_0 for which the claim holds is closed, or simply by (1.13). □

12. Moments

In this section we prove Theorems 1.3 and 1.12 on moments of $X_n(\alpha)$ and of the limits $Y(\alpha)$. The section is largely based on Fill and Kapur [19] and [21], and uses the methods of [17], also presented in [26, Section VI.10].

We assume for simplicity throughout this section that ξ has span 1. The general case follows by minor modifications of standard type.

12.1. **More notation and preliminaries.** Recall that $\mathcal{T}$ is the random Galton–Watson tree defined by the offspring distribution ξ. Let $p_k := \mathbb{P}(\xi = k)$ denote the values of the probability mass function for ξ, and let Φ be its probability generating function:

$$\Phi(z) := \mathbb{E}\, z^{\xi} = \sum_{k=0}^{\infty} p_k z^k. \tag{12.1}$$

Similarly, let $q_n := \mathbb{P}(|\mathcal{T}| = n)$, and let y denote the corresponding probability generating function:

$$y(z) := \mathbb{E}\, z^{|\mathcal{T}|} = \sum_{n=1}^{\infty} \mathbb{P}\bigl(|\mathcal{T}| = n\bigr) z^n = \sum_{n=1}^{\infty} q_n z^n. \tag{12.2}$$

If $\mathcal{T}$ has root degree k, denote the subtrees rooted at the children of the root by $\mathcal{T}_1, \dots, \mathcal{T}_k$; note that, conditioned on k, these are independent copies of $\mathcal{T}$. By conditioning on the root degree, we thus obtain the standard formula

$$\begin{aligned} y(z) &= \sum_{k=0}^{\infty} p_k\, \mathbb{E}\bigl[z^{1+|\mathcal{T}_1|+\dots+|\mathcal{T}_k|}\bigr] = \sum_{k=0}^{\infty} p_k z \bigl(\mathbb{E}[z^{|\mathcal{T}|}]\bigr)^k = z \sum_{k=0}^{\infty} p_k y(z)^k \\ &= z\Phi\bigl(y(z)\bigr). \end{aligned} \tag{12.3}$$

A Δ-domain is a complex domain of the type

$$\{z : |z| < R,\ z \neq 1,\ |\arg(z-1)| > \theta\} \tag{12.4}$$

where $R > 1$ and $0 < \theta < \pi/2$, see [26, Section VI.3]. A function is Δ-*analytic* if it is analytic in some Δ-domain (or can be analytically continued to such a domain). Under our standing assumptions $\mathbb{E}\,\xi = 1$ and $0 < \operatorname{Var}\xi < \infty$, the generating function $y(z)$ is Δ-analytic; moreover, as $z \to 1$ in some Δ-domain,

$$y(z) = 1 - \sqrt{2}\sigma^{-1}(1-z)^{1/2} + o\bigl(|1-z|^{1/2}\bigr), \tag{12.5}$$

see [35, Lemma A.2]. This is perhaps more well-known if ξ has some exponential moment, and then (12.5) may be improved to a full asymptotic expansion, and in particular

$$y(z) = 1 - \sqrt{2}\sigma^{-1}(1-z)^{1/2} + O\bigl(|1-z|\bigr), \tag{12.6}$$

see e.g. [26, Theorem VI.6]. In fact, (12.6) holds provided only $\mathbb{E}\,\xi^3 < \infty$. This follows easily from (12.3), see Lemma 12.15.

In the present section, asymptotic estimates similar to (12.5) and (12.6) should always be interpreted as holding when $z \to 1$ in a suitable Δ-domain, even when not said so explicitly; the domain may be different each time.

Remark 12.1. In most parts of the present section, we will only use the assumption $\mathbb{E}\,\xi^2 < \infty$ and the general (12.5). If we assume the $\mathbb{E}\,\xi^3 < \infty$, and thus (12.6) holds, then the error estimates below can be improved, and explicit error estimates can be obtained in Theorem 1.12; see [21] where this is done in detail for a special ξ using similar arguments. In fact, it can be checked that if $\mathbb{E}\,\xi^3 < \infty$, then all o terms in the proof below can be shown to be of (at most) the same order as the bounds given in

[21] for the corresponding terms. Further, when ξ has an exponential moment, a full asymptotic expansion of the mean is derived in [17, Section 5.2]; it seems possible that this can be extended to higher moments, but we have not pursued this. □

In some formulas below, certain unspecified polynomials appear as "error terms". (These are best regarded as polynomials in $1-z$.) Let $\mathcal{P}$ be the set of all polynomials, and, for any real a, let

$$\mathcal{P}_a := \{P(z) \in \mathcal{P} : \deg(P(z)) < a\}. \tag{12.7}$$

Note that if $a \leqslant 0$, then $\mathcal{P}_a = \{0\}$, and thus terms in $\mathcal{P}_a$ vanish and can be ignored. In the formulas below, a restriction of the type $P(z) \in \mathcal{P}_a$, i.e., $\deg(P(z)) < a$, will always be a triviality, since higher powers of $1-z$ can be absorbed in an O or o term.

Recall that the polylogarithm function is defined, for $\alpha \in \mathbb{C}$, by

$$\mathrm{Li}_\alpha(z) := \sum_{n=1}^{\infty} n^{-\alpha} z^n, \qquad |z| < 1; \tag{12.8}$$

see [26, Section VI.8], [47, §25.12], or Appendix B. It is well known that $\mathrm{Li}_\alpha(z)$ is Δ-analytic; in fact, it can be analytically continued to $\mathbb{C}\backslash[1,\infty)$. Moreover, if $\alpha \notin \{1,2,\dots\}$, then, as $z \to 1$,

$$\mathrm{Li}_\alpha(z) = \Gamma(1-\alpha)(1-z)^{\alpha-1} + P(z) + O\bigl(|1-z|^{\operatorname{Re}\alpha}\bigr), \qquad P(z) \in \mathcal{P}_{\operatorname{Re}\alpha}, \tag{12.9}$$

see [26, Theorem VI.7] or [23], where a complete asymptotic expansion is given; see also Appendix B. In particular, if $\operatorname{Re}\alpha \leqslant 0$, then $P(z)$ vanishes and so (12.9) simplifies.

Recall also that the *Hadamard product* $A(z) \odot B(z)$ of two power series $A(z) = \sum_{n=0}^{\infty} a_n z^n$ and $B(z) = \sum_{n=0}^{\infty} b_n z^n$ is defined by

$$A(z) \odot B(z) := \sum_{n=0}^{\infty} a_n b_n z^n. \tag{12.10}$$

As a simple example, for any complex α and β,

$$\mathrm{Li}_\alpha(z) \odot \mathrm{Li}_\beta(z) = \mathrm{Li}_{\alpha+\beta}(z). \tag{12.11}$$

We will use some results on Hadamard products, essentially taken from [17]. In the next lemma, Part (i) is [17, Propositions 9 and 10(i)], and (ii) follows by the same arguments; the proof of Δ-analyticity of the Hadamard product given for [17, Proposition 9] holds for any Δ-analytic functions. (For the case $a+b+1 \in \{0,1,2,\dots\}$, see [17] and [26].)

Lemma 12.2 ([17])**.** *If $g(z)$ and $h(z)$ are Δ-analytic, then $g(z) \odot h(z)$ is Δ-analytic. Moreover, suppose that a and b are real with $a+b+1 \notin \{0,1,2,\dots\}$; then the following holds, as $z \to 1$ in a suitable Δ-domain.*

(i) *If $g(z) = O(|1-z|^a)$ and $h(z) = O(|1-z|^b)$, then*

$$g(z) \odot h(z) = P(z) + O\bigl(|1-z|^{a+b+1}\bigr), \qquad P(z) \in \mathcal{P}_{a+b+1}. \tag{12.12}$$

(ii) *If* $g(z) = O(|1-z|^a)$ *and* $h(z) = o(|1-z|^b)$*, then*
$$g(z) \odot h(z) = P(z) + o\bigl(|1-z|^{a+b+1}\bigr), \qquad P(z) \in \mathcal{P}_{a+b+1}. \tag{12.13}$$

The next lemma is a simplified version of [17, Proposition 8]; that proposition gives (when $\alpha, \beta, \alpha+\beta \notin \mathbb{Z}$) a complete asymptotic expansion, and in particular a more explicit error term for our (12.14).

Lemma 12.3 ([17])**.** *Suppose that* $\operatorname{Re}\alpha + \operatorname{Re}\beta + 1 \notin \{0, 1, 2, \dots\}$*. Then, as* $z \to 1$ *in a suitable* Δ*-domain,*
$$\begin{aligned}(1-z)^\alpha \odot (1-z)^\beta &\\ = \frac{\Gamma(-\alpha-\beta-1)}{\Gamma(-\alpha)\Gamma(-\beta)}(1-z)^{\alpha+\beta+1} &+ P(z) + o\bigl(|1-z|^{\operatorname{Re}\alpha+\operatorname{Re}\beta+1}\bigr),\\ &P(z) \in \mathcal{P}_{\operatorname{Re}\alpha+\operatorname{Re}\beta+1}.\end{aligned} \tag{12.14}$$

Proof. The case when none of $\alpha, \beta, \alpha+\beta$ is an integer is part of [17, Proposition 8].

In general, we use arguments from [17]. If neither α nor β is a non-negative integer, the result follows easily from (12.9), (12.11), and Lemma 12.2, which then imply that
$$\begin{aligned}&\Gamma(-\alpha)(1-z)^\alpha \odot \Gamma(-\beta)(1-z)^\beta\\ &= \bigl(\operatorname{Li}_{\alpha+1}(z) + P_1(z) + o\bigl(|1-z|^{\operatorname{Re}\alpha}\bigr)\bigr) \odot \bigl(\operatorname{Li}_{\beta+1}(z) + P_2(z) + o\bigl(|1-z|^{\operatorname{Re}\beta}\bigr)\bigr)\\ &= \operatorname{Li}_{\alpha+\beta+2}(z) + P_3(z) + o\bigl(|1-z|^{\operatorname{Re}\alpha+\operatorname{Re}\beta+1}\bigr)\\ &= \Gamma(-\alpha-\beta-1)(1-z)^{\alpha+\beta+1} + P_4(z) + o\bigl(|1-z|^{\operatorname{Re}\alpha+\operatorname{Re}\beta+1}\bigr),\end{aligned} \tag{12.15}$$
where $P_i(z)$ are polynomials. [Note that $P(z) \odot f(z)$ is a polynomial for any polynomial P and analytic f, and that we may assume $\deg(P_4(z)) < \operatorname{Re}\alpha + \operatorname{Re}\beta + 1$ by the comment after (12.7).]

Finally, if α is a non-negative integer, then $(1-z)^\alpha$ is a polynomial and thus the left-hand side of (12.14) is a polynomial, so (12.14) holds trivially [with $1/\Gamma(-\alpha) = 0$]. The same holds if β is a non-negative integer. ∎

12.2. **Generating functions.** Let $(b_n)_1^\infty$ be a given sequence of constants and consider the toll function $f(T) := b_{|T|}$ and the corresponding additive functional $F(T)$ given by (1.1). We are mainly interested in the case $b_n = n^\alpha$, but will also consider $b_n = n^\alpha - c$ below for a suitable constant c. In the present subsection, b_n can be arbitrary if we regard the generating functions as formal power series; if we assume $b_n = O(n^K)$ for some K, then the generating functions below converge and are analytic at least in the unit disc.

We are interested in the random variable $F(\mathcal{T}_n)$. We denote its moments by
$$m_n^{(\ell)} := \mathbb{E}[F(\mathcal{T}_n)^\ell] \tag{12.16}$$
for integer $\ell \geqslant 0$. Define the generating functions
$$M_\ell(z) := \mathbb{E}\bigl[F(\mathcal{T})^\ell z^{|\mathcal{T}|}\bigr] = \sum_{n=1}^\infty q_n\, \mathbb{E}\bigl[F(\mathcal{T})^\ell z^{|\mathcal{T}|} \mid |\mathcal{T}| = n\bigr] = \sum_{n=1}^\infty q_n m_n^{(\ell)} z^n. \tag{12.17}$$

Note that $M_0(z) = y(z)$, see (12.2).

The generating functions M_ℓ can be calculated recursively as follows, using Hadamard products and the generating function

$$B(z) := \sum_{n=1}^{\infty} b_n z^n. \tag{12.18}$$

Lemma 12.4. *For every $\ell \geqslant 1$,*

$$M_\ell(z) = \frac{zy'(z)}{y(z)} \sum_{m=0}^{\ell} \frac{1}{m!} \sum\nolimits^{**} \binom{\ell}{\ell_0, \dots, \ell_m} B(z)^{\odot \ell_0} \odot \bigl[zM_{\ell_1}(z) \cdots M_{\ell_m}(z)\Phi^{(m)}\bigl(y(z)\bigr)\bigr], \tag{12.19}$$

where $\sum^{**}$ is the sum over all $(m+1)$-tuples $(\ell_0, \dots, \ell_m)$ of non-negative integers summing to ℓ such that $1 \leqslant \ell_1, \dots, \ell_m < \ell$.

Proof. Condition on the root degree k of $\mathcal{T}$, and let $\mathcal{T}_1, \dots, \mathcal{T}_k$ be the principal subtrees as at the beginning of Section 12.1. Then (1.2) can be written

$$F(\mathcal{T}) = f(\mathcal{T}) + \sum_{i=1}^{k} F(\mathcal{T}_i) = b_{|\mathcal{T}|} + \sum_{i=1}^{k} F(\mathcal{T}_i). \tag{12.20}$$

Hence, the multinomial theorem yields the following, where for each k we let $\sum$ denote the sum over all $(k+1)$-tuples $(\ell_0, \dots, \ell_k)$ summing to ℓ such that each $\ell_i \geqslant 0$, and furthermore $\mathcal{T}_1, \dots, \mathcal{T}_k$ are independent copies of $\mathcal{T}$, and $|\mathcal{T}|$ is $1+|\mathcal{T}_1|+\dots+|\mathcal{T}_k|$:

$$\begin{aligned}
M_\ell(z) &= \sum_{k=0}^{\infty} p_k \, \mathbb{E}\Bigl[z^{|\mathcal{T}|}\Bigl(b_{|\mathcal{T}|} + \sum_{i=1}^{k} F(\mathcal{T}_i)\Bigr)^{\ell}\Bigr] \\
&= \sum_{k=0}^{\infty} p_k \sum \binom{\ell}{\ell_0, \dots, \ell_k} \mathbb{E}\Bigl[z^{|\mathcal{T}|} b_{|\mathcal{T}|}^{\ell_0} F(\mathcal{T}_1)^{\ell_1} \cdots F(\mathcal{T}_k)^{\ell_k}\Bigr] \\
&= \sum_{k=0}^{\infty} p_k \sum \binom{\ell}{\ell_0, \dots, \ell_k} B(z)^{\odot \ell_0} \odot \mathbb{E}\Bigl[z^{|\mathcal{T}|} F(\mathcal{T}_1)^{\ell_1} \cdots F(\mathcal{T}_k)^{\ell_k}\Bigr] \\
&= \sum_{k=0}^{\infty} p_k \sum \binom{\ell}{\ell_0, \dots, \ell_k} B(z)^{\odot \ell_0} \odot \mathbb{E}\Bigl[z \prod_{i=1}^{k} \bigl(z^{|\mathcal{T}_i|} F(\mathcal{T}_i)^{\ell_i}\bigr)\Bigr] \\
&= \sum_{k=0}^{\infty} p_k \sum \binom{\ell}{\ell_0, \dots, \ell_k} B(z)^{\odot \ell_0} \odot \Bigl[z \prod_{i=1}^{k} \mathbb{E}\bigl[z^{|\mathcal{T}_i|} F(\mathcal{T}_i)^{\ell_i}\bigr]\Bigr] \\
&= \sum_{k=0}^{\infty} p_k \sum \binom{\ell}{\ell_0, \dots, \ell_k} B(z)^{\odot \ell_0} \odot \Bigl[z \prod_{i=1}^{k} M_{\ell_i}(z)\Bigr].
\end{aligned} \tag{12.21}$$

We consider the terms where $\ell_i = \ell$ for some $1 \leqslant i \leqslant k$ separately. In this case, $\ell_0 = 0$ and $\ell_j = 0$ for $j \neq i$, and thus the combined contribution of these k terms is, recalling $M_0(z) = y(z)$ and (12.1),

$$\sum_{k=1}^{\infty} p_k k\bigl[zM_\ell(z)y(z)^{k-1}\bigr] = zM_\ell(z)\sum_{k=1}^{\infty} p_k k y(z)^{k-1} = zM_\ell(z)\Phi'(y(z)). \tag{12.22}$$

