Identities and periodic oscillations of divide-and-conquer recurrences splitting at half[*]

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Abstract

We study divide-and-conquer recurrences of the form

$$f(n) = \alpha f\big(\big\lfloor \tfrac{n}{2} \big\rfloor\big) + \beta f\big(\big\lceil \tfrac{n}{2} \big\rceil\big) + g(n) \qquad (n \geqslant 2),$$

with $g(n)$ and $f(1)$ given, where $\alpha, \beta \geqslant 0$ with $\alpha + \beta > 0$; such recurrences appear often in analysis of computer algorithms, numeration systems, combinatorial sequences, and related areas. We show that the solution satisfies always the simple *identity*

$$f(n) = n^{\log_2(\alpha+\beta)} P(\log_2 n) - Q(n)$$

under an optimum (iff) condition on $g(n)$. This form is not only an identity but also an asymptotic expansion because $Q(n)$ is of a smaller order. Explicit forms for the *continuity* of the periodic function P are provided, together with a few other smoothness properties. We show how our results can be easily applied to many dozens of concrete examples collected from the literature, and how they can be extended in various directions. Our method of proof is surprisingly simple and elementary, but leads to the strongest types of results for all examples to which our theory applies.

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Contents

1 Introduction

This paper is a sequel to [27], where we studied the case $(\alpha, \beta) = (1, 1)$ of the following recurrence

$$f(n) = \alpha f\left(\left\lfloor \tfrac{n}{2} \right\rfloor\right) + \beta f\left(\left\lceil \tfrac{n}{2} \right\rceil\right) + g(n) \qquad (n \geqslant 2), \tag{1.1}$$

with $f(1)$ and $\{g(n)\}_{n \geqslant 2}$ given; we focus here on the general case of two given constants $\alpha, \beta > 0$. (The case when $\alpha \leqslant 0$ or $\beta \leqslant 0$ is briefly discussed in Section 6.) As in [27], our aim in this paper will be

- to establish *optimum iff-conditions* for the identity

$$f(n) = n^\varrho P(\log_2 n) - Q(n) \qquad (n \geqslant 1), \tag{1.2}$$

which is also an asymptotic expansion, where

$$\varrho := \log_2(\alpha + \beta), \tag{1.3}$$

P is a bounded, *continuous*, *periodic* function and $Q(n)$ is of a smaller order $o(n^\varrho)$; and

- to explore the usefulness of such a result by examining other associated properties and applying to many concrete examples.

In addition, we also examine further *smoothness properties* of the periodic function P, and introduce and explore a new notion to describe the *equivalence of different recurrences*.

An elementary interpolation approach. The crucial step of our approach is to identify a (generally nonlinear) interpolation function $\varphi(x)$ such that the sequence $f(n)$ as defined by (1.1) for positive integers n can be extended to a *continuous* function $f(x)$ defined for all real $x \geqslant 1$ in the way that $f(x)$ equals the original sequence $f(n)$ when $x = n$, a positive integer, and there is a version of the recurrence (1.1) valid for all x; see Section 2 for details.

Such an interpolation-based analysis for (1.1) will then be extended (in Section 7) to the more general q-ary recurrence ($q \geqslant 2$) of the form

$$f(n) = \sum_{0 \leqslant j < q} \alpha_j f\left(\left\lfloor \tfrac{n+j}{q} \right\rfloor\right) + g(n) \qquad (n \geqslant q), \tag{1.4}$$

for some given constants $\alpha_0, \ldots, \alpha_{q-1}$, and a result of the form (1.2) will also be derived under some conditions. Typical situations where (1.4) arises is the application of divide-and-conquer into q parts whose sizes are as evenly as possible. The special case when $\alpha_j = 1$ for $0 \leqslant j < q$ was already discussed in [27]. Another special case is $\alpha_j = 0$ for $1 \leqslant j \leqslant q - 2$, which yields (with $\alpha = \alpha_0$ and $\beta = \alpha_{q-1}$) the recursion $f(n) = \alpha f(\lfloor \tfrac{n}{q} \rfloor) + \beta f(\lceil \tfrac{n}{q} \rceil) + g(n)$ considered in [11, Theorem 4.1] and [33] although they allow also non-integer $q > 1$.

Recurrences with or without floors and ceilings. While the divide-and-conquer paradigm with evenly divided parts is widely used in computer algorithms, our formulation of the divide-and-conquer recurrence (1.4), as well as the very precise identity (1.2), is surprisingly rare in the computer algorithm literature; instead one finds predominantly a recurrence of the form

$$f(n) = (\alpha + \beta) f\left(\tfrac{n}{2}\right) + g(n), \tag{1.5}$$

or more generally

$$f(n) = \sum_{0 \leqslant j < d} \alpha_j f\left(\tfrac{n}{q_j}\right) + g(n), \tag{1.6}$$

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where $\alpha, \alpha_j > 0$, $d = 1, 2, \ldots$ and $q_j > 1$. According to the first edition of Cormen et al.'s widely used textbook on Algorithms [10, p. 54]: "*When we state and solve recurrences, we often omit floors, ceilings, and boundary conditions. We forge ahead without these details and later determine whether or not they matter. . . . we shall address some of these details to show the fine points of recurrence solution methods.*"

Such a simplifying approach also appears in most publications on Algorithms. One of our aims in this paper is to show that *retaining floors and ceilings is not much more complicated than omitting them, and with various advantages that are mostly unnoticed in the literature.* This suggests that one main reason of omitting floors and ceilings in handling a divide-and-conquer recurrence lies more in methodological deficiencies than simply technical conventions; the approach proposed in this paper will then complete to some extent the required methodological developments.

More precisely, when $\alpha, \beta, g(n) \geqslant 0$, typical approaches adopted in the computer algorithms community to solving the recurrence (1.1) include

- dropping floor and ceiling in (1.1) by assuming n to be a power of 2, resulting in the closed-form expression

$$f(2^m) = \sum_{1 \leqslant j \leqslant m} (\alpha + \beta)^{m-j} g(2^j) + (\alpha + \beta)^m f(1), \tag{1.7}$$

  and

- lower- and upper-bounding f by keeping only floor and only ceiling function in (1.1), leading to

$$f_-(n) := (\alpha + \beta) f_- \left(\lfloor \tfrac{n}{2} \rfloor \right) + g(n), \tag{1.8}$$

  and

$$f_+(n) := (\alpha + \beta) f_+ \left(\lceil \tfrac{n}{2} \rceil \right) + g(n), \tag{1.9}$$

  respectively.

While simple and effective in estimating the growth order of $f(n)$ for large n, both approaches suffer from subtle oversights and intrinsic limitations, with different shortcomings.

Monotonicity. First, in either of the approaches (1.7) and (1.8)–(1.9), the next crucial property used in estimating the asymptotic growth of $f(n)$ is *monotonicity*, which in the first approach is of the form $f(2^m) \leqslant f(2^m + \ell) \leqslant f(2^{m+1})$ for $0 \leqslant \ell \leqslant 2^m$, and $f_-(n) \leqslant f(n) \leqslant f_+(n)$ in the second approach. However, these inequalities may not hold in general due to the periodic nature of the recurrences (1.1), (1.8) and (1.9). For example, take $\alpha = \beta = 1$ and $g(n) = 1 + \mathbf{1}_{n \text{ odd}}$ with $f(1) = f_+(1) = 0$. Then $8 = f(7) > f(8) = 7$, and $f(15) = 17 > f_+(15) = 16$. Thus the use of the monotonicity is more subtle than it is generally taken to be.

On the other hand, when $g(n) = \mathbf{1}_{n \text{ odd}}$ with $f(1) = 0$ (which yields A296062 in the *On-Line Encyclopedia of Integer Sequences* database [34] with many combinatorial interpretations), then $f(n)$ oscillates between 0 and $\Theta(n)$. Thus not only monotonicity fails but also the growth order oscillates violently, although our result yields that $f(n) = n P(\log_2 n)$ for some continuous periodic function; see (5.7).

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Discontinuity. Apart from monotonicity, there is yet another deeper reason why the original sequence (1.1) is preferred to its simplified one-sided versions (1.8) and (1.9): the periodic function P in (1.2) is always *continuous*, while the corresponding one for the solution of either (1.8) or (1.9) is almost always *discontinuous*; more precisely, in typical cases the function $P(t)$ is discontinuous at every t such that 2^t is a dyadic rational, i.e., has a finite binary representation (see Section 6.1 and, for details in the case $\alpha + \beta = 2$, [27, Section 8]). But why does continuity matters here? The reason is because the periodic function P when evaluated at $\log_2 n$ (see (1.2)), involves indeed functions evaluated at the dyadic rational $n/2^{\lfloor \log_2 n \rfloor}$ (see (2.22) and (2.30)), so that discontinuity causes the sequence (or the original cost function) to have more violent jumps even for neighbouring input sizes. Thus simplifying the recurrence (1.1) to either (1.8) or (1.9) has the advantage of being easily solvable by iteration, but suffers from structural discontinuities, or more rough oscillations.

Master theorems. Another commonly used approach to solve (1.5) is to apply the so-called "master theorems" (see [2, 11]), which are generally effective and user-friendly, but does not provide more precise asymptotic approximations. For example, in the case of (1.1), if $g(n) = O(n^{\varrho - \varepsilon})$, $\varepsilon > 0$, then the master theorem [11, § 4.5] or [33] gives $f(n) = \Theta(n^\varrho)$, where ϱ is defined in (1.3), while under the same growth order of g, our approach (see Corollary 2.14) again guarantees (1.2) with explicitly computable functions P and Q. There do exist finer master theorems that give more precise asymptotics under stronger assumptions on $g(n)$ (such as monotonicity; see [13]), but none of them is as precise as our identity (1.2).

Discrete and continuous master theorems. An additional feature of our interpolation-based analysis is that we always work on the same real function $f(x)$, which coincides with the original sequence $f(n)$ at integer parameters, unlike general master theorems that distinguish between "discrete master theorems" and "continuous master theorems"; for example, according to [33]:

> *To distinguish the two situations, we call the master theorem without floors and ceilings the continuous master theorem and the master theorem with floors and ceilings the discrete master theorem.*

The subtleties of the two different versions (together with other issues) are only very recently thoroughly examined in [33], where they write:

> *Several academic works provide proofs and proof sketches of the discrete master theorem. To the best of our knowledge, however, all of these proofs are either incomplete, incorrect, or require sophisticated mathematics.*

See also the long paper (more than 300 pages) [4] for other delicate issues arising from divide-and-conquer recurrences. Additionally, the chapter on "Recurrences" (Chapter I.4) in the first edition of Cormen et al.'s book [10] is now largely expanded and updated in the latest, very recent, edition [11, Ch. I.4], more than three decades after its first edition and following the corresponding developments in clarifying the subtleties; see [4, 11].

For more information and references on master theorems and divide-and-conquer recurrences, see, for example, [13, 23, 26, 33, 35]. See also [27] for more references on other approaches (including complex-analytic, Tauberian, renewal, fractal geometry, and Ansatz or guess-and-prove, called the substitution method in [11]) used in the literature for solving (1.1).

Periodic equivalence of sequences. For a more canonical way to group or classify the diverse periodic functions P, we will introduce in Section 4.1 a useful notion called "periodic equivalence", roughly meaning that different sequences share, modulo amplitude and scale, the same oscillating part. For example, denote by $S_{2,1}(n)$ (A006046) the total number of odd entries in the first n rows of Pascal triangle; then $f(n) = S_{2,1}(n)$ satisfies (1.1) with $(\alpha, \beta) = (2, 1)$, $g(n) = 0$ and $f(1) = 1$. We have $S_{2,1}(n) = n^{\log_2 3} P(\log_2 n)$; see Example 4.9. Then the following OEIS sequences, all satisfying (1.1) with $(\alpha, \beta) = (2, 1)$, are periodically equivalent to $S_{2,1}(n)$ (involving, up to scale and amplitude, the same P):

OEIS id.	$g(n)$	$f(1)$	$f(n)$
A051679	$\frac{1}{8}n^2 - \begin{cases} \frac{n}{4}, & n \text{ even} \\ \frac{1}{8}, & n \text{ odd} \end{cases}$	0	$\binom{n+1}{2} - S_{2,1}(n)$
A080978	-2	3	$2S_{2,1}(n) + 1$
A159912	$\lfloor \frac{n}{2} \rfloor$	1	$2S_{2,1}(n) - n$
A171378	$\lceil \frac{n}{2} \rceil^2 - \mathbf{1}_{n \text{ odd}}$	0	$n^2 - S_{2,1}(n)$
A267700	$\lfloor \frac{n}{2} \rfloor$	0	$S_{2,1}(n) - n$

See Example 4.9 for more periodically equivalent sequences.

Smoothness of the periodic function. While the periodic equivalence is introduced to identify the same fluctuating part of different sequences satisfying the *same* recurrence, we also examine the varying smoothness nature exhibited by *different* recurrences. The main motivating observation is that most periodic functions we obtain have visible cusps (likely to be non-differentiable points), and there are other classes of functions (such as Lipshitz and Hölder continuous) between the class of continuous functions and that of continuously differentiable functions. We will thus clarify the different, characteristic, inherent types of Hölder continuity of the interpolated function $f(x)$ and the periodic function $P(t)$ when (α, β) varies.

For example, in Figure 1, we plot the periodic functions P (as defined in (1.2)) when f satisfies (1.1) with $g(n) = 0$ and $f(1) = 1$, and with $(\alpha, \beta) = (\alpha, 1)$, $\alpha = 2, 3, 4, 5$ (the lower blue curves) and $(\alpha, \beta) = (1, \beta)$, $\beta = 2, 3, 4, 5$ (the upper green curves). Our results show that these periodic functions are Hölder continuous with exponent $\log_2(1 + \alpha^{-1})$ and $\log_2(1 + \beta^{-1})$, respectively, and we conjecture that these exponents are the best possible; see Section 3 for details. Thus for these recurrences, the larger the values of α or β, the "less smooth" the periodic functions.

Moreover, we show that many of the periodic functions appearing in our analysis are indeed not continuously differentiable at all points in $(0, 1)$. Several examples are piecewise differentiable with jumps in the derivative at some points (see, for example, Remark 2.8 and Examples 5.4 and 5.5). Other examples are less smooth; for brevity of presentation, we discuss only one case (A006581) in detail in Example 5.6 and show (in Appendix D) that the periodic function is continuous but nowhere differentiable, leaving such a deeper property for other sequences to the interested reader.

A generating function viewpoint. To see how the general cases differ from the special case $(\alpha, \beta) = (1, 1)$ or more generally $\alpha = \beta$, we consider the generating function

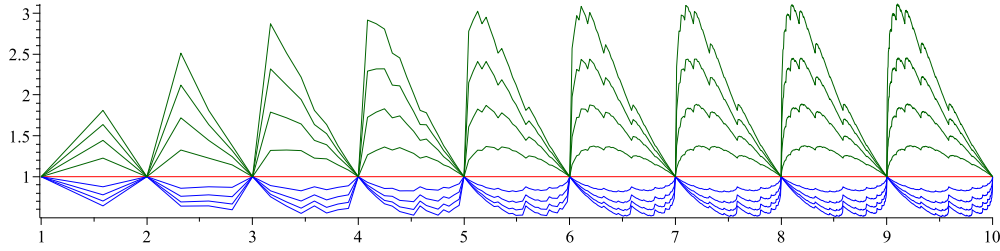$$A(z) := \sum_{n \geqslant 1} f(n) z^n, \tag{1.10}$$

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Figure 1: Periodic fluctuations of the functions $P(\log_2 n) = f(n)n^{-\log_2(\alpha+\beta)}$ for $n = 2, \ldots, 1024$ when f satisfies (1.1) *with $g(n) = 0$, $f(1) = 1$ and with (the blue curves) $\beta = 1$ and $\alpha = 2, 3, 4, 5$ (from top to bottom), and (the green curves) $\alpha = 1$ and $\beta = 2, 3, 4, 5$ (from bottom to top).*

which, by (1.1) satisfies the functional equation

$$A(z) = \frac{\beta + (\alpha + \beta)z + \alpha z^2}{z} A(z^2) + B(z), \tag{1.11}$$

where

$$B(z) := f(1)(1 - \beta)z + \sum_{n \geqslant 2} g(n)z^n. \tag{1.12}$$

We see that if $\alpha = \beta$, then the functional equation becomes

$$A(z) = \frac{\alpha(1 + z)^2}{z} A(z^2) + B(z), \tag{1.13}$$

so that the generating function $\bar{A}(z) := \frac{(1-z)^2}{z} A(z)$ of the second difference of $f(n)$ satisfies the simpler equation

$$\bar{A}(z) = \alpha \bar{A}(z^2) + \bar{B}(z), \tag{1.14}$$

where $\bar{B}(z) := \frac{(1-z)^2}{z} B(z)$. Assuming for simplicity (without real loss of generality) that $\bar{B}(0) = 0$, we then obtain, by iteration, the exact solution

$$A(z) = \frac{z}{(1 - z)^2} \sum_{k \geqslant 0} \alpha^k \bar{B}(z^{2^k}). \tag{1.15}$$

Such a neat representation is the basis of the analytic approach introduced in [20], but is not available in general when $\alpha \neq \beta$. We will therefore use a different method.

This paper is structured as follows. We develop in Section 2 the required technicalities in order to prove (1.2), and then address the smoothness properties of P in Section 3. These two sections provide a theoretical foundation to the resolution of the divide-and-conquer recurrences of the form (1.1). Applications of our theory to concrete examples are discussed in Section 4 for $\alpha \neq \beta$ and in Section 5 for $\alpha = \beta$. We then extend very briefly our analysis to nonpositive α or β in Section 6 and to general q-ary recurrence (1.4) in Section 7. For completeness, an appendix on the connection of our approach to Mellin transforms is given, together with two others providing detailed proofs of some results in the paper.

