A CENTRAL LIMIT THEOREM FOR m-DEPENDENT VARIABLES

SVANTE JANSON

Abstract. We give a simple and general central limit theorem for a triangular array of m-dependent variables. The result requires only a Lindeberg condition and avoids unnecessary extra conditions that have been used earlier. The result applies also to increasing $m = m(n)$, provided the Lindeberg condition is modified accordingly. This improves earlier results by several authors.

1. Introduction

Central limit theorems for m-dependent variables under various conditions have a long history. Pioneering results, for a fixed m, were given by Hoeffding and Robbins [10] and Diananda [5] (for an m-dependent sequence), and Orey [12] (more generally, and also for a triangular array). The results were extended to the case of increasing $m = m(n)$, see for example Bergström [1], Berk [2], Romano and Wolf [16].

The purpose of the present paper is to give a simple and general central limit theorem which includes several previous results, but to our knowledge has not been stated before in this form. We state first the case of a fixed m, where we only have to assume the usual Lindeberg condition. For notation, see Section 2.

Theorem 1.1. *Let $m \geqslant 0$ be fixed. Suppose that $(X_{ni})_{n\geqslant 1, 1\leqslant i\leqslant N_n}$ is an m-dependent triangular array and denote its row sums by $S_n := \sum_{i=1}^{N_n} X_{ni}$. Suppose further that the variables X_{ni} have finite second moments and $\mathbb{E}\, X_{ni} = 0$. Let*

$$\sigma_n^2 := \operatorname{Var} S_n, \tag{1.1}$$

and assume that $\sigma_n^2 > 0$ for all large n. Finally, assume the usual Lindeberg condition: for every $\varepsilon > 0$, as $n \to \infty$,

$$\frac{1}{\sigma_n^2} \sum_{i=1}^{N_n} \mathbb{E}\big[X_{ni}^2 \mathbf{1}\{|X_{ni}| > \varepsilon\sigma_n\}\big] \to 0. \tag{1.2}$$

Then

$$S_n/\sigma_n \xrightarrow{\mathrm{d}} N(0,1) \qquad \text{as } n \to \infty. \tag{1.3}$$

Remark 1.2. The case $m = 0$ of Theorem 1.1, i.e., an independent array (X_{ni}), is the classical central limit theorem with Lindeberg's condition; see e.g. [6, Theorem XV.6.1 and Problem XV.29], [11, Theorem 5.12] or [7, Theorem 7.2.4]. Moreover, in this case the Lindeberg condition (1.2) in necessary under a weak extra condition, see

---

Date: 27 August, 2021.

2020 *Mathematics Subject Classification.* 60F05.

Key words and phrases. m-dependent; central limit theorem; asymptotic normality; Lindeberg condition; Lyapunov condition.

Partly supported by the Knut and Alice Wallenberg Foundation.

[6, Theorem XV.6.2] and [11, Theorem 5.12]; hence we cannot expect a more general theorem for m-dependent variables without this condition (or something stronger).

Theorem 1.1 is only a minor generalization of the result by Orey [12], where the main theorem essentially (ignoring some technical details) shows the same result under the extra condition

$$\sum_{i=1}^{N_n} \operatorname{Var} X_{ni} = O\bigl(\sigma_n^2\bigr). \tag{1.4}$$

(See also [17, Theorem 13.1] which gives another proof of Orey's result, now stated similarly to our Theorem 1.1 with the extra condition (1.4).) The condition (1.4) is satisfied in most applications, but it is easy to see that there are cases where (1.4) does not hold but Theorem 1.1 applies, see Example 5.1.

Note also that this result by Orey [12] for m-dependent variables extends to much more mixing conditions. As shown by Peligrad [13, Theorem 2.1], Theorem 1.1 holds also if we replace "m-dependent" by "strongly mixing", and add (1.4) and the condition $\lim \bar\rho_n^* < 1$. (See [13] for definition, and note that in the m-dependent case this is trivial since then $\bar\rho_n^* = 0$ when $n > m$.) We will not consider mixing conditions further, but we state a problem. (There is a large literature on asymptotic normality under various mixing conditions. See e.g. [3], which however mainly considers only the case of stationary sequences, and the references there.)

Problem 1.3. Does Theorem 1.1 extend to suitable mixing conditions? In particular, does [13, Theorem 2.1] hold also without the assumption (1.4)?

More generally, we can allow m to depend on n. (Then mixing results such as [13] do not apply. However, more complicated results such as [15] may apply in this case, see Remark 4.5.) The statement is almost the same in this case; we only have to modify the Lindeberg condition.

Theorem 1.4. *Let $(m_n)_n$ be a given sequence of integers with $m_n \geqslant 1$. Suppose that $(X_{ni})_{n\geqslant 1, 1\leqslant i\leqslant N_n}$ is a (m_n)-dependent triangular array and denote its row sums by $S_n := \sum_{i=1}^{N_n} X_{ni}$. Suppose further that the variables X_{ni} have finite second moments and $\mathbb{E}\, X_{ni} = 0$. Let*

$$\sigma_n^2 := \operatorname{Var} S_n, \tag{1.5}$$

and assume that $\sigma_n^2 > 0$ for all large n. Finally, assume the following version of the Lindeberg condition: for every $\varepsilon > 0$, as $n \to \infty$,

$$\frac{m_n}{\sigma_n^2} \sum_{i=1}^{N_n} \mathbb{E}\Bigl[X_{ni}^2 \mathbf{1}\Bigl\{|X_{ni}| > \frac{\varepsilon \sigma_n}{m_n}\Bigr\}\Bigr] \to 0. \tag{1.6}$$

Then

$$S_n/\sigma_n \xrightarrow{\mathrm{d}} N(0,1) \qquad \text{as } n \to \infty. \tag{1.7}$$

Remark 1.5. The assumption $m_n \geqslant 1$ in Theorem 1.4 is just for convenience. (Otherwise we would have to replace m_n by $m_n + 1$ or $m_n \vee 1$ in (1.6).) It is no real loss of generality, since we may replace any $m_n = 0$ by 1. It is then obvious that Theorem 1.1 is a special case of Theorem 1.4.