Let $\sum^*$ denote the sum over the remaining terms, i.e., the terms with $\ell_1,\dots,\ell_k < \ell$, and define

$$R_\ell(z) := \sum_{k=0}^{\infty} p_k \sum\nolimits^* \binom{\ell}{\ell_0,\dots,\ell_k} B(z)^{\odot \ell_0} \odot \Bigl[z\prod_{i=1}^{k} M_{\ell_i}(z)\Bigr]. \tag{12.23}$$

Using (12.22)–(12.23), we can write (12.21) as

$$M_\ell(z) = z\Phi'\bigl(y(z)\bigr)M_\ell(z) + R_\ell(z). \tag{12.24}$$

Moreover, differentiating (12.3) yields

$$y'(z) = \Phi\bigl(y(z)\bigr) + z\Phi'\bigl(y(z)\bigr)y'(z) \tag{12.25}$$

and thus, using (12.3) again,

$$\bigl(1 - z\Phi'\bigl(y(z)\bigr)\bigr)y'(z) = \Phi\bigl(y(z)\bigr) = y(z)/z. \tag{12.26}$$

Hence, (12.24) yields

$$M_\ell(z) = \frac{R_\ell(z)}{1 - z\Phi'\bigl(y(z)\bigr)} = \frac{zy'(z)}{y(z)}R_\ell(z). \tag{12.27}$$

Finally, in each term in the sum $\sum^*$ in (12.23), let $m \geqslant 0$ be the number of $\ell_1,\dots,\ell_k$ that equal 0. By symmetry, we may assume that $\ell_1,\dots,\ell_m \geqslant 1$ and $\ell_{m+1} = \dots \ell_k = 0$, and multiply by the symmetry factor $\binom{k}{m}$. Thus,

$$\begin{aligned}
R_\ell(z) &= \sum_{k=0}^{\infty} p_k \sum_{m=0}^{k}\binom{k}{m}\sum\nolimits^{**}\binom{\ell}{\ell_0,\dots,\ell_m} B(z)^{\odot\ell_0}\odot\Bigl[z\Bigl(\prod_{i=1}^{m} M_{\ell_i}(z)\Bigr)y(z)^{k-m}\Bigr]\\
&= \sum_{m=0}^{\infty}\sum\nolimits^{**}\binom{\ell}{\ell_0,\dots,\ell_m} B(z)^{\odot\ell_0}\odot\Bigl[z\Bigl(\prod_{i=1}^{m} M_{\ell_i}(z)\Bigr)\sum_{k=m}^{\infty} p_k\binom{k}{m}y(z)^{k-m}\Bigr]\\
&= \sum_{m=0}^{\infty}\sum\nolimits^{**}\binom{\ell}{\ell_0,\dots,\ell_m} B(z)^{\odot\ell_0}\odot\Bigl[z\Bigl(\prod_{i=1}^{m} M_{\ell_i}(z)\Bigr)\frac{1}{m!}\Phi^{(m)}\bigl(y(z)\bigr)\Bigr].
\end{aligned} \tag{12.28}$$

The result (12.19) follows from (12.27) and (12.28), noting that the sum $\sum^{**}$ is empty if $m > \ell$. ☐

12.3. **The mean.** For $\ell = 1$, (12.19) contains only the term $m = 0$ and thus $\ell_0 = \ell = 1$. Hence, Lemma 12.4 yields, recalling (12.3),

$$M_1(z) = \frac{zy'(z)}{y(z)}\cdot\bigl(B(z)\odot\bigl[z\Phi(y(z))\bigr]\bigr) = \frac{zy'(z)}{y(z)}\cdot\bigl(B(z)\odot y(z)\bigr). \tag{12.29}$$

Let us first consider the factor $zy'(z)/y(z)$. It follows from (12.3) that $y(z) = 0$ implies $z = 0$, and thus $z/y(z)$ is analytic in any domain where $y(z)$ is. Hence,

$zy'(z)/y(z)$ is Δ-analytic, since $y(z)$ is. Moreover, by Cauchy's estimates as in [17, Theorem 6], (12.5) implies, as $z \to 1$,

$$y'(z) = 2^{-1/2}\sigma^{-1}(1-z)^{-1/2} + o\big(|1-z|^{-1/2}\big). \tag{12.30}$$

Consequently,

$$\frac{zy'(z)}{y(z)} = 2^{-1/2}\sigma^{-1}(1-z)^{-1/2} + o\big(|1-z|^{-1/2}\big). \tag{12.31}$$

We turn to the second factor $B(z) \odot y(z)$. We consider first the case

$$f(n) = b_n = n^\alpha, \qquad n \geqslant 1, \tag{12.32}$$

for some $\alpha \in \mathbb{C}$; then $F = F_\alpha$ and, by (1.3),

$$F(\mathcal{T}_n) = X_n(\alpha). \tag{12.33}$$

By (12.32) and (12.8), $B(z) = \operatorname{Li}_{-\alpha}(z)$, a polylogarithm function, and thus (12.9) yields, at least for $\operatorname{Re}\alpha > -1$,

$$B(z) = \Gamma(1+\alpha)(1-z)^{-\alpha-1} + o\big(|1-z|^{-\operatorname{Re}\alpha-1}\big). \tag{12.34}$$

Furthermore, by the definitions,

$$B(z) \odot y(z) = \sum_{n=1}^{\infty} b_n q_n z^n = \sum_{n=1}^{\infty} q_n n^\alpha z^n = \mathbb{E}\big[|\mathcal{T}|^\alpha z^{|\mathcal{T}|}\big]. \tag{12.35}$$

Lemma 12.5. *Let* $\operatorname{Re}\alpha > \frac12$ *and let* $b_n := n^\alpha$. *Then, as* $z \to 1$ *in some* Δ*-domain,*

$$M_1(z) = \frac{\sigma^{-2}}{2\sqrt{\pi}}\Gamma\big(\alpha-\tfrac12\big)(1-z)^{-\alpha} + o\big(|1-z|^{-\operatorname{Re}\alpha}\big). \tag{12.36}$$

Proof. By (12.34) and (12.5) together with Lemmas 12.2 and 12.3, and the fact that $B(z) \odot 1 = 0$,

$$\begin{aligned} B(z) \odot y(z) &= -\Gamma(1+\alpha)(1-z)^{-\alpha-1} \odot \sqrt{2}\sigma^{-1}(1-z)^{1/2} + o\big(|1-z|^{-\operatorname{Re}\alpha+\frac12}\big) \\ &= -2^{1/2}\sigma^{-1}\frac{\Gamma(\alpha-\frac12)}{\Gamma(-\frac12)}(1-z)^{-\alpha+\frac12} + o\big(|1-z|^{-\operatorname{Re}\alpha+\frac12}\big). \end{aligned} \tag{12.37}$$

The result follows by (12.29) and (12.31). ∎

12.4. **The mean when** $0 < \operatorname{Re}\alpha < \frac12$**.** Consider now the case $\operatorname{Re}\alpha < \frac12$. If we still take $b_n = n^\alpha$ as in (12.32), then (12.35) and (2.6) show that $B(z)\odot y(z)$ is continuous in the closed unit disc, and a comparison with (1.12) yields

$$(B \odot y)(1) = \mathbb{E}\,|\mathcal{T}|^\alpha = \mu(\alpha). \tag{12.38}$$

Hence, (12.37) cannot hold, since the right-hand side tends to 0 as $z \to 1$. Actually, it follows from the arguments below that the leading term in $B(z) \odot y(z)$ is the constant $\mu(\alpha)$, which by (12.29) and singularity analysis corresponds to the fact that the leading term in (1.16) is $\mu(\alpha)n$. We recall from Section 1 that when $\operatorname{Re}\alpha < \frac12$, we want to subtract this term. In the present setting, we achieve this by modifying (12.32) and instead taking

$$f(n) = b_n := n^\alpha - \mu(\alpha). \tag{12.39}$$

Then (12.33) is modified to

$$F(\mathcal{T}_n) = \sum_{v\in\mathcal{T}_n} \bigl[|\mathcal{T}_{n,v}|^\alpha - \mu(\alpha)\bigr] = X_n(\alpha) - \mu(\alpha)n, \tag{12.40}$$

and (12.35) is modified to

$$B(z)\odot y(z) = \sum_{n=1}^{\infty} q_n[n^\alpha - \mu(\alpha)]z^n = \mathbb{E}\bigl[|\mathcal{T}|^\alpha z^{|\mathcal{T}|}\bigr] - \mu(\alpha)y(z). \tag{12.41}$$

In particular,

$$(B\odot y)(1) = \mathbb{E}\,|\mathcal{T}|^\alpha - \mu(\alpha) = 0. \tag{12.42}$$

Lemma 12.6. *Let $0 < \operatorname{Re}\alpha < \frac12$ and let $b_n := n^\alpha - \mu(\alpha)$. As $z\to 1$ in some Δ-domain,*

$$M_1(z) = \frac{\sigma^{-2}}{2\sqrt{\pi}}\Gamma\bigl(\alpha-\tfrac12\bigr)(1-z)^{-\alpha} + o\bigl(|1-z|^{-\operatorname{Re}\alpha}\bigr). \tag{12.43}$$

Proof. We now have, by (12.39) and (12.9),

$$\begin{aligned} B(z) &= \operatorname{Li}_{-\alpha}(z) - \mu(\alpha)z(1-z)^{-1} \\ &= \Gamma(1+\alpha)(1-z)^{-\alpha-1} + o\bigl(|1-z|^{-\operatorname{Re}\alpha-1}\bigr), \end{aligned} \tag{12.44}$$

just as in (12.34). Then, arguing as for (12.37) using (12.5) and Lemmas 12.2 and 12.3 now yields

$$\begin{aligned} B(z)\odot y(z) &= -\Gamma(1+\alpha)(1-z)^{-\alpha-1}\odot\sqrt{2}\sigma^{-1}(1-z)^{1/2} + P_1(z) + o\bigl(|1-z|^{-\operatorname{Re}\alpha+\frac12}\bigr) \\ &= -2^{1/2}\sigma^{-1}\frac{\Gamma(\alpha-\frac12)}{\Gamma(-\frac12)}(1-z)^{-\alpha+\frac12} + P_2(z) + o\bigl(|1-z|^{-\operatorname{Re}\alpha+\frac12}\bigr), \end{aligned} \tag{12.45}$$

where $P_1(z), P_2(z) \in \mathcal{P}_{\frac12-\operatorname{Re}\alpha}$ and thus are constants. Letting $z\to1$ in (12.45) shows that $P_2(z) = (B\odot y)(1) = 0$, by (12.42). Hence, the result in (12.37) holds in the present case too, and the result follows again by (12.29) and (12.31). ∎

12.5. **Higher moments.** In the remainder of this Section 12, we assume that $\operatorname{Re}\alpha > 0$, and that we have chosen b_n by (12.32) or (12.39) so that

$$b_n := \begin{cases} n^\alpha, & \operatorname{Re}\alpha \geqslant \frac12, \\ n^\alpha - \mu(\alpha), & 0 < \operatorname{Re}\alpha < \frac12. \end{cases} \tag{12.46}$$

In the present subsection we also assume $\operatorname{Re}\alpha \neq \frac12$.

We need one more general lemma.

Lemma 12.7. *Under our standing assumptions $\mathbb{E}\,\xi = 1$ and $0 < \operatorname{Var}\xi < \infty$, the function $\Phi^{(m)}(y(\cdot))$ is Δ-analytic for every $m\geqslant 0$, and as $z\to1$ in some Δ-domain,*

$$\Phi^{(m)}\bigl(y(z)\bigr) = \begin{cases} O(1), & m\leqslant 2, \\ o\bigl(|1-z|^{1-\frac{m}{2}}\bigr), & m\geqslant 3. \end{cases} \tag{12.47}$$

Proof. As noted at the beginning of Section 12.1, $y(z)$ is Δ-analytic. It follows from (12.5) that for some Δ-domain Δ_1, if $z \in \Delta_1$ with $|1-z|$ small enough, then

$$|y(z)| < 1 - c|1-z|^{1/2}. \tag{12.48}$$

Moreover, the definition (12.2) implies that $|y(z)| \leqslant 1$ for $|z| = 1$ with strict inequality unless $z = 1$. Hence, by continuity, for some $\delta, \eta > 0$, $|y(z)| \leqslant 1 - \eta$ when $z \in \Delta_1$, $|1-z| \geqslant \varepsilon$, and $|z| \leqslant 1 + \delta$. It follows that (12.48) holds (with a new $c > 0$) for all z in the Δ-domain $\Delta_2 := \{z \in \Delta_1 : |z| < 1 + \delta\}$.

In particular, $|y(z)| < 1$ in Δ_2 and thus $\Phi^{(m)}\bigl(y(z)\bigr)$ is analytic in Δ_2.

The assumption $\mathbb{E}\,\xi^2 < \infty$ implies that Φ, Φ' and Φ'' are bounded and continuous functions on the closed unit disc. Hence, (12.47) holds for $m \leqslant 2$.

Now suppose $m \geqslant 3$. Since Φ'' is continuous, we have $\Phi''(z) - \Phi''(1) = o(1)$ as $z \to 1$ with $|z| < 1$. Hence it folows from Cauchy's estimates that

$$\Phi^{(m)}(z) = o\bigl((1-|z|)^{2-m}\bigr) \qquad \text{as } z \to 1 \text{ with } |z| < 1. \tag{12.49}$$

The result (12.47) for $m \geqslant 3$ follows from (12.49) and (12.48). □

Lemma 12.8. *Assume* $\operatorname{Re}\alpha \in (0, \frac12) \cup (\frac12, \infty)$ *and that* (12.46) *holds. Then, for every* $\ell \geqslant 1$, $M_\ell(z)$ *is* Δ*-analytic, and as* $z \to 1$ *in some* Δ*-domain,*

$$M_\ell(z) = \chi_\ell \sigma^{-\ell-1}(1-z)^{-\ell(\alpha+\frac12)+\frac12} + o\bigl(|1-z|^{-\ell(\operatorname{Re}\alpha+\frac12)+\frac12}\bigr), \tag{12.50}$$

where the constants χ_ℓ *are given recursively by*

$$\chi_1 = \frac{1}{2\sqrt{\pi}}\Gamma\bigl(\alpha - \tfrac12\bigr), \tag{12.51}$$

$$\chi_\ell = 2^{-3/2}\sum_{j=1}^{\ell-1}\binom{\ell}{j}\chi_j\chi_{\ell-j} + 2^{-1/2}\ell\frac{\Gamma\bigl(\ell(\alpha+\frac12)-1\bigr)}{\Gamma\bigl((\ell-1)(\alpha+\frac12)-\frac12\bigr)}\chi_{\ell-1}. \tag{12.52}$$

The Δ-domain may depend on ℓ. We write χ_ℓ in (12.51)–(12.52) as $\chi_\ell(\alpha)$ when we want to emphasize the dependence on α.

Proof. We use induction on ℓ, based on Lemma 12.4. First, this shows that M_ℓ is Δ-analytic, using the fact that B and, by Lemma 12.7, $\Phi^{(m)}(y(\cdot))$ are, together with Lemma 12.2.

To show (12.50) by induction, we note that the base case $\ell = 1$ is Lemmas 12.5 and 12.6.