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Notation. For convenience, we introduce the operator $\Lambda_{\alpha,\beta}$ as follows:

$$\Lambda_{\alpha,\beta}[f](n) := f(n) - \alpha f\left(\left\lfloor \tfrac{n}{2} \right\rfloor\right) - \beta f\left(\left\lceil \tfrac{n}{2} \right\rceil\right). \tag{1.16}$$

Let, for $x > 0$,

$$L_x := \lfloor \log_2 x \rfloor, \qquad\qquad \theta_x := \{\log_2 x\} = \log_2 x - L_x \in [0, 1), \tag{1.17}$$

and $L_0 := 0$.

For real x and y, we let $x \vee y := \max(x, y)$.

2 The recurrence $\Lambda_{\alpha,\beta}[f] = g$

Here and throughout this section, we assume that $\alpha, \beta > 0$, and define $g(1) := 0$. The recurrence (1.1) can be rewritten as

$$\begin{aligned} f(2n) &= (\alpha + \beta)f(n) + g(2n), \\ f(2n + 1) &= \alpha f(n) + \beta f(n + 1) + g(2n + 1), \end{aligned} \tag{2.1}$$

for $n \geqslant 1$, with given $f(1)$ and $g(n)$, $n \geqslant 2$.

Extending the sequence $f(n)$ to a function $f(x)$ in $\mathbb{R}^+$. We extend the sequence $f(n)$ to a continuous function $f(x)$ defined for real $x \geqslant 1$ by interpolation between the integers with scaled copies of a certain function $\varphi : [0, 1] \to [0, 1]$ constructed below:

$$\begin{aligned} f(n + t) &:= f(n) + \varphi(t)\left(f(n + 1) - f(n)\right) \\ &= (1 - \varphi(t))\,f(n) + \varphi(t)f(n + 1), \end{aligned} \tag{2.2}$$

for $n \geqslant 1$ and $0 \leqslant t \leqslant 1$; in a similar way, we construct $g(x)$ from $g(n)$. The function φ is constructed so that we will have

$$f(x) = (\alpha + \beta)f\left(\frac{x}{2}\right) + g(x), \qquad x \geqslant 2. \tag{2.3}$$

We further define $g(x) := 0$ for $x \in [0, 1)$; thus $g(x)$ is defined for $x \geqslant 0$.

The function $\varphi(t)$ depends on α and β; we sometimes write it as $\varphi_{\alpha,\beta}(t)$ to emphasise the dependence on parameters.

Construction of the interpolation function φ. We first give relations on φ that imply the functional equation (2.3). The existence of such a φ is shown later.

Lemma 2.1. *Let $\varphi(t) = \varphi_{\alpha,\beta}(t)$ be a function on $[0, 1]$ such that $\varphi(0) = 0$, $\varphi(1) = 1$ and*

$$\varphi(t) = \begin{cases} \frac{\beta}{\alpha+\beta}\,\varphi(2t), & \text{if } t \in [0, \tfrac{1}{2}], \\ \frac{\alpha}{\alpha+\beta}\,\varphi(2t - 1) + \frac{\beta}{\alpha+\beta}, & \text{if } t \in [\tfrac{1}{2}, 1]. \end{cases} \tag{2.4}$$

Assume that (2.1) holds. Then (2.3) holds if we extend $f(n)$ and $g(n)$ by (2.2) to $f(x)$ and $g(x)$, respectively, for all real $x \geqslant 1$.

Proof. If $n \geqslant 1$ and $0 \leqslant t \leqslant \frac{1}{2}$, then, using the interpolation (2.2), the recurrences (2.1) and the assumption (2.4), we have

$$
\begin{aligned}
f(2n + 2t) &- g(2n + 2t) \\
&= (1 - \varphi(2t))(f(2n) - g(2n)) + \varphi(2t)(f(2n+1) - g(2n+1)) \\
&= (1 - \varphi(2t))(\alpha + \beta)f(n) + \varphi(2t)\big(\alpha f(n) + \beta f(n+1)\big) \\
&= (\alpha + \beta)f(n) - \varphi(2t)\beta f(n) + \varphi(2t)\beta f(n+1) \\
&= (\alpha + \beta)f(n) - (\alpha + \beta)\varphi(t)f(n) + (\alpha + \beta)\varphi(t)f(n+1) \\
&= (\alpha + \beta)\big((1 - \varphi(t))f(n) + \varphi(t)f(n+1)\big) \\
&= (\alpha + \beta)f(n + t).
\end{aligned}
\tag{2.5}
$$

Similarly, the lower part of (2.4) can be rewritten as

$$
(\alpha + \beta)\,\varphi\left(\tfrac{1}{2} + t\right) = \alpha\varphi(2t) + \beta, \qquad t \in [0, \tfrac{1}{2}],
\tag{2.6}
$$

and thus, still for $n \geqslant 1$ and $0 \leqslant t \leqslant \frac{1}{2}$,

$$
\begin{aligned}
f(2n + 1 + 2t) &- g(2n + 1 + 2t) \\
&= (1 - \varphi(2t))(f(2n+1) - g(2n+1)) + \varphi(2t)(f(2n+2) - g(2n+2)) \\
&= (1 - \varphi(2t))(\alpha f(n) + \beta f(n+1)) + \varphi(2t)(\alpha + \beta)f(n+1) \\
&= \alpha f(n) + \beta f(n+1) - \varphi(2t)\alpha f(n) + \varphi(2t)\alpha f(n+1) \\
&= \alpha f(n) + \beta f(n+1) - \big((\alpha + \beta)\varphi\left(\tfrac{1}{2} + t\right) - \beta\big)\big(f(n) - f(n+1)\big) \\
&= (\alpha + \beta)f(n) - (\alpha + \beta)\varphi\left(\tfrac{1}{2} + t\right)f(n) + (\alpha + \beta)\varphi\left(\tfrac{1}{2} + t\right)f(n+1) \\
&= (\alpha + \beta)\big((1 - \varphi\left(\tfrac{1}{2} + t\right))f(n) + \varphi\left(\tfrac{1}{2} + t\right)f(n+1)\big) \\
&= (\alpha + \beta)f\left(n + \tfrac{1}{2} + t\right).
\end{aligned}
\tag{2.7}
$$

Combining (2.5) and (2.7), we obtain

$$
f(n + 2t) = (\alpha + \beta)f\left(\tfrac{n}{2} + t\right) + g(n + 2t)
\tag{2.8}
$$

for $n \geqslant 2$ and $0 \leqslant t \leqslant \frac{1}{2}$, and thus (2.3) holds. $\qquad\square$

Remark 2.2. Conversely, it is immediate that if (2.3) holds for the extensions of f and g defined by (2.2), then (2.1) holds.

In the case $\alpha = \beta = 1$ treated in [27], the system of equations (2.4) has the solution $\varphi(t) = t$. In general, it is not obvious that a continuous solution to (2.4) exists. We now show that this is the case, and that this φ is unique.

Lemma 2.3. *If $\alpha, \beta > 0$, then there exists a unique continuous function $\varphi(t) = \varphi_{\alpha,\beta}(t)$ on $[0, 1]$ such that $\varphi(0) = 0$, $\varphi(1) = 1$ and (2.4) holds. Moreover, φ is strictly increasing.*

Proof. The equation (2.4) and the conditions $\varphi(0) = 0$, $\varphi(1) = 1$ define recursively $\varphi(t)$ uniquely for dyadic rational $t \in [0, 1]$; hence, there is at most one continuous φ satisfying these requirements. The existence of such a φ can be proved in several different ways.

First proof (probabilistic). Let φ be the distribution function $\varphi(t) := \Pr(X \leqslant t)$ of the random variable $X \in [0, 1]$ defined by the binary expansion $X = 0.B_1 B_2 \ldots$, where the bits $B_1, B_2, \ldots$ are independent and with $\Pr(B_i = 1) = \frac{\alpha}{\alpha+\beta}$. Since the random variable $X' := 0.B_2 B_3 \cdots = 2X - B_1$ has the same distribution as X, we see that

$$
\begin{aligned}
\varphi(t) &:= \Pr(X \leqslant t) \\
&= \begin{cases} \Pr(B_1 = 0) \Pr(X' \leqslant 2t), & t \in [0, \tfrac{1}{2}], \\ \Pr(B_1 = 0) + \Pr(B_1 = 1) \Pr(X' \leqslant 2t - 1), & t \in [\tfrac{1}{2}, 1], \end{cases} \\
&= \begin{cases} \frac{\beta}{\alpha+\beta}\, \varphi(2t), & t \in [0, \tfrac{1}{2}], \\ \frac{\beta}{\alpha+\beta} + \frac{\alpha}{\alpha+\beta}\, \varphi(2t - 1), & t \in [\tfrac{1}{2}, 1], \end{cases}
\end{aligned} \tag{2.9}
$$

which verifies (2.4). Furthermore, φ is continuous and strictly increasing on $[0, 1]$, with $\varphi(0) = 0$ and $\varphi(1) = 1$.

Second proof (digital sums). An alternative approach begins with an explicit construction, which will also be useful later. For $t = \sum_{j \geqslant 1} b_j 2^{-j} \in [0, 1]$, where $b_j \in \{0, 1\}$, define

$$
\varphi(t) := \sum_{j \geqslant 1} b_j \left(\frac{\alpha}{\beta} \right)^{b_1 + \cdots + b_{j-1}} \left(\frac{\beta}{\alpha + \beta} \right)^j. \tag{2.10}
$$

Equivalently, define, for $t = \sum_{k \geqslant 1} 2^{-e_k} \in [0, 1]$, where $1 \leqslant e_1 < e_2 < \cdots$ is a finite or infinite sequence,

$$
\varphi\left(t = \sum_{k \geqslant 1} 2^{-e_k} \right) := \sum_{k \geqslant 1} \frac{\alpha^{k-1} \beta^{e_k - k + 1}}{(\alpha + \beta)^{e_k}}. \tag{2.11}
$$

We first show that both (2.10) and (2.11) are well defined for dyadic rational t with two different representations, namely,

$$
\sum_{1 \leqslant i \leqslant k} 2^{-e_i} = \sum_{1 \leqslant j < k} 2^{-e_j} + \sum_{j \geqslant 1} 2^{-(e_k + j)}. \tag{2.12}
$$

The value in (2.11) for the last term on the left-hand side of (2.12) is $\frac{\alpha^{k-1} \beta^{e_k - k + 1}}{(\alpha + \beta)^{e_k}}$; the value for the sum on the right-hand side equals

$$
\sum_{j \geqslant 1} \frac{\alpha^{(k+j-1)-1} \beta^{e_k + j - (k+j-1)+1}}{(\alpha + \beta)^{e_k + j}} = \frac{\alpha^{k-1} \beta^{e_k - k + 1}}{(\alpha + \beta)^{e_k}} \sum_{j \geqslant 1} \frac{\alpha^{j-1} \beta}{(\alpha + \beta)^j} = \frac{\alpha^{k-1} \beta^{e_k - k + 1}}{(\alpha + \beta)^{e_k}}, \tag{2.13}
$$

since the final geometric series sums to 1. Hence, the two representations in (2.12) yield the same sum (2.11), which always equals the sum (2.10), and thus φ is well-defined on $[0, 1]$.

Clearly, $\varphi(0) = 0$ and $\varphi(1) = 1$, again by summing the same geometric sum. Moreover, (2.4) follows easily from (2.10), considering the cases $b_1 = 0$ and $b_1 = 1$ separately.

Next, if $t = \sum_{1 \leqslant j \leqslant N} b_j 2^{-j}$ is a dyadic rational in $[0, 1)$, and $s := \sum_{1 \leqslant j \leqslant N} b_j$, then, by (2.4) and induction on $N \geqslant 0$, we have

$$
\varphi\left(t + 2^{-N} \right) - \varphi(t) = \frac{\alpha^s \beta^{N-s}}{(\alpha + \beta)^N}. \tag{2.14}
$$

It follows from (2.14) that φ is strictly increasing on the set of dyadic rationals in $[0, 1]$. Further-more, suppose that $t_1 = j2^{-N}$ is a dyadic rational in $[0, 1)$, and let $t_2 := t + 2^{-N}$. If $t \in [t_1, t_2]$, then both t and t_2 have binary expansions beginning with the expansion of t_1 (choosing the infinite representation for t_2), and it follows from (2.11) that $\varphi(t_1) \leqslant \varphi(t) \leqslant \varphi(t_2)$. Now, suppose $0 \leqslant t < u \leqslant 1$, and choose N such that $2^{1-N} < u - t$. Then there exist j and k with $j2^{-N} \leqslant t \leqslant (j+1)2^{-N} < k2^{-N} \leqslant u \leqslant (k+1)2^{-N}$. Hence, by what was just shown, $\varphi(t) \leqslant \varphi\big((j+1)2^{-N}\big) < \varphi\big(k2^{-N}\big) \leqslant \varphi(u)$. Thus, φ is strictly increasing.

The monotonicity implies that any discontinuity of φ must be a jump. Define

$$\Delta_+\varphi(t) := \lim_{u \searrow t} \varphi(u) - \varphi(t), \qquad t \in [0, 1), \tag{2.15}$$

the jump to the right at t, and let $R := \sup_{t \in [0,1)} \Delta_+(t)$. It follows from (2.4) that $R = \frac{\alpha \vee \beta}{\alpha + \beta} R$, and thus $R = 0$. Hence, φ is right-continuous. Similarly, φ is left-continuous.
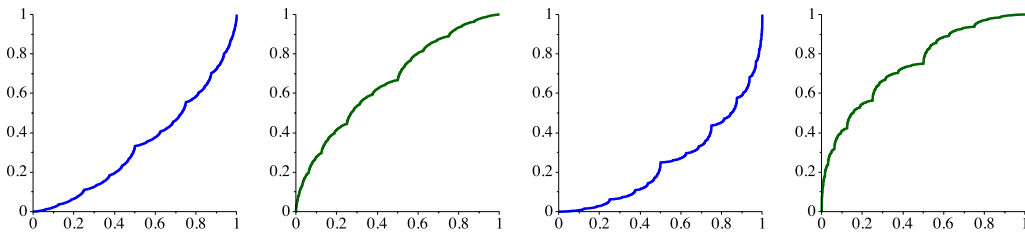

Figure 2: The function φ for $(\alpha, \beta) = (2, 1), (1, 2), (3, 1), (1, 3)$ (from left to right).

Third proof (recursive construction). Yet another alternative, similar to the construction of Koch's snowflake curve [41], is to define $\varphi_0(t) := t, t \in [0, 1]$, and then recursively let φ_{k+1} consist of two suitably scaled copies of φ_k; more precisely

$$\varphi_{k+1}(t) := \begin{cases} \frac{\beta}{\alpha+\beta} \varphi_k(2t), & \text{if } t \in [0, \frac{1}{2}], \\ \frac{\alpha}{\alpha+\beta} \varphi_k(2t - 1) + \frac{\beta}{\alpha+\beta}, & \text{if } t \in [\frac{1}{2}, 1]; \end{cases} \tag{2.16}$$

see Figure 3 for an illustration. Then, by induction, $|\varphi_{k+1}(t) - \varphi_k(t)| \leqslant \left(\frac{\alpha \vee \beta}{\alpha + \beta}\right)^k$ for $k \geqslant 0$ and $t \in [0, 1]$; consequently, the functions $\varphi_k(t)$ converge uniformly on $[0, 1]$ to a continuous function φ satisfying (2.4), with $\varphi(0) = 0$ and $\varphi(1) = 1$. It follows also that φ is weakly increasing; it is then easy to show, using (2.4), that such φ is strictly increasing, but we omit this, since we have already shown this using the other constructions. $\square$

Remark 2.4. If $\alpha = \beta$, then (2.4) is solved by $\varphi(t) = t$, just as in the case $\alpha = \beta = 1$ treated in [27], so $f(x)$ is defined in (2.2) by linear interpolation. If $\alpha \neq \beta$, then this is not the case; a linear interpolation would not yield (2.3).

Figure 2 shows that φ has a typical fractal shape when $\alpha \neq \beta$. We will show in Lemma 3.1 and Remark 3.5 below that φ is Hölder continuous but not Lipschitz continuous when $\alpha \neq \beta$.

Remark 2.5. It follows from (2.4) that

$$\varphi_{\beta,\alpha}(t) = 1 - \varphi_{\alpha,\beta}(1 - t), \tag{2.17}$$

so the graph of $\varphi_{\beta,\alpha}$ equals the graph of $\varphi_{\alpha,\beta}$ rotated about $(\frac{1}{2}, \frac{1}{2})$. See Figures 2 and 3 for a graphical illustration of some examples.
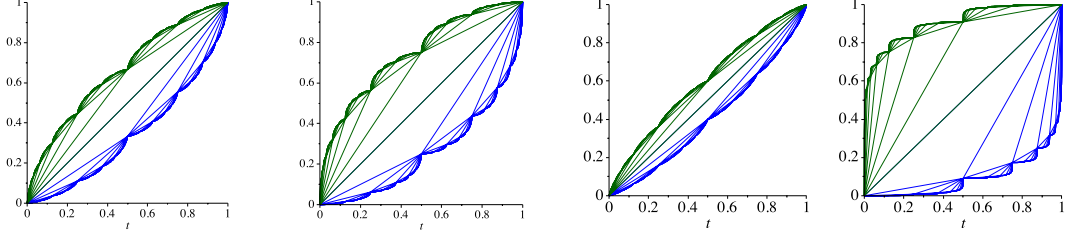
Figure 3: The functions φ_k for $k = 0, \ldots, 15$, where $(\alpha, \beta) = (2, 1), (1, 2)$ (left), $(\alpha, \beta) = (3, 1), (1, 3)$ (middle-left), $(\alpha, \beta) = (3, 2), (2, 3)$ (middle-right), and $(\alpha, \beta) = (10, 1), (1, 10)$ (right).