We will see in Example 5.2 that (1.6) is the natural version of the Lindeberg condition when m is allowed to depend on n, and that it cannot be weakened. In particular, (1.2) is not enough if $m_n \to \infty$.

As immediate corollaries, the Lindeberg conditions (1.2) and (1.6) can be replaced by corresponding Lyapunov conditions; se Section 4. We will also compare this to the results of [2] and [16], and in particular show that their main results follow from Theorem 1.4 and that some of their conditions are not needed.

Remark 1.6. There is a large number of papers on various aspects of limits for m-dependent random variables not discussed here. In particular, we mention results on rate of convergence and Berry–Essen type estimates, see for example [14; 9; 18; 4].

2. Notation

We recall some standard notions, and give our notation for them.

Let $m \geqslant 0$ be an integer. A (finite or infinite) sequence $(X_i)_i$ of random variables is *m-dependent* if the two families $\{X_i\}_{i \leqslant k}$ and $\{X_i\}_{i > k+m}$ of random variables are independent of each other for every k. In particular, 0-dependent is the same as independent.

A *triangular array* is an array of random variables $(X_{ni})_{n \geqslant 1,\, 1 \leqslant i \leqslant N_n}$, for some given sequence $N_n \geqslant 1$; it is assumed that the variables $(X_{ni})_i$ in a single row are defined on the same probability space. (No relation is required between variables in different rows.)

The row lengths N_n are supposed to be given; we often omit them from the notation and write e.g. $\sum_i X_{ni}$ for the row sum $\sum_{i=1}^{N_n} X_{ni}$.

If $m \geqslant 0$ is a fixed integer, we say that the triangular array (X_{ni}) is *m-dependent* if each row $(X_{ni})_i$ is m-dependent. More generally, given a sequence $(m_n)_1^\infty$ with $m_n \geqslant 0$, we say that (X_{ni}) is *(m_n)-dependent* if, for every $n \geqslant 1$, the row $(X_{ni})_i$ is m_n-dependent.

For a random variable X, $\|X\|_2 := \big(\mathbb{E}[X^2]\big)^{1/2}$.

Convergence in probability and distribution is denoted by $\overset{\mathrm{p}}{\longrightarrow}$ and $\overset{\mathrm{d}}{\longrightarrow}$, respectively. Unspecified limits are as $n \to \infty$.

3. Proof of Theorems 1.1 and 1.4

We begin with a special case of Theorem 1.4. The general case will then follow by a simple truncation argument.

Lemma 3.1. *In addition to the assumptions in Theorem 1.4, assume also that $\sigma_n^2 \to 1$ as $n \to \infty$, and that $(\varepsilon_n)_n$ is a sequence with $\varepsilon_n \to 0$ such that*

$$|X_{ni}| \leqslant \varepsilon_n / m_n \qquad a.s., \tag{3.1}$$

for all n and i. Then

$$S_n \overset{\mathrm{d}}{\longrightarrow} N(0,1) \qquad as\ n \to \infty. \tag{3.2}$$

Proof. The idea of the proof is to approximate, for each n, the sequence of partial sums $\sum_{i=1}^k X_{ni}$ by a martingale $(M_{nk})_{k=0}^{N_n}$ with $M_{n0} = 0$ and $M_{nN_n} = S_n$, see (3.7) below, and then use a martingale central limit theorem for M_{nk}. (Note that in the independent case, the sequence of partial sums is a martingale, but in the m-dependent case it is in general not; the proof shows that the martingale (3.7) is a good approximation.)

The martingale limit theorem that we use is [8, Theorem 3.2 with Remarks, pp. 58–59], which shows that the conclusion (3.2) follows provided we show that, with $\Delta_{nk} := M_{n,k} - M_{n,k-1}$,

$$\max_k |\Delta_{nk}| \xrightarrow{\text{p}} 0, \tag{3.3}$$

$$\sum_k \Delta_{nk}^2 \xrightarrow{\text{p}} 1, \tag{3.4}$$

$$\mathbb{E}\big[\max_k \Delta_{nk}^2\big] \leqslant C. \tag{3.5}$$

We separate the proof into several steps. For notational convenience, we define $X_{ni} := 0$ for $i \leqslant 0$ and $i > N_n$.