Assume thus $\ell \geqslant 2$, and let $A := -\ell(\alpha+\frac12)+\frac12 < -\frac12$ be the exponent of $1-z$ in (12.50). Consider one of the terms in (12.19). By the induction hypothesis and Lemma 12.7, we have

$$\begin{aligned} zM_{\ell_1}(z)\cdots M_{\ell_m}(z)\Phi^{(m)}\bigl(y(z)\bigr) &= O\bigl(|1-z|^{-\sum_{i=1}^m \ell_i(\operatorname{Re}\alpha+\frac12)+\frac{m}{2}}\Phi^{(m)}\bigl(y(z)\bigr)\bigr) \\ &= \begin{cases} O\bigl(|1-z|^{-(\ell-\ell_0)(\operatorname{Re}\alpha+\frac12)+\frac{m}{2}}\bigr), & m \leqslant 2, \\ o\bigl(|1-z|^{-(\ell-\ell_0)(\operatorname{Re}\alpha+\frac12)+1}\bigr), & m \geqslant 3. \end{cases} \end{aligned} \tag{12.53}$$

Since $\ell - \ell_0 \geqslant m$, the exponent here is

$$-(\ell-\ell_0)(\operatorname{Re}\alpha+\tfrac12) + \frac{m\wedge 2}{2} \leqslant -m(\operatorname{Re}\alpha+\tfrac12) + \frac{m\wedge 2}{2} \leqslant -m\operatorname{Re}\alpha \leqslant 0. \tag{12.54}$$

Furthermore, (12.34) and (12.44) show that, for both $\operatorname{Re}\alpha > \frac12$ and $\operatorname{Re}\alpha < \frac12$,

$$B(z) = O\bigl(|1-z|^{-\operatorname{Re}\alpha-1}\bigr) \tag{12.55}$$

and thus Lemma 12.2 applies ℓ_0 times and yields

$$\begin{aligned} &B(z)^{\odot\ell_0} \odot \bigl[zM_{\ell_1}(z)\cdots M_{\ell_m}(z)\Phi^{(m)}\bigl(y(z)\bigr)\bigr] \\ &\qquad = \begin{cases} O\bigl(|1-z|^{-(\ell-\ell_0)(\operatorname{Re}\alpha+\frac12)+\frac{m}{2}-\ell_0\operatorname{Re}\alpha}\bigr), & m \leqslant 2, \\ o\bigl(|1-z|^{-(\ell-\ell_0)(\operatorname{Re}\alpha+\frac12)+1-\ell_0\operatorname{Re}\alpha}\bigr), & m \geqslant 3. \end{cases} \end{aligned} \tag{12.56}$$

The exponent here is

$$-(\ell-\ell_0)(\operatorname{Re}\alpha+\tfrac12) + \frac{m\wedge 2}{2} - \ell_0\operatorname{Re}\alpha = -\ell(\operatorname{Re}\alpha+\tfrac12) + \frac{\ell_0 + m\wedge 2}{2}. \tag{12.57}$$

For $m \geqslant 3$, this is at least $-\ell(\operatorname{Re}\alpha+\frac12)+1 = \operatorname{Re}A+\frac12$, and thus the term is

$$o\bigl(|1-z|^{\operatorname{Re}A+\frac12}\bigr). \tag{12.58}$$

We will see that this contributes only to the error term in (12.50), so such terms may be ignored. Similarly, for every term with $m \leqslant 2$ and $\ell_0 + m > 2$, the exponent considered in (12.57) is strictly larger than $\operatorname{Re}A+\frac12$, and thus such terms also satisfy (12.58) and may be ignored.

If $m = 1$, then $\ell_1 < \ell$, and thus $\ell_0 \geqslant 1$. Hence, the only remaining terms to consider are (1) $m = 0$ and thus $\ell_0 = \ell$; (2) $m = 1$ and $\ell_0 = 1$; (3) $m = 2$ and $\ell_0 = 0$.

Furthermore, also the term with $m = 0$ can be ignored, since it is

$$\begin{aligned} B(z)^{\odot\ell} \odot \bigl[z\Phi\bigl(y(z)\bigr)\bigr] &= B(z)^{\odot\ell} \odot y(z) \\ &= B(z)^{\odot\ell} \odot 1 + B(z)^{\odot\ell} \odot \bigl(y(z)-1\bigr), \end{aligned} \tag{12.59}$$

where $B(z)^{\odot\ell}\odot 1$ vanishes and $y(z)-1 = O\bigl(|1-z|^{\frac12}\bigr) = o\bigl(|1-z|^0\bigr)$ by (12.5); hence Lemma 12.2(ii) yields

$$B(z)^{\odot\ell} \odot \bigl[z\Phi\bigl(y(z)\bigr)\bigr] = o\bigl(|1-z|^{-\ell\operatorname{Re}\alpha}\bigr) = o\bigl(|1-z|^{\operatorname{Re}A+\frac12}\bigr). \tag{12.60}$$

Consequently, recalling (12.28), we have

$$\begin{aligned} R_l(z) = \ell B(z) \odot \bigl[zM_{\ell-1}(z)\Phi'\bigl(y(z)\bigr)\bigr] + \frac12\sum_{j=1}^{\ell-1}\binom{\ell}{j} zM_j(z)M_{\ell-j}(z)\Phi''\bigl(y(z)\bigr) \\ + o\bigl(|1-z|^{\operatorname{Re}A+\frac12}\bigr). \end{aligned} \tag{12.61}$$

Since Φ' is continuous in the unit disc with $\Phi'(1) = 1$, the induction hypothesis implies that

$$zM_{\ell-1}(z)\Phi'\bigl(y(z)\bigr) = \chi_{\ell-1}\sigma^{-\ell}(1-z)^{-(\ell-1)(\alpha+\frac12)+\frac12} + o\bigl(|1-z|^{-(\ell-1)(\operatorname{Re}\alpha+\frac12)+\frac12}\bigr), \tag{12.62}$$

Hence, (12.34), (12.44), and Lemmas 12.2 and 12.3 yield

$$B(z) \odot \bigl[zM_{\ell-1}(z)\Phi'\bigl(y(z)\bigr)\bigr]$$

$$= \chi_{\ell-1}\sigma^{-\ell}\frac{\Gamma\big(\ell(\alpha+\frac12)-1\big)}{\Gamma\big((\ell-1)(\alpha+\frac12)-\frac12\big)}(1-z)^{A+\frac12}+o\big(|1-z|^{\operatorname{Re}A+\frac12}\big). \quad (12.63)$$

Similarly, the induction hypothesis yields, using $\Phi''(1)=\sigma^2$,

$$\sum_{j=1}^{\ell-1}\binom{\ell}{j}zM_j(z)M_{\ell-j}(z)\Phi''\big(y(z)\big)$$
$$= \sum_{j=1}^{\ell-1}\binom{\ell}{j}\chi_j\chi_{\ell-j}\sigma^{-\ell}(1-z)^{A+\frac12}+o\big(|1-z|^{\operatorname{Re}A+\frac12}\big). \quad (12.64)$$

The result (12.50) now follows from (12.27), (12.31), and (12.61)–(12.64). ☐

12.6. **Mixed moments. Proof of Theorem 1.12.** We may extend Theorem 1.12 to mixed moments of $X_n(\alpha_1),\dots,X_n(\alpha_m)$, for several given $\alpha_1,\dots,\alpha_m$, using the same arguments with only notational differences. For convenience, define

$$\mathbf{X}_n(\alpha) := \begin{cases} n^{-\alpha-\frac12}X_n(\alpha), & \operatorname{Re}\alpha>\frac12,\\ n^{-\alpha-\frac12}\big(X_n(\alpha)-\mu(\alpha)n\big), & 0<\operatorname{Re}\alpha<\frac12. \end{cases} \quad (12.65)$$

We consider for simplicity only two different values of α; the general case is similar but left to the reader.

Theorem 12.9. *Let* $\operatorname{Re}\alpha_1,\operatorname{Re}\alpha_2\in(0,\frac12)\cup(\frac12,\infty)$, *and write* $\alpha_i' := \alpha_i+\frac12$. *Then, for any integers* $\ell_1,\ell_2\geqslant 0$ *with* $\ell_1+\ell_2\geqslant 1$,

$$\sigma^{\ell_1+\ell_2}\,\mathbb{E}\big[\mathbf{X}_n(\alpha_1)^{\ell_1}\mathbf{X}_n(\alpha_2)^{\ell_2}\big]\to\mathbb{E}\big[Y(\alpha_1)^{\ell_1}Y(\alpha_2)^{\ell_2}\big]=\frac{\sqrt{2\pi}}{\Gamma(\ell_1\alpha_1'+\ell_2\alpha_2'-\frac12)}\chi_{\ell_1,\ell_2}, \quad (12.66)$$

where $\chi_{1,0}=\chi_1(\alpha_1)$ *and* $\chi_{0,1}=\chi_1(\alpha_2)$ *are given by* (12.51), *and, for* $\ell_1+\ell_2\geqslant 2$,

$$\begin{aligned}\chi_{\ell_1,\ell_2} = {}& 2^{-3/2}\sum_{0<j_1+j_2<\ell_1+\ell_2}\binom{\ell_1}{j_1}\binom{\ell_2}{j_2}\chi_{j_1,j_2}\chi_{\ell_1-j_1,\ell_2-j_2}\\ &+2^{-1/2}\ell_1\frac{\Gamma\big(\ell_1\alpha_1'+\ell_2\alpha_2'-1\big)}{\Gamma\big(\ell_1\alpha_1'+\ell_2\alpha_2'-1-\alpha_1\big)}\chi_{\ell_1-1,\ell_2}\\ &+2^{-1/2}\ell_2\frac{\Gamma\big(\ell_1\alpha_1'+\ell_2\alpha_2'-1\big)}{\Gamma\big(\ell_1\alpha_1'+\ell_2\alpha_2'-1-\alpha_2\big)}\chi_{\ell_1,\ell_2-1}. \end{aligned} \quad (12.67)$$

Proof of Theorems 1.12 and 12.9. For a given α, we continue to use the choice (12.46) of b_n. This yields (12.33) ($\operatorname{Re}\alpha\geqslant\frac12$) or (12.40) ($\operatorname{Re}\alpha<\frac12$), i.e., now writing $\check F_\alpha$ for F,

$$\check F_\alpha(\mathcal{T}_n) = \begin{cases} X_n(\alpha), & \operatorname{Re}\alpha\geqslant\frac12,\\ X_n(\alpha)-\mu(\alpha)n, & 0<\operatorname{Re}\alpha<\frac12. \end{cases} \quad (12.68)$$

Hence, in both cases, $\mathbf{X}_n(\alpha)=n^{-\alpha-\frac12}\check F_\alpha(\mathcal{T}_n)$, and Theorem 1.10 yields

$$\mathbf{X}_n(\alpha)=n^{-\alpha-\frac12}\check F_\alpha(\mathcal{T}_n)\xrightarrow{\mathrm{d}}\sigma^{-1}Y(\alpha); \quad (12.69)$$

moreover, this holds jointly for any number of α by the proof of Theorem 1.10.

The asymptotic formula (12.50) yields, by (12.17) and standard singularity analysis [26, Chapter VI],

$$q_n m_n^{(\ell)} = [z^n]M_\ell(z) \sim \chi_\ell \sigma^{-\ell-1} \frac{1}{\Gamma\bigl(\ell(\alpha+\frac12)-\frac12\bigr)} n^{\ell(\alpha+\frac12)-\frac32}. \tag{12.70}$$

Together with (2.6) (with $h = 1$) for q_n, this yields

$$m_n^{(\ell)} \sim \frac{\sqrt{2\pi}\chi_\ell\sigma^{-\ell}}{\Gamma\bigl(\ell(\alpha+\frac12)-\frac12\bigr)} n^{\ell(\alpha+\frac12)}. \tag{12.71}$$

Recall that $m_n^{(\ell)} := \mathbb{E}\, \breve{F}_\alpha(\mathcal{T}_n)^\ell$ by (12.16). Hence, (12.71) can be written as

$$\sigma^\ell\, \mathbb{E}\, \mathbf{X}_n(\alpha)^\ell \to \frac{\sqrt{2\pi}}{\Gamma\bigl(\ell(\alpha+\frac12)-\frac12\bigr)} \chi_\ell =: \kappa_\ell, \tag{12.72}$$

where we thus denote the right-hand side by κ_ℓ. The recursion (1.25)–(1.26) then follows from (12.51)–(12.52).

This shows most parts of Theorem 1.12, but it remains to show that the κ_ℓ's (the limits of moments) are the moments of the limit (in distribution) $Y(\alpha)$ of $\sigma\mathbf{X}_n(\alpha)$. For real α, this follows from (12.72) by a standard argument, but for general complex α we want to consider absolute moments, so we postpone the proof of this, and first turn to Theorem 12.9.

Define, in analogy with (12.16)–(12.17),

$$m_n^{(\ell_1,\ell_2)} := \mathbb{E}\bigl[\breve{F}_{\alpha_1}(\mathcal{T}_n)^{\ell_1} \breve{F}_{\alpha_2}(\mathcal{T}_n)^{\ell_2}\bigr], \tag{12.73}$$

$$M_{\ell_1,\ell_2}(z) := \mathbb{E}\bigl[\breve{F}_{\alpha_1}(\mathcal{T})^{\ell_1} \breve{F}_{\alpha_2}(\mathcal{T})^{\ell_2} z^{|\mathcal{T}|}\bigr] = \sum_{n=1}^{\infty} q_n m_n^{(\ell_1,\ell_2)} z^n. \tag{12.74}$$

It is straightforward to extend Lemma 12.4 to the following, valid for every $\ell, r \geqslant 0$ with $\ell + r \geqslant 1$:

$$M_{\ell,r}(z) = \frac{zy'(z)}{y(z)} \sum_{m=1}^{\ell+r} \frac{1}{m!} \sum\nolimits^{**} \binom{\ell}{\ell_0,\dots,\ell_m}\binom{r}{r_0,\dots,r_m} B_{\alpha_1}(z)^{\odot \ell_0} \odot B_{\alpha_2}(z)^{\odot r_0} \odot \bigl[zM_{\ell_1,r_1}(z)\cdots M_{\ell_m,r_m}(z)\Phi^{(m)}\bigl(y(z)\bigr)\bigr], \tag{12.75}$$

where $\sum^{**}$ is the sum over all pairs of $(m+1)$-tuples $(\ell_0,\dots,\ell_m)$ and $(r_0,\dots,r_m)$ of non-negative integers that sum to ℓ and r, respectively, such that $1 \leqslant \ell_i + r_i < \ell + r$ for every i.

Then, the inductive proof of Lemma 12.8 is easily extended to show that in some Δ-domain (possibly depending on ℓ_1 and ℓ_2)

$$M_{\ell_1,\ell_2}(z) = \chi_{\ell_1,\ell_2}\sigma^{-\ell_1-\ell_2-1}(1-z)^{-\ell_1\alpha_1'-\ell_2\alpha_2'+\frac12} + o\bigl(|1-z|^{-\ell_1\operatorname{Re}\alpha_1'-\ell_2\operatorname{Re}\alpha_2'+\frac12}\bigr), \tag{12.76}$$

with χ_{ℓ_1,ℓ_2} given by (12.51) and (12.67). Singularity analysis yields, as for the special case (12.71),

$$\sigma^{\ell_1+\ell_2}\,\mathbb{E}\bigl[\mathbf{X}_n(\alpha_1)^{\ell_1}\mathbf{X}_n(\alpha_2)^{\ell_2}\bigr] \to \frac{\sqrt{2\pi}}{\Gamma\bigl(\ell_1\alpha_1'+\ell_2\alpha_2'-\frac12\bigr)}\chi_{\ell_1,\ell_2} =: \kappa_{\ell_1,\ell_2}. \tag{12.77}$$

In particular, for any α in the domain, we may take $\alpha_1 := \alpha$ and $\alpha_2 := \overline{\alpha}$. Then (12.77) shows, in particular, that for any integer $\ell \geqslant 1$, $\mathbb{E}\bigl|\mathbf{X}_n(\alpha)\bigr|^{2\ell} = \mathbb{E}\bigl[\mathbf{X}_n(\alpha)^{\ell}\mathbf{X}_n(\overline{\alpha})^{\ell}\bigr]$ converges as $n \to \infty$.