Remark 2.6. For later use we note that

$$\int_0^1 \varphi(x)\,\mathrm{d}x = \frac{\beta}{\alpha + \beta}. \tag{2.18}$$

To see this, we may denote the integral by I and integrate both sides of (2.4), which then yields

$$I = \frac{\beta}{\alpha + \beta} \cdot \frac{1}{2}I + \frac{\alpha}{\alpha + \beta} \cdot \frac{1}{2}I + \frac{\beta}{\alpha + \beta} \cdot \frac{1}{2} = \frac{1}{2}I + \frac{1}{2}\frac{\beta}{\alpha + \beta}; \tag{2.19}$$

solving I gives (2.18). $\qquad\square$

Identities. By iterating the functional equation (2.3), we obtain, for $x \geqslant 1$,

$$f(x) = \sum_{0 \leqslant k < m} (\alpha + \beta)^k g(2^{-k}x) + (\alpha + \beta)^m f(2^{-m}x), \qquad 0 \leqslant m \leqslant L_x. \tag{2.20}$$

This leads to the following identity.

Lemma 2.7. *Assume that f satisfies (2.1) and (2.2). Then, for $x \geqslant 1$,*

$$x^{-\varrho} f(x) = \sum_{k \geqslant 0} (2^{-k}x)^{-\varrho} g(2^{-k}x) + f(1)P_0\left(\log_2 x\right), \tag{2.21}$$

where $\varrho := \log_2(\alpha + \beta)$ and

$$P_0(t) := \left(1 + (\alpha + \beta - 1)\varphi\left(2^{\{t\}} - 1\right)\right)(\alpha + \beta)^{-\{t\}} \tag{2.22}$$

is a continuous 1-periodic function satisfying $P_0(0) = P_0(1) = 1$.

Since $g(x) = 0$ for $x \leqslant 1$, the sum in (2.21) is indeed finite.

Proof. First, (2.22) yields $P_0(0) = 1$ and $\lim_{t \nearrow 1} P_0(t) = \left(1 + (\alpha + \beta - 1)\varphi(1)\right)(\alpha + \beta)^{-1} = (\alpha + \beta)(\alpha + \beta)^{-1} = 1 = P_0(1)$. Furthermore, P_0 is continuous on $[0, 1)$ since φ is continuous, and P_0 is 1-periodic. It follows that P_0 is continuous on $\mathbb{R}$.

By (2.1), $f(2) = (\alpha + \beta)f(1) + g(2)$. Thus, for $1 \leqslant x < 2$, recalling $g(1) = 0$,

$$\begin{aligned}
f(x) &= f(1) + \varphi(x - 1)(f(2) - f(1)) \\
&= f(1) + \varphi(x - 1)\left((\alpha + \beta - 1)f(1) + g(2)\right) \\
&= f(1)\left(1 + (\alpha + \beta - 1)\varphi(x - 1)\right) + g(x) \\
&= f(1)P_0\left(\log_2 x\right)(\alpha + \beta)^{\log_2 x} + g(x). \tag{2.23}
\end{aligned}$$

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Now consider $x \geqslant 1$. Take $m = L_x$ in (2.20) and use (2.23) with x replaced by $2^{-L_x}x = 2^{\theta_x} \in [1, 2)$; we then obtain, by the relation $\alpha + \beta = 2^{\varrho}$,

$$
\begin{aligned}
f(x) &= \sum_{0 \leqslant k < L_x} (\alpha + \beta)^k g(2^{-k}x) + (\alpha + \beta)^{L_x} f(2^{-L_x}x) \\
&= \sum_{0 \leqslant k \leqslant L_x} (\alpha + \beta)^k g(2^{-k}x) + (\alpha + \beta)^{L_x + \theta_x} f(1) P_0(\theta_x) \\
&= \sum_{k \geqslant 0} 2^{\varrho k} g(2^{-k}x) + 2^{\varrho \log_2 x} f(1) P_0(\log_2 x).
\end{aligned}
\tag{2.24}
$$

This implies (2.21). $\qquad\square$

Remark 2.8. If $\alpha = \beta$, then $\varphi(t) = t$ by Remark 2.4, and thus (2.22) yields

$$
P_0(t) := \left(1 + (\alpha + \beta - 1)(2^{\{t\}} - 1)\right)(\alpha + \beta)^{-\{t\}}.
\tag{2.25}
$$

In the case $\alpha = \beta = 1$ studied in [27], and also in the case $\alpha = \beta = \frac{1}{2}$, this yields $P_0(t) \equiv 1$. In all other cases with $\alpha = \beta > 0$, the periodic function $P_0(t)$ is infinitely differentiable in $(0, 1)$, but a simple calculation shows that the derivative has a jump at the integers; hence, P_0 is Lipschitz but not continuously differentiable.

Remark 2.9. If $\beta = 1$, then (2.22) and (2.4) yield also

$$
P_0(t) = (\alpha + 1)^{1 - \{t\}} \varphi(2^{\{t\} - 1}).
\tag{2.26}
$$

We can now give an extension of Theorem 2 in our previous paper [27].

Theorem 2.10. *Suppose that f and g are given by* (2.1) *and* (2.2)*. The following are equivalent.*

(i) $n^{-\varrho} f(n) = P(\log_2 n) + o(1)$ *as $n \to \infty$, for some continuous 1-periodic function P on $\mathbb{R}$.*

(ii) $x^{-\varrho} f(x) = P(\log_2 x) + o(1)$ *as $x \to \infty$, for some continuous 1-periodic function P on $\mathbb{R}$.*

(iii)

$$
x^{-\varrho} f(x) = P(\log_2 x) + o(1) \qquad \textit{as } x \to \infty,
\tag{2.27}
$$

for some 1-periodic function P on $\mathbb{R}$.

(iv) *The sum*

$$
Q(x) := \sum_{m \geqslant 1} 2^{-\varrho m} g(2^m x)
\tag{2.28}
$$

converges uniformly for $x \in [1, 2]$.

Furthermore, when these conditions hold,

$$f(x) = x^\varrho P(\log_2 x) - Q(x), \qquad x \geqslant 1, \tag{2.29}$$

where $Q(x) = o(x^\varrho)$ as $x \to \infty$, $Q(x)$ is defined by (2.28) for all $x > 0$, and the continuous periodic function $P(t)$ is given by

$$P(t) = \sum_{m \in \mathbb{Z}} 2^{-\varrho(m+t)} g(2^{m+t}) + f(1) P_0(t), \qquad t \in \mathbb{R}, \tag{2.30}$$

with $P_0(t)$ given by (2.22).

Note that (2.29) is not only an identity but also an asymptotic expansion.

Before proving Theorem 2.10, we give two partial results.

Proposition 2.11. *Suppose that $h(x)$ is a function such that $h(x)$ lies between $h(\lfloor x \rfloor)$ and $h(\lceil x \rceil)$. Then, the following are equivalent.*

(i) *$n^{-\varrho} h(n) = P(\log_2 n) + o(1)$ as $n \to \infty$, for some continuous 1-periodic function P on $\mathbb{R}$.*

(ii) *$x^{-\varrho} h(x) = P(\log_2 x) + o(1)$ as $x \to \infty$, for some continuous 1-periodic function P on $\mathbb{R}$.*

Proof. (ii) $\Longrightarrow$ (i) is trivial.

The proof of (i) $\Longrightarrow$ (ii) is very similar to the proof of (i) $\Longrightarrow$ (ii) in [27, Theorem 2], so we omit some details. First, $\log_2 x - \log_2 \lfloor x \rfloor = O(1/x) = o(1)$ for large x and P is uniformly continuous, so $P(\log_2 x) = P(\log_2 \lfloor x \rfloor) + o(1)$. Hence, (i) implies that

$$\lfloor x \rfloor^{-\varrho} h(\lfloor x \rfloor) = P(\log_2 \lfloor x \rfloor) + o(1) = P(\log_2 x) + o(1) \tag{2.31}$$

and consequently, $x^{-\varrho} h(\lfloor x \rfloor) = P(\log_2 x) + o(1)$. Similarly, $x^{-\varrho} h(\lceil x \rceil) = P(\log_2 x) + o(1)$, and (ii) follows. $\qquad \square$

Proposition 2.12. *Suppose that $g(x)$ is a continuous function on $(0, \infty)$ with $g(x) = 0$ for $x \leqslant 1$. Define*

$$h(x) := \sum_{k \geqslant 0} 2^{k\varrho} g(2^{-k} x). \tag{2.32}$$

Then, the following are equivalent.

(i) *$x^{-\varrho} h(x) = P_1(\log_2 x) + o(1)$ as $x \to \infty$, for some continuous 1-periodic function P_1 on $\mathbb{R}$.*

(ii) *$x^{-\varrho} h(x) = P_1(\log_2 x) + o(1)$ as $x \to \infty$, for some 1-periodic function P_1 on $\mathbb{R}$.*

(iii) *$Q(x) := \sum_{k \geqslant 1} 2^{-\varrho k} g(2^k x)$ converges uniformly for $x \in [1, 2]$.*

Furthermore, when these conditions hold, $Q(x)$ is defined for all $x > 0$, $Q(x) = o(x^\varrho)$ as $x \to \infty$,

$$P_1(t) = \sum_{m \in \mathbb{Z}} 2^{-\varrho(m+t)} g(2^{m+t}), \qquad t \in \mathbb{R}, \tag{2.33}$$

and

$$h(x) = x^\varrho P_1(\log_2 x) - Q(x), \qquad x > 0. \tag{2.34}$$

Proof. Again, the proof differs mainly notationally from the proofs of the corresponding implications in [27, Theorem 2], and we omit some details. Let $G_m(x) := \sum_{k=0}^m 2^{-\varrho k} g(2^k x)$, and note that (iii) is equivalent to the property that $G_m(x)$ converges uniformly on $[1, 2]$ to $G(x) := Q(x) + g(x)$.

(i) $\implies$ (ii). Trivial.

(ii) $\implies$ (iii). Suppose that $y \in [1, 2]$ and $m \geqslant 0$. Then (2.32) yields

$$h(2^m y) = \sum_{0 \leqslant k \leqslant m} 2^{k\varrho} g(2^{-k+m} y) = \sum_{0 \leqslant j \leqslant m} 2^{(m-j)\varrho} g(2^j y) = 2^{m\varrho} G_m(y). \tag{2.35}$$

Hence, taking $x = 2^m y$, (ii) implies, as $m \to \infty$, uniformly for $y \in [1, 2]$,

$$y^{-\varrho} G_m(y) = (2^m y)^{-\varrho} h(2^m y) = P_1(m + \log_2 y) + o(1) = P_1(\log_2 y) + o(1). \tag{2.36}$$

Hence $y^{-\varrho} G_m(y)$ converges uniformly on $[1, 2]$, and thus $G_m(y)$ converges uniformly on $[1, 2]$.

(iii) $\implies$ (i). Conversely, (2.35) now yields

$$(2^m y)^{-\varrho} h(2^m y) = y^{-\varrho} G_m(y) = y^{-\varrho} G(y) + o(1) \tag{2.37}$$

as $m \to \infty$, uniformly for $y \in [1, 2]$. Next, we show that

$$P_1(t) := \sum_{m \in \mathbb{Z}} 2^{-\varrho(m+t)} g(2^{m+t}). \tag{2.38}$$

is a well-defined 1-periodic function. First, if $t \in [0, 1]$, then all terms with $m < 0$ vanish and thus

$$P_1(t) := \sum_{m \geqslant 0} 2^{-\varrho(m+t)} g(2^{m+t}) = 2^{-\varrho t} G(2^t), \tag{2.39}$$

where the sum converges uniformly for $t \in [0, 1]$ by assumption. This shows that the sum in (2.38) converges for $t \in [0, 1]$, and that $P_1(t)$ is continuous there. Furthermore, the sum in (2.38) is 1-periodic in t, and thus the sum converges for all real t and defines a 1-periodic continuous function $P_1(t)$.

Taking $m = L_x$ and $y = 2^{\theta_x}$ in (2.37) yields as $x \to \infty$, using (2.39) and the periodicity of P_1,

$$x^{-\varrho} h(x) = 2^{-\varrho \theta_x} G(2^{\theta_x}) + o(1) = P_1(\theta_x) + o(1) = P_1(\log_2 x) + o(1). \tag{2.40}$$

Hence (i) holds.

Finally, (2.33) is (2.38), which also shows that the sum defining $Q(x)$ converges for all $x > 0$, and that

$$\begin{aligned} h(x) + Q(x) &= \sum_{k \geqslant 0} 2^{k\varrho} g(2^{-k} x) + \sum_{k \geqslant 1} 2^{-\varrho k} g(2^k x) \\ &= \sum_{k \in \mathbb{Z}} 2^{-k\varrho} g(2^k x) \\ &= x^{\varrho} P_1(\log_2 x). \end{aligned} \tag{2.41}$$

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Moreover,

$$(2^m x)^{-\varrho} Q(2^m x) = x^{-\varrho} \sum_{k \geqslant 1} 2^{-\varrho(k+m)} g(2^{k+m} x) = x^{-\varrho} \sum_{k \geqslant m+1} 2^{-\varrho k} g(2^k x), \qquad (2.42)$$

which by (iii) converges to 0 as $m \to \infty$, uniformly for $x \in [1,2]$. Hence, $x^{-\varrho} Q(x) \to 0$ as $x \to \infty$.

This completes the proof of the proposition. $\qquad\square$

Proof of Theorem 2.10. First, (i) $\iff$ (ii) follows from Proposition 2.11, with $h(x) = f(x)$.

Next, define $h(x) := f(x) - x^{\varrho} f(1) P_0(\log_2 x)$, and note that (2.21) implies (2.32). Thus (ii) $\iff$ (iii) $\iff$ (iv) and the last sentence of the statement follow from Proposition 2.12, with $P(t) := f(1) P_0(t) + P_1(t)$. $\qquad\square$

A more practical condition than uniform convergence is the following.

Corollary 2.13. *Define*

$$A_m := \sup_{2^m \leqslant n \leqslant 2^{m+1}} |g(n)|. \qquad (2.43)$$

If $\sum_m 2^{-m\varrho} A_m < \infty$ *then* (2.27) *and* (2.29) *hold, where* P *is continuous, periodic and given by* (2.30). $\qquad\square$

Corollary 2.14. *Suppose that* $g(n) = O(n^{\varrho - \varepsilon})$ *for some* $\varepsilon > 0$. *Then* (2.27) *and* (2.29) *hold with* P *continuous, periodic and given by* (2.30), *and* $Q(x) = O(x^{\varrho - \varepsilon})$. $\qquad\square$

Example 2.15. One simple but important case is when $g(n) = 0$, $n \geqslant 2$, i.e., the recurrence $\Lambda_{\alpha,\beta}[f] = 0$. By suitable normalisations, we may assume $f(1) = 1$. As the sequence f satisfying $\Lambda_{\alpha,\beta}[f] = 0$ with $f(1) = 1$ plays a fundamental role in most of our applications, *we denote the solution by* $S_{\alpha,\beta}(n)$ *throughout this paper.*

Theorem 2.10 applies trivially, with $Q(x) = 0$ and $P(t) = P_0(t)$, and thus

$$S_{\alpha,\beta}(n) = f(n) = n^{\log_2(\alpha+\beta)} P_0(\log_2 n), \qquad n \geqslant 1, \qquad (2.44)$$

where $P_0(t)$ is given by (2.22); thus also, more explicitly,

$$S_{\alpha,\beta}(n) = \big(1 + (\alpha + \beta - 1)\varphi(2^{\{\log_2 n\}} - 1)\big)(\alpha + \beta)^{\lfloor \log_2 n \rfloor}. \qquad (2.45)$$

(In the case $\alpha = \beta$, when $\varphi(x) = x$ by Remark 2.4, (2.44) was found, in an equivalent form, by [14].) $\qquad\square$

Example 2.16. Another simple but important case is when $g(2m) = 0$ for all $m \geqslant 1$, i.e., $g(n)$ is non-zero only for odd n. Suppose also, for example, that $g(n) = O(n^{\varrho - \varepsilon})$ so that Corollary 2.14 applies. Then, (2.28) shows that $Q(n) = 0$ for every integer n, and thus (2.29) yields the identity $f(n) = n^{\varrho} P(\log_2 n)$, $n \geqslant 1$. $\qquad\square$

Remark 2.17. We have here studied the case with o-estimates and convergence in e.g. Theorem 2.10. Analogously, we note that it follows easily from Lemma 2.7 and (2.35) that

$$f(n) = O(n^{\varrho}) \iff f(x) = O(x^{\varrho}) \quad \text{for } x \geqslant 1$$

$$\iff \sum_{0 \leqslant k \leqslant m} 2^{-\varrho k} g(2^k x) = O(1) \quad \text{for } x \in [1,2] \text{ and } m \geqslant 0. \qquad (2.46)$$

$\qquad\square$

Remark 2.18. We concentrate in this paper on the case when $g(n)$ grows more slowly than n^ϱ (and (2.28) converges), and thus the sums in (2.20) and (2.21) are dominated by terms with k large (more precisely, $k = L_x + O(1)$ in (2.21)). One might also consider the opposite case, when $g(n)$ grows more rapidly that n^ϱ. In this case, the sums in (2.20) and (2.21) are dominated by their first terms, which shows that (typically, at least) $f(n)$ grows at the same rate as $g(n)$, and has the same smoothness properties. (In particular, there is no smoothening effect as we can see in (2.29).) We consider one example as part of Example 5.10, but we otherwise leave this case to a future study. (Except for several examples where we reduce to a slower growing g by subtracting a polynomial.) $\qquad\square$

3 Smoothness properties of the periodic function P

We prove in this section that under certain conditions on $g(n)$ stronger than those in Theorem 2.10, the periodic function P is Hölder continuous, and has an absolutely convergent Fourier series expansion. We also show that the interpolating function φ is Hölder continuous, and thus the interpolated function $f(x)$ is not only continuous but always locally Hölder continuous.

In this section, starting from the recursion (1.1), we tacitly assume that $\alpha, \beta > 0$ and $g(1) = 0$; also P_0, P_1 and P are functions defined as in Section 2; furthermore, we define

$$\lambda := \log_2 \frac{\alpha + \beta}{\alpha \vee \beta} \in (0, 1]. \tag{3.1}$$

Note that $\lambda = 1$ if and only if $\alpha = \beta$.