Step 1: The martingale. Let $\mathcal{F}_{nk}$ be the σ-field generated by $X_{n1}, \dots, X_{nk}$, and define

$$W_{nik} := \mathbb{E}\big(X_{ni} \mid \mathcal{F}_{nk}\big), \tag{3.6}$$

$$M_{nk} := \mathbb{E}\big(S_n \mid \mathcal{F}_{nk}\big) = \sum_i W_{nik}. \tag{3.7}$$

Thus $(M_{nk})_{k=0}^{N_n}$ is a martingale for each n, with $M_{n0} = \mathbb{E}\, S_n = 0$, $M_{nN_n} = S_n$, and martingale differences

$$\Delta_{nk} := M_{n,k} - M_{n,k-1} = \sum_i \big(W_{ni,k} - W_{ni,k-1}\big). \tag{3.8}$$

If $i \leqslant k$, then X_{ni} is $\mathcal{F}_{nk}$-measurable, and thus

$$W_{nik} = X_{ni}, \qquad i \leqslant k. \tag{3.9}$$

In particular, if $i \leqslant k-1$, then $W_{ni,k} - W_{ni,k-1} = X_{ni} - X_{ni} = 0$. Furthermore, if $i > k+m$, then the m-dependence shows that X_{ni} is independent of $\mathcal{F}_{nk}$, and thus

$$W_{nik} = \mathbb{E}(X_{ni} \mid \mathcal{F}_{nk}) = \mathbb{E}\, X_{ni} = 0, \qquad i > k+m. \tag{3.10}$$

Hence, (3.8) simplifies to

$$\Delta_{nk} = \sum_{i=k}^{k+m} \big(W_{ni,k} - W_{ni,k-1}\big). \tag{3.11}$$

Similarly, by (3.9) and (3.10) again,

$$M_{nk} = \sum_{i=1}^{k} X_{ni} + \sum_{i=k+1}^{k+m} W_{nik}. \tag{3.12}$$

We have also, by the martingale property and $M_{nN_n} = S_n$,

$$\mathbb{E} \sum_k \Delta_{nk}^2 = \sum_k \mathbb{E}\, \Delta_{nk}^2 = \mathbb{E}\, M_{nN_n}^2 = \mathbb{E}\, S_n^2. \tag{3.13}$$

Step 2: Proof of (3.3) *and* (3.5). The assumption (3.1) and (3.6) yield

$$|W_{nik}| \leqslant \varepsilon_n / m_n \qquad \text{a.s.} \tag{3.14}$$

There are $2(m_n+1)$ variables W in the sum in (3.11), and thus (3.14) yields

$$|\Delta_{nk}| \leqslant 2(m_n+1)\frac{\varepsilon_n}{m_n} \leqslant 4\varepsilon_n \qquad \text{a.s.} \tag{3.15}$$

Since $\varepsilon_n \to 0$, both (3.3) and (3.5) follow (trivially) from (3.15).

Step 3: Proof of (3.4). Let

$$Q_n := \sum_k \Delta_{nk}^2, \qquad q_{nk} := \mathbb{E}\,\Delta_{nk}^2, \qquad T_{nk} := \sum_{i=1}^{k} X_{ni}. \tag{3.16}$$

Then $\mathbb{E}\,Q_n = \mathbb{E}\,S_n^2$ by (3.13). Furthermore,

$$\begin{aligned} \operatorname{Var} Q_n &= \mathbb{E}\left[\sum_{i=1}^{N_n}\bigl(\Delta_{ni}^2 - q_{ni}\bigr)\right]^2 \\ &= \sum_{i=1}^{N_n}\mathbb{E}\bigl[\bigl(\Delta_{ni}^2 - q_{ni}\bigr)^2\bigr] + 2\sum_{i=1}^{N_n}\sum_{j=i+1}^{N_n}\mathbb{E}\bigl[\bigl(\Delta_{ni}^2 - q_{ni}\bigr)\Delta_{nj}^2\bigr]. \end{aligned} \tag{3.17}$$

First, using (3.15) and (3.13),

$$\sum_i \mathbb{E}\bigl[\bigl(\Delta_{ni}^2 - q_{ni}\bigr)^2\bigr] = \sum_i \operatorname{Var}\bigl[\Delta_{ni}^2\bigr] \leqslant \sum_i \mathbb{E}\bigl[\Delta_{ni}^4\bigr] \leqslant 16\varepsilon_n^2 \sum_i \mathbb{E}\,\Delta_{ni}^2 = 16\varepsilon_n^2\,\mathbb{E}\,S_n^2. \tag{3.18}$$

For the double sum in (3.17), we note that since Δ_{ni} are martingale differences, if $i<j<k$, then $\mathbb{E}\bigl[\bigl(\Delta_{ni}^2 - q_{ni}\bigr)\Delta_{nj}\Delta_{nk}\bigr] = 0$; by symmetry, the same holds if $i<k<j$. Hence,

$$\begin{aligned} \sum_{i=1}^{N_n}\sum_{j=i+1}^{N_n}\mathbb{E}\bigl[\bigl(\Delta_{ni}^2 - q_{ni}\bigr)\Delta_{nj}^2\bigr] &= \sum_{i=1}^{N_n}\sum_{j=i+1}^{N_n}\sum_{k=i+1}^{N_n}\mathbb{E}\bigl[\bigl(\Delta_{ni}^2 - q_{ni}\bigr)\Delta_{nj}\Delta_{nk}\bigr] \\ &= \sum_{i=1}^{N_n}\mathbb{E}\sum_{j=i+1}^{N_n}\sum_{k=i+1}^{N_n}\bigl(\Delta_{ni}^2 - q_{ni}\bigr)\Delta_{nj}\Delta_{nk} \\ &= \sum_i \mathbb{E}\bigl[\bigl(\Delta_{ni}^2 - q_{ni}\bigr)\bigl(M_{nN_n} - M_{ni}\bigr)^2\bigr]. \end{aligned} \tag{3.19}$$