By a standard argument, see e.g. [27, Theorems 5.4.2 and 5.5.9], this implies uniform integrability of each smaller power of $\mathbf{X}_n(\alpha)$, which together with (12.69) implies convergence of all lower moments to the moments of the limit $\sigma^{-1}Y(\alpha)$. This completes the proof of (1.23)–(1.24) with

$$\kappa_\ell := \mathbb{E}\,Y(\alpha)^\ell = \frac{\sqrt{2\pi}}{\Gamma\bigl(\ell(\alpha+\frac12)-\frac12\bigr)}\chi_\ell. \tag{12.78}$$

Similarly, using Hölder's inequality, the sequence $\mathbf{X}_n(\alpha_1)^{\ell_1}\mathbf{X}_n(\alpha_2)^{\ell_2}$ is uniformly integrable for every fixed $\alpha_1, \alpha_2, \ell_1, \ell_2$, and (12.66) follows from the joint convergence in (12.69) and (12.77). ☐

Example 12.10. Taking $\ell_1 = \ell_2 = 1$ in (12.67), we obtain, with obvious notation and using (12.51),

$$\begin{aligned}\chi_{1,1}(\alpha,\beta) &= 2^{-\frac12}\chi_1(\alpha)\chi_1(\beta) + 2^{-\frac12}\frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)}\chi_1(\beta) + 2^{-\frac12}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}\chi_1(\alpha)\\ &= \frac{2^{-\frac52}}{\pi}\Gamma(\alpha-\tfrac12)\Gamma(\beta-\tfrac12) + \frac{2^{-\frac32}}{\sqrt{\pi}}\frac{\Gamma(\alpha+\beta)\Gamma(\beta-\frac12)}{\Gamma(\beta)} + \frac{2^{-\frac32}}{\sqrt{\pi}}\frac{\Gamma(\alpha+\beta)\Gamma(\alpha-\frac12)}{\Gamma(\alpha)}.\end{aligned} \tag{12.79}$$

In particular, taking $\beta = \overline{\alpha}$ and using (12.66) and (12.77),

$$\begin{aligned}\mathbb{E}\,|Y(\alpha)|^2 = \kappa_{1,1}(\alpha,\overline{\alpha}) &= \frac{\sqrt{2\pi}}{\Gamma(2\operatorname{Re}\alpha+\frac12)}\chi_{1,1}(\alpha,\overline{\alpha})\\ &= \frac{|\Gamma(\alpha-\frac12)|^2}{4\sqrt{\pi}\Gamma(2\operatorname{Re}\alpha+\frac12)} + \frac{\Gamma(2\operatorname{Re}\alpha)}{\Gamma(2\operatorname{Re}\alpha+\frac12)}\operatorname{Re}\frac{\Gamma(\alpha-\frac12)}{\Gamma(\alpha)}.\end{aligned} \tag{12.80}$$

☐

Example 12.11. As mentioned in Example 1.25, for the case of joint moments of $Y(1)$ and $Y(2)$, Theorem 12.9 yields the recursion formula given in [34]; the method used there is related to the one used here, but seems to apply only for integer α. ☐

Remark 12.12. The mixed moments of $Y(\alpha)$ and $\overline{Y(\alpha)} = Y(\overline{\alpha})$ determine the distribution of $Y(\alpha)$ uniquely, for any $\alpha \neq \frac12$ with $\operatorname{Re}\alpha > 0$. In fact, there exists $C(\alpha) > 0$ such that for every $\ell \geqslant 1$,

$$\mathbb{E}\,|Y(\alpha)|^\ell \leqslant C(\alpha)^\ell \ell!, \tag{12.81}$$

and thus $(\operatorname{Re}Y(\alpha), \operatorname{Im}Y(\alpha))$ has a finite moment generating function in a neighborhood of the origin. The estimate (12.81) was shown for real α in [21, Lemma

3.4] (with proof in [20]); the general case is similar, considering even ℓ and using induction and (12.67).

The constant $C(\alpha)$ in (12.81) can be taken uniformly bounded on compact subsets of $H_+\setminus\{\frac12\}$. Moreover, (12.81) obviously implies the same estimate for $\widetilde{Y}(\alpha) = Y(\alpha) - \mathbb{E}\,Y(\alpha)$ [with $C(\alpha)$ replaced by $2C(\alpha)$], and then we can argue using analyticity as in the proof of Lemma 12.21 below and conclude that (12.81) holds also for $\widetilde{Y}(\frac12)$, which thus also is determined by its moments, as noted in [21]. ☐

12.7. **Uniform estimates.** In this Section 12, we have so far estimated moments for a fixed α, or mixed moments for a fixed set of different α. We turn to uniform estimates for α in suitable sets. This is rather straightforward if $\operatorname{Re}\alpha$ stays away from $\frac12$. However, we want uniformity also for $\operatorname{Re}\alpha$ approaching (or equalling) $\frac12$, and this is more complicated. For our proofs, we assume throughout the present subsection the weak moment condition

$$\mathbb{E}\,\xi^{2+\delta} < \infty, \tag{12.82}$$

for some $\delta > 0$. Throughout this subsection, δ is fixed; we assume without loss of generality that $\delta \leqslant 1$.

Problem 12.13. Do Lemmas 12.18–12.21 and Theorem 1.3 hold without the extra condition (12.82)? (Cf. Remark 1.4.)

We begin with some preliminaries. We start with a standard estimate, included for completeness.

Lemma 12.14. *If* (12.82) *holds with* $0 < \delta \leqslant 1$, *then*

$$\Phi(z) = z + \tfrac12\sigma^2(1-z)^2 + O\bigl(|1-z|^{2+\delta}\bigr), \qquad |z| \leqslant 1. \tag{12.83}$$

Proof. Let $z = 1 - w$, with $|z| \leqslant 1$. Taylor's theorem yields the two estimates, uniformly for $|z| \leqslant 1$ and $k \geqslant 0$,

$$z^k = (1-w)^k = 1 - kw + O\bigl(k^2|w|^2\bigr) = 1 - kw + \binom{k}{2}w^2 + O\bigl(k^2|w|^2\bigr), \tag{12.84}$$

$$z^k = (1-w)^k = 1 - kw + \binom{k}{2}w^2 + O\bigl(k^3|w|^3\bigr), \tag{12.85}$$

and thus, taking a geometric mean of the O terms in (12.84) and (12.85),

$$z^k = 1 - kw + \binom{k}{2}w^2 + O\bigl(k^{2+\delta}|w|^{2+\delta}\bigr). \tag{12.86}$$

Hence, (12.1) yields, using the assumption (12.82),

$$\Phi(z) = \sum_{k=1}^{\infty} p_k\Bigl[1 - kw + \binom{k}{2}w^2 + O\bigl(k^{2+\delta}|w|^{2+\delta}\bigr)\Bigr] = 1 - w + \frac{\sigma^2}{2}w^2 + O\bigl(|w|^{2+\delta}\bigr), \tag{12.87}$$

which is (12.83). ☐

This enables us to improve (12.5).

Lemma 12.15. *If* (12.82) *holds with* $0 < \delta \leqslant 1$, *then, for* z *in some* Δ*-domain,*

$$y(z) = 1 - \sqrt{2}\sigma^{-1}(1-z)^{1/2} + O\bigl(|1-z|^{\frac{1}{2}+\frac{\delta}{2}}\bigr). \tag{12.88}$$

Proof. By [35, Lemma A.2], $y(z)$ is analytic in some Δ-domain Δ such that $|y(z)| < 1$ for $z \in \Delta$ and (12.5) holds as $z \to 1$ in Δ. To show the improvement (12.88), it suffices to consider $z \in \Delta$ close to 1, since the estimate is trivial when $|1-z|$ is bounded below.

Let $w := 1 - y(z)$. By (12.5) we have $|w| = \Theta\bigl(|1-z|^{\frac{1}{2}}\bigr)$. The functional equation (12.3) and Lemma 12.14 yield

$$y(z)/z = \Phi\bigl(y(z)\bigr) = y(z) + \frac{\sigma^2}{2}w^2 + O\bigl(|w|^{2+\delta}\bigr) = y(z) + \frac{\sigma^2}{2}w^2\bigl[1 + O\bigl(|w|^{\delta}\bigr)\bigr] \tag{12.89}$$

and thus, for $|1-z|$ small,

$$\frac{\sigma^2}{2}w^2 = \frac{1-z}{z}y(z)\bigl[1 + O\bigl(|w|^{\delta}\bigr)\bigr] = (1-z)\bigl[1 + O\bigl(|1-z|^{\delta/2}\bigr)\bigr]. \tag{12.90}$$

The result (12.88) follows. ☐

We need also a uniform version of Lemma 12.2(i). We state it in a rather general form.

Lemma 12.16. *Let* $\mathcal{I}$ *be an arbitrary index set, and suppose that* a_ι, b_ι, $\iota \in \mathcal{I}$, *are real numbers such that* $\sup_{\mathcal{I}} |a_\iota| < \infty$, $\sup_{\mathcal{I}} |b_\iota| < \infty$ *and* $\sup_{\mathcal{I}}(a_\iota + b_\iota + 1) < 0$. *Suppose that* $g_\iota(z)$ *and* $h_\iota(z)$ *are* Δ*-analytic functions such that, in some fixed* Δ*-domain* Δ, $g_\iota(z) = O\bigl(|1-z|^{a_\iota}\bigr)$ *and* $h_\iota(z) = O\bigl(|1-z|^{b_\iota}\bigr)$, *uniformly in* ι. *Then*

$$g_\iota(z) \odot h_\iota(z) = O\bigl(|1-z|^{a_\iota + b_\iota + 1}\bigr), \tag{12.91}$$

in some fixed Δ*-domain* Δ', *uniformly in* ι.

Proof. This follows from the proof of [17, Proposition 9], taking there the same integration contour for all ι. ☐

As a final preparation, we state a uniform version of a special case of the asymptotic expansion of polylogarithms by Flajolet [23], cf. (12.9). A proof is given in Appendix B.

Lemma 12.17. *For every* Δ*-domain* Δ *and every compact set* $K \subset \mathbb{C}\backslash\{1, 2, \dots\}$ *we have*

$$\operatorname{Li}_\alpha(z) = \Gamma(1-\alpha)(1-z)^{\alpha-1} + O\bigl(|1-z|^{\operatorname{Re}\alpha} + 1\bigr) \tag{12.92}$$

uniformly for $z \in \Delta$ *and* $\alpha \in K$.

We continue to assume (12.46). We now denote the generating function (12.18) by $B_\alpha(z)$; thus

$$B_\alpha(z) = \begin{cases} \operatorname{Li}_{-\alpha}(z), & \operatorname{Re}\alpha \geqslant \frac{1}{2}, \\ \operatorname{Li}_{-\alpha}(z) - \mu(\alpha)z(1-z)^{-1}, & \operatorname{Re}\alpha < \frac{1}{2}. \end{cases} \tag{12.93}$$

The following lemma is the central step to establishing uniformity in the estimates above. (Cf. Lemmas 12.5 and 12.6.) Note that the lemma does not hold for $\alpha = \frac{1}{2}$; it is easily seen from (2.6) that $B_{1/2}(z) \odot y(z) = \Theta\bigl(|\log|1-z||\bigr)$ as $z \nearrow 1$.

Lemma 12.18. *Assume that* $\mathbb{E}\,\xi^{2+\delta} < \infty$. *Let* K *be a compact subset of* $\{\alpha : \operatorname{Re}\alpha > 0\}\setminus\{\frac{1}{2}\}$. *Then,*

$$B_\alpha(z) \odot y(z) = O\bigl(|1-z|^{\frac{1}{2}-\operatorname{Re}\alpha}\bigr) \tag{12.94}$$

in some fixed Δ*-domain, uniformly for* $\alpha \in K$.

Proof. We consider three different cases, and therefore define $K_1 := \{\alpha \in K : \operatorname{Re}\alpha \geqslant \frac{1}{2} + \frac{\delta}{4}\}$, $K_2 := \{\alpha \in K : \frac{1}{2} \leqslant \operatorname{Re}\alpha < \frac{1}{2} + \frac{\delta}{4}\}$, $K_3 := \{\alpha \in K : \operatorname{Re}\alpha < \frac{1}{2}\}$. Estimates of the type $O\bigl(|1-z|^a\bigr)$ below are valid in some fixed Δ-domain, which may change from line to line.

Case 1: $\operatorname{Re}\alpha \geqslant \frac{1}{2} + \frac{\delta}{4}$. In this range, we have by Lemma 12.17

$$B_\alpha(z) = \operatorname{Li}_{-\alpha}(z) = O\bigl(|1-z|^{-\operatorname{Re}\alpha-1}\bigr), \tag{12.95}$$

uniformly in $\alpha \in K_1$. Furthermore, $y(z) = 1 + O\bigl(|1-z|^{\frac{1}{2}}\bigr)$ by (12.5) (or Lemma 12.15), and $\frac{1}{2} - \operatorname{Re}\alpha \leqslant -\frac{\delta}{4}$ for $\alpha \in K_1$. Hence, Lemma 12.16 yields

$$B_\alpha(z) \odot y(z) = B_\alpha(z) \odot \bigl(y(z)-1\bigr) = O\bigl(|1-z|^{\frac{1}{2}-\operatorname{Re}\alpha}\bigr), \tag{12.96}$$

uniformly in $\alpha \in K_1$.

Case 2 and 3: $0 < \operatorname{Re}\alpha < \frac{1}{2} + \frac{\delta}{4}$. We have, by (12.88) and (12.9),

$$y(z) = 1 - c_1(1-z)^{1/2} + O\bigl(|1-z|^{\frac{1}{2}+\frac{\delta}{2}}\bigr) = 1 + c_2\operatorname{Li}_{3/2}(z) + P(z) + O\bigl(|1-z|^{\frac{1}{2}+\frac{\delta}{2}}\bigr), \tag{12.97}$$

where $P(z)$ is a polynomial that can be assumed to have degree less than $\frac{1}{2} + \frac{\delta}{2}$, and thus $P(z) = C_1$, a constant. Let

$$h(z) := y(z) - c_2\operatorname{Li}_{3/2}(z) - 1 - C_1 = O\bigl(|1-z|^{\frac{1}{2}+\frac{\delta}{2}}\bigr). \tag{12.98}$$

Let ϑ denote the differential operator $z\frac{\mathrm{d}}{\mathrm{d}z}$. Note the identity $\vartheta(g_1(z) \odot g_2(z)) = \vartheta g_1(z) \odot g_2(z)$ and that $\vartheta\operatorname{Li}_\alpha(z) = \operatorname{Li}_{\alpha-1}(z)$. Thus,

$$\vartheta\bigl(\operatorname{Li}_{-\alpha}(z) \odot h(z)\bigr) = \vartheta\operatorname{Li}_{-\alpha}(z) \odot h(z) = \operatorname{Li}_{-\alpha-1}(z) \odot h(z). \tag{12.99}$$

We have $\operatorname{Li}_{-\alpha-1}(z) = O\bigl(|1-z|^{-\operatorname{Re}\alpha-2}\bigr)$ uniformly in $\alpha \in K$ by Lemma 12.17, which together with (12.98) and Lemma 12.16 yields

$$\vartheta\bigl(\operatorname{Li}_{-\alpha}(z) \odot h(z)\bigr) = O\bigl(|1-z|^{-\operatorname{Re}\alpha-\frac{1}{2}+\frac{\delta}{2}}\bigr), \tag{12.100}$$

uniformly in $\alpha \in K_2 \cup K_3$.

Furthermore, by Lemma 12.17,

$$\begin{aligned}\vartheta\bigl(\operatorname{Li}_{-\alpha}(z) \odot \operatorname{Li}_{3/2}(z)\bigr) &= \vartheta\operatorname{Li}_{-\alpha+3/2}(z) = \operatorname{Li}_{-\alpha+1/2}(z)\\ &= \Gamma(\alpha+\tfrac{1}{2})(1-z)^{-\alpha-\frac{1}{2}} + O\bigl(|1-z|^{-\operatorname{Re}\alpha+\frac{1}{2}} + 1\bigr)\end{aligned} \tag{12.101}$$

uniformly in $\alpha \in K_2 \cup K_3$.

The exponent $-\operatorname{Re}\alpha - \frac{1}{2} + \frac{\delta}{2}$ in (12.100) lies in $[-1 + \frac{\delta}{4}, 0)$, and thus (12.100) and (12.101) yield, after division by z,

$$\frac{\mathrm{d}}{\mathrm{d}z}\bigl(\operatorname{Li}_{-\alpha}(z) \odot y(z)\bigr) = c_2 \frac{\mathrm{d}}{\mathrm{d}z}\bigl(\operatorname{Li}_{-\alpha}(z) \odot \operatorname{Li}_{3/2}(z))\bigr) + \frac{\mathrm{d}}{\mathrm{d}z}\bigl(\operatorname{Li}_{-\alpha}(z) \odot h(z))\bigr)$$
$$= c_2 \Gamma(\alpha + \tfrac{1}{2})(1-z)^{-\alpha-\frac{1}{2}} + O\bigl(|1-z|^{-\operatorname{Re}\alpha - \frac{1}{2} + \frac{\delta}{2}}\bigr), \quad (12.102)$$

again uniformly in $\alpha \in K_2 \cup K_3$.

We now consider Cases 2 and 3 separately.

Case 2: $\frac{1}{2} \leqslant \operatorname{Re}\alpha < \frac{1}{2} + \frac{\delta}{4}$. By integrating (12.102) along a suitable contour, for example from 0 along the negative real axis to $-|z|$ and then along the circle with radius $|z|$ to z,

$$B_\alpha(z) \odot y(z) = \operatorname{Li}_{-\alpha}(z) \odot y(z) = c_2 \Gamma(\alpha - \tfrac{1}{2})(1-z)^{-\alpha+\frac{1}{2}} + O(1), \quad (12.103)$$

uniformly in $\alpha \in K_2$, which implies (12.94).