Bounded variation, Lipschitz continuity and Hölder continuity. We recall some standard definitions. A function ϕ is *Lipschitz continuous* on an interval $[a, b]$ if there exists a positive number C such that

$$|\phi(x) - \phi(y)| \leqslant C|x - y| \quad (x, y \in [a, b]). \tag{3.2}$$

This definition extends to *Hölder continuity* by replacing the last inequality by

$$|\phi(x) - \phi(y)| \leqslant C|x - y|^\gamma \quad (x, y \in [a, b]), \tag{3.3}$$

for some $0 < \gamma \leqslant 1$. Let $H_\gamma[a, b]$ be the space of functions ϕ on $[a, b]$ such that the seminorm

$$\|\phi\|'_{H_\gamma[a,b]} := \sup_{a \leqslant x < y \leqslant b} \frac{|\phi(x) - \phi(y)|}{|x - y|^\gamma} \tag{3.4}$$

is finite; this is a Banach space with the norm

$$\|\phi\|_{H_\gamma[a,b]} := \|\phi\|'_{H_\gamma[a,b]} + \sup_{x \in [a,b]} |\phi(x)|. \tag{3.5}$$

A function ϕ is of *bounded variation* on $[a, b]$ if its total variation is bounded. Such a function is differentiable almost everywhere. Let $\mathrm{BV}[a, b]$ be the space of functions on $[a, b]$ of bounded variation, with the norm

$$\|\phi\|_{\mathrm{BV}[a,b]} := V(\phi; a, b) + \sup_{[a,b]} |\phi|, \tag{3.6}$$

where V denotes the total variation.

Note that both $\mathrm{BV}[a, b]$ and $H_\gamma[a, b]$ are Banach algebras. Also $H_1 \subset \mathrm{BV}$, namely, Lipschitz continuity implies bounded variation.

Smoothness of φ and P. We prove first that the interpolating function φ is Hölder continuous. Note that φ is trivially of bounded variation since it is monotone.

Lemma 3.1. $\varphi \in \mathrm{H}_\lambda[0,1]$.

Proof. First, let $x = m2^{-N}$ and $y = (m+1)2^{-N}$ for some integers $N \geqslant 0$ and $m < 2^N$. Then (2.14) implies

$$\varphi(y) - \varphi(x) \leqslant \left(\frac{\alpha \vee \beta}{\alpha + \beta}\right)^N = \left(2^{-\lambda}\right)^N = 2^{-N\lambda}. \tag{3.7}$$

For general x and y with $0 \leqslant x < y \leqslant 1$, let $N := \lceil -\log_2 (y - x)\rceil$ and $k := \lfloor x2^N \rfloor$. Then $2^{-N} \leqslant y - x \leqslant 2^{1-N}$ and $k2^{-N} \leqslant x < y \leqslant (k+3)2^{-N}$. Hence, using (3.7) thrice,

$$\varphi(y) - \varphi(x) \leqslant \varphi\big((k+3)2^{-N}\big) - \varphi\big(k2^{-N}\big) \leqslant 3 \cdot 2^{-N\lambda} \leqslant 3|y - x|^\lambda. \tag{3.8}$$

This proves that $\varphi \in \mathrm{H}_\lambda[0,1]$. $\qquad\square$

Lemma 3.2. $P_0 \in \mathrm{H}_\lambda[0,1] \cap \mathrm{BV}[0,1]$.

Proof. For $t \in [0,1]$, we may replace $\{t\}$ by t in (2.22). It then follows from Lemma 3.1 that the first factor in (2.22) belongs to $\mathrm{H}_\lambda[0,1]$, and so does the second factor since it has a bounded derivative. Hence, $P_0 \in \mathrm{H}_\lambda[0,1]$.

Similarly, both factors in (2.22) then are monotone on $[0,1)$, and therefore of bounded variation. Hence $P_0 \in \mathrm{BV}[0,1]$. $\qquad\square$

To treat the function P, we need a smoothness assumption on the sequence $g(n)$. Let

$$A_m := \max_{2^m \leqslant n \leqslant 2^{m+1}} |g(n)| \quad \text{and} \quad B_m := \max_{2^m \leqslant n < 2^{m+1}} |g(n+1) - g(n)|. \tag{3.9}$$

Lemma 3.3. *If $\varrho > 0$ and*

$$\sum_{m \geqslant 0} 2^{(1-\varrho)m} B_m < \infty, \tag{3.10}$$

then

$$\sum_{m \geqslant 0} 2^{-\varrho m} A_m < \infty. \tag{3.11}$$

Note that $\varrho > 0$ is equivalent to $\alpha + \beta > 1$.

Proof. Let $m \geqslant 1$. For every n lying in the interval $2^m \leqslant n \leqslant 2^{m+1}$,

$$|g(n)| \leqslant |g(2)| + \sum_{2 \leqslant j < n} |g(j+1) - g(j)|$$

$$\leqslant |g(2)| + \sum_{1 \leqslant k \leqslant m} 2^k B_k, \tag{3.12}$$

implying that

$$A_m \leqslant |g(2)| + \sum_{1 \leqslant k \leqslant m} 2^k B_k. \tag{3.13}$$

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Thus,

$$\sum_{m \geqslant 1} 2^{-\varrho m} A_m \leqslant \sum_{m \geqslant 1} 2^{-\varrho m} \left(|g(2)| + \sum_{1 \leqslant k \leqslant m} 2^k B_k \right)$$

$$= |g(2)| \frac{2^{-\varrho}}{1 - 2^{-\varrho}} + \frac{1}{1 - 2^{-\varrho}} \sum_{k \geqslant 1} 2^{(1-\varrho)k} B_k. \qquad (3.14)$$

This proves the lemma. $\qquad \square$

Lemma 3.4. *If* $\varrho > 0$ *and* (3.10) *holds, then* $P \in \mathrm{H}_\lambda[0,1] \cap \mathrm{BV}[0,1]$.

Proof. Let $g_m(x) := g(2^m x)$. Then, by (2.30) and $g(x) = 0$ for $x \leqslant 1$, we have

$$P(t) = 2^{-\varrho t} \sum_{m \geqslant 0} 2^{-\varrho m} g_m(2^t) + f(1) P_0(t) \qquad (t \in [0,1]). \qquad (3.15)$$

Since the function $2^{-\varrho t}$ belongs to $\mathrm{BV}[0,1]$, which is a Banach algebra, we see that

$$\| P(t) \|_{\mathrm{BV}[0,1]} \leqslant C_1 \sum_{m \geqslant 0} 2^{-\varrho m} \| g_m(2^t) \|_{\mathrm{BV}[0,1]} + |f(1)| \, \| P_0(t) \|_{\mathrm{BV}[0,1]}, \qquad (3.16)$$

for some constant $C_1 > 0$. Furthermore, by the monotonicity of the interpolating function φ, we obtain

$$\| g_m(2^t) \|_{\mathrm{BV}[0,1]} = \| g_m(x) \|_{\mathrm{BV}[1,2]}$$

$$= \| g(x) \|_{\mathrm{BV}[2^m, 2^{m+1}]}$$

$$= \sup_{2^m \leqslant x \leqslant 2^{m+1}} |g(x)| + \sum_{2^m \leqslant n < 2^{m+1}} |\Delta g(n)|$$

$$\leqslant A_m + 2^m B_m, \qquad (3.17)$$

where $\Delta g(n) := g(n+1) - g(n)$. It follows from (3.16)–(3.17), Lemma 3.2, (3.10) and Lemma 3.3 that $P(t) \in \mathrm{BV}[0,1]$.

Similarly, for the Hölder norm, we have

$$\| P_1(t) \|_{\mathrm{H}_\lambda[0,1]} \leqslant C_2 \sum_{m \geqslant 0} 2^{-\varrho m} \| g_m(2^t) \|_{\mathrm{H}_\lambda[0,1]}$$

$$\leqslant C_3 \sum_{m \geqslant 0} 2^{-\varrho m} \| g_m(x) \|_{\mathrm{H}_\lambda[1,2]}, \qquad (3.18)$$

where, furthermore, by the definitions (3.4)–(3.5) and (3.9),

$$\| g_m(x) \|_{\mathrm{H}_\lambda[1,2]} = \| g_m(x) \|'_{\mathrm{H}_\lambda[1,2]} + \sup_{x \in [1,2]} |g_m(x)|$$

$$= 2^{\lambda m} \| g(x) \|'_{\mathrm{H}_\lambda[2^m, 2^{m+1}]} + \sup_{x \in [2^m, 2^{m+1}]} |g(x)|$$

$$= 2^{\lambda m} \| g(x) \|'_{\mathrm{H}_\lambda[2^m, 2^{m+1}]} + A_m. \qquad (3.19)$$

In order to bound $\| g \|'_{\mathrm{H}_\lambda[2^m, 2^{m+1}]}$, we estimate $|g(y) - g(x)|$ for $2^m \leqslant x \leqslant y \leqslant 2^{m+1}$. By splitting the interval $[x, y]$ into $[x, \lceil x \rceil]$, $[\lceil x \rceil, \lfloor y \rfloor]$ and $[\lfloor y \rfloor, y]$, it suffices (up to a constant

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factor in the norm) to consider the two cases $n \leqslant x \leqslant y \leqslant n+1$ and $x = n$, $y = n + \eta$, where n and η are integers.

In the first case, $n \leqslant x \leqslant y \leqslant n+1$ with $2^m \leqslant n < 2^{m+1}$, we have $g(y) - g(x) = \Delta g(n) \, (\varphi(y) - \varphi(x))$. Since $\varphi \in \mathrm{H}_\lambda[0, 1]$ by Lemma 3.1,

$$|g(y) - g(x)| \leqslant C_4 |\Delta g(n)| \, |y - x|^\lambda \leqslant C_4 B_m |y - x|^\lambda. \tag{3.20}$$

In the second case,

$$|g(y) - g(x)| = |g(n + \eta) - g(n)| \leqslant \sum_{0 \leqslant i < \eta} |\Delta g(n + i)| \leqslant \eta B_m \leqslant B_m 2^{m(1-\lambda)} \eta^\lambda. \tag{3.21}$$

Combining the two cases, we obtain from (3.20) and (3.21)

$$\|g\|'_{\mathrm{H}_\lambda[2^m, 2^{m+1}]} \leqslant C_5 B_m 2^{(1-\lambda)m}. \tag{3.22}$$

Consequently, (3.19) yields

$$\|g_m\|_{\mathrm{H}_\lambda[1,2]} \leqslant C_5 2^m B_m + A_m. \tag{3.23}$$

It follows from (3.18), (3.23), (3.10) and Lemma 3.3 that $P_1 \in \mathrm{H}_\lambda[0, 1]$, and then $P \in \mathrm{H}_\lambda[0, 1]$ by Lemma 3.2. $\qquad\square$

Remark 3.5. Lemma 3.1 is best possible: $\varphi \notin \mathrm{H}_\gamma[0, 1]$ for $\gamma > \lambda$. In particular, φ is not Lipschitz continuous (and not differentiable) unless $\alpha = \beta$. To see this, it suffices to note that (2.14) yields $\varphi(2^{-j}) - \varphi(0) = \left(\frac{\beta}{\alpha+\beta}\right)^j$ and $\varphi(1) - \varphi(1 - 2^{-j}) = \left(\frac{\alpha}{\alpha+\beta}\right)^j$ for all $j \geqslant 1$, and one of these equals $\left(\frac{\alpha \vee \beta}{\alpha+\beta}\right)^j = \left(2^{-j}\right)^\lambda$.

Hence, (2.22) shows that if $\alpha + \beta \neq 1$, then also Lemma 3.2 is best possible: $P_0 \notin \mathrm{H}_\gamma[0, 1]$ for $\gamma > \lambda$. Furthermore, (2.30) shows that typically also $P \notin \mathrm{H}_\gamma[0, 1]$ for $\gamma > \lambda$. (Also in the case $f(1) = 0$, since $g(x)$ has the same smoothness as φ.) However, note that P may be more smooth in special cases (which means that there is cancellation of non-smoothness in (2.30)); for example, we may take any continuously differentiable periodic function $P(t)$ and let $f(n) := n^\varrho P(\log_2 n)$ and then define $g(n)$ by (2.1).

Fourier series. The periodic function $P(t)$ may be described by its Fourier coefficients; these are given by the following formula, where we use the notation

$$\chi_k := \frac{2k\pi}{\log 2} i, \qquad k \in \mathbb{Z}. \tag{3.24}$$

Theorem 3.6. (i) *If Theorem 2.10(iv) holds (and thus all statements in Theorem 2.10), then the Fourier coefficients $\widehat{P}(k) := \int_0^1 P(t) e^{-2k\pi it} \, \mathrm{d}t$ of $P(t)$ are given by*

$$\widehat{P}(k) = \frac{1}{\log 2} \int_1^\infty \frac{g(u)}{u^{\varrho + \chi_k + 1}} \, \mathrm{d}u + \frac{f(1)}{\log 2} \int_0^1 \frac{1 + (\alpha + \beta - 1)\varphi(u)}{(1 + u)^{\varrho + \chi_k + 1}} \, \mathrm{d}u, \tag{3.25}$$

where the first integral is (at least) conditionally convergent.

(ii) *If (3.11) holds, then (3.25) holds, with absolutely convergent integrals.*

(iii) *If $\alpha + \beta > 1$ and (3.10) holds, then $P(t)$ has an absolutely convergent Fourier series $P(t) = \sum_{k \in \mathbb{Z}} \widehat{P}(k) e^{2\pi ikt}$, for $t \in \mathbb{R}$, where the coefficients are given by (3.25), with absolutely convergent integrals.*

Proof. (i): The Fourier coefficients of $P(t)$ are given by, using the definitions (2.30), (1.3) and (2.22),

$$
\begin{aligned}
\widehat{P}(k) &:= \int_0^1 P(t) e^{-2k\pi it}\, \mathrm{d}t \\
&= \sum_{m\in\mathbb{Z}} \int_0^1 g\left(2^{m+t}\right) 2^{-\varrho(m+t)} e^{-2k\pi it}\, \mathrm{d}t \\
&\qquad + f(1) \int_0^1 \left(1 + (\alpha+\beta-1)\varphi\left(2^t-1\right)\right)(\alpha+\beta)^{-t} e^{-2k\pi it}\, \mathrm{d}t \\
&= \int_{-\infty}^{\infty} g(2^t) 2^{-\varrho t} e^{-2k\pi it}\, \mathrm{d}t \\
&\qquad + f(1)\int_0^1 \left(1+(\alpha+\beta-1)\varphi\left(2^t-1\right)\right) 2^{-\varrho t} e^{-2k\pi it}\, \mathrm{d}t, \qquad (3.26)
\end{aligned}
$$

where the sum and the integral over $(-\infty,\infty)$ are (conditionally) convergent because the sum in (2.30) converges uniformly, and $g(x) = 0$ for $x \leqslant 1$. (The integrals over $[0,1]$ are, trivially, absolutely convergent.) Finally, (3.25) follows by the changes of variables $u = 2^t$ and $u = 2^t - 1$, respectively, in the two integrals.

(ii): Corollary 2.13 shows that all statements in Theorem 3.6 hold, and thus part (i) applies, which gives (3.25). The absolute convergence of the first integral follows by (3.11).

(iii): By Lemma 3.3, we have (3.11), and thus (ii) applies. Since P is a continuous 1-periodic function, the absolute convergence of the Fourier series follows directly from Lemma 3.4 by a theorem of Zygmund (see [43, p. 241, VI.(3.6)] or [30, p. 35]). ☐

The formula (3.25) shows a connection with Mellin transforms; this is explored further in Appendix A.

The conditions of Theorem 3.6(iii) are, of course, not necessary for absolute convergence of the Fourier series of $P(t)$. Another simple case is the following.

Example 3.7. If $\alpha, \beta > 0$ and $g(n) = 0$, $n \geqslant 2$, we may normalise by $f(1) = 1$ as in Example 2.15; thus $f(n) = S_{\alpha,\beta}(n)$ and (2.44) holds. $P(t) = P_0(t)$ has an absolutely convergent Fourier series, given by (3.25) with $g(x) = 0$. This follows as in the proof of Theorem 3.6, now using Lemma 3.2. ☐

Many other examples of $P(t)$ with absolutely convergent Fourier series are given in Sections 4–5. An example where the Fourier series is not absolutely convergent is part of Example 5.10 (for some α).

In the case $\alpha = \beta$, the integrals in (3.25) can easily be evaluated explicitly, and we obtain the following extension of the case $\alpha = \beta = 1$ in [27, Theorem 3]. Let, as in [27],

$$
D(s) := \sum_{n\geqslant 2} g(n)\left((n-1)^{-s} - 2n^{-s} + (n+1)^{-s}\right), \qquad (3.27)
$$

for all complex s such the sum is convergent; note that if $g(n) = O(n^{\varrho-\varepsilon})$ for some $\varepsilon \geqslant 0$ (this holds at least with $\varepsilon = 0$ whenever Theorem 2.10 applies), then the sum in (3.27) is absolutely convergent in the half-space $\Re s > \varrho - 1 - \varepsilon$, and thus $D(s)$ is analytic there. Note

also (as in [27]) that if $\Re s$ is large enough, (3.27) can be rearranged as a Dirichlet series (using $g(0) = g(1) = 0$)

$$D(s) = \sum_{n \geqslant 1} \Big(g(n+1) - 2g(n) + g(n-1) \Big) n^{-s}. \tag{3.28}$$

Corollary 3.8. *If $\alpha = \beta > 0$ and Theorem 2.10(iv) holds (and thus all statements in Theorem 2.10), then, assuming $\varrho + \chi_k \neq 0, 1$,*

$$\widehat{P}(k) = \frac{1}{(\varrho + \chi_k)(\varrho - 1 + \chi_k) \log 2} \Big(D(\varrho - 1 + \chi_k) + \frac{(2\alpha - 1)(\alpha - 1)}{\alpha} f(1) \Big). \tag{3.29}$$

$\square$

The formula (3.29) is used (often tacitly) in numerous examples below.