Recall that $M_{nN_n} = S_n = T_{nN_n}$. The conjugate rule gives

$$\begin{aligned} \bigl(M_{nN_n} - M_{ni}\bigr)^2 &= \bigl(T_{nN_n} - M_{ni}\bigr)^2 \\ &= \bigl(T_{nN_n} - T_{n,i+m_n}\bigr)^2 + \bigl(2T_{nN_n} - T_{n,i+m_n} - M_{ni}\bigr)\bigl(T_{n,i+m_n} - M_{ni}\bigr). \end{aligned} \tag{3.20}$$

Hence,

$$\begin{aligned} \mathbb{E}\bigl[\bigl(\Delta_{ni}^2 - q_{ni}\bigr)\bigl(M_{nN_n} - M_{ni}\bigr)^2\bigr] &= \mathbb{E}\bigl[\bigl(\Delta_{ni}^2 - q_{ni}\bigr)\bigl(T_{nN_n} - T_{n,i+m_n}\bigr)^2\bigr] \\ &\quad + \mathbb{E}\bigl[\bigl(\Delta_{ni}^2 - q_{ni}\bigr)\bigl(T_{n,i+m_n} - M_{ni}\bigr)\bigl(2T_{nN_n} - T_{n,i+m_n} - M_{ni}\bigr)\bigr]. \end{aligned} \tag{3.21}$$

For the first term on the right-hand side of (3.21), we note that Δ_{ni} is $\mathcal{F}_i$-measurable, and thus the m-dependence of $(X_{ni})_i$ implies that $T_{nN_n} - T_{n,i+m_n} = \sum_{i+m_n+1}^{N_n} X_{nk}$ is independent of $\Delta_{ni}^2 - q_{ni}$. Furthermore, $\mathbb{E}[\Delta_{ni}^2 - q_{ni}] = 0$, and thus

$$\mathbb{E}\bigl[\bigl(\Delta_{ni}^2 - q_{ni}\bigr)\bigl(T_{nN_n} - T_{n,i+m_n}\bigr)^2\bigr] = \mathbb{E}\bigl[\Delta_{ni}^2 - q_{ni}\bigr]\,\mathbb{E}\bigl[\bigl(T_{nN_n} - T_{n,i+m_n}\bigr)^2\bigr] = 0. \tag{3.22}$$

Similarly, $T_{nN_n} - T_{n,i+2m_n}$ is independent of $\bigl(\Delta_{ni}^2 - q_{ni}\bigr)\bigl(T_{n,i+m_n} - M_{ni}\bigr)$, and $\mathbb{E}\bigl(T_{nN_n} - T_{n,i+2m_n}\bigr) = 0$; hence,

$$\mathbb{E}\bigl[\bigl(\Delta_{ni}^2 - q_{ni}\bigr)\bigl(T_{n,i+m_n} - M_{ni}\bigr)\bigl(2T_{nN_n} - 2T_{n,i+2m_n}\bigr)\bigr] = 0. \tag{3.23}$$

Consequently, we obtain from (3.21)–(3.23)

$$\begin{aligned}\mathbb{E}\bigl[\bigl(\Delta_{ni}^2-q_{ni}\bigr)\bigl(M_{nN_n}-M_{ni}\bigr)^2\bigr]\\ =\mathbb{E}\bigl[\bigl(\Delta_{ni}^2-q_{ni}\bigr)\bigl(T_{n,i+m_n}-M_{ni}\bigr)\bigl(2T_{n,i+2m_n}-T_{n,i+m_n}-M_{ni}\bigr)\bigr].\end{aligned} \tag{3.24}$$

The assumption (3.1) implies $|T_{n,i+2m_n}-T_{n,i+m_n}|\leqslant\varepsilon_n$, and also, using (3.12) and (3.14),

$$|T_{n,i+m_n}-M_{ni}|=\left|\sum_{j=i+1}^{i+m_n}\bigl(X_{nj}-W_{nji}\bigr)\right|\leqslant 2\varepsilon_n. \tag{3.25}$$

Hence,

$$\bigl|2T_{n,i+2m_n}-T_{n,i+m_n}-M_{ni}\bigr|\leqslant 2|T_{n,i+2m_n}-T_{n,i+m_n}|+|T_{n,i+m_n}-M_{ni}|\leqslant 4\varepsilon_n, \tag{3.26}$$

and (3.24)–(3.26) yield

$$\mathbb{E}\bigl[\bigl(\Delta_{ni}^2-q_{ni}\bigr)\bigl(M_{nN_n}-M_{ni}\bigr)^2\bigr]\leqslant 8\varepsilon_n^2\,\mathbb{E}\bigl|\Delta_{ni}^2-q_{ni}\bigr|\leqslant 16\varepsilon_n^2\,\mathbb{E}\,\Delta_{ni}^2. \tag{3.27}$$

Combining (3.19) and (3.27) yields, using again (3.13),

$$\sum_{i=1}^{N_n}\sum_{j=i+1}^{N_n}\mathbb{E}\bigl[\bigl(\Delta_{ni}^2-q_{ni}\bigr)\Delta_{nj}^2\bigr]\leqslant 16\varepsilon_n^2\sum_i\mathbb{E}\,\Delta_{ni}^2=16\varepsilon_n^2\,\mathbb{E}\,S_n^2. \tag{3.28}$$

Finally, (3.17), (3.18) and (3.28) yield the estimate

$$\operatorname{Var}\bigl[Q_n\bigr]\leqslant 48\varepsilon_n^2\,\mathbb{E}\,S_n^2=48\varepsilon_n^2\sigma_n^2\to 0, \tag{3.29}$$

recalling $\varepsilon_n\to 0$ and $\sigma_n^2\to 1$.