Case 3: $0 < \operatorname{Re}\alpha < \frac{1}{2}$. Recall that now

$$B_\alpha(z) \odot y(z) = \operatorname{Li}_{-\alpha}(z) \odot y(z) - \mu(\alpha) y(z), \quad (12.104)$$

see (12.93) and (12.41). The estimate (12.5) implies, in a smaller Δ-domain,

$$y'(z) = O\bigl(|1-z|^{-\frac{1}{2}}\bigr). \quad (12.105)$$

Furthermore, $\mu(\alpha) = O(1)$ on K_3, as a consequence of Theorem 10.7. Hence (12.104), (12.102), and (12.105) imply

$$\frac{\mathrm{d}}{\mathrm{d}z}\bigl(B_\alpha(z) \odot y(z)\bigr) = c_2 \Gamma(\alpha + \tfrac{1}{2})(1-z)^{-\alpha-\frac{1}{2}} + O\bigl(|1-z|^{-((\operatorname{Re}\alpha + \frac{1}{2} - \frac{\delta}{2}) \vee \frac{1}{2})}\bigr). \quad (12.106)$$

We now have $(B_\alpha \odot y)(1) = 0$ by (12.42), and thus (12.94) follows from (12.106) by integration, noting that the exponents in (12.106) stay away from -1 for $\alpha \in K_3$. ☐

Lemma 12.19. *Assume that* $\mathbb{E}\,\xi^{2+\delta} < \infty$. *Let* K *be a compact subset of* $\{\alpha : \operatorname{Re}\alpha > 0\} \backslash \{\frac{1}{2}\}$. *Then, with notations as in* (12.17) *and* (12.74), *for every* $\ell \geqslant 1$,

$$M_\ell(z) = O\bigl(|1-z|^{-\ell(\operatorname{Re}\alpha + \frac{1}{2}) + \frac{1}{2}}\bigr) \quad (12.107)$$

in some fixed Δ*-domain (depending on* ℓ*) uniformly for all* $\alpha \in K$. *More generally,*

$$M_{\ell_1,\ell_2}(z) = O\bigl(|1-z|^{-\ell_1 \operatorname{Re}\alpha_1' - \ell_2 \operatorname{Re}\alpha_2' + \frac{1}{2}}\bigr), \quad (12.108)$$

in some fixed Δ*-domain (depending on* ℓ_1, ℓ_2*), uniformly for all* $\alpha_1, \alpha_2 \in K$.

Proof. For (12.107), the case $\ell = 1$ follows from (12.29), (12.31), and Lemma 12.18. We then proceed by induction as in the proof of Lemma 12.8. [But the induction is now simpler; it suffices to note that (12.57) is at least $\operatorname{Re} A + \frac{1}{2}$.]

The proof of (12.108) is essentially the same, see the proof of Theorem 12.9. ☐

Lemma 12.20. *Assume that* $\mathbb{E}\,\xi^{2+\delta} < \infty$. *Let* K *be a compact subset of* $\{\alpha : \operatorname{Re}\alpha > 0\} \backslash \{\frac{1}{2}\}$. *Then, for every fixed* $r > 0$,

$$\mathbb{E}\bigl[|X_n(\alpha) - n\mu(\alpha)|^r\bigr] = O\bigl(n^{r(\operatorname{Re}\alpha + \frac{1}{2})}\bigr), \quad (12.109)$$

uniformly for all $\alpha \in K$ *with* $\operatorname{Re}\alpha < \frac{1}{2}$*, and*

$$\mathbb{E}\big[|X_n(\alpha)|^r\big] = O\big(n^{r(\operatorname{Re}\alpha+\frac{1}{2})}\big), \tag{12.110}$$

uniformly for all $\alpha \in K$ *with* $\operatorname{Re}\alpha \geqslant \frac{1}{2}$.

Proof. Using the notation (12.68), (12.109) and (12.110) can be combined as

$$\mathbb{E}\,|\check{F}_\alpha(\mathcal{T}_n)|^r = O\big(n^{r(\operatorname{Re}\alpha+\frac{1}{2})}\big), \tag{12.111}$$

uniformly in $\alpha \in K$. By Hölder's (or Lyapounov's) inequality, it suffice to prove (12.111) when $r = 2\ell$, an even integer. In this case, we let $\alpha_1 = \alpha$, $\alpha_2 = \overline{\alpha}$ and $\ell_1 = \ell_2 = \ell$; then (12.73)–(12.74) show that, using also (2.6),

$$\mathbb{E}\,|\check{F}_\alpha(\mathcal{T}_n)|^{2\ell} = \mathbb{E}\big[\check{F}_\alpha(\mathcal{T}_n)^\ell \check{F}_{\overline{\alpha}}(\mathcal{T}_n)^\ell\big] = m_n^{(\ell,\ell)} = q_n^{-1}[z^n]M_{\ell,\ell}(z) \leqslant Cn^{3/2}[z^n]M_{\ell,\ell}(z), \tag{12.112}$$

and the desired result (12.111) (with $r = 2\ell$) follows from (12.112) and (12.108) by standard singularity analysis, see [26, Proof of Theorem VI.3, p. 390–392]. ☐

Lemma 12.21. *Assume that* $\mathbb{E}\,\xi^{2+\delta} < \infty$*. Let* K *be a compact subset of* $\{\alpha : \operatorname{Re}\alpha > 0\}$*. Then, for every* $r > 0$*,*

$$\mathbb{E}\big[|X_n(\alpha) - \mathbb{E}\,X_n(\alpha)|^r\big] = O\big(n^{r(\operatorname{Re}\alpha+\frac{1}{2})}\big), \tag{12.113}$$

uniformly for all $\alpha \in K$.

Proof. It suffices to show this for $r \geqslant 1$. Let L^r be the Banach space of all complex random variables X defined on our underlying probability space such that

$$\|X\|_r := \big(\mathbb{E}\,|X|^r\big)^{1/r} < \infty. \tag{12.114}$$

Case 1: $\frac{1}{2} \notin K$. In this case, Lemma 12.20 applies and thus (12.109) and (12.110) hold, uniformly for α in the specified sets. We may write these as $\|X_n(\alpha) - n\mu(\alpha)\|_r \leqslant Cn^{\operatorname{Re}\alpha+\frac{1}{2}}$ and $\|X_n(\alpha)\|_r \leqslant Cn^{\operatorname{Re}\alpha+\frac{1}{2}}$, respectively. As is well known, for any (complex) random variable X,

$$\|X - \mathbb{E}\,X\|_r \leqslant \|X\|_r + |\mathbb{E}\,X| \leqslant 2\|X\|_r. \tag{12.115}$$

Hence we obtain in both cases, and thus uniformly for all $\alpha \in K$,

$$\|X_n(\alpha) - \mathbb{E}\,X_n(\alpha)\|_r \leqslant Cn^{\operatorname{Re}\alpha+\frac{1}{2}}, \tag{12.116}$$

which is equivalent to (12.113).

Case 2: $\frac{1}{2} \in K$. Consider first the special case $K_1 := \{\alpha \in \mathbb{C} : |\alpha - \frac{1}{2}| \leqslant 0.1\}$ and let $K_2 := \partial K_1 = \{\alpha \in \mathbb{C} : |\alpha - \frac{1}{2}| = 0.1\}$. Then Case 1 applies to K_2. Moreover, recalling the notation (1.9), we can write (12.113) and (12.116) as

$$\|\widetilde{Y}_n(\alpha)\|_r \leqslant C, \tag{12.117}$$

where $\widetilde{Y}_n(\alpha) = n^{-\alpha-\frac{1}{2}}\big(X_n(\alpha) - \mathbb{E}\,X_n(\alpha)\big)$ is, for each $n \geqslant 1$, an L^r-valued analytic function of α. [Recall that for a fixed n, there are only finitely many choices for the tree $\mathcal{T}_n$, and for each choice, (1.3) is an entire function of α.] The maximum modulus principle holds for Banach space valued analytic functions, see e.g. [13, p. 230], and thus, using (12.117) for K_2,

$$\sup_{\alpha\in K_1} \|\widetilde{Y}_n(\alpha)\|_r = \sup_{\alpha\in K_2} \|\widetilde{Y}_n(\alpha)\|_r \leqslant C. \tag{12.118}$$

Hence, (12.117) holds uniformly for $\alpha \in K_1$, and thus so does (12.113).

For a general compact set K, Case 1 applies to $\{\alpha \in K : |\alpha - \frac{1}{2}| \geqslant 0.1\}$, which together with the case K_1 just proved yields the result (12.113) uniformly for all $\alpha \in K$. ☐

Proof of Theorem 1.3. We give the proof for ordinary moments, i.e., (1.11). The other cases are similar, with mainly notational differences.

Let $\ell \geqslant 1$ and choose $r := \ell + 1$. First, consider a fixed α with $\operatorname{Re}\alpha > 0$. Then Lemma 12.21 shows that $\mathbb{E}\,|\widetilde{Y}_n(\alpha)|^r = O(1)$, and thus the sequence $\widetilde{Y}_n(\alpha)^\ell$ is uniformly integrable, which together with (1.10) implies (1.11). (See again [27, Theorems 5.4.2 and 5.5.9].)

To show uniform convergence on compact sets of α, consider first a convergent sequence (α_k) in H_+ with $\alpha_k \to \alpha_\infty \in H_+$ as $k \to \infty$, and a sequence $n_k \to \infty$. By Theorem 1.2, $\widetilde{Y}_n(\alpha) \xrightarrow{\mathrm{d}} \sigma^{-1}\widetilde{Y}(\alpha)$ in $\mathcal{H}(H_+)$, and by the Skorohod coupling theorem [40, Theorem 4.30], we may assume that a.s. $\widetilde{Y}_n(\alpha) \to \sigma^{-1}\widetilde{Y}(\alpha)$ in $\mathcal{H}(H_+)$, i.e., uniformly on compact sets. It then follows that $\widetilde{Y}_{n_k}(\alpha_k) \xrightarrow{\mathrm{a.s.}} \sigma^{-1}\widetilde{Y}(\alpha_\infty)$ as $k \to \infty$. Furthermore, Lemma 12.21 applies to the compact set $\{\alpha_1, \alpha_2, \dots\} \cup \{\alpha_\infty\}$, and thus (12.117) holds and shows that $\mathbb{E}\,|\widetilde{Y}_{n_k}(\alpha_k)|^r \leqslant C$. Hence, similarly to the case of a fixed α, the sequence $\widetilde{Y}_{n_k}(\alpha_k)^\ell$ is uniformly integrable, and

$$\mathbb{E}\,\widetilde{Y}_{n_k}(\alpha_k)^\ell \to \sigma^{-\ell}\,\mathbb{E}\,\widetilde{Y}(\alpha_\infty)^\ell, \qquad \text{as } k \to \infty. \tag{12.119}$$

This holds for any sequence $n_k \to \infty$. In particular, we may for each k, using (1.11) which we just have proved for each fixed α, choose n_k so large that $\bigl|\mathbb{E}\,\widetilde{Y}_{n_k}(\alpha_k)^\ell - \sigma^{-\ell}\,\mathbb{E}\,\widetilde{Y}(\alpha_k)^\ell\bigr| < 1/k$ for each k. Then (12.119) implies

$$\mathbb{E}\,\widetilde{Y}(\alpha_k)^\ell \to \mathbb{E}\,\widetilde{Y}(\alpha_\infty)^\ell \qquad \text{as } k \to \infty. \tag{12.120}$$

Since this hold for any sequence $\alpha_k \to \alpha_\infty$, (12.120) shows that $\mathbb{E}\,\widetilde{Y}(\alpha)^\ell$ is a continuous function of $\alpha \in H_+$.

Moreover, (12.119) and (12.120) show that for any convergent sequence (α_k) in H_+, and any $n_k \to \infty$,

$$\mathbb{E}\,\widetilde{Y}_{n_k}(\alpha_k)^\ell - \sigma^{-\ell}\,\mathbb{E}\,\widetilde{Y}(\alpha_k)^\ell \to 0. \tag{12.121}$$

Let $K \subset H_+$ be compact. We claim that $\mathbb{E}\,\widetilde{Y}_n(\alpha)^\ell \to \sigma^{-\ell}\,\mathbb{E}\,\widetilde{Y}(\alpha)^\ell$ uniformly for $\alpha \in K$. Suppose not. Then there exists $\varepsilon > 0$, a subsequence $n_k \to \infty$ and a sequence $(\alpha_k) \in K$ such that $\bigl|\mathbb{E}\,\widetilde{Y}_{n_k}(\alpha_k)^\ell - \sigma^{-\ell}\,\mathbb{E}\,\widetilde{Y}(\alpha_k)^\ell\bigr| > \varepsilon$ for every k. Since K is compact, we may by selecting a subsequence assume that $\alpha_k \to \alpha_\infty$ for some $\alpha_\infty \in K$. But then (12.121) holds, which is a contradiction. This shows the claimed uniform convergence on K.

Finally $\mathbb{E}\,\widetilde{Y}(\alpha)^\ell$ is an analytic function of $\alpha \in H_+$ since it is the uniform limit on compact sets of the sequence of analytic functions $\mathbb{E}\,\widetilde{Y}_n(\alpha)^\ell$. ☐

12.8. **Final remark.**

Remark 12.22. In this Section 12 we have only considered the case $\operatorname{Re}\alpha > 0$. It seems likely that similar arguments can be used to show moment convergence in Theorem 1.1 for $\operatorname{Re}\alpha < 0$, but we have not pursued this, and we leave it as an open problem. ☐

Appendix A. Some examples of $\mu(\alpha)$

Although $\mu(\alpha)$ easily can be evaluated numerically for a given ξ by (2.3) or perhaps (10.6), neither formula seems to yield exact values for a given α in any simple form, not even for, e.g., $\alpha = -1$. We give here alternative formulas that can be used to find exact values in some important examples when α is a negative integer.

Let $U \sim U(0,1)$ and $E := -\log U \sim \operatorname{Exp}(1)$ be independent of $\mathcal{T}$. Define the random variable

$$V := U^{1/|\mathcal{T}|} = e^{-E/|\mathcal{T}|}. \tag{A.1}$$

Then $0 < V < 1$, and V has the distribution function, for $0 \leqslant x \leqslant 1$,

$$\mathbb{P}(V \leqslant x) = \mathbb{P}\bigl(U \leqslant x^{|T|}\bigr) = \sum_{n=1}^{\infty} \mathbb{P}(|\mathcal{T}| = n)x^n =: g(x), \tag{A.2}$$

the probability generating function of $|\mathcal{T}|$. Hence, the density function of V is, for $0 \leqslant x < 1$,

$$g'(x) = \sum_{n=1}^{\infty} n\,\mathbb{P}(|\mathcal{T}| = n)x^{n-1} = \sum_{n=1}^{\infty} \mathbb{P}(S_n = n-1)x^{n-1}. \tag{A.3}$$

Since $-\log V = E/|\mathcal{T}|$, we have, for $\operatorname{Re}\alpha < \frac{1}{2}$,

$$\mathbb{E}(-\log V)^{-\alpha} = \mathbb{E}(E/|\mathcal{T}|)^{-\alpha} = \mathbb{E}\,E^{-\alpha}\,\mathbb{E}\,|\mathcal{T}|^{\alpha} = \Gamma(1-\alpha)\mu(\alpha) \tag{A.4}$$

and thus

$$\mu(\alpha) = \frac{1}{\Gamma(1-\alpha)}\,\mathbb{E}(-\log V)^{-\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^1 (-\log x)^{-\alpha}\,\mathrm{d}g(x). \tag{A.5}$$

This can also be written as

$$\mu(\alpha) = \frac{1}{\Gamma(1-\alpha)} \int_0^1 (-\log x)^{-\alpha} g'(x)\,\mathrm{d}x = \frac{1}{\Gamma(1-\alpha)} \int_0^{\infty} y^{-\alpha} g'(e^{-y})e^{-y}\,\mathrm{d}y. \tag{A.6}$$

Define the generating function

$$H(z) := \sum_{k=0}^{\infty} \mu(-k)z^k, \tag{A.7}$$

which converges absolutely for $|z| < 1$, since $|\mu(-k)| = \mu(-k) = \mathbb{E}\,|\mathcal{T}|^{-k} \leqslant 1$. Then (A.5) yields, for $z \in [0,1)$ say, using an integration by parts in the final equality,

$$H(z) = \sum_{k=0}^{\infty} \frac{1}{k!}\,\mathbb{E}(-\log V)^k z^k = \mathbb{E}\,e^{-z\log V} = \mathbb{E}\,V^{-z} \tag{A.8}$$

$$= \int_0^1 x^{-z} g'(x)\,\mathrm{d}x = 1 + z\int_0^1 x^{-z-1} g(x)\,\mathrm{d}x. \tag{A.9}$$

Note that both integrals in (A.9) converge for all z with $\operatorname{Re} z < 1$; hence, (A.9) shows that $H(z)$ extends analytically to this halfplane.

We will see below several examples where $H(z)$ can be found explicitly; then $\mu(-k)$ can be found by extracting Taylor coefficients. In particular, by (A.7) and (A.9),

$$\mu(-1) = H'(0) = \int_0^1 \frac{g(x)}{x}\,\mathrm{d}x, \tag{A.10}$$

which also follows directly from (1.12) and (A.2).