Remark 3.9. The two exceptional cases are $k = 0$ and either $\alpha = \beta = \frac{1}{2}$ or $\alpha = \beta = 1$; in these cases, (3.29) is of the form $0/0$ and is replaced by a suitable limit form. The case $\alpha = 1$ is included in [27, Theorem 3]; the case $\alpha = \frac{1}{2}$ is similar, but we omit the details. $\square$

Proof. Theorem 3.6 applies and yields (3.25); we treat the two integrals in (3.25) separately. For the first integral, [27, (2.23) in the proof of Theorem 3] holds whenever $\alpha = \beta$, and shows that

$$\int_1^\infty \frac{g(u)}{u^{\varrho + \chi_k + 1}} \, \mathrm{d}u = \frac{D(\varrho - 1 + \chi_k)}{(\varrho + \chi_k)(\varrho - 1 + \chi_k)}, \tag{3.30}$$

with the sum (3.27) converging at least conditionally.

Since $\alpha = \beta > 0$, we have $\varphi(u) = u$ and thus the second integral in (3.25) is, by a simple calculation using $2^{\varrho + \chi_k} = 2^\varrho = 2\alpha$,

$$\int_0^1 \frac{1 + (2\alpha - 1)u}{(1 + u)^{\varrho + \chi_k + 1}} \, \mathrm{d}u = \int_0^1 \Big(\frac{2\alpha - 1}{(1 + u)^{\varrho + \chi_k}} + \frac{2 - 2\alpha}{(1 + u)^{\varrho + \chi_k + 1}} \Big) \, \mathrm{d}u$$

$$= \frac{(2\alpha - 1)(\alpha - 1)}{\alpha} \cdot \frac{1}{(\varrho + \chi_k)(\varrho - 1 + \chi_k)}. \tag{3.31}$$

Using (3.30)–(3.31) in (3.25) yields (3.29). $\square$

Remark 3.10. For later use we note that, still assuming $\alpha = \beta$, if the condition Theorem 2.10(iv) holds, then [27, (2.23)] more generally yields the Mellin transform

$$\int_1^\infty \frac{g(u)}{u^{s+1}} \, \mathrm{d}u = \frac{D(s-1)}{s(s-1)} \tag{3.32}$$

with the integral converging absolutely at least for every complex s with $\Re s > \varrho$. $\square$

Example 3.11. Let $\alpha = \beta > 0$, and consider $P_0(t)$ given by (2.25) in Remark 2.8. By Corollary 3.8, with $g(n) = 0$ and $f(1) = 1$ (and thus $D(s) = 0$), we have the Fourier coefficients, assuming $\varrho + \chi_k \neq 0, 1$:

$$\widehat{P}_0(k) = \frac{(2\alpha - 1)(\alpha - 1)}{\alpha \log 2} \cdot \frac{1}{(\varrho + \chi_k)(\varrho - 1 + \chi_k)}, \qquad k \in \mathbb{Z}. \tag{3.33}$$

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In particular, (3.33) verifies that $P_0(t)$ has an absolutely convergent Fourier series, as shown more generally in Example 3.7. If $\alpha \in \{\frac{1}{2}, 1\}$ (so $\varrho = 0$ or 1), (3.33) shows that $\widehat{P}_0(k)$ vanishes for every $k \neq 0$, which is obvious since then $P_0(t) \equiv 1$ by Remark 2.8. In all other cases, (3.33) yields $\left|\widehat{P}_0(k)\right| = \Theta(k^{-2})$ as $k \to \pm\infty$, which agrees well with the fact that $P_0(t)$ is Lipschitz but not C^1; see again Remark 2.8. $\qquad\square$

Remark 3.12. Again, by the recursive relations in (2.4), we can express the second integral in (3.25) in the series form

$$\frac{(\varrho + \chi_k)(\alpha + \beta)}{\alpha + \beta - 1} \int_0^1 \frac{1 + (\alpha + \beta - 1)\varphi(t)}{(1 + t)^{\varrho + \chi_k + 1}} \, dt$$
$$= 1 + \sum_{m \geq 0} \sum_{0 \leq j < 2^m} \alpha^{\nu(j)} \beta^{m+1-\nu(j)} \left(\frac{1}{(2^m + j + \frac{1}{2})^{\varrho + \chi_k}} - \frac{1}{(2^m + j + 1)^{\varrho + \chi_k}} \right), \quad (3.34)$$

which is more useful for numerical purposes, where $\nu(j)$ denotes the number of 1s in the binary expansion of j. See Appendix B for a proof. $\qquad\square$

4 Applications, I. $\alpha \neq \beta$

We discuss applications of our results in this section, grouping them according to the growth order of g. Most examples are taken from OEIS, sometimes with a shift of the index (which for simplicity of presentation is not explicitly specified in this paper). For example, if

$$f(n) = \alpha f\left(\left\lfloor \tfrac{n+d}{2} \right\rfloor\right) + \beta f\left(\left\lceil \tfrac{n+d}{2} \right\rceil\right) + g(n) \qquad (n \geq n_0 \geq 1), \qquad (4.1)$$

then $\bar{f}(n) := f(n + d)$ satisfies

$$\bar{f}(n) = \alpha \bar{f}\left(\left\lfloor \tfrac{n}{2} \right\rfloor\right) + \beta \bar{f}\left(\left\lceil \tfrac{n}{2} \right\rceil\right) + g(n + d) \qquad (n \geq n_0 - d). \qquad (4.2)$$

Note also that if (1.1) holds only for $n \geq n_0$, we can make it hold for all $n \geq 2$ by redefining $g(n)$ for $2 \leq n < n_0$. See A294456 (contained in Example 5.3) for an example.

Some of the examples are defined in OEIS by a recursion of the form (1.1); in other examples, such a recursion is stated as a property; in yet other examples below, no such recursion is given explicitly in OEIS, but can be concluded from other properties given there. Of course, every sequence $f(n)$ satisfies $\Lambda_{\alpha,\beta}[f] = g$ for some sequence $g(n)$; we are only interested in cases when $g(n)$ has a simple explicit form, and in particular does not grow too fast. We regard polynomial terms in $f(n)$ as essentially trivial, so we also include examples when they dominate f and the periodic fluctuations constitute a lower-order term. (In such cases our theorems apply only after subtracting a suitable polynomial.)

To avoid trivialities, we discard in our discussions sequences from OEIS whose generating functions are rational with all singularities on the unit circle. (For example, polynomials.) Such sequences are in the thousands in OEIS.

For notational convenience, we insert the subscript to φ by writing $\varphi_{\alpha,\beta}(t)$ whenever necessary. The symbols $f(n), g(n)$ and $P(t)$ are all generic and may differ from one instance to the other; we also specify explicitly them as $f_{\text{A006046}}(n)$, $g_{\text{A006046}}(n)$, and $P_{\text{A006046}}(t)$ if needed. Note that our indexing of the sequence f may differ from that on OEIS by a shift; for example,

$f_{\text{A006581}}(n) = \text{A006581}(n+1)$ for $n \geqslant 1$. Also the format $g(n) = \begin{cases} \cdots \\ \cdots \end{cases}$ in the tables without explicit mention always means the values of $g(n)$ in the even and odd cases, respectively.

We continue to assume $\alpha, \beta > 0$, and consider in this section cases with $\alpha \neq \beta$. On the other hand, examples in the special cases when $\alpha = \beta$ exhibit more structural properties and explicit expressions, and will be discussed in Section 5. The properties are very similar to the case when $\alpha = \beta = 1$ that we already examined in detail in [27], although there are also subtle differences on the smoothness of the periodic functions.

4.1 Periodic equivalence

We introduce a simple notion here, very useful in identifying the relation between sequences. The main case is when two sequences f_1 and f_2 both satisfy Theorem 2.10 (with the same ϱ), and the corresponding periodic functions P_1 and P_2 are the same, or more generally proportional. It will be convenient to be a bit more general, and regard polynomial terms as trivial. We thus define:

Definition 4.1. Two sequences f_1 and f_2 are said to be *periodically equivalent* if, for some ϱ, $f_j(n) = n^\varrho P_j(\log_2 n) + p_j(n)$, $j = 1, 2$, where $p_j(n)$ are polynomials and P_j are periodic functions such that $P_1(k) = cP_2(k)$ for some constant $c \neq 0$. For simplicity, we write $f_1 \asymp f_2$.

It is possible to extend the definition to *asymptotically periodic equivalence* by allowing $f_j(n) = n^\varrho P_j(\log_2 n) + p_j(n) + o(n^\varrho)$ for $j = 1, 2$, but the above definition without $o(1)$-term is sufficient for our use in this paper.

We begin with the simplest cases when $\Lambda_{\alpha,\beta}[f](n) = g(n) = 0$. Such cases cover also the situation when $g(n) = c$ for $n \geqslant 2$ because normalising f by $\bar{f}(n) := f(n) + \frac{c}{\alpha+\beta-1}$ yields the recurrence $\Lambda_{\alpha,\beta}[\bar{f}] = 0$ with $\bar{f}(1) = f(1) + \frac{c}{\alpha+\beta-1}$. For convenience of reference, we state this observation, in a somewhat generalised form, as a lemma.

Lemma 4.2. *If $\alpha + \beta > 1$, then two sequences defined by $\Lambda_{\alpha,\beta}[f_1] = 0$ with $f_1(1) \neq 0$ and $\Lambda_{\alpha,\beta}[f_2] = c$ are connected by*

$$f_2(n) = \left(f_2(1) + \frac{c}{\alpha + \beta - 1} \right) \frac{f_1(n)}{f_1(1)} - \frac{c}{\alpha + \beta - 1}. \tag{4.3}$$

Hence, if $f_2(1) + \frac{c}{\alpha+\beta-1} \neq 0$, the sequences $f_1(n)$ and $f_2(n)$ are periodically equivalent with the underlying periodic function P satisfying $P \in \mathrm{H}_\lambda[0,1] \cap \mathrm{BV}[0,1]$, where λ is defined in (3.1).

Proof. It is easy to verify that (4.3) satisfies the recursion $\Lambda_{\alpha,\beta}[f_2] = c$. and thus (4.3) holds by induction. Thus $f_1 \asymp f_2$ if $f_2(1) + \frac{c}{\alpha+\beta-1} \neq 0$. The Hölder continuity and bounded variation of P follow from Lemma 3.4. □

Lemma 4.3. *Write $f \sim \widehat{\Lambda}_{\alpha,\beta}[c; d, e]$ if $\Lambda_{\alpha,\beta}[f] = cn + \begin{cases} d, & n \text{ even} \\ e, & n \text{ odd} \end{cases}$ for $n \geqslant 2$. Suppose that $\alpha + \beta > 2$, and that*

$$f_1 \sim \widehat{\Lambda}_{\alpha,\beta}[c; d, e] \quad \text{and} \quad f_2 \sim \widehat{\Lambda}_{\alpha,\beta}\left[0; 0, \frac{(\alpha - \beta)c}{\alpha + \beta - 2} - d + e \right], \tag{4.4}$$

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where $f_2(1) = f_1(1) + \frac{2c}{\alpha+\beta-2} + \frac{d}{\alpha+\beta-1} \neq 0$. Then $f_1 \eqsim f_2$. Furthermore,

$$f_1(n) = n^\varrho P(\log_2 n) - \frac{2cn}{\alpha+\beta-2} - \frac{d}{\alpha+\beta-1} \qquad (n \geqslant 1) \tag{4.5}$$

for a periodic function $P \in \mathrm{H}_\lambda[0,1] \cap \mathrm{BV}[0,1]$.

Proof. The normalised sequence

$$\bar{f}(n) := f_1(n) + \frac{2cn}{\alpha+\beta-2} + \frac{d}{\alpha+\beta-1} \tag{4.6}$$

satisfies

$$\Lambda_{\alpha,\beta}[\bar{f}] = \left(\frac{(\alpha-\beta)c}{\alpha+\beta-2} - d + e\right) \cdot \mathbf{1}_{n \text{ is odd}}, \tag{4.7}$$

with the initial condition $\bar{f}(1) = f_1(1) + \frac{2c}{\alpha+\beta-2} + \frac{d}{\alpha+\beta-1}$. Thus, $\bar{f} = f_2$. Furthermore, $f_2(n) = n^\varrho P(\log_2 n)$ for a periodic function P by Example 2.16. This yields (4.5). Again, the Hölder continuity and bounded variation of P follow from Lemma 3.4. $\qquad \square$

4.2 $\quad \Lambda_{\alpha,\beta}[f] = 0$

Example 4.4 (Generating polynomial of the sum-of-digits function). As an immediate application of Theorem 3.6, we consider the following partial sum

$$f(n) := \sum_{0 \leqslant k < n} \alpha^{\nu(k)} \qquad (n \geqslant 0), \tag{4.8}$$

where $\alpha > 0$ and $\nu(n)$ denotes the number of 1s in the binary expansion of n. Such sums with various α have been encountered and studied in a large number of different contexts; see the recent survey [6] and the references therein for more information. Then by (4.8) and the recurrence relation

$$\nu(n) = \nu\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + \mathbf{1}_{n \text{ is odd}} \qquad (n \geqslant 1), \tag{4.9}$$

we see that f satisfies $\Lambda_{\alpha,1}[f] = 0$ with $f(1) = 1$, or (see Example 2.15)

$$f(n) = S_{\alpha,1}(n). \tag{4.10}$$

Thus $g(n) \equiv 0$ for all n, so (3.10) is trivial and Theorem 3.6(iii) applies. Furthermore, (2.30) yields $P(t) = P_0(t)$ given by (2.22), and Theorem 2.10 or Example 2.15 shows that

$$f(n) = n^{\log_2(\alpha+1)} P_0(\log_2 n) \qquad (n \geqslant 1), \tag{4.11}$$

where, by Theorem 3.6(iii) and (2.26),

$$P_0(t) := (\alpha+1)^{-\{t\}}\left(1 + \alpha\varphi_{\alpha,1}\left(2^{\{t\}} - 1\right)\right) = (\alpha+1)^{1-\{t\}}\varphi_{\alpha,1}\left(2^{\{t\}-1}\right) \tag{4.12}$$

has an absolutely convergent Fourier series for $\alpha > 0$. This extends and improves the result established in [25] for α in the range $(\sqrt{2}-1, \sqrt{2}+1)$, where a completely different approach

was employed: instead of Zygmund's theorem applied above (see the proof of Theorem 3.6), the proof in [25] used a theorem of Bernstein saying that a periodic function $P \in H_\lambda$ with $\lambda > \frac{1}{2}$ has an absolutely convergent Fourier series. Note that $\lambda > \frac{1}{2}$ when $\alpha \in (\sqrt{2} - 1, \sqrt{2} + 1)$.

The Fourier coefficients $\widehat{P_0}(k)$ are by Theorem 3.6 given by, with $\chi_k := \frac{2k\pi i}{\log 2}$,

$$\widehat{P_0}(k) = \frac{1}{\log 2} \int_0^1 \frac{1 + \alpha\varphi_{\alpha,1}(u)}{(1 + u)^{\log_2(\alpha+1)+\chi_k+1}} \, \mathrm{d}u \qquad (k \in \mathbb{Z}). \tag{4.13}$$

The same type of results also hold for the recurrence $\Lambda_{1,\alpha}[f] = 0$, the only difference being replacing the underlying interpolation function $\varphi_{\alpha,1}$ by $\varphi_{1,\alpha}$. $\qquad\square$

Example 4.5 (Recurrence with minimisation or maximisation)**.** A class of sequences satisfying recurrences of the form

$$\mu(n) = \min_{1 \leqslant k \leqslant \lfloor \frac{n}{2} \rfloor} \{\alpha\mu(k) + \beta\mu(n - k)\} \qquad (n \geqslant 2) \tag{4.14}$$

with $\mu(1) = 1$ was studied in [5] to solve the AND-OR Problem. It is proved there that if $\beta \geqslant \alpha$ are positive integers, then the minimum in (4.14) is reached at $k = \lfloor \frac{n}{2} \rfloor$, so that $\mu(n) = f(n) = S_{\alpha,\beta}(n)$. In this case, Theorem 2.10 or Example 2.15 and Example 3.7 imply that $\mu(n) = n^{\log_2(\alpha+\beta)} P_0(\log_2 n)$, where P_0 is a periodic function with an absolutely convergent Fourier series, given by (2.22). We can extend this to the cases (*i*) $\beta \geqslant \alpha$, $\beta \geqslant 1$ and (*ii*) $\alpha \geqslant \beta$, $\alpha + \beta \leqslant 1$; see Appendix C.

On the other hand, it is not difficult to see that if $\beta \leqslant 1$, $\alpha + \beta \geqslant 1$, and $\alpha \geqslant \beta^2$, then the minimum in (4.14) is attained at $k = 1$, and we get a simple geometric expression for $f(n)$. (Apart from the trivial case $\alpha + \beta = 1$, when $\mu(n) = 1$, these are the only (α, β) for which the minimum always is attained at $k = 1$.) The behaviour of the recursion (4.14) for the remaining (α, β) seems to be more complicated. (For example, the case $1 < \beta < \alpha$.)

In Appendix C, we show in a similar manner that the solution to the corresponding recurrence with maximisation

$$\mu(n) = \max_{1 \leqslant k \leqslant \lfloor \frac{n}{2} \rfloor} \{\alpha\mu(k) + \beta\mu(n - k)\} \qquad (n \geqslant 2) \tag{4.15}$$

with $\mu(1) = 1$ is given by $\mu(n) = f(n) = S_{\alpha,\beta}(n)$ whenever $\alpha \geqslant \beta$, $\beta \leqslant 1$ and $\alpha + \beta \geqslant 1$. We again obtain, by Examples 2.15 and 3.7, that $f(n) = n^{\log_2(\alpha+\beta)} P_0(\log_2 n)$, where P_0 is the periodic function in (2.22), with an absolutely convergent Fourier series. The case $\alpha > \beta = 1$ of (4.15) was solved (in an equivalent form) in [32].

For more recurrences with minimisation or maximisation, see [22, 28] and the references therein. $\qquad\square$

Example 4.6 (OEIS sequences satisfying $\Lambda_{\alpha,\beta}[f] = 0$ with $f(1) = 1$ and thus $f(n) = S_{\alpha,\beta}(n)$)**.** We collect OEIS sequences of this category in Table 1, where f and P are both generic symbols, not necessarily the same in each occurrence; some of the "popular" sequences will be discussed in detail below.