Consequently, $Q_n-\mathbb{E}\,Q_n\xrightarrow{\mathrm{p}}0$. Since $\mathbb{E}\,Q_n=\mathbb{E}\,S_n=\sigma_n^2\to 1$ by (3.13) and assumption, we obtain

$$Q_n\xrightarrow{\mathrm{p}}1, \tag{3.30}$$

which is (3.4). (Recall the definition (3.16).)

Step 4: Conclusion. We have verified (3.3)–(3.5), and, as said above, the asymptotic normality (3.2) of $S_n=M_{nN_n}$ follows by [8, Theorem 3.2 with Remarks, pp. 58–59]. ☐

Proof of Theorem 1.4. First, by replacing X_{ni} by X_{ni}/σ_n (possibly ignoring some small n with $\sigma_n=0$), we may and will assume that $\sigma_n=1$ for all n.

Next, since (1.6) holds for every fixed $\varepsilon>0$, it holds also for some sequence $\varepsilon_n\to 0$; i.e., there exists a sequence $\varepsilon_n\to 0$ such that

$$m_n\sum_{i=1}^{N_n}\mathbb{E}\bigl[X_{ni}^2\mathbf{1}\{|X_{ni}|>\varepsilon_n/m_n\}\bigr]\to 0. \tag{3.31}$$

We fix such a sequence ε_n, and use it to truncate the variables: define

$$X'_{ni}:=X_{ni}\mathbf{1}\{|X_{ni}|\leqslant\varepsilon_n/m_n\}-\upsilon_{ni},\quad X''_{ni}:=X_{ni}\mathbf{1}\{|X_{ni}|>\varepsilon_n/m_n\}+\upsilon_{ni}, \tag{3.32}$$

where

$$\upsilon_{ni}:=\mathbb{E}\bigl[X_{ni}\mathbf{1}\{|X_{ni}|\leqslant\varepsilon_n/m_n\}\bigr]=-\,\mathbb{E}\bigl[X_{ni}\mathbf{1}\{|X_{ni}|>\varepsilon_n/m_n\}\bigr]. \tag{3.33}$$

Clearly, both $(X'_{ni})_{n,i}$ and $(X''_{ni})_{n,i}$ are triangular arrays with (m_n)-dependent rows and means 0. Denote the corresponding row sums by S'_n and S''_n.

We will estimate $\mathbb{E}(S''_n)^2$ in (3.35) below; this is an instance of an estimate in [17, Lemma 13.1], but for completeness we include the simple proof. For any two square-integrable random variables Y and Z, we have by the Cauchy–Schwarz inequality and the arithmetic-geometric inequality

$$|\operatorname{Cov}(Y,Z)| \leqslant \bigl(\mathbb{E}[Y^2]\,\mathbb{E}[Z^2]\bigr)^{1/2} \leqslant \tfrac{1}{2}\bigl(\mathbb{E}[Y^2]+\mathbb{E}[Z^2]\bigr). \tag{3.34}$$

Hence, by (3.32)–(3.33), for convenience again defining $X_{ni} := 0$ for $i \leqslant 0$ and $i > N_n$,

$$\begin{aligned}
\mathbb{E}\bigl[(S''_n)^2\bigr] &= \sum_{i,j} \operatorname{Cov}\bigl(X''_{ni}, X''_{nj}\bigr) = \sum_i \sum_{j=i-m_n}^{i+m_n} \operatorname{Cov}\bigl(X''_{ni}, X''_{nj}\bigr) \\
&\leqslant \sum_i \sum_{j=i-m_n}^{i+m_n} \tfrac{1}{2}\bigl(\mathbb{E}[|X''_{ni}|^2] + \mathbb{E}[|X''_{nj}|^2]\bigr) \leqslant (2m_n+1)\sum_i \mathbb{E}[|X''_{ni}|^2] \\
&\leqslant (2m_n+1)\sum_i \mathbb{E}\bigl[X_{ni}^2 \mathbf{1}\{|X_{ni}| > \varepsilon_n/m_n\}\bigr].
\end{aligned} \tag{3.35}$$

Consequently, (3.31) implies

$$\mathbb{E}\bigl[(S''_n)^2\bigr] \to 0. \tag{3.36}$$

In other words, $\|S''_n\|_2 \to 0$, and since $\|S_n\|_2 = \sigma_n = 1$ by assumption (recalling $\mathbb{E}\, S_n = 0$), we obtain from Minkowski's inequality $\|S'_n\|_2 = \|S_n - S''_n\|_2 \to 1$, and thus

$$\operatorname{Var} S'_n = \mathbb{E}\bigl[(S'_n)^2\bigr] = \|S'_n\|_2^2 \to 1. \tag{3.37}$$

Furthermore, (3.32)–(3.33) imply $|X'_{ni}| \leqslant 2\varepsilon_n/m_n$. Consequently, Lemma 3.1 applies to (X'_{ni}) (with $2\varepsilon_n$), which yields

$$S'_n \xrightarrow{\mathrm{d}} N(0,1). \tag{3.38}$$

Since also $S''_n \xrightarrow{\mathrm{p}} 0$ by (3.36), the conclusion (1.7) follows by the Cramér–Slutsky theorem [7, Theorem 5.11.4]. □

Proof of Theorem 1.1. Theorem 1.1 is, as said in Remark 1.5, a special case of Theorem 1.4. □

4. Lyapunov conditions

It is an immediate corollary of Theorems 1.1 and 1.4 that instead of the Lindeberg conditions (1.2) and (1.6), we may use a Lyapunov type condition. This is often more convenient for applications. We state such a version of Theorem 1.4.