Example A.1 (labelled trees; Po(1))**.** Consider uniformly random labelled trees; this is the case $\xi \sim \mathrm{Po}(1)$. Then $S_n \sim \mathrm{Po}(n)$, and thus (A.2) and (2.2) give

$$g(x) = \sum_{n=1}^{\infty} \frac{1}{n}\,\mathbb{P}(S_n = n-1)x^n = \sum_{n=1}^{\infty} \frac{n^{n-1}e^{-n}}{n\cdot(n-1)!}x^n = \sum_{n=1}^{\infty} \frac{n^{n-1}}{n!}(x/e)^n = T(x/e), \tag{A.11}$$

where T is the well-known tree function, satisfying

$$T(x)e^{-T(x)} = x, \qquad |x| \leqslant e^{-1}. \tag{A.12}$$

Since V has the distribution function g,

$$U \overset{\mathrm{d}}{=} g(V) = T(V/e) \tag{A.13}$$

and thus, using (A.12),

$$V/e = T(V/e)e^{-T(V/e)} \overset{\mathrm{d}}{=} Ue^{-U}. \tag{A.14}$$

Hence,

$$\log V \overset{\mathrm{d}}{=} 1 + \log U - U \tag{A.15}$$

and

$$\mathbb{E}(-\log V)^{-\alpha} = \mathbb{E}(U - \log U - 1)^{-\alpha} = \int_0^1 (u - \log u - 1)^{-\alpha}\,\mathrm{d}u. \tag{A.16}$$

Consequently, by (A.5), for $\operatorname{Re}\alpha < \frac{1}{2}$,

$$\mu(\alpha) = \frac{1}{\Gamma(1-\alpha)}\int_0^1 (u - \log u - 1)^{-\alpha}\,\mathrm{d}u$$

$$= \frac{1}{\Gamma(1-\alpha)}\int_0^\infty \bigl(e^{-x} - 1 + x\bigr)^{-\alpha} e^{-x}\,\mathrm{d}x. \tag{A.17}$$

In particular, when α is a negative integer, $\mu(\alpha)$ can be evaluated as a finite combination of gamma integrals, yielding a rational value. For example, $\mu(0) = 1$

(as always!), $\mu(-1) = 1/2$, $\mu(-2) = 5/12$, $\mu(-3) = 7/18$. $\mu(-4) = 1631/4320$, $\mu(-5) = 96547/259200$.

In this example, by (A.8) and (A.15),

$$\begin{aligned} H(z) &= \mathbb{E}\, e^{-z(1+\log U-U)} = e^{-z}\,\mathbb{E}\big(U^{-z}e^{zU}\big) = e^{-z}\int_0^1 u^{-z}e^{zu}\,\mathrm{d}u \\ &= e^{-z}\Gamma(1-z)\gamma^*(1-z,-z) = e^{-z}(-z)^{z-1}\gamma(1-z,-z), \end{aligned} \tag{A.18}$$

where γ is an incomplete gamma function and γ^* is closely related, see [47, §8.2(i)] for both. ☐

Example A.2 (Ordered trees; Ge(1/2))**.** For uniformly random ordered trees we have $\xi \sim \mathrm{Ge}(1/2)$, with $\mathbb{P}(\xi = k) = 2^{-k-1}$, $k \geqslant 0$. Thus S_n has a Negative Binomial distribution, and, using (2.2),

$$\begin{aligned} \mathbb{P}(|\mathcal{T}| = n) &= \frac{1}{n}\,\mathbb{P}(S_n = n-1) = \frac{1}{n}2^{1-2n}\binom{2n-2}{n-1} = 2^{1-2n}\frac{(2n-2)!}{n!\,(n-1)!} \\ &= (-1)^{n-1}\binom{\frac12}{n}. \end{aligned} \tag{A.19}$$

Hence, the distribution function $g(x)$ of V is by (A.2)

$$g(x) = \sum_{n=1}^{\infty}(-1)^{n-1}\binom{\frac12}{n}x^n = 1-(1-x)^{1/2} \tag{A.20}$$

and the density function is

$$g'(x) = \frac12(1-x)^{-1/2}. \tag{A.21}$$

Thus, V has a Beta distribution: $V \sim B(1,\frac12)$.

By (A.9) and (A.21),

$$H(z) = \frac12\int_0^1 x^{-z}(1-x)^{-1/2}\mathrm{d}x = \frac12 B\big(1-z,\tfrac12\big) = \frac{\Gamma(1-z)\Gamma(\frac32)}{\Gamma(\frac32-z)}. \tag{A.22}$$

By repeated differentiations we obtain for example, assisted by [47, §5.15] and Maple, and using again $\psi(x) := \Gamma'(x)/\Gamma(x)$,

$$\mu(-1) = H'(0) = \psi\big(\tfrac32\big) - \psi(1) = 2 - 2\log 2 \doteq 0.6137, \tag{A.23}$$

$$\begin{aligned} \mu(-2) &= \frac12 H''(0) = \frac12\Big(\big(\psi\big(\tfrac32\big)-\psi(1)\big)^2 - \big(\psi'\big(\tfrac32\big)-\psi'(1)\big)\Big) \\ &= 2\log^2 2 - 4\log 2 - \tfrac16\pi^2 + 4 \doteq 0.5434, \end{aligned} \tag{A.24}$$

$$\begin{aligned} \mu(-3) &= \tfrac13(\log 2 - 1)\pi^2 - \tfrac43\log^3 2 + 4\log^2 2 - 8\log 2 - 2\zeta(3) + 8 \\ &\doteq 0.5190, \end{aligned} \tag{A.25}$$

$$\begin{aligned} \mu(-4) &= -\tfrac{1}{40}\pi^4 + \big(-\tfrac13\log^2 2 + \tfrac23\log 2 - \tfrac23\big)\pi^2 + \tfrac23\log^4 2 \\ &\qquad - \tfrac83\log^3 2 + 8\log^2 2 - 16\log 2 + (4\log 2 - 4)\zeta(3) + 16 \\ &\doteq 0.5088. \end{aligned} \tag{A.26}$$

☐

Example A.3 (Binary trees; $\mathrm{Bi}(2,\frac12)$)**.** Uniformly random binary trees is an example with $\xi\sim\mathrm{Bi}(2,\frac12)$, Thus $S_n\sim\mathrm{Bi}(2n,\frac12)$ and, using (2.2),

$$\begin{aligned}\mathbb{P}(|\mathcal{T}|=n)&=\frac1n\,\mathbb{P}(S_n=n-1)=\frac1n2^{-2n}\binom{2n}{n-1}=2^{-2n}\frac{(2n)!}{n!\,(n+1)!}\\&=2(-1)^n\binom{\frac12}{n+1}.\end{aligned}\tag{A.27}$$

Hence,

$$g(x)=\sum_{n=1}^{\infty}2(-x)^n\binom{\frac12}{n+1}=\frac{2}{-x}\bigl((1-x)^{1/2}-1+\tfrac12x\bigr)=\frac{2-x-2\sqrt{1-x}}{x}\tag{A.28}$$

and (A.9) yields, first for $z<-1$ and then for $\operatorname{Re}z<1$ by analytic continuation,

$$\begin{aligned}H(z)&=1+z\int_0^1\Bigl(2x^{-z-2}-x^{-z-1}-2x^{-z-2}(1-x)^{1/2}\Bigr)\,\mathrm{d}x\\&=1+\frac{2z}{-z-1}-\frac{z}{-z}-2z\frac{\Gamma(-z-1)\Gamma(\frac32)}{\Gamma(\frac12-z)}\\&=\frac{2}{1+z}-\Gamma(\tfrac12)\frac{\Gamma(1-z)}{(1+z)\Gamma(\frac12-z)}.\end{aligned}\tag{A.29}$$

Taking Taylor coefficients at 0 yields, for example, again using [47, §5.15] and Maple,

$$\mu(-1)=H'(0)=-1+\psi(1)-\psi(\tfrac12)=2\log2-1\doteq0.3863,\tag{A.30}$$

$$\mu(-2)=\tfrac16\pi^2-2\log^2 2-2\log2+1\doteq0.2977.\tag{A.31}$$

□

Example A.4 (Full binary trees; $2\,\mathrm{Bi}(1,\frac12)$)**.** Uniformly random full binary trees is an example with $\xi/2\sim\mathrm{Bi}(1,\frac12)$, i.e., $\mathbb{P}(\xi=0)=\mathbb{P}(\xi=2)=\frac12$. Thus $S_n/2\sim\mathrm{Bi}(n,\frac12)$ and, using (2.2), if $n=2m+1$ is odd,

$$\begin{aligned}\mathbb{P}(|\mathcal{T}|=n)&=\frac1n\,\mathbb{P}(S_n=n-1)=\frac1n2^{-n}\binom{n}{m}=2^{-2m-1}\frac{(2m)!}{m!\,(m+1)!}\\&=(-1)^m\binom{\frac12}{m+1}.\end{aligned}\tag{A.32}$$

Hence,

$$g(x)=\sum_{k=0}^{\infty}(-1)^m x^{2m+1}\binom{\frac12}{m+1}=\frac{1-\sqrt{1-x^2}}{x}\tag{A.33}$$

and (A.9) yields, similarly to (A.29), omitting some details,

$$H(z)=1+z\int_0^1x^{-z-2}\bigl(1-\sqrt{1-x^2}\bigr)\,\mathrm{d}x=\frac{1}{1+z}+\frac{\Gamma(-\frac{1+z}{2})\Gamma(\frac32)}{\Gamma(-\frac z2)}.\tag{A.34}$$

This yields, for example,

$$\mu(-1)=\frac{\pi}{2}-1\doteq0.5708,\tag{A.35}$$

$$\mu(-2) = 1 - \tfrac{1}{2}(1-\log 2)\pi \doteq 0.5180. \tag{A.36}$$

□

Appendix B. Polylogarithms

As said in (12.8), the polylogarithm function is defined, for $\alpha \in \mathbb{C}$, by

$$\operatorname{Li}_\alpha(z) := \sum_{n=1}^{\infty} n^{-\alpha} z^n, \qquad |z| < 1; \tag{B.1}$$

the function is then extended analytically to $z \in \mathbb{C}\setminus[0,\infty)$, for example by the integral formula [26, (VI.48)]. As a bivariate function, $\operatorname{Li}_\alpha(z)$ is analytic in both variables $(\alpha, z) \in \mathbb{C} \times (\mathbb{C}\setminus[0,\infty))$.

Let $U := \{z \in \mathbb{C}\setminus(-\infty, 0] : |\log z| < 2\pi\}$ (where $\log z$ denotes the principal value), and note that U is a neighborhood of 1. In particular, U contains, for example, the disc $U_1 := \{z : |z-1| < \frac{1}{2}\}$. If $\alpha \notin \{1, 2, \dots\}$, $z \notin [1,\infty)$, and furthermore $z \in U' := U\setminus[0,\infty)$, then, see [47, 25.12.2] and [14, (1.11.8)],

$$\operatorname{Li}_\alpha(z) = \Gamma(1-\alpha)(-\log z)^{\alpha-1} + \sum_{n=0}^{\infty} \zeta(\alpha-n)\frac{(\log z)^n}{n!}. \tag{B.2}$$

We denote the infinite sum in (B.2) by $h_\alpha(z)$, and note that it converges absolutely for $z \in U$, and thus is analytic there, since the reflection formula for the Riemann zeta function [47, 25.4.2] easily implies

$$\frac{|\zeta(\alpha-n)|}{n!} = O\Big(\frac{(2\pi)^{-n}\Gamma(n+1-\alpha)\zeta(n+1-\alpha)}{n!}\Big) = O\Big(n^{-\operatorname{Re}\alpha}(2\pi)^{-n}\Big). \tag{B.3}$$

for each fixed complex α and $n \geqslant \operatorname{Re}\alpha + 1$.

Moreover, we define the analytic function

$$G(z) := \frac{-\log z}{1-z}, \qquad z \in U_1, \tag{B.4}$$

where by continuity $G(1) = 1$. Since $G(z) \neq 0$ in U_1,

$$g(z) := \log(G(z)), \qquad z \in U_1, \tag{B.5}$$

also defines an analytic function in U_1, with $g(1) = 0$. Then, for $z \in U_1' := U_1\setminus[1,\infty)$,

$$(-\log z)^{\alpha-1} = \big((1-z)G(z)\big)^{\alpha-1} = \big((1-z)e^{g(z)}\big)^{\alpha-1} = (1-z)^{\alpha-1}e^{(\alpha-1)g(z)}. \tag{B.6}$$

Consequently, (B.2) yields

$$\operatorname{Li}_\alpha(z) = \Gamma(1-\alpha)(1-z)^{\alpha-1}e^{(\alpha-1)g(z)} + h_\alpha(z), \qquad z \in U_1'. \tag{B.7}$$

The functions $e^{(\alpha-1)g(z)}$ and $h_\alpha(z)$ are analytic functions of $z \in U_1$, and can thus be expanded as Taylor series in $1-z$. Hence, (B.7) yields, for $z \in U_1'$, an absolutely convergent expansion

$$\operatorname{Li}_\alpha(z) = \sum_{j=0}^{\infty} a_j(\alpha)(1-z)^{\alpha-1+j} + \sum_{k=0}^{\infty} b_k(\alpha)(1-z)^k \tag{B.8}$$

for some coefficients $a_j(\alpha)$ and $b_k(\alpha)$. This is the asymptotic expansion given in Flajolet [23] and [26, Theorem VI.7]; we see now that the expansion actually converges for $z \in U_1'$.

The coefficients $a_j(\alpha)$ and $b_k(\alpha)$ can be found from the formulas above by repeated differentiations at $z = 1$, or (as in [23] and [26]) by substitution in (B.7) of

$$\log z = \log\bigl(1-(1-z)\bigr) = -\sum_{k=1}^{\infty}\frac{(1-z)^k}{k} = -(1-z)\sum_{k=0}^{\infty}\frac{(1-z)^k}{k+1} \tag{B.9}$$

and its consequence

$$g(z) = \log\Bigl[1+\sum_{k=1}^{\infty}\frac{(1-z)^k}{k+1}\Bigr] = \sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}\Bigl[\sum_{k=1}^{\infty}\frac{(1-z)^k}{k+1}\Bigr]^m, \tag{B.10}$$

followed by rearrangements into single power series. Note that $a_j(\alpha)$ and $b_k(\alpha)$ are analytic functions of $\alpha \in \mathbb{C}\setminus\{1,2,\dots\}$.

In particular, $a_0(\alpha) = \Gamma(1-\alpha)$, and thus by keeping only the first term in the first sum in (B.8), we obtain (12.9).

Proof of Lemma 12.17. It is easily checked that the estimate (B.3) holds uniformly for $\alpha \in K$ and large enough n. Hence, uniformly for $\alpha \in K$ and $z \in U_1$,

$$|h_\alpha(z)| = O(1), \tag{B.11}$$

Similarly, since $g(1) = 0$, we have $g(z) = O(|1-z|)$ in U_1, and

$$e^{(\alpha-1)g(z)} = 1 + O\bigl(|1-z|\bigr), \tag{B.12}$$

again uniformly for $\alpha \in K$ and $z \in U_1$. Hence, for $z \in U_1'$, (12.92) follows from (B.7), with the O term uniform for $\alpha \in K$. The case $z \in \Delta\setminus U_1'$ is trivial, since $|1-z|$ is bounded above and below in that set, and $\operatorname{Li}_\alpha(z)$ is uniformly bounded in the compact set $\overline{\Delta}\setminus U_1$ by continuity. ∎

In the same way we see that we may expand the two sums in (B.8) to any number of finite terms, and the resulting expansion will have error terms that are uniform in $\alpha \in K$, for any compact $K \subset \mathbb{C}\setminus\{1,2,\dots\}$.

Appendix C. The limit as $\alpha \to 0$

We show here the claim in Remark 1.19 about limits (in distribution) of $Y(\alpha)$ as $\alpha \to 0$ (with $\operatorname{Re}\alpha > 0$; recall that $Y(\alpha)$ is defined only for such α). It turns out that the limit depends on how α appoaches 0. We consider for simplicity only the case when α approaches on a straight line, i.e., with constant argument (necessarily with $|\arg\alpha| < \pi/2$). In this case, $\alpha^{-1}Y(\alpha)$ has a complex normal limiting distribution, but the limit depends on $\arg\alpha$.