OEIS id.	(α, β)	Description	$f(n)$

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A000027	$(1,1)$	Natural numbers	n
A064194	$(1,2)$	Gates in AND/OR problem [5]	$n^{\log_2(3)}P(\log_2 n)$
A006046	$(2,1)$	Odd entries in Pascal's triangle	$n^{\log_2(3)}P(\log_2 n)$
A268524	$(1,3)$	$\Lambda_{1,3}[f]=0$ with $f(1)=1$	$n^2 P(\log_2 n)$
A130665	$(3,1)$	$S_{3,1}(n)$ (see (4.8)); also (4.15)	$n^2 P(\log_2 n)$
A073121	$(2,2)$	appeared in [14]	$n^2 P(\log_2 n)$
A268527	$(1,4)$	$\Lambda_{1,4}[f]=0$ with $f(1)=1$	$n^{\log_2(5)}P(\log_2 n)$
A116520	$(4,1)$	(4.15) with $(\alpha,\beta)=(4,1)$	$n^{\log_2(5)}P(\log_2 n)$
A268526	$(2,3)$	$\Lambda_{2,3}[f]=0$ with $f(1)=1$	$n^{\log_2(5)}P(\log_2 n)$
A268525	$(3,2)$	$\Lambda_{3,2}[f]=0$ with $f(1)=1$	$n^{\log_2(5)}P(\log_2 n)$
A130667	$(5,1)$	(4.15) with $(\alpha,\beta)=(5,1)$	$n^{\log_2(6)}P(\log_2 n)$
A116522	$(6,1)$	Limit of the power of a matrix	$n^{\log_2(7)}P(\log_2 n)$
A161342	$(7,1)$	3-D cellular automaton	$n^3 P(\log_2 n)$
A116526	$(8,1)$	Limit of the power of a matrix	$n^{\log_2(9)}P(\log_2 n)$
A116525	$(10,1)$	Limit of the power of a matrix	$n^{\log_2(11)}P(\log_2 n)$
A116524	$(12,1)$	$\Lambda_{12,1}[f]=0$ with $f(1)=1$	$n^{\log_2(13)}P(\log_2 n)$
A116523	$(16,1)$	$\Lambda_{16,1}[f]=0$ with $f(1)=1$	$n^{\log_2(17)}P(\log_2 n)$

Table 1: OEIS sequences of the form $S_{\alpha,\beta}(n)$ (Example 4.6).

We will see that these sequences play to some extent a prototypical role for more general recurrences with nonzero g.

On the other hand, the only periodic function in this table that admits a closed-form expression in terms of elementary functions is when $(\alpha,\beta)=(2,2)$ for which we have $P(t)=2^{-\{t\}}\big(3-2^{1-\{t\}}\big)$, by (2.25). $\qquad\square$

We note also the following example, where $\Lambda_{\alpha,\beta}[f]$ is not zero, but a constant.

Example 4.7. We can generalise Example 4.4 by considering the partial sum, for $\alpha,\beta>0$,

$$f(n):=\sum_{0\leqslant k<n}\alpha^{\nu(k)}\beta^{\nu_0(k)}\qquad(n\geqslant 0),\tag{4.16}$$

where $\nu(n)$ is as above, and similarly $\nu_0(n)$ denotes the number of 0s in the binary expansion of n (with $\nu_0(0):=0$, A080791).

In analogy with (4.9), we have the recurrence relation

$$\nu_0(n)=\nu_0\big(\big\lfloor\tfrac{n}{2}\big\rfloor\big)+\mathbf{1}_{n\text{ is even}}\qquad(n\geqslant 1),\tag{4.17}$$

and it follows easily that f satisfies $f(1)=1$ and

$$\Lambda_{\alpha,\beta}[f](n)=g(n)=1-\beta\qquad(n\geqslant 2).\tag{4.18}$$

Assume $\alpha+\beta>1$. Then, by (4.18) and Lemma 4.2,

$$f(n)=\frac{\alpha}{\alpha+\beta-1}S_{\alpha,\beta}(n)+\frac{\beta-1}{\alpha+\beta-1}\qquad(n\geqslant 1).\tag{4.19}$$

Equivalently,

$$S_{\alpha,\beta}(n) = \frac{\alpha + \beta - 1}{\alpha} \sum_{0 \leqslant k < n} \alpha^{\nu(k)} \beta^{\nu_0(k)} - \frac{\beta - 1}{\alpha} \qquad (n \geqslant 1). \tag{4.20}$$

By (4.19) and Example 2.15, we have

$$f(n) = n^{\log_2(\alpha+\beta)} P(\log_2 n) + \frac{\beta - 1}{\alpha + \beta - 1} \qquad (n \geqslant 1), \tag{4.21}$$

with $P(t) = P_0(t)$ given by (2.22). Furthermore, by Theorem 3.6 or Example 3.7, $P(t)$ has an absolutely convergent Fourier series. □

4.3 $(\alpha, \beta) = (1, 2)$ and $(\alpha, \beta) = (2, 1)$

The large number of concrete sequences discussed here and in the following sections show the usefulness and power of the notion of "periodic equivalence".

Example 4.8 ($\Lambda_{1,2}[f] = g$)**.** We begin with A064194 in Example 4.6, which satisfies $\Lambda_{1,2}[f] = 0$ and thus equals $S_{1,2}(n)$. This sequence enumerates the number of gates in the AND/OR problem (Example 4.14) [5]. It also counts the number of multiplications needed to multiply two degree n polynomials using Karatsuba's algorithm [31, Exercise 4.3.3-17], as well as the total number of odd entries in the $(n+1) \times (n+1)$ matrix $\left[\binom{i+j}{i}\right]_{i,j=0}^{n}$. Another interpretation in terms of Sierpiński-like arrays can be found on Peter Karpov's webpage at

In this case, $f(n) = n^{\log_2 3} P(\log_2 n)$ for $n \geqslant 1$, where, by Lemma 2.7 or Theorem 2.10,

$$P(t) = 3^{-\{t\}} \left(1 + 2\varphi_{1,2}\left(2^{\{t\}} - 1 \right) \right). \tag{4.22}$$

Here the interpolation function $\varphi_{1,2}$ has by (2.11) the form (see Figure 2)

$$\varphi_{1,2}\left(\sum_{k \geqslant 1} 2^{-e_k} \right) = \sum_{k \geqslant 1} 2^{e_k - k + 1} 3^{-e_k} \qquad (1 \leqslant e_1 < e_2 < \cdots). \tag{4.23}$$

Some other sequences periodically equivalent to $S_{1,2}(n) = f_{\text{A064194}}(n)$ are given in Table 2, where AND denotes the bitwise logic AND operator; we also denote by $b_j(n)$ the $(j+1)$st bit (from right to left) in the binary expansion of n: $b_j(n) = \lfloor \frac{n}{2^j} \rfloor - 2\lfloor \frac{n}{2^{j+1}} \rfloor$ for $j = 0, 1, \ldots, L_n$, and $\nu_0(n)$ is as in Example 4.7 the number of 0s in the binary expansion of n.

Observe that

$$f_{\text{A325103}}(n) - f_{\text{A325103}}(n-1) = \text{A115378}(n) = 2^{\nu_0(n)} - 1, \tag{4.24}$$

and this provides a proof for the recurrence $\Lambda_{1,2}[f] = \lceil \frac{n}{2} \rceil - 1$ satisfied by $f_{\text{A325103}}(n)$. The consideration of the other two sequences A325102 and A325104 is similar. On the other hand, the sequence $2^{\nu_0(n)}$ corresponds to A080100 whose partial sum satisfies $\Lambda_{1,2}[f] = -1$ with $f(1) = 1$. Then $f(n) = \frac{1}{2}(S_{1,2}(n) + 1)$.

Another example with $(\alpha, \beta) = (1, 2)$ is the sequence A086845, which counts the number of comparators used in Bose and Nelson's sorting networks [3], and satisfies $\Lambda_{1,2}[f] = \lfloor \frac{n}{2} \rfloor$ with $f(1) = 0$. Lemma 4.3 applies with $(c, d, e) = (\frac{1}{2}, 0, -\frac{1}{2})$, giving $f(n) = n^{\log_2 3} P(\log_2 n) - n$; see [25]. In fact, Lemma 4.3 and its proof show that the periodically equivalent sequence $h(n) := f(n) + n$ satisfies $\Lambda_{1,2}[h] = -\mathbf{1}_{n \text{ odd}}$ with $h(1) = 1$, and thus $h(n) = n^{\log_2 3} P(\log_2 n)$ by Example 2.16. □

OEIS id.	Description	$g(n)$	$f(n)$
A080572	$\displaystyle\sum_{0\leqslant k,j<n}\mathbf{1}_{k\text{ AND }j\neq 0}$	$\lfloor\frac{n}{2}\rfloor^2-\mathbf{1}_{n\text{ odd}}$	$n^2-S_{1,2}(n)$
A268514	$\displaystyle\sum_{1\leqslant j<n}2^{\nu_0(j)}$	1	$\frac{1}{2}(S_{1,2}(n)-1)$
A325102	#(pairs (k,m), $1\leqslant k,m\leqslant n$ such that $(b_j(k),b_j(m))\neq(1,1)$, $0\leqslant j\leqslant\min\{L_k,L_m\}$)	$2\lceil\frac{n}{2}\rceil-2$	$S_{1,2}(n)-2n+1$
A325103	#(pairs (k,m), $1\leqslant k<m\leqslant n$ such that $(b_j(k),b_j(m))\neq(1,1)$, $0\leqslant j\leqslant L_k$)	$\lceil\frac{n}{2}\rceil-1$	$\frac{1}{2}S_{1,2}(n)-n+\frac{1}{2}$
A325104	#(pairs (k,m), $1\leqslant k<m\leqslant n$ such that $(b_j(k),b_j(m))=(1,1)$ for some $0\leqslant j\leqslant L_k$)	$\binom{\lfloor\frac{n}{2}\rfloor+1}{2}-1$	$\frac{n^2-n+1}{2}-\frac{1}{2}S_{1,2}(n)$

Table 2: Sequences periodically equivalent to $S_{1,2}(n)$ (Example 4.8).

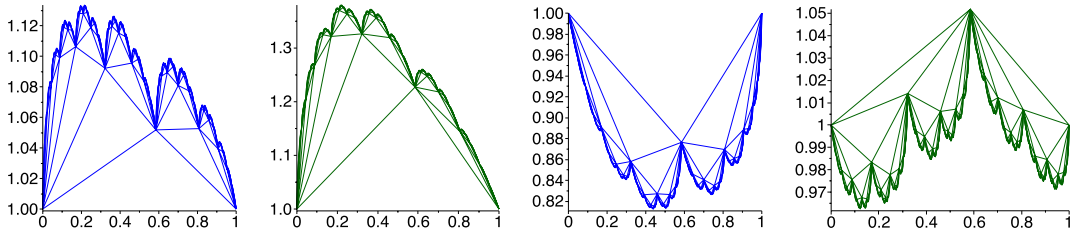*Figure 4: The periodic functions arising from the four sequences (in left to right order) $\Lambda_{1,2}[f]=\lfloor\frac{n}{2}\rfloor$, $\Lambda_{1,2}[f]=\lceil\frac{n}{2}\rceil$, $\Lambda_{2,1}[f]=\lfloor\frac{n}{2}\rfloor$, $\Lambda_{2,1}[f]=\lceil\frac{n}{2}\rceil$ all with $f(1)=0$, as approximated by the fractional part $\left\{\frac{f(2^k+j)+2^k+j}{(2^k+j)^{\log_2 3}}\right\}$ for $0\leqslant j<2^k$ and $k=1,2,\ldots,10$.*

Example 4.9 ($\Lambda_{2,1}[f]=g$). A006046(n) is the total number of odd entries in first n rows of Pascal's triangle. This sequence $f(n)$ equals $S_{2,1}(n)$ (defined in Example 2.15) and has a rich literature with different extensions and connections; for example, it equals (4.8) with $\alpha=2$; see below, the OEIS page, the survey papers [38, 6] and Finch's book [16, §2.16] for more information. By Example 2.15, we have $f(n)=n^{\log_2 3}P_0(\log_2 n)$, where $P_0\in\mathrm{H}_{\log_2 3-1}[0,1]$ by Lemma 3.2; see [20, 25]. Some periodically equivalent sequences (possibly with a shift) are given as follows.

- A051679: Total number of even entries in the first n rows of Pascal's triangle, namely, $f(n)=\binom{n+1}{2}-S_{2,1}(n)$. Then $f(1)=0$ and

$$\Lambda_{2,1}[f]=\frac{1}{8}n^2-\begin{cases}\frac{n}{4}, & \text{if } n \text{ is even};\\ \frac{1}{8}, & \text{if } n \text{ is odd}.\end{cases} \tag{4.25}$$

- A064406: The accumulation of the number of even entries (A048967) over the number of odd entries (A001316) in row n of Pascal's triangle (A007318); in other words, A051679(n) − A006046(n). Thus $f(n)=\binom{n+1}{2}-2S_{2,1}(n)$. Then f satisfies the same recurrence (4.25) but with the different initial condition $f(1)=-1$. The sequence is positive except for the first 18 terms.

- A074330: $S_{2,1}(n+1)-1$.

- A080978: $f(n) = 2S_{2,1}(n) + 1$. Then $\Lambda_{2,1}[f] = -2$ with $f(1) = 3$.

- A151788: $f(n) := \frac{1}{2}(3S_{2,1}(n) - 1) = \frac{3}{2}S_{2,1}(n) - \frac{1}{2}$ and satisfies $\Lambda_{2,1}[f] = 1$ with $f(1) = 1$.

- A159912: $f(n) = \sum_{j<n}\left(2^{\nu(2j+1)} - 1\right)$ satisfies $\Lambda_{2,1}[f] = \lfloor \frac{n}{2} \rfloor$ and the relation $f(n) = 2S_{2,1}(n) - n$.

- A160720: Number of "ON" cells in a certain 2-dimensional cellular automaton: $f(1) = 1$ and
$$\Lambda_{2,1}[f] = 2n - 2 - 2 \times \mathbf{1}_{n \text{ odd}}. \tag{4.26}$$
One has $f(n) = 4(S_{2,1}(n) - n) + 1$.

- A160722: Number of "ON" cells in a certain 2-dimensional cellular automaton based on Sierpiński triangles. Then $\Lambda_{2,1}[f] = 2\lfloor \frac{n}{2} \rfloor$ with $f(1) = 1$ and $f(n) = 3S_{2,1}(n) - 2n$.

- A171378: $f(n) = n^2 - S_{2,1}(n)$. Then $\Lambda_{2,1}[f] = \lceil \frac{n}{2} \rceil^2 - \mathbf{1}_{n \text{ odd}}$ with $f(1) = 0$.

- A193494: Worst case of an unbalanced recursive algorithm over all n-node binary trees; $2A(n-1) + 1$ satisfies the max-recursion (4.15) with $(\alpha, \beta) = (2, 1)$. Thus the sequence $f(n) := A(n-1) = \frac{1}{2}(S_{2,1}(n) - 1)$ satisfies $\Lambda_{2,1}[f] = 1$ with $f(1) = 0$; see Example 4.5 and Proposition C.3.

- A256256: Number of "ON" cells in a cellular automaton on triangular grid, which is $6S_{2,1}(n)$ and satisfies the recurrence $\Lambda_{2,1}[f] = 0$ with $f(1) = 6$.

- A262867: Total number of "ON" cells in a cellular automaton. $f(n) = n^2 - S_{2,1}(n) + 1 = f_{\text{A171378}}(n) + 1$, which satisfies $\Lambda_{2,1}[f] = \lceil \frac{n}{2} \rceil^2 - 2 - \mathbf{1}_{n \text{ odd}}$.

- A266532: Number of Y-toothpicks in a cellular automaton. We then get the recurrence $\Lambda_{2,1}[f] = 3\lfloor \frac{n}{2} \rfloor - 2$ and $f(n) = 3(S_{2,1}(n) - n) + 1$.

- A267610: Accumulated number of "OFF" cells in a cellular automaton. This is $f(n) = S_{2,1}(n) - 2n + 1$, and $\Lambda_{2,1}[f] = 2\lfloor \frac{n}{2} \rfloor$ with $f(1) = 0$.

- A267700: "Tree" sequence in a 90 degree sector of some cellular automaton. (Also the partial sum of A038573.) This sequence satisfies $\Lambda_{2,1}[f] = \lfloor \frac{n}{2} \rfloor$ with $f(1) = 0$, so that $f(n) = S_{2,1}(n) - n$.

A different example with $(\alpha, \beta) = (2, 1)$ is the sequence A137294, which arises in a polynomial-time algorithm for a sowing game; see [15, p. 289] for more information. It satisfies $\Lambda_{2,1}[f] = 1 + \mathbf{1}_{n \text{ odd}}$ with $f(1) = 0$. By Example 2.16 applied to $f(n) + \frac{1}{2}$, we have $f(n) = n^{\log_2 3}P(\log_2 n) - \frac{1}{2}$. See Figure 5.

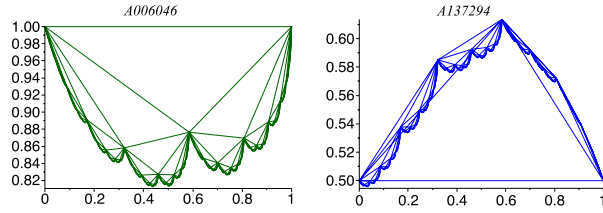$\square$

Figure 5: The periodic functions arising from A006046 and A137294 (see Example 4.9), respectively.

4.4 Sequences satisfying $\Lambda_{\alpha,\beta}[f] = g$ with $\alpha + \beta \geqslant 4$

Example 4.10 ($\alpha + \beta = 4$)**.** The sequence A268524 satisfies $\Lambda_{1,3}[f] = 0$ with $f(1) = 1$ and thus equals our $S_{1,3}(n)$, as listed in Example 4.6.