Theorem 4.1. *Suppose that $(X_{ni})_{n\geqslant 1, 1\leqslant i\leqslant N_n}$ is a (m_n)-dependent triangular array with $\mathbb{E}\, X_{ni} = 0$. Let $S_n := \sum_{i=1}^{N_n} X_{ni}$ and $\sigma_n^2 := \operatorname{Var} S_n$, and assume that $\sigma_n^2 > 0$ for all large n. Assume also that for some fixed $r > 2$, as $n \to \infty$,*

$$\frac{m_n^{r-1}}{\sigma_n^r} \sum_{i=1}^{N_n} \mathbb{E}\,|X_{ni}|^r \to 0. \tag{4.1}$$

Then

$$S_n/\sigma_n \xrightarrow{\mathrm{d}} N(0,1) \qquad \text{as } n \to \infty. \tag{4.2}$$

Proof. We have

$$\mathbb{E}\Big[X_{ni}^2\mathbf{1}\Big\{|X_{ni}|>\frac{\varepsilon\sigma_n}{m_n}\Big\}\Big]\leqslant\Big(\frac{m_n}{\varepsilon\sigma_n}\Big)^{r-2}\mathbb{E}\,|X_{ni}|^r \tag{4.3}$$

and thus

$$\frac{m_n}{\sigma_n^2}\sum_{i=1}^{N_n}\mathbb{E}\Big[X_{ni}^2\mathbf{1}\Big\{|X_{ni}|>\frac{\varepsilon\sigma_n}{m_n}\Big\}\Big]\leqslant\varepsilon^{2-r}\frac{m_n^{r-1}}{\sigma_n^r}\sum_{i=1}^{N_n}\mathbb{E}\,|X_{ni}|^r. \tag{4.4}$$

Hence, (1.6) follows from (4.1), and Theorem 1.4 applies. □

Remark 4.2. In the classical case with independent summands, the Lyapunov condition gets stronger as the exponent r increases, so the most general result is obtained with r small (i.e., close to 2). However, this is not always the case here; see Examples 5.1 and 5.3. Thus different values of r yield incomparable conditions, so in an application r may have to be adapted to the problem.

We next compare Theorem 4.1 to the results of Berk [2] and Romano and Wolf [16], and show that their theorems follow from Theorem 4.1. We will see in Example 5.3 that the implications are strict; there are examples where Theorem 4.1 applies but not [2] or [16].

Example 4.3. Berk [2, Theorem(i)(iii)(iv)] assumes, in our notation, for some $\delta>0$ and constants C and c,

$$\mathbb{E}\,|X_{ni}|^{2+\delta}\leqslant C, \tag{4.5}$$

$$\sigma_n^2/N_n\to c>0 \tag{4.6}$$

$$m_n^{2+2/\delta}=o(N_n). \tag{4.7}$$

With $r:=2+\delta$, we obtain from (4.5)–(4.6) (for large n)

$$\frac{m_n^{r-1}}{\sigma_n^r}\sum_{i=1}^{N_n}\mathbb{E}\,|X_{ni}|^r\leqslant C_1\frac{m_n^{1+\delta}}{N_n^{(2+\delta)/2}}N_n=C_1\frac{m_n^{1+\delta}}{N_n^{\delta/2}}=C_1\left(\frac{m_n^{2+2/\delta}}{N_n}\right)^{\delta/2}. \tag{4.8}$$

Hence, (4.1) follows from (4.7). Consequently, the theorem in [2] is a special case of Theorem 4.1. (Note that we have not used the assumption (ii) in [2]; thus Theorem 4.1 is stronger, and more convenient to apply.)

Example 4.4. Romano and Wolf [16] show that their theorem extends the result by Berk [2] discussed in Example 4.3. Romano and Wolf [16, Theorem 2.1(1)(3)(5)(6)] assume, in our notation, for some $\delta>0$ and $\gamma\in[-1,1)$, and some Δ_n and L_n,

$$\mathbb{E}\,|X_{ni}|^{2+\delta}\leqslant\Delta_n, \tag{4.9}$$

$$\sigma_n^2/(N_nm_n^\gamma)\geqslant L_n, \tag{4.10}$$

$$\Delta_n/L_n^{(2+\delta)/2}=O(1), \tag{4.11}$$

$$m_n^{1+(1-\gamma)(1+2/\delta)}/N_n\to 0. \tag{4.12}$$

With $r:=2+\delta$, we obtain by (4.9), (4.11), (4.10), (4.12), for some constant C,

$$\frac{m_n^{r-1}}{\sigma_n^r}\sum_{i=1}^{N_n}\mathbb{E}\,|X_{ni}|^r\leqslant\frac{m_n^{1+\delta}}{\sigma_n^{2+\delta}}N_n\Delta_n\leqslant C\frac{m_n^{1+\delta}}{\sigma_n^{2+\delta}}N_nL_n^{(2+\delta)/2}\leqslant C\frac{m_n^{1+\delta}}{(N_nm_n^\gamma)^{(2+\delta)/2}}N_n$$

$$= C\frac{m_n^{(1+\delta/2)(1-\gamma)+\delta/2}}{N_n^{\delta/2}} = C\left(\frac{m_n^{(2/\delta+1)(1-\gamma)+1}}{N_n}\right)^{\delta/2} \to 0. \tag{4.13}$$

Hence, (4.1) follows. Consequently, the theorem in [16] is a special case of Theorem 4.1. (Note that we did not use assumptions (2) and (4) in [16]. Also, our condition is simpler and seems easier to apply.)