Theorem C.1. *Let $\alpha = re^{i\theta}$ with $|\theta| < \pi/2$, and let $r \to 0$ with θ fixed. Then*

$$\alpha^{-1/2}Y(\alpha) \xrightarrow{\mathrm{d}} \zeta, \tag{C.1}$$

where ζ is a centered complex normal variable, which is characterized by the covariance matrix

$$\operatorname{Cov}\begin{pmatrix}\operatorname{Re}\zeta\\ \operatorname{Im}\zeta\end{pmatrix} = \frac{1-\log 2}{\cos\theta}\begin{pmatrix}1+\cos\theta & 0\\ 0 & 1-\cos\theta\end{pmatrix}. \tag{C.2}$$

In other words, $\operatorname{Re}\zeta$ *and* $\operatorname{Im}\zeta$ *are independent centered normal variables with respective variances* $(1-\log 2)[(1/\cos\theta)\pm 1]$*; equivalently, with*

$$\mathbb{E}\,\zeta^2 = 2(1-\log 2) \qquad \textit{and} \qquad \mathbb{E}\,|\zeta|^2 = 2(1-\log 2)/\cos\theta. \tag{C.3}$$

The case α real, i.e., $\theta = 0$, was noted in [21, Remark 3.6(e)]. As stated in (1.29), then ζ is a real normal variable $N\bigl(0, 2(1-\log 2)\bigr)$.

We prove Theorem C.1 by the method of moments, using Theorem 12.9. We procced via a series of lemmas that are stated for somewhat more general situations.

Lemma C.2. *As* $\alpha,\beta\to 0$*, with* $\operatorname{Re}\alpha, \operatorname{Re}\beta > 0$*, we have*

$$\mathbb{E}\,Y(\alpha) = \kappa_1(\alpha) = \frac{\sqrt{2\pi}}{\Gamma(\alpha)}\chi_1(\alpha) \sim -\sqrt{2\pi}\,\alpha, \tag{C.4}$$

$$\mathbb{E}\bigl[Y(\alpha)Y(\beta)\bigr] = \kappa_{1,1}(\alpha,\beta) \sim \sqrt{2}\chi_{1,1}(\alpha,\beta) \sim 4(1-\log 2)\frac{\alpha\beta}{\alpha+\beta}. \tag{C.5}$$

Proof. All asymptotic notions in the proof are as $\alpha,\beta\to 0$. We assume throughout that $|\alpha|$ and $|\beta|$ are small.

Recall again the standard notation

$$\psi(x) := \frac{\mathrm{d}}{\mathrm{d}x}\log\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}. \tag{C.6}$$

First, by (12.51) and (C.6),

$$\chi_1(\alpha) = \frac{\Gamma(\alpha-\frac12)}{2\sqrt{\pi}} = -\frac{\Gamma(-\frac12+\alpha)}{\Gamma(-\frac12)} = -\bigl(1+\psi(-\tfrac12)\alpha + O(|\alpha|^2)\bigr). \tag{C.7}$$

In particular, $\chi_1(\alpha)\sim -1$ and thus (C.4) follows by (12.72) [or (1.25)].

For the second moment (C.5), we first note that by (12.66) and (12.77),

$$\mathbb{E}\bigl[Y(\alpha)Y(\beta)\bigr] = \kappa_{1,1}(\alpha,\beta) = \frac{\sqrt{2\pi}}{\Gamma(\frac12+\alpha+\beta)}\chi_{1,1}(\alpha,\beta) \sim \sqrt{2}\chi_{1,1}(\alpha,\beta). \tag{C.8}$$

Finally, by (12.67), as in (12.79),

$$\begin{aligned}\sqrt{2}\chi_{1,1}(\alpha,\beta) &= \chi_1(\alpha)\chi_1(\beta) + \frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)}\chi_1(\beta) + \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}\chi_1(\alpha)\\ &= \chi_1(\alpha)\chi_1(\beta)\Bigl[1 + \frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)\chi_1(\alpha)} + \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\chi_1(\beta)}\Bigr].\end{aligned} \tag{C.9}$$

We have, using (C.6),

$$\begin{aligned}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)} &= \frac{\alpha}{\alpha+\beta}\cdot\frac{\Gamma(1+\alpha+\beta)}{\Gamma(1+\alpha)} = \frac{\alpha}{\alpha+\beta}\Bigl[1+\psi(1+\alpha)\beta + O(|\beta|^2)\Bigr]\\ &= \frac{\alpha}{\alpha+\beta}\Bigl[1+\psi(1)\beta + O(|\alpha\beta|+|\beta|^2)\Bigr],\end{aligned} \tag{C.10}$$

which together with (C.7) yields

$$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\chi_1(\beta)} = -\frac{\alpha}{\alpha+\beta}\Big[1+\big(\psi(1)-\psi(-\tfrac12)\big)\beta + O(|\alpha\beta|+|\beta|^2)\Big]. \tag{C.11}$$

Using (C.11), and the same with α and β interchanged, in (C.9) we obtain, recalling $\chi_1(\alpha)\sim\chi_1(\beta)\sim -1$,

$$\begin{aligned}\sqrt{2}\chi_{1,1}(\alpha,\beta) &\sim 1+\frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)\chi_1(\alpha)}+\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\chi_1(\beta)}\\ &= -2\frac{\alpha\beta}{\alpha+\beta}\Big[\psi(1)-\psi(-\tfrac12)+O\big(|\alpha|+|\beta|\big)\Big].\end{aligned} \tag{C.12}$$

The result (C.5) now follows because $\psi(1)=-\gamma$ and

$$\psi(-\tfrac12)=\psi(\tfrac12)+2=-\gamma-2\log 2+2, \tag{C.13}$$

see [47, 5.4.12–13 and 5.5.2]. ☐

Lemma C.3. *Let $\alpha = re^{\mathrm{i}\theta_1}$ and $\beta = re^{\mathrm{i}\theta_2}$ with $\theta_1,\theta_2\in(-\frac{\pi}{2},\frac{\pi}{2})$, and let $r\to 0$ with θ_1,θ_2 fixed. Then, for every fixed $\ell_1,\ell_2\geqslant 0$ with $\ell_1+\ell_2\geqslant 2$,*

$$r^{-(\ell_1+\ell_2)/2}\chi_{\ell_1,\ell_2}(\alpha,\beta)\to\varpi_{\ell_1,\ell_2}, \tag{C.14}$$

where ϖ_{ℓ_1,ℓ_2} is given recursively by

$$\varpi_{0,0}=0, \tag{C.15}$$

$$\varpi_{\ell_1,\ell_2}=0, \qquad \text{when } \ell_1+\ell_2=1, \tag{C.16}$$

$$\varpi_{2,0}=\sqrt{2}(1-\log 2)e^{\mathrm{i}\theta_1}, \tag{C.17}$$

$$\varpi_{0,2}=\sqrt{2}(1-\log 2)e^{\mathrm{i}\theta_2}, \tag{C.18}$$

$$\varpi_{1,1}=2\sqrt{2}(1-\log 2)\frac{e^{\mathrm{i}(\theta_1+\theta_2)}}{e^{\mathrm{i}\theta_1}+e^{\mathrm{i}\theta_2}}, \tag{C.19}$$

$$\varpi_{\ell_1,\ell_2}=2^{-3/2}\sum_{j_1,j_2}\binom{\ell_1}{j_1}\binom{\ell_2}{j_2}\varpi_{j_1,j_2}\varpi_{\ell_1-j_1,\ell_2-j_2}, \qquad \text{when } \ell_1+\ell_2\geqslant 3. \tag{C.20}$$

Moreover,

$$\varpi_{\ell_1,\ell_2}=0, \qquad \text{when } \ell_1+\ell_2 \text{ is odd.} \tag{C.21}$$

Proof. We define for convenience $\varpi_{\ell_1,\ell_2}:=0$ for $\ell_1+\ell_2\leqslant 1$, and note that then (C.15)–(C.16) hold, but not (C.14).

For $\ell_1+\ell_2=2$, (C.17)–(C.19) hold by Lemma C.2.

It remains to treat the case $\ell_1+\ell_2\geqslant 3$, where we use induction on $\ell_1+\ell_2$. We use (12.67). In the double sum there, the two terms with $(j_1,j_2)=(1,0)$ and $(j_1,j_2)=(\ell_1-1,\ell_2)$ are equal, and together, using (C.7), sum to

$$2^{-1/2}\ell_1\chi_1(\alpha)\chi_{\ell_1-1,\ell_2}(\alpha,\beta)=-2^{-1/2}\ell_1\big[1+O(r)\big]\chi_{\ell_1-1,\ell_2}(\alpha,\beta). \tag{C.22}$$

On the other hand, the second of the three terms on the right in (12.67) is

$$2^{-1/2}\ell_1\frac{\Gamma((\ell_1+\ell_2-2)/2+\ell_1\alpha+\ell_2\beta)}{\Gamma((\ell_1+\ell_2-2)/2+(\ell_1-1)\alpha+\ell_2\beta)}\chi_{\ell_1-1,\ell_2}(\alpha,\beta)$$

$$= 2^{-1/2}\ell_1\bigl[1+O(r)\bigr]\chi_{\ell_1-1,\ell_2}(\alpha,\beta). \tag{C.23}$$

Hence the main terms of the contributions (C.22) and (C.23) cancel, and together, using the induction hypothesis, (C.22) and (C.23) sum to

$$O(r)\cdot\chi_{\ell_1-1,\ell_2}(\alpha,\beta) = O\bigl(r^{1+(\ell_1-1+\ell_2)/2}\bigr) = o\bigl(r^{(\ell_1+\ell_2)/2}\bigr). \tag{C.24}$$

Similarly, the terms in the double sum with $(j_1, j_2) = (0, 1)$ and $(\ell_1, \ell_2 - 1)$ together cancel the last term in (12.67) up to another error $o\bigl(r^{(\ell_1+\ell_2)/2}\bigr)$.

This shows that (12.67) yields

$$\chi_{\ell_1,\ell_2}(\alpha,\beta) = 2^{-3/2}\sum_{2\leqslant j_1+j_2\leqslant \ell_1+\ell_2-2}\binom{\ell_1}{j_1}\binom{\ell_2}{j_2}\chi_{j_1,j_2}\chi_{\ell_1-j_1,\ell_2-j_2} + o\bigl(r^{(\ell_1+\ell_2)/2}\bigr), \tag{C.25}$$

and (C.14) together with (C.20) follows by the induction hypothesis, noting that the terms in (C.20) with $j_1 + j_2 \leqslant 1$ or $j_1 + j_2 \geqslant \ell_1 + \ell_2 - 1$ vanish by (C.15)–(C.16).

The conclusion (C.21) follows from (C.16) and (C.20) by induction, since each of the terms in (C.20) vanishes. □

Recall that if $\ell = 2k$ is an even integer, then

$$(\ell-1)!! = (2k-1)!! := 1\cdot 3\cdot\dots\cdot(2k-1) = \frac{(2k)!}{2^k k!} = 2^k\frac{\Gamma(k+\frac12)}{\Gamma(\frac12)}. \tag{C.26}$$

Lemma C.4. *Let $\alpha = re^{\mathrm{i}\theta_1}$ and $\beta = re^{\mathrm{i}\theta_2}$ with $\theta_1, \theta_2 \in (-\frac{\pi}{2}, \frac{\pi}{2})$, and let $r \to 0$ with θ_1, θ_2 fixed. Let t and u be fixed complex numbers. Then, for every $\ell \geqslant 1$,*

$$r^{-\ell/2}\,\mathbb{E}\bigl(tY(\alpha)+uY(\beta)\bigr)^\ell \to \begin{cases} 0, & \ell \text{ is odd},\\ (\ell-1)!!\,\Sigma^{\ell/2}, & \ell \text{ is even,}\end{cases} \tag{C.27}$$

where

$$\Sigma = 2(1-\log 2)\Bigl(t^2e^{\mathrm{i}\theta_1} + u^2e^{\mathrm{i}\theta_2} + 4tu\frac{e^{\mathrm{i}(\theta_1+\theta_2)}}{e^{\mathrm{i}\theta_1}+e^{\mathrm{i}\theta_2}}\Bigr). \tag{C.28}$$

Remark C.5. If $\Sigma \geqslant 0$, then the limits in (C.27) are the moments of a normal distribution $N(0, \Sigma)$. Hence, if $tY(\alpha) + uY(\beta)$ is a real random variable (and $\Sigma \neq 0$), then Lemma C.4 implies asymptotic normality by the method of moments. However, in general, $tY(\alpha) + uY(\beta)$ is a complex random variable and Σ is complex. Nevertheless, the right-hand side can be interpreted as the moments of a complex normal random variable, since the relation $\mathbb{E}\,\zeta^{2\ell} = (2\ell-1)!!\,(\mathbb{E}\,\zeta^2)^\ell$ holds for arbitrary centered complex normal variables, see e.g. [32, Theorem 1.28 and Section I.4]. □

Proof. Theorem 12.9 and Lemma C.3 imply that, if $\ell_1 + \ell_2 = \ell \geqslant 2$, then

$$r^{-\ell/2}\,\mathbb{E}\bigl[Y(\alpha)^{\ell_1}Y(\beta)^{\ell_2}\bigr] = r^{-\ell/2}\kappa_{\ell_1,\ell_2}(\alpha,\beta) \to \frac{\sqrt{2\pi}}{\Gamma((\ell-1)/2)}\varpi_{\ell_1,\ell_2}. \tag{C.29}$$

For $\ell_1+\ell_2=1$, (C.14) does not hold, but a direct appeal to (1.25) yields

$$\mathbb{E}\,Y(\alpha)=\frac{\Gamma(\alpha-\frac12)}{\sqrt{2}\Gamma(\alpha)}=O(|\alpha|)=O\bigl(r\bigr), \tag{C.30}$$

and similarly $\mathbb{E}\,Y(\beta)=O(r)$; hence, (C.29) holds in the case $\ell=1$ too, with the limit 0. (Recall that $1/\Gamma(0)=0$.)

By the binomial formula,

$$\mathbb{E}(tY(\alpha)+uY(\beta))^\ell=\sum_{\ell_1+\ell_2=\ell}\binom{\ell}{\ell_1}t^{\ell_1}u^{\ell_2}\kappa_{\ell_1,\ell_2}(\alpha,\beta) \tag{C.31}$$

which together with (C.29) yields, for every $\ell\geqslant 1$,

$$\begin{aligned} r^{-\ell/2}\,\mathbb{E}(tY(\alpha)+uY(\beta))^\ell &\to \sum_{\ell_1+\ell_2=\ell}\binom{\ell}{\ell_1}t^{\ell_1}u^{\ell_2}\frac{\sqrt{2\pi}}{\Gamma((\ell-1)/2)}\varpi_{\ell_1,\ell_2}\\ &=\frac{\sqrt{2\pi}}{\Gamma((\ell-1)/2)}\tau_\ell(t,u), \end{aligned} \tag{C.32}$$

where we define

$$\tau_\ell(t,u):=\sum_{\ell_1+\ell_2=\ell}\binom{\ell}{\ell_1}t^{\ell_1}u^{\ell_2}\varpi_{\ell_1,\ell_2}. \tag{C.33}$$

We have $\tau_\ell(t,u)=0$ when ℓ is odd or $\ell=0$, by (C.33) together with (C.21) and (C.15). Hence (C.27) for odd ℓ follows from (C.32).