When $(\alpha, \beta) = (3, 1)$, the prototype sequence $S_{3,1}(n)$ corresponds to A130665, which satisfies $\Lambda_{3,1}[f] = 0$ and $f(1) = 1$; see Example 4.6. Some variants of this sequence from OEIS, all having $\Lambda_{1,3}[f] = g$ constant, are given in the following table; they arise mostly from the combinatorics of the Ulam–Warburton cellular automaton.

OEIS id.	$(f(1), g(n))$	$f(n)$	OEIS id.	$(f(1), g(n))$	$f(n)$
A147562	$(1, 1)$	$\frac{1}{3}(4S_{3,1}(n) - 1)$	A151914	$(4, -4)$	$\frac{4}{3}(2S_{3,1}(n) + 1)$
A151917	$(1, -1)$	$\frac{1}{3}(2S_{3,1}(n) + 1)$	A151920	$(0, 1)$	$\frac{1}{3}(S_{3,1}(n) - 1)$
A160410	$(4, 0)$	$4S_{3,1}(n)$	A160412	$(3, 0)$	$3S_{3,1}(n)$

Table 3: Sequences periodically equivalent to $S_{3,1}(n)$ (Example 4.10).

Three other sequences satisfying $\Lambda_{3,1}[f] = g$ are given below.

OEIS id.	Context	$g(n)$	$f(1)$	$f(n)$
A183060	Cellular automaton	$-n + \begin{cases} 2 \\ 3 \end{cases}$	1	$n^2 P(\log_2 n) + n - \frac{2}{3}$
A183126	Toothpick sequence	$-4n + \begin{cases} 3 \\ 7 \end{cases}$	7	$n^2 P(\log_2 n) + 4n - 1$
A183148	Toothpick sequence	$-3n + \begin{cases} 3 \\ 6 \end{cases}$	4	$n^2 P(\log_2 n) + 3n - 1$

Table 4: Further sequences with $\Lambda_{3,1}[f] = g$ (Example 4.10).

Although very different in appearance, these sequences are all periodically equivalent to $S_{3,1}(n)$ because, by Lemma 4.3, the right-hand side of (4.4) are all of the form $\Lambda_{3,1}[f; 0, 0, 0]$ when $(\alpha, \beta) = (3, 1)$ and $(c, d, e) = (-1, 2, 3), (-4, 3, 7)$ and $(-3, 3, 6)$, respectively. More

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precisely, we have the relations ($f_{\text{A130665}}(n) = S_{3,1}(n)$)

$$f_{\text{A183060}}(n) = \tfrac{2}{3} f_{\text{A130665}}(n) + n - \tfrac{2}{3}, \tag{4.27}$$

$$f_{\text{A183126}}(n) = 4 f_{\text{A130665}}(n) + 4n - 1, \tag{4.28}$$

$$f_{\text{A183148}}(n) = 2 f_{\text{A130665}}(n) + 3n - 1. \tag{4.29}$$

$\square$

Example 4.11 ($\alpha + \beta = 5$). The sequence A268527 in Example 4.6 satisfies $\Lambda_{1,4}[f] = 0$ with $f(1) = 1$, and thus equals our $S_{1,4}(n)$.

Three other sequences were found with $(\alpha, \beta) = (4, 1)$. The first is A116520 which satisfies $\Lambda_{4,1}[f] = 0$ and equals $S_{4,1}(n)$; see Example 4.6. Another sequence A151790 equals $\tfrac{1}{4}(5 S_{4,1}(n) - 1)$. It satisfies $\Lambda_{4,1}[f] = 1$ with $f(1) = 1$; as a check, Lemma 4.2 and (4.3) yield

$$f_{\text{A151790}}(n) = \tfrac{5}{4} f_{\text{A116520}}(n) - \tfrac{1}{4} = \tfrac{5}{4} S_{4,1}(n) - \tfrac{1}{4}. \tag{4.30}$$

The last sequence with the pattern $(\alpha, \beta) = (4, 1)$ we found is A273578, which is the total number of "ON" cells in a 2-D cellular automaton. It satisfies $f(1) = 1$ and

$$\Lambda_{4,1}[f] = \begin{cases} \tfrac{1}{2} n^3 + \tfrac{n}{2} - 4, & \text{if } n \text{ is even;} \\ \tfrac{1}{2} n^3 + \tfrac{3}{2} n^2 - 2n - 4, & \text{if } n \text{ is odd.} \end{cases} \tag{4.31}$$

This sequence is also periodically equivalent to $f_{\text{A116520}}(n)$. To see this, we consider the difference $\Delta(n) = \tfrac{4}{3} n^3 - \tfrac{1}{3} n + 1 - f(n)$, which satisfies $\Lambda_{4,1}[\Delta] = 0$ with $\Delta(1) = 1$; thus $\Delta(n) = S_{4,1}(n) = f_{\text{A116520}}(n)$. From this we deduce the identity

$$f(n) = \tfrac{4}{3} n^3 + n^{\log_2 5} P(\log_2 n) - \tfrac{1}{3} n + 1 \qquad (n \geqslant 1), \tag{4.32}$$

and the relation

$$f_{\text{A273578}}(n) = \tfrac{4}{3} n^3 - \tfrac{1}{3} n + 1 - f_{\text{A116520}}(n). \tag{4.33}$$

Two other sequences with $\alpha + \beta = 5$ are given in Example 4.6: A268526, which satisfies $\Lambda_{2,3}[f] = 0$, and A268525, which satisfies $\Lambda_{3,2}[f] = 0$, both with $f(1) = 1$. $\square$

Example 4.12 ($\alpha + \beta = 6$). Sequences in OEIS of this type have to do either with digital sums or cellular automata. They include A130667 ($= S_{5,1}(n)$) from Example 4.6.

The sequence $S_{4,2}(n)$ is not in OEIS, but the sequence A270106 (which comes from a cellular automaton) equals the sum (4.16) with $(\alpha, \beta) = (4, 2)$, as follows from the discussion in OEIS of its increments A189007. Hence, Example 4.7 shows that

$$f_{A270106}(n) = \tfrac{4}{5} S_{4,2}(n) + \tfrac{1}{5}. \tag{4.34}$$

The six sequences above lead only to two periodically distinct ones (say, A130667 and A270106) because, by Lemma 4.2 and Lemma 4.3, we have (with $f_{\text{A130667}}(n) = S_{5,1}(n)$)

$$f_{\text{A151781}}(n) = \tfrac{6}{5} f_{\text{A130667}}(n) - \tfrac{1}{5} \tag{4.35}$$

$$f_{\text{A186410}}(n) = \tfrac{4}{5} f_{\text{A130667}}(n) + n - \tfrac{4}{5}, \tag{4.36}$$

OEIS id.	(α, β)	$g(n)$	$f(1)$	$f(n)$
A130667	$(5,1)$	0	1	$n^{\log_2 6}P(\log_2 n)$
A151781	$(5,1)$	1	1	$n^{\log_2 6}P(\log_2 n) - \frac{1}{5}$
A186410	$(5,1)$	$-4\lfloor \frac{n}{2} \rfloor + 4$	1	$n^{\log_2 6}P(\log_2 n) + n - \frac{4}{5}$
A270106	$(4,2)$	-1	1	$n^{\log_2 6}P(\log_2 n) + \frac{1}{5}$
A273500	$(4,2)$	$\begin{cases} \frac{1}{3}n^3 + \frac{2}{3}n - 4, \\ \frac{1}{3}n^3 + n^2 - \frac{7}{3}n - 4, \end{cases}$	1	$\begin{aligned} &\frac{4}{3}n^3 + n^{\log_2 6}P(\log_2 n) \\ &\qquad -\frac{1}{3}n + \frac{4}{5} \end{aligned}$
A273562	$(4,2)$	$\begin{cases} \frac{1}{6}n^3 - \frac{2}{3}n + 1, \\ \frac{1}{6}n^3 + \frac{1}{2}n^2 - \frac{13}{6}n + \frac{3}{2}, \end{cases}$	0	$\begin{aligned} &\frac{2}{3}n^3 + n^{\log_2 6}P(\log_2 n) \\ &\qquad +\frac{1}{3}n - \frac{1}{5} \end{aligned}$

Table 5: Sequences with $\Lambda_{\alpha,\beta}[f] = g$ with $\alpha + \beta = 6$ (Example 4.12).

and, recall also (4.34),

$$f_{\text{A273500}}(n) = \tfrac{4}{3}n^3 - \tfrac{1}{3}n + 1 - f_{\text{A270106}}(n) \tag{4.37}$$
$$f_{\text{A273562}}(n) = \tfrac{2}{3}n^3 + \tfrac{1}{3}n - f_{\text{A270106}}(n), \tag{4.38}$$

so that

$$f_{\text{A273500}}(n) = \tfrac{2}{3}n^3 - \tfrac{2}{3}n + 1 + f_{\text{A27356}}(n). \tag{4.39}$$

$\square$

Example 4.13 ($\alpha + \beta \geqslant 7$). For $\alpha + \beta \geqslant 7$, we found the following examples with $\beta = 1$. (The ones with $g(n) = 0$ and $f(1) = 1$ appear also in Example 4.6.)

OEIS id.	(α, β)	$g(n)$	$f(1)$	$f(n)$	$xS_{\alpha,1}(n) + y$
A116522	$(6,1)$	0	1	$n^{\log_2 7}P(\log_2 n)$	$S_{6,1}(n)$
A151792	$(6,1)$	1	1	$n^{\log_2 7}P(\log_2 n) - \frac{1}{6}$	$\frac{7}{6}S_{6,1}(n) - \frac{1}{6}$
A151793	$(7,1)$	1	1	$n^3 P(\log_2 n) - \frac{1}{7}$	$\frac{8}{7}S_{7,1}(n) - \frac{1}{7}$
A160428	$(7,1)$	0	8	$n^3 P(\log_2 n)$	$8S_{7,1}(n)$
A161342	$(7,1)$	0	1	$n^3 P(\log_2 n)$	$S_{7,1}(n)$
A116526	$(8,1)$	0	1	$n^{\log_2 9}P(\log_2 n)$	$S_{8,1}(n)$
A255764	$(8,1)$	1	1	$n^{\log_2 9}P(\log_2 n) - \frac{1}{8}$	$\frac{9}{8}S_{8,1}(n) - \frac{1}{8}$
A255765	$(9,1)$	1	1	$n^{\log_2 10}P(\log_2 n) - \frac{1}{9}$	$\frac{10}{9}S_{9,1}(n) - \frac{1}{9}$
A116525	$(10,1)$	0	1	$n^{\log_2 11}P(\log_2 n)$	$S_{10,1}(n)$
A255766	$(10,1)$	1	1	$n^{\log_2 11}P(\log_2 n) - \frac{1}{10}$	$\frac{11}{10}S_{10,1}(n) - \frac{1}{10}$
A116524	$(12,1)$	0	1	$n^{\log_2 13}P(\log_2 n)$	$S_{12,1}(n)$
A116523	$(16,1)$	0	1	$n^{\log_2 17}P(\log_2 n)$	$S_{16,1}(n)$

Table 6: Sequences with $\Lambda_{\alpha,\beta}[f] = g$ for $\alpha + \beta \geqslant 7$ (Example 4.13).

The only example we found in OEIS with $\beta \neq 1$ is A269589 with $(\alpha, \beta) = (3,4)$:

$$\Lambda_{3,4}[f] = -6 \cdot 2^{\nu_0(n)}\mathbf{1}_{n\text{ odd}}, \tag{4.40}$$

with $f(1) = 1$; this sequence enumerates the number of triples $(i, j, k) \in [0, n-1]^3$ such that their bitwise AND is zero. By Example 2.16, we have $f(n) = n^{\log_2 7} P(\log_2 n)$ for some periodic function P. $\qquad\square$

5   Applications II. $\alpha = \beta$

We group in this section examples satisfying the recurrence $\Lambda_{\alpha,\alpha}[f] = g$. Since the interpolating function $\varphi_{\alpha,\alpha}(t) = t$ for every $\alpha > 0$, this is similar to the case $\alpha = \beta = 1$ treated in [27]. In particular, $\varphi_{\alpha,\alpha}$ is linear on $[0, 1]$, and therefore we can derive in many cases a closed-form solution in terms of elementary functions. In cases when the periodic functions do not have simple explicit forms, we can often derive explicit Fourier expansions in terms of known functions such as Riemann's or Hurwitz's zeta functions. As the situations and analysis are very similar to the case when $(\alpha, \beta) = (1, 1)$, we omit most of the details, which can be found in [27]. Note, however, one difference between the cases $\alpha > 1$ and $\alpha = 1$: as discussed in Remark 2.8, the periodic function $P_0(t)$ in Lemma 2.7 is not continuously differentiable when $\alpha > 1$; hence, typically, the periodic function $P(t)$ in Theorem 2.10 also is not continuously differentiable.

We begin with two examples for a general α.

Example 5.1. Consider the sum $f(n) := \sum_{1 \leqslant k < n} \alpha^{L_k}$. The case $\alpha = 2$ is A063915. Since $L_k = \nu(k) + \nu_0(k) - 1$ for $k \geqslant 1$, we have by Example 4.7 and (4.19)

$$f(n) = \sum_{1 \leqslant k < n} \alpha^{L_k} = \frac{1}{\alpha}\left(\sum_{0 \leqslant k < n} \alpha^{\nu(k) + \nu_0(k)} - 1\right) = \frac{1}{2\alpha - 1}\big(S_{\alpha,\alpha}(n) - 1\big), \qquad (5.1)$$

at least provided $\alpha > \frac{1}{2}$; the result holds indeed for any $\alpha \neq \frac{1}{2}$ since both sides are polynomials in α for fixed n. It follows easily (cf. (4.18)), that $\Lambda_{\alpha,\alpha}[f](n) = g(n) = 1$. $\qquad\square$

Example 5.2. A more general pattern that we found in several OEIS sequences (all with $\alpha = \beta = 2$; see Table 8 below) is of the form in Lemma 4.3, i.e.,

$$\Lambda_{\alpha,\alpha}[f] = g(n) = cn + \begin{cases} d, & \text{if } n \text{ is even;} \\ e, & \text{if } n \text{ is odd.} \end{cases} \qquad (5.2)$$

Assume for simplicity $\alpha > 1$, so Lemma 4.3 applies. Then (4.5) yields the solution

$$f(n) = n^\varrho P(\log_2 n) - \frac{cn}{\alpha - 1} - \frac{d}{2\alpha - 1} \qquad (n \geqslant 1). \qquad (5.3)$$

Furthermore, it follows, from the proof of Lemma 4.3, that the Fourier coefficients of the periodic function P are given by (assuming first $\alpha > 2$ so that $\varrho > 2$),

$$\begin{aligned}
\widehat{P}(k) &= \frac{1}{(\varrho + \chi_k)(\varrho + \chi_k - 1)\log 2}\left(\frac{(2\alpha - 1)(\alpha - 1)}{\alpha}\bar{f}(1)\right. \\
&\quad \left. + (e - d)\sum_{j \geqslant 1}\Big((2j)^{-(\varrho + \chi_k - 1)} - 2(2j+1)^{-(\varrho + \chi_k - 1)} + (2j+2)^{-(\varrho + \chi_k - 1)}\Big)\right) \\
&= \frac{(2\alpha - 1)(\alpha - 1)f(1) + (2\alpha - 1)(c + e) - \alpha d + 2(e - d)(2 - \alpha)\zeta(\varrho - 1 + \chi_k)}{\alpha(\varrho + \chi_k)(\varrho - 1 + \chi_k)\log 2}.
\end{aligned}$$
$$(5.4)$$

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By analytic continuation (temporarily allowing complex α and ϱ), (5.4) holds for all α with $\alpha > 1$ (so $\varrho > 1$), but we have to be careful when $\alpha = 2$ and thus $\varrho = 2$, since then ζ has a pole at $\varrho - 1 + \chi_0 = 1$, and we have to interpret (by continuity) $(2 - \alpha)\zeta(\varrho - 1 + \chi_0) = -2\log 2$. The formula simplifies when $\alpha = 2$, since then all other terms with ζ disappears. See further Example 5.4, where an explicit formula for $P(t)$ is given for $\alpha = 2$.

From a generating function viewpoint, the fact that $\alpha = 2$ is special may be due to the identity

$$\sum_{k \geqslant 0} \frac{\alpha^k z^{2^k}}{1 + z^{2^k}} = \frac{z}{1 - z} \quad \text{iff} \quad \alpha = 2. \tag{5.5}$$

$\square$

5.1 $(\alpha, \beta) = (1, 1)$

Example 5.3 (Sequences not in [27]). As this case has already been discussed in detail in [27], we only list sequences (together with their closed-form expressions) that are not included in [27]. We use the pattern $f(n) = nP(\log_2 n) - Q(n)$. (Some recursions start at some $n > 2$.)

OEIS	$g(n)$	**Initials**	$P(t)$	$-Q(n)$
A277267 (binary trees)	1	$\{f(j)\}_{2 \leqslant j \leqslant 3}$ $= \{0, 0\}$	$\max \left\{ \begin{array}{c} 2^{-1-\{t\}} \\ 1 - 2^{-\{t\}} \end{array} \right\}$	-1
A279521 (binary trees)	1	$\{f(j)\}_{2 \leqslant j \leqslant 3}$ $= \{0, 1\}$	$\min \left\{ \begin{array}{c} 1 - 2^{-1-\{t\}} \\ 2^{-\{t\}} \end{array} \right\}$	-1
A294456 (recursion)	2	$\{f(j)\}_{1 \leqslant j \leqslant 2}$ $= \{0, 1\}$	$\min \left\{ \begin{array}{c} 2 - 2^{-1-\{t\}} \\ 1 + 2^{-\{t\}} \end{array} \right\}$	-2
A295513 (binary length)	n	$f(1) = -1$	$1 - \{t\} - 2^{1-\{t\}}$	$n \log_2 n$
A296062 (binary trees)	$\mathbf{1}_{n \text{ odd}}$	$f(1) = 0$	see below	0
A296349 (digital sum)	$n - 2$	$f(1) = 1$	$1 - \{t\} - 2^{1-\{t\}}$	$n \log_2 n + 2$
A297531 (subword complexity)	0	$\{f(j)\}_{4 \leqslant j \leqslant 7}$ $= \{13, 17, 21, 24\}$	$\min \left\{ \begin{array}{c} 4 - 3 \cdot 2^{-2-\{t\}} \\ 3 + 3 \cdot 2^{-2-\{t\}} \\ 2 + 5 \cdot 2^{-1-\{t\}} \end{array} \right\}$	0
A301336 (digital sum)	$2 - \mathbf{1}_{n \text{ odd}}$	$f(1) = -1$	see below	-2
A303548 (Hamming weight)	$\mathbf{1}_{n \equiv 3 \bmod 4}$	$f(1) = 0$	see below	0
A316936 (word complexity)	$\frac{1}{2}n^2 - 2n$ $+ \left\{ \begin{array}{c} 1 \\ \frac{1}{2} \end{array} \right.$	$f(1) = 3$	$-1 + 2\{t\} + 2^{2-\{t\}}$	$n^2 - 2n \log_2 n$ -1

Table 7: Some sequences with $\Lambda_{1,1}[f] = g$ (Example 5.3).