Remark 4.5. Rio [15, Corollary 1] is a result stated much more generally for strongly mixing triangular arrays, where the mixing rate may depend on n. In the special case of an (m_n)-dependent array, we have (in the notation of [15]) $\alpha_{(n)}^{-1}(x) \leqslant m_n+1$, and using this it is easy to see that, assuming $m_n \geqslant 1$, condition (b) in the corollary in [15] holds if

$$\frac{m_n}{\sigma_n^2}\sum_i \mathbb{E}\Big[X_{ni}^2\Big(\frac{m_n}{\sigma_n}|X_n| \wedge 1\Big)\Big] \to 0. \tag{4.14}$$

Furthermore, it can be seen that (4.14) also implies condition (a) in the corollary, and thus the corollary then yields asymptotic normality.

Note that the condition (4.14) is intermediate between (1.6) and (4.1) for $r=3$. More precisely, it is easily seen that (4.14) implies (1.6) (and thus this special case of [15, Corollary 1] follows from Theorem 1.4); on the other hand, (4.1) with $r=3$ implies (4.14), and thus the case $r=3$ of Theorem 4.1 follows from [15, Corollary 1].

Finally, we note that in Example 5.1 below, it follows from (5.3), (5.5) and (5.8) that (4.14) holds only if $\alpha > 1/3$. Hence, Theorems 1.1 and 1.4 do not follow from this special case of [15].

5. Examples

We give some examples illustrating the various conditions.

Example 5.1. Let ξ_i and η_i, $i \geqslant 0$, be i.i.d. random variables with $\mathbb{P}(\xi_i = \pm 1) = \mathbb{P}(\eta_i = \pm 1) = \frac{1}{2}$. Let $N_n := n$, let $0 < \alpha < \frac{1}{2}$, and define

$$X_{ni} := n^{-1/2}\xi_i + n^{-\alpha}\big(\eta_i - \eta_{i-1}\big), \qquad 1 \leqslant i \leqslant n. \tag{5.1}$$

Then

$$S_n = n^{-1/2}\sum_{i=1}^n \xi_i + n^{-\alpha}\big(\eta_n - \eta_0\big). \tag{5.2}$$

It follows that we have

$$\sigma_n^2 = \operatorname{Var} S_n = 1 + 2n^{-2\alpha} \to 1, \tag{5.3}$$

and, by the standard central limit theorem,

$$S_n \xrightarrow{\mathrm{d}} N(0,1). \tag{5.4}$$

The triangular array (X_{ni}) is 1-dependent, and (1.2) is trivial since

$$|X_{ni}| \leqslant n^{-1/2} + 2n^{-\alpha} \to 0. \tag{5.5}$$

Thus Theorem 1.1 applies and yields (5.4). However,

$$\sum_{i=1}^n \operatorname{Var} X_{ni} = n\big(n^{-1} + 2n^{-2\alpha}\big) = 1 + 2n^{1-2\alpha} \to \infty, \tag{5.6}$$

so (1.4) does not hold. Thus, as said in Section 1, Theorem 1.1 is more general than previous versions assuming also (1.4).

Furthermore, let us check the Lyapunov condition (4.1). We have by (5.1), since $n^{-1/2} \ll n^{-\alpha}$,

$$\mathbb{E}\,|X_{ni}|^r \sim n^{-r\alpha}\,\mathbb{E}\,|\eta_i - \eta_{i-1}|^r = cn^{-r\alpha} \tag{5.7}$$

for some constant $c > 0$. (In fact, $c = 2^{r-1}$.) Hence,

$$\frac{1}{\sigma_n^r}\sum_{i=1}^{n}\mathbb{E}\,|X_{ni}|^r \sim cn^{1-r\alpha}. \tag{5.8}$$

Since $m = 1$, (5.8) shows that (4.1) holds if $r > 1/\alpha$, but not if $2 < r < 1/\alpha$. Hence, in this example, the Lyapunov condition gets weaker if r is increased, and not stronger as in the independent case.

Example 5.2. Let $m_n \geqslant 1$ be a given sequence, and let (Y_{ni}) be a triangular array with independent rows. Define the array (X_{ni}) by repeating each random variable Y_{ni} m_n times, and dividing it by m_n. In other words, we define $X_{ni} := m_n^{-1}Y_{n,\lceil i/m_n\rceil}$. Then, denoting the row-wise sums by S_n^X and S_n^Y, we have $S_n^X = S_n^Y$. Moreover, for any $\varepsilon > 0$,

$$\begin{aligned}\sum_i \mathbb{E}\big[X_{ni}^2\mathbf{1}\{|X_{ni}| > \varepsilon\sigma_n/m_n\}\big] &= \sum_j m_n\,\mathbb{E}\big[(Y_{nj}/m_n)^2\mathbf{1}\{|Y_{nj}| > \varepsilon\sigma_n\}\big] \\ &= m_n^{-1}\sum_j \mathbb{E}\big[Y_{nj}^2\mathbf{1}\{|Y_{nj}| > \varepsilon\sigma_n\}\big]. \end{aligned} \tag{5.9}$$

Hence, the condition (1.6) is equivalent to the usual Lindeberg condition on (Y_{nj}). This shows that (1.6) is a natural version of the Lindeberg condition for (m_n)-dependent arrays, and that it cannot be weakened.

Similarly, the left-hand side of (4.1) is the same for (X_{ni}) and for (Y_{nj}); this shows that (4.1) is a natural version of the Lyapunov condition for (m_n)-dependent arrays.