Moreover, if $\ell\geqslant 3$, then (C.33) (thrice) and the recursion (C.20) imply

$$\begin{aligned} 2^{3/2}\tau_\ell(t,u) &= \sum_{\ell_1+\ell_2=\ell}\sum_{j_1,j_2}\binom{\ell}{\ell_1}\binom{\ell_1}{j_1}\binom{\ell_2}{j_2}t^{\ell_1}u^{\ell_2}\varpi_{j_1,j_2}\varpi_{\ell_1-j_1,\ell_2-j_2}\\ &=\sum_j\sum_{j_1+j_2=j}\sum_{\ell_1+\ell_2=\ell}\binom{\ell}{j}\binom{j}{j_1}\binom{\ell-j}{\ell_1-j_1}t^{\ell_1}u^{\ell_2}\varpi_{j_1,j_2}\varpi_{\ell_1-j_1,\ell_2-j_2}\\ &=\sum_j\binom{\ell}{j}\tau_j(t,u)\tau_{\ell-j}(t,u). \end{aligned} \tag{C.34}$$

Since $\tau_\ell(t,u)=0$ when ℓ is odd or $\ell=0$, (C.34) yields

$$2^{3/2}\tau_{2\ell}(t,u)=\sum_{j=1}^{\ell-1}\binom{2\ell}{2j}\tau_{2j}(t,u)\tau_{2(\ell-j)}(t,u). \tag{C.35}$$

The recursion (C.35) is easily solved, by defining

$$d_\ell:=2^{-3/2}\tau_{2\ell}(t,u)/(2\ell)!, \tag{C.36}$$

$$e_\ell:=d_1^{-\ell}d_\ell. \tag{C.37}$$

Then (C.35) yields

$$d_\ell:=\sum_{j=1}^{\ell-1}d_jd_{\ell-j}\qquad\text{and}\qquad e_\ell:=\sum_{j=1}^{\ell-1}e_je_{\ell-j},\qquad \ell\geqslant 2. \tag{C.38}$$

This is a version of the Catalan recursion, and since $e_1 = 1$, it is solved by

$$e_\ell = C_{\ell-1} = \frac{(2\ell-2)!}{(\ell-1)!\,\ell!}, \qquad \ell \geqslant 1; \tag{C.39}$$

and thus, by (C.36) and (C.37),

$$\tau_{2\ell}(t,u) = 2^{3/2}\frac{(2\ell)!\,(2\ell-2)!}{(\ell-1)!\,\ell!}d_1^\ell. \tag{C.40}$$

Hence, (C.32) yields

$$\begin{aligned} r^{-\ell}\,\mathbb{E}(tY(\alpha)+uY(\beta))^{2\ell} \to \frac{\sqrt{2\pi}}{\Gamma(\ell-\frac12)}\tau_{2\ell}(t,u) &= \frac{4\sqrt{\pi}}{\Gamma(\ell-\frac12)}\frac{(2\ell)!\,(2\ell-2)!}{(\ell-1)!\,\ell!}d_1^\ell \\ &= 2^{2\ell}\frac{(2\ell)!}{\ell!}d_1^\ell = (2\ell-1)!!\,(8d_1)^\ell. \end{aligned} \tag{C.41}$$

This proves (C.27) for even ℓ with, recalling (C.36) and (C.33),

$$\Sigma := 8d_1 = \sqrt{2}\tau_2(t,u) = \sqrt{2}\big(t^2\varpi_{2,0} + u^2\varpi_{0,2} + 2tu\varpi_{1,1}\big). \tag{C.42}$$

Finally, (C.28) follows from (C.17)–(C.19). □

Proof of Theorem C.1. We apply Lemma C.4 with $\beta := \overline{\alpha}$ and thus $\theta_1 = \theta$ and $\theta_2 = -\theta$. Let $t \in \mathbb{C}$ and take $u := \bar{t}$. Then $tY(\alpha) + uY(\beta) = 2\operatorname{Re}\big(tY(\alpha)\big)$ is a real random variable, and thus (C.27) shows by the method of moments that

$$2r^{-1/2}\operatorname{Re}\big(tY(\alpha)\big) \overset{\mathrm{d}}{\longrightarrow} N(0,\Sigma), \tag{C.43}$$

with $\Sigma = \Sigma(t)$ (now real) given by (C.28). Since $t \in \mathbb{C}$ is arbitrary and $\operatorname{Re}\big(tY(\alpha)\big)$ can be regarded as the (real) scalar product of $\bar{t}$ and $Y(\alpha)$ if we identify $\mathbb{C}$ and $\mathbb{R}^2$, (C.43) and the Cramér–Wold device show that

$$2r^{-1/2}Y(\alpha) \overset{\mathrm{d}}{\longrightarrow} \zeta', \tag{C.44}$$

for some centered complex normal variable ζ'. Consequently,

$$\alpha^{-1/2}Y(\alpha) = e^{-\mathrm{i}\theta/2}r^{-1/2}Y(\alpha) \overset{\mathrm{d}}{\longrightarrow} \zeta := \frac{e^{-\mathrm{i}\theta/2}}{2}\zeta', \tag{C.45}$$

which proves (C.1). Moreover, the argument above shows that (C.44) holds with all moments (including mixed moments with the complex conjugate), and thus so does (C.45). Taking $t = 1$, $u = 0$ and $\ell = 2$ in (C.27)–(C.28) yields

$$\mathbb{E}\big(\alpha^{-1/2}Y(\alpha)\big)^2 = e^{-\mathrm{i}\theta}r^{-1}\,\mathbb{E}\,Y(\alpha)^2 \to 2(1-\log 2). \tag{C.46}$$

Similarly, by extracting the tu terms in (C.27) and (C.28),

$$\mathbb{E}\big|\alpha^{-1/2}Y(\alpha)\big|^2 = r^{-1}\,\mathbb{E}\big|Y(\alpha)\big|^2 \to 2(1-\log 2)\frac{2}{e^{\mathrm{i}\theta}+e^{-\mathrm{i}\theta}} = \frac{2(1-\log 2)}{\cos\theta}. \tag{C.47}$$

This shows (C.3), and (C.2) follows by elementary calculations. □

Appendix D. The limit towards the imaginary axis

Let $\alpha = a + \mathrm{i}b \to \mathrm{i}t$ in the right half-plane, i.e., with $a = \operatorname{Re}\alpha > 0$. The case $t = 0$ is treated in Appendix C; recall that then, if say α is real for simplicity, $Y(\alpha) \xrightarrow{\mathrm{p}} 0$ and that $a^{-1/2}Y(\alpha)$ converges in distribution to a normal limit; see also Remark 1.19 and [21, Remark 3.6(e)].

Assume in the sequel $t \neq 0$. In this case, we have instead $|Y(\alpha)| \xrightarrow{\mathrm{p}} \infty$, and we obtain a complex normal limit by the following normalization. (Note that, unlike the case $t = 0$ in Theorem C.1, here α can approach its limit $\mathrm{i}t$ in any way, as long as $\operatorname{Re}\alpha > 0$.)

Theorem D.1. *Let $a \searrow 0$ and $b \to t \neq 0$. Then*

$$a^{1/2}Y(a+\mathrm{i}b) \xrightarrow{\mathrm{d}} \zeta, \tag{D.1}$$

where ζ is a symmetric complex normal variable with

$$\mathbb{E}\,|\zeta|^2 = \frac{1}{2\sqrt{\pi}} \operatorname{Re} \frac{\Gamma(\mathrm{i}t - \frac{1}{2})}{\Gamma(\mathrm{i}t-1)} > 0. \tag{D.2}$$

That ζ is symmetric complex normal means that $\zeta \overset{\mathrm{d}}{=} \omega\zeta$ for every complex constant ω with $|\omega| = 1$; equivalently, $\mathbb{E}\,\zeta = 0$ and the real and imaginary parts are independent and have the same variance. (See e.g. [32, Proposition 1.31].)

Proof. We use the method of moments, and argue similarly as for the related Theorem C.1. Take $\alpha_1 := a + \mathrm{i}b$ and $\alpha_2 := \overline{\alpha_1} = a - \mathrm{i}b$ in Theorem 12.9. We claim that, for any $\ell_1, \ell_2 \geqslant 0$,

$$a^{(\ell_1+\ell_2)/2}\chi_{\ell_1,\ell_2}(\alpha_1,\alpha_2) \to \rho_{\ell_1,\ell_2}, \tag{D.3}$$

where

$$\rho_{\ell_1,\ell_2} = 0 \qquad \text{if } \ell_1 \neq \ell_2, \tag{D.4}$$

$$\rho_{1,1} = \frac{1}{\sqrt{8\pi}} \operatorname{Re} \frac{\Gamma(\mathrm{i}t - \frac{1}{2})}{\Gamma(\mathrm{i}t-1)}, \tag{D.5}$$

$$\rho_{\ell,\ell} = 2^{-3/2} \sum_{j=1}^{\ell-1} \binom{l}{j}^2 \rho_{j,j}\rho_{\ell-j,\ell-j}, \qquad l \geqslant 2. \tag{D.6}$$

We prove this using induction on $\ell_1 + \ell_2$. First, if $\ell_1 + \ell_2 = 1$, so $(\ell_1, \ell_2) = (1, 0)$ or $(0, 1)$, then (12.51) shows that χ_{ℓ_1,ℓ_2} is bounded (and converges) as $\alpha \to \mathrm{i}t$, so (D.3) holds with $\rho_{\ell_1,\ell_2} = 0$ as stated in (D.4).

If $\ell_1 + \ell_2 \geqslant 2$, we use (12.67). We have

$$\ell_1\alpha_1' + \ell_2\alpha_2' - 1 \to \ell_1\bigl(\mathrm{i}t + \tfrac{1}{2}\bigr) + \ell_2\bigl(-\mathrm{i}t + \tfrac{1}{2}\bigr) - 1 = \bigl(\ell_1+\ell_2\bigr)/2 - 1 + \bigl(\ell_1 - \ell_2\bigr)\mathrm{i}t. \tag{D.7}$$

If $\ell_1 + \ell_2 \geqslant 3$, or if $\ell_1 \neq \ell_2$, the limit is not a pole of $\Gamma(z)$, and thus the factor $\Gamma\bigl(\ell_1\alpha_1' + \ell_2\alpha_2' - 1\bigr) = O(1)$; hence, (12.67) together with the induction hypothesis yields (D.3) with (D.4) and (D.6).

In the remaining case $\ell_1 = \ell_2 = 1$, $\Gamma\bigl(\ell_1\alpha_1' + \ell_2\alpha_2' - 1\bigr) = \Gamma(2a) \sim (2a)^{-1}$, and (12.67) yields, using (12.51),

$$a\chi_{1,1} = 2^{-1/2}\frac{1}{2\Gamma(-\mathrm{i}t)}\chi_{0,1} + 2^{-1/2}\frac{1}{2\Gamma(\mathrm{i}t)}\chi_{1,0} + o(1) \to 2^{-1/2}\operatorname{Re}\frac{\Gamma\bigl(\mathrm{i}t-\frac12\bigr)}{2\sqrt{\pi}\Gamma(\mathrm{i}t)}, \tag{D.8}$$

which verifies (D.3) with (D.5).

This proves (D.3)–(D.6). The recursion (D.6) is similar to (C.35) and can be solved in the same way. We now define, instead of (C.36),

$$d_\ell := 2^{-3/2}\rho_{\ell,\ell}/\ell!^2. \tag{D.9}$$

With (C.37) as above, we again have (C.38)–(C.39), Hence, using (C.37) and (D.9),

$$\rho_{\ell,\ell} = 2^{3/2}\ell!^2 d_1^\ell e_\ell = 2^{3/2}\frac{\ell!\,(2\ell-2)!}{(\ell-1)!}d_1^\ell. \tag{D.10}$$

Finally, (12.66) and (D.3) yield, using the duplication formula for the Gamma function,

$$\mathbb{E}\bigl[a^\ell|Y(\alpha)|^{2\ell}\bigr] \to \frac{\sqrt{2\pi}}{\Gamma(\ell-\frac12)}\rho_{\ell,\ell} = 4\sqrt{\pi}\frac{\ell!\,\Gamma(2\ell-1)}{\Gamma(\ell-\frac12)\Gamma(\ell)}d_1^\ell = 2^{2\ell}d_1^\ell\ell! = (4d_1)^\ell\ell!, \tag{D.11}$$

and, whenever $\ell_1 \neq \ell_2$,

$$\mathbb{E}\bigl[a^{(\ell_1+\ell_2)/2}Y(\alpha)^{\ell_1}\overline{Y(\alpha)}^{\ell_2}\bigr] \to 0. \tag{D.12}$$

These moment limits are the moments of a symmetric complex normal variable with

$$\mathbb{E}\,|\zeta|^2 = 4d_1, \tag{D.13}$$

(See e.g. [32, Theorem 1.28].) Hence, (D.1) follows by the method of moments, with (D.2) following by (D.13), (D.9), and (D.5).

It remains to prove that the expression in (D.2) in non-zero. (It can obviously not be negative by the case $\ell = 1$ in the argument above.) In other words, we must show that $\Gamma(\mathrm{i}t-\frac12)/\Gamma(\mathrm{i}t-1)$ cannot be imaginary when $t \neq 0$. To see this, we first use the reflection formula for the Gamma function [47, 5.5.3] to obtain

$$\frac{\Gamma(\mathrm{i}t-\frac12)}{\Gamma(\mathrm{i}t-1)} = \frac{\Gamma(2-\mathrm{i}t)\sin((\mathrm{i}t-1)\pi)}{\Gamma(\frac32-\mathrm{i}t)\sin((\mathrm{i}t-\frac12)\pi)} = \frac{\mathrm{i}\sinh(\pi t)}{\cosh(\pi t)}\cdot\frac{\Gamma(2-\mathrm{i}t)}{\Gamma(\frac32-\mathrm{i}t)}. \tag{D.14}$$

Hence, it is enough to show that $\Gamma(2-\mathrm{i}t)/\Gamma(\frac32-\mathrm{i}t)$ is not real for $t \neq 0$. Since $(\log\Gamma(z))' = \Gamma'(z)/\Gamma(z) = \psi(z)$, we have

$$\arg\frac{\Gamma(2-\mathrm{i}t)}{\Gamma(\frac32-\mathrm{i}t)} = \operatorname{Im}\log\frac{\Gamma(2-\mathrm{i}t)}{\Gamma(\frac32-\mathrm{i}t)} = \operatorname{Im}\int_{3/2}^2\psi(s-\mathrm{i}t)\,\mathrm{d}s. \tag{D.15}$$

Moreover, see [47, 5.7.6],

$$\psi(s+\mathrm{i}t) = -\gamma + \sum_{k=0}^\infty\Bigl(\frac{1}{k+1} - \frac{1}{k+s+\mathrm{i}t}\Bigr), \tag{D.16}$$

and thus

$$\operatorname{Im}\psi(s+\mathrm{i}t) = -\sum_{k=0}^{\infty} \operatorname{Im}\frac{1}{k+s+\mathrm{i}t} = \sum_{k=0}^{\infty}\frac{t}{(k+s)^2+t^2}. \tag{D.17}$$

Hence, if $s \geqslant 1$ and $t > 0$, then

$$0 < \operatorname{Im}\psi(s+\mathrm{i}t) < \int_{s-1}^{\infty}\frac{t}{x^2+t^2}\,\mathrm{d}x \leqslant \int_0^{\infty}\frac{t}{x^2+t^2}\,\mathrm{d}x = \frac{\pi}{2}. \tag{D.18}$$

Consequently, if $t < 0$, then (D.15) yields $0 < \arg\bigl(\Gamma(2-\mathrm{i}t)/\Gamma(\frac{3}{2}-\mathrm{i}t)\bigr) < \pi/4$, and thus $\Gamma(2-\mathrm{i}t)/\Gamma(\frac{3}{2}-\mathrm{i}t)$ is not real. The case $t > 0$ follows by conjugation. As said above, using (D.14), this completes the proof that $\mathbb{E}\,|\zeta|^2 > 0$. ☐

Remark D.2. A similar argument shows that if also $\alpha' = a' + \mathrm{i}b' \to \mathrm{i}t'$, for some $t' \notin \{0, \pm t\}$, then the covariances $\operatorname{Cov}\bigl(Y(\alpha), Y(\alpha')\bigr)$ and $\operatorname{Cov}\bigl(Y(\alpha), \overline{Y(\alpha')}\bigr) = \operatorname{Cov}\bigl(Y(\alpha), Y(\overline{\alpha}')\bigr)$ are $O(1)$, and thus after normalization as in (D.1), the covariances tend to 0. It follows that we have joint convergence in (D.1) with independent complex normal limits, for any number of $\alpha_k = a_k + \mathrm{i}b_k \to \mathrm{i}t_k$ with $t_k > 0$. We thus find as limits an uncountable family of independent complex normal variables. ☐

As a corollary to Theorem D.1 we see that $|Y(\alpha)| \overset{\mathrm{p}}{\longrightarrow} \infty$ as $\alpha \to \mathrm{i}t$, with $t \neq 0$.

Problem D.3. For $t \neq 0$, does $|Y(\alpha)| \overset{\mathrm{a.s.}}{\longrightarrow} \infty$ as $\alpha \to \mathrm{i}t$?

Nevertheless, the divergence in probability is enough to show the following.

Corollary D.4. *Almost surely, the imaginary axis is a natural boundary for the analytic functions $Y(\cdot)$ and $\widetilde{Y}(\cdot)$.*

Proof. Let $t \neq 0$. Then Theorem D.1 implies that $|Y(s+\mathrm{i}t)| \overset{\mathrm{p}}{\longrightarrow} \infty$ as $s \searrow 0$. Hence, there exists a sequence $s_n \to 0$ such that $|Y(s_n + \mathrm{i}t)| \to \infty$ a.s. In particular, a.s. $Y(\alpha)$ cannot be extended analytically to a neighbourhood of $\mathrm{i}t$.

Almost surely, this holds for every rational $t \neq 0$, and thus $Y(\cdot)$ cannot be extended analytically across the imaginary axis at any point. The same holds for $\widetilde{Y}(\cdot)$ by (1.20). ☐

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