Example 5.3. Let ξ_i, $i \geqslant 1$, and η be i.i.d. $N(0,1)$ variables. Let $m_n \to \infty$ with $m_n = o(n^{1/2})$, and take $N_n := n + m_n \sim n$. Define

$$X_{ni} := \begin{cases}\xi_i, & 1 \leqslant i \leqslant n,\\ \eta, & n < i \leqslant n + m_n.\end{cases} \tag{5.10}$$

Then (X_{ni}) is an (m_n)-dependent triangular array. Furthermore,

$$\sigma_n^2 = n + m_n^2 \sim n. \tag{5.11}$$

Moreover, $S_n \in N(0, \sigma_n^2)$, so (1.3) is trivial. The left-hand side of (4.1) is

$$\sim \frac{m_n^{r-1}}{n^{r/2}}N_n\,\mathbb{E}\,|\xi_1|^r \sim c_r\frac{m_n^{r-1}}{n^{r/2-1}} \tag{5.12}$$

for some constant $c_r > 0$, and thus (4.1) holds if and only if

$$m_n = o\big(n^{(r-2)/(2(r-1))}\big). \tag{5.13}$$

Note that the exponent in (5.13) increases with r. Consequently, as in Example 5.1 but for another reason, the Lyapunov condition (4.1) gets weaker if r is increased. In the present example, choosing a larger r means weakening the restriction on m_n. On the other hand, if we modify the example and let ξ_i and η have some other (centred) distribution, a larger r also means a stronger moment condition on the variables, so there might be a trade-off.

We note also that this example does not satisfy the conditions in [16], and thus not the stronger conditions in [2]. To see this, note that conditions (2), (4) and (3) in [16] (choosing $k = m_n$ and $a = n + 1$) imply

$$m_n^2 = \operatorname{Var} \sum_{i=n+1}^{m_m+n} X_{ni} \leqslant C m_n^{1+\gamma} \frac{\sigma_n^2}{N_n m_n^\gamma} \sim C m_n \tag{5.14}$$

and thus $m_n = O(1)$, contradicting our assumptions.

References

[1] Harald Bergström: A comparison method for distribution functions of sums of independent and dependent random variables. (Russian). *Teor. Verojatnost. i Primenen.* **15** (1970), 442–468, 750; English transl. *Theor. Probab. Appl.* **15** (1970), 430–457, 727. MR 0283850, MR 0281245

[2] Kenneth N. Berk: A central limit theorem for m-dependent random variables with unbounded m. *Ann. Probab.* **1** (1973), 352–354. MR 0350815

[3] Richard C. Bradley: *Introduction to Strong Mixing Conditions. Vol. 1–3.* Kendrick Press, Heber City, UT, 2007. MR 2325294–2325296

[4] Louis H. Y. Chen & Qi-Man Shao: Normal approximation under local dependence. *Ann. Probab.* **32** (2004), no. 3A, 1985–2028. MR 2073183

[5] P. H. Diananda: The central limit theorem for m-dependent variables. *Proc. Cambridge Philos. Soc.* **51** (1955), 92–95. MR 0067396

[6] William Feller, *An Introduction to Probability Theory and its Applications, Volume II*, 2nd ed., Wiley, New York, 1971. MR 0270403

[7] Allan Gut: *Probability: A Graduate Course*, 2nd ed., Springer, New York, 2013. MR 2977961

[8] P. Hall & C. C. Heyde: *Martingale Limit Theory and its Application.* Academic Press, New York, 1980. MR 0624435

[9] Lothar Heinrich: Nonuniform estimates and asymptotic expansions of the remainder in the central limit theorem for m-dependent random variables. *Math. Nachr.* **115** (1984), 7–20. MR 0755264

[10] Wassily Hoeffding & Herbert Robbins: The central limit theorem for dependent random variables. *Duke Math. J.* **15** (1948), 773–780. MR 0026771

[11] Olav Kallenberg: *Foundations of Modern Probability.* 2nd ed., Springer, New York, 2002. MR 1876169

[12] Steven Orey: A central limit theorem for m-dependent random variables. *Duke Math. J.* **25** (1958), 543–546. MR 0097841

[13] Magda Peligrad: On the asymptotic normality of sequences of weak dependent random variables. *J. Theoret. Probab.* **9** (1996), no. 3, 703–715. MR 1400595

[14] V. V. Petrov: On the central limit theorem for m-dependent quantities. (Russian) *Proc. All-Union Conf. Theory Prob. and Math. Statist. (Erevan, 1958)*, pp. 38–44, Izdat. Akad. Nauk Armjan. SSR, Erevan, 1960. MR 0200963

[15] Emmanuel Rio: About the Lindeberg method for strongly mixing sequences. *ESAIM Probab. Statist.* **1** (1995/97), 35–61. MR 1382517

[16] Joseph P. Romano & Michael Wolf: A more general central limit theorem for m-dependent random variables with unbounded m. *Statist. Probab. Lett.* **47** (2000), no. 2, 115–124. MR 1747098

[17] Bengt Rosén: On the central limit theorem for sums of dependent random variables. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **7** (1967), 48–82. MR 0208656

[18] V. V. Shergin: On the convergence rate in the central limit theorem for m-dependent random variables. *Teor. Veroyatnost. i Primenen.* **24** (1979), no. 4, 781–794; English transl. *Theory Probab. Appl.* **24** (1980), no. 4, 782–796. MR 0550533

Department of Mathematics, Uppsala University, PO Box 480, SE-751 06 Uppsala, Sweden

Email address: svante.janson@math.uu.se

URL: http://www.math.uu.se/svante-janson