In particular, the sequence A297531 gives the maximum possible subword complexity over

all binary overlap-free words of a given length, A301336 counts the difference between the total number of 1s and the total number of 0s in the binary expansions of $\{0, \ldots, n-1\}$, and A303548 equals the sum of the distances from n to the nearest number with a given Hamming weight.

Note that some of these sequences are discussed in [27] and were subsequently added to OEIS:

OEIS	A294456	A296062	A303548
Example in [27]	3.2	3.6	3.7

Here $A295513(n) = A001855(n) - 1$, $A296349(n) = A083652(n-1)$, and

$$f_{A296349}(n) = f_{A295513}(n) + 2, \quad f_{A301336}(n) = n - 2 - f_{A296062}(n). \tag{5.6}$$

On the other hand, with $\chi_k := \frac{2k\pi i}{\log 2}$ as above, using [27, Theorem 3 and Example 3.6],

$$P_{A296062}(t) = 2 - \log_2 \pi + \frac{1}{\log 2} \sum_{k \neq 0} \frac{1 + 2\zeta(\chi_k)}{\chi_k(1 + \chi_k)} e^{2k\pi it} \tag{5.7}$$

$$P_{A301336}(t) = 1 - P_{A296062}(t) \tag{5.8}$$

$$P_{A303548}(t) = \tfrac{1}{2} \log_2 \pi - 2 \log_2 \Gamma\left(\tfrac{3}{4}\right) + \frac{1}{\log 2} \sum_{k \neq 0} \frac{\zeta(\chi_k) - 2\zeta(\chi_k, \tfrac{3}{4})}{\chi_k(1 + \chi_k)} e^{2k\pi it}. \tag{5.9}$$

Finally, the sequence A330038, satisfying $\Lambda_{1,1}[f] = \lfloor \frac{n}{2} \rfloor$ with $f(1) = 1$, is nothing but A000788 (best case of mergesort or partial sum of $\nu(n)$) plus n; see [27, Example 5.2]. $\square$

5.2 $(\alpha, \beta) = (2, 2)$

While most analysis here parallels that in [27], we will see that there are subtle differences in the genesis of periodic oscillations. In particular, as remarked above, the periodic function $P(t)$ is generally not continuously differentiable.

Example 5.4 (Periodic functions differentiable except at integers). Consider the simple case when $\Lambda_{2,2}[f] = c$ for $n \geqslant 2$. Then, by Lemma 4.2, Example 2.15 and (2.22) (or directly by (2.21) and (2.22)), we deduce that

$$f(n) = \left(f(1) + \tfrac{1}{3}c\right) S_{2,2}(n) - \tfrac{1}{3}c = \left(f(1) + \tfrac{1}{3}c\right) 2^{L_n}\left(3n - 2^{L_n+1}\right) - \tfrac{1}{3}c \qquad (n \geqslant 1), \tag{5.10}$$

so that $f(n) = n^2 P(\log_2 n) - \tfrac{1}{3}c$, where

$$P(t) = \left(f(1) + \tfrac{1}{3}c\right) P_0(t) = \left(f(1) + \tfrac{1}{3}c\right) 2^{-\{t\}}(3 - 2^{1-\{t\}}), \tag{5.11}$$

is continuously differentiable on $[0, 1]$, but not at integer t, where the derivative $P'(t)$ has a jump.

A more general pattern that we identified from the OEIS sequences is of the form (see Lemma 4.3 and Example 5.2)

$$\Lambda_{2,2}[f] = g(n) = cn + \begin{cases} d, & \text{if } n \text{ is even;} \\ e, & \text{if } n \text{ is odd.} \end{cases} \tag{5.12}$$

Then the solution is given by $f(n) = n^2 P(\log_2 n) - cn - \frac{1}{3} d$ for $n \geqslant 1$, where

$$P(t) = d - e + \big(f(1) + c - \tfrac{2}{3} d + e\big) 2^{-\{t\}} \big(3 - 2^{1-\{t\}}\big), \tag{5.13}$$

which easily is verified using Lemma 4.3, (4.6)–(4.7), and the special case $\Lambda_{2,2}[n^2] = -\mathbf{1}_{n \text{ is odd}}$.

This can be compared with the recursion $\Lambda_{\alpha,\alpha}[f] = g(n)$ for a general α (with the same $g(n)$) studied in Example 5.2, where it is readily checked that the Fourier coefficients (5.4) (with $\alpha = 2$) agree with (5.13) and (3.33).

The prototype sequence in this category is A073121:

$$f_{\text{A073121}}(n) = S_{2,2}(n) = n^2 P_{\text{A073121}}(\log_2 n), \tag{5.14}$$

where $P_{\text{A073121}}(t) = P_0(t) := 2^{-\{t\}} \big(3 - 2^{1-\{t\}}\big)$, which arises as an upper bound for the cardinality of some regular expressions; see [14] for the more general form $\Lambda_{\alpha,\alpha}[f] = 0$. The Fourier coefficients of $P_0(t)$ are, by (3.33), of the form
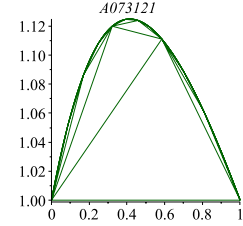
$$\widehat{P}_0(k) = \frac{3}{2 \log 2 \cdot (1 + \chi_k)(2 + \chi_k)}. \tag{5.15}$$

With $S_{2,2}(n)$, the solution to (5.12) can alternatively be written as

$$f(n) = (d - e)n^2 + \big(f(1) + c - \tfrac{2}{3} d + e\big) S_{2,2}(n) - cn - \tfrac{1}{3} d. \tag{5.16}$$

Table 8 gives more examples from OEIS that are periodically equivalent to $S_{2,2}(n)$. $\qquad\square$

OEIS id.	Context	$g(n)$	$f(1)$	$f(n)$
A063915	$\sum_{1 \leqslant k < n} 2^{L_k}$	1	0	$\frac{1}{3}(S_{2,2}(n) - 1)$
A073121	Regular expressions	0	1	$S_{2,2}(n)$
A181497	Combinatorial sequence	-1	1	$\frac{1}{3}(2S_{2,2}(n) + 1)$
A236305	Nim game	$\begin{cases} 0 \\ -3 \end{cases}$	1	$3n^2 - 2S_{2,2}(n)$
A255748	Cellular automaton	$\begin{cases} -\frac{1}{2}n + 2 \\ -\frac{1}{2}n + \frac{5}{2} \end{cases}$	0	$-\frac{1}{2}n(n-1) + \frac{2}{3}(S_{2,2}(n) - 1)$
A256249	Josephus problem	$\begin{cases} 1 \\ 0 \end{cases}$	0	$n^2 - \frac{1}{3}(2S_{2,2}(n) + 1)$
A256250	Cellular automaton	$\begin{cases} 1 \\ -3 \end{cases}$	1	$4n^2 - \frac{1}{3}(8S_{2,2}(n) + 1)$
A256266	Cellular automaton	$\begin{cases} -3n + 12 \\ -3n + 15 \end{cases}$	0	$-3n(n-1) + 4(S_{2,2}(n) - 1)$
A256534	Cellular automaton	$\begin{cases} 0 \\ -12 \end{cases}$	4	$4(3n^2 - 2S_{2,2}(n))$

|				
A261692	Cellular automaton	$\begin{cases} 1 \\ 2 \end{cases}$	0	$-n^2 + \frac{1}{3}(4S_{2,2}(n) - 1)$
A262620	Cellular automaton	$\begin{cases} 1 \\ 5 \end{cases}$	1	$-4n^2 + \frac{1}{3}(16S_{2,2}(n) - 1)$
A266538	Josephus problem	$\begin{cases} 2 \\ 0 \end{cases}$	0	$2n^2 - \frac{2}{3}(2S_{2,2}(n) + 1)$

Table 8: Sequences satisfying $\Lambda_{2,2}[f] = g$ and their relations to $S_{2,2}(n)$ (Example 5.4).

Example 5.5 (Piecewise differentiable periodic functions). When the recurrence $\Lambda_{2,2}[f] = g$ is satisfied only for $n \geqslant n_0 > 1$, the resulting periodic function is specified according to the initial conditions $f(n)$ with $n \leqslant n_0$. For simplicity, we consider the sequence A080075 (Proth numbers), which denotes the numbers of the form $(2r+1)2^k + 1$ for $k \geqslant 1$ and $2r + 1 < 2^k$ and has many variants. The sequence $f(n) = $ A080075$(n-1)$ then satisfies $\Lambda_{2,2}[f] = -3$ with the initial conditions $f(2) = 3$ and $f(3) = 5$. The solution is then given by $f(n) = n^2 P(\log_2 n) + 1$ for $n \geqslant 2$ with

$$P(t) = \begin{cases} 2^{-\{t\}}\left(1 - 2^{-1-\{t\}}\right), & \text{if } t \in [0, \log_2 \frac{3}{2}]; \\ 2^{1-\{t\}}\left(1 - 2^{-\{t\}}\right), & \text{if } t \in [\log_2 \frac{3}{2}, 1]; \end{cases} \tag{5.17}$$

see [27, Examples 3.1 and 3.2] for similar behaviour.

OEIS id.	$g(n)$	$(f(2), f(3))$	$f(n)$
A080075	-3	$(3, 5)$	
A082662	0	$(1, 2)$	$\frac{1}{2}(f_{\text{A080075}}(n) - 1)$
A112714	3	$(1, 3)$	$f_{\text{A080075}}(n) - 2$
A116882	0	$(2, 4)$	$f_{\text{A080075}}(n) - 1$
A260711	0	$(8, 16)$	$4(f_{\text{A080075}}(n) - 1)$

$\square$
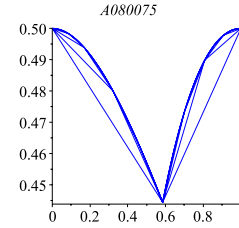
Example 5.6 (Non-differentiable periodic functions). A few sequences defined as the partial sum of the bitwise operator between j and $n - j$ or their complements satisfy $\Lambda_{2,2}[f] = g$ with different g; see Table 9 for a summary; in all cases $f(1) = 0$. Note that the NOT operator $\bar{j}$ uses the full number of bits $L_n + 1$ for each $0 \leqslant j < n$ and equals

$$(\underbrace{1 \ldots 1}_{L_n - s} 0\bar{b}_{s-1} \cdots \bar{b}_0)_2 \text{ if } j = (1 b_{s-1} \cdots b_0)_2. \tag{5.18}$$

OEIS id.	Description	$g(n)$	$f(n)$
A006581	$\displaystyle\sum_{1 \leqslant j \leqslant n-2} \left(j \text{ AND } (n-1-j)\right)$	$\begin{cases} 0 \\ \frac{n-1}{2} \end{cases}$	$n^2 P(\log_2 n)$
A006582	$\displaystyle\sum_{1 \leqslant j \leqslant n-2} \left(j \text{ XOR } (n-1-j)\right)$	$\begin{cases} 3n-6 \\ 2n-6 \end{cases}$	$n^2 P(\log_2 n) - 3n + 2$

A006583	$\displaystyle\sum_{1\leqslant j\leqslant n-2}\big(j\ \mathsf{OR}\ (n-1-j)\big)$	$\begin{cases}3n-6\\[2pt]\frac{5}{2}n-\frac{13}{2}\end{cases}$	$n^2 P(\log_2 n)-3n+2$
A090889	$\displaystyle\sum_{1\leqslant j<n} j\,v_2(j)(n-j)$	$\begin{cases}\frac{1}{12}n^3-\frac{1}{3}n\\[2pt]\frac{1}{12}n^3-\frac{1}{12}n\end{cases}$	$\frac{1}{6}n^3+n^2 P(\log_2 n)+\frac{n}{3}$
A099027	$\displaystyle\sum_{0\leqslant j<n}\big(\bar{j}\ \mathsf{AND}\ (n-1-j)\big)$	$\begin{cases}\frac{n}{2}\\[2pt]0\end{cases}$	$n^2 P(\log_2 n)-\frac{n}{2}$

Table 9: Sequences satisfying $\Lambda_{2,2}[f]=g$ with non-smooth periodic functions (Example 5.6).

Here $v_2(n)$ denotes the dyadic valuation of n (exponent of the highest power of 2 dividing n). Note that the recurrence provided on OEIS for A090889 is incorrect (and the generating function misses a factor of 2).

These apparently different sequences are all periodically equivalent. Indeed, by the recurrences and induction, we can prove the relations

$$
\begin{aligned}
f_{\mathrm{A006582}}(n) &= (n-1)(n-2)-2f_{\mathrm{A006581}}(n)\\
f_{\mathrm{A006583}}(n) &= (n-1)(n-2)-f_{\mathrm{A006581}}(n)\\
f_{\mathrm{A090889}}(n) &= \tfrac{1}{6}n(n-1)(n-2)+f_{\mathrm{A006581}}(n)\\
f_{\mathrm{A099027}}(n) &= \tfrac{1}{2}n(n-1)-f_{\mathrm{A006581}}(n).
\end{aligned}
\tag{5.19}
$$

The Fourier expansion of $P_{\mathrm{A006581}}(t)$ is, by (3.29) and standard calculations, given by

$$
P_{\mathrm{A006581}}(t) = \frac{1}{2}-\frac{1}{4\log 2}+\frac{1}{\log 2}\sum_{k\neq 0}\frac{\zeta(\chi_k)}{(1+\chi_k)(2+\chi_k)}\,e^{2k\pi i t}.
\tag{5.20}
$$

We prove in Appendix D that the continuous function P_{A006581} is nowhere differentiable, and that it is not Lipschitz.

Another sequence A048641 is defined as the sum $\sum_{k<n}\gamma(k)$, where $\gamma(k)=k\ \mathsf{XOR}\ \lfloor\frac{1}{2}k\rfloor$ denotes the numerical value of the binary reflected Gray code of k (A003188), and satisfies the recurrence $\Lambda_{2,2}[f]=\frac{1}{2}\big(n-\sin\frac{1}{2}n\pi\big)$ with $f(1)=0$. We then obtain $f(n)=n^2P(\log_2 n)-\frac{1}{2}n$ with

$$
P(t) = \frac{\pi}{8\log 2}+\frac{1}{4\log 2}\sum_{k\neq 0}\frac{\zeta\big(1+\chi_k,\frac{1}{4}\big)-\zeta\big(1+\chi_k,\frac{3}{4}\big)}{(1+\chi_k)(2+\chi_k)}\,e^{2k\pi i t}.
\tag{5.21}
$$

We leave the question whether this function is nowhere differentiable as an open problem.

Yet another related example is A022560; this is the sum $f(n):=\sum_{1\leqslant k<n}2^{v_2(k)}(n-k)$, which satisfies the recurrence $\Lambda_{2,2}[f]=\lfloor\frac{1}{4}n^2\rfloor$. (Note that one of the recursions given in OEIS is incorrect.) In this case $g(n)$ grows too rapidly for Theorem 2.10 to be directly applicable, since the series (2.28) diverges. A modification of our arguments, see Example A.3 for details, shows that we have

$$
f_{\mathrm{A022560}}(n) = \tfrac{1}{4}n^2\log_2 n + n^2 P_{\mathrm{A022560}}(\log_2 n)
\tag{5.22}
$$

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with a logarithmic factor in the leading term, and periodic fluctuations given by

$$P_{\text{A022560}}(t) = \frac{3}{8} + \frac{2\gamma - 3}{8\log 2} + \frac{1}{2\log 2} \sum_{k \neq 0} \frac{\zeta(1 + \chi_k)}{(1 + \chi_k)(2 + \chi_k)} e^{2k\pi it}. \tag{5.23}$$
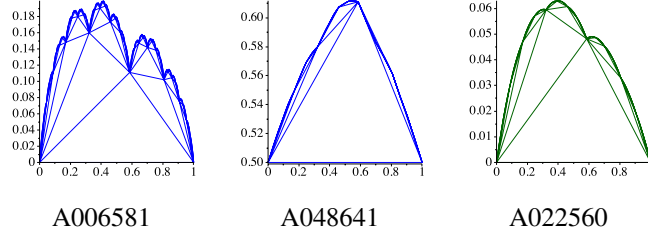
See Figure 6. □

A006581   A048641   A022560

Figure 6: Fluctuations (properly normalised) in the three periodically distinct cases in Example 5.6. At least the first is nowhere differentiable.

Example 5.7 (Sensitivity test: small variations inducing big differences)**.** While the previous examples show that many different toll functions lead to periodically equivalent oscillations, the same recurrence also exhibits the opposite sensitivity property, as we now examine. We begin with the difference $f(n) = f_{\text{A048641}}(n) - \binom{n}{2}$, which gives A048644 and satisfies the recurrence $\Lambda_{2,2}[f] = \mathbf{1}_{n \bmod 4 \equiv 3}$ with the solution $f(n) = n^2 P(\log_2 n)$, where $P(t) = P_{\text{A048641}}(t) - \frac{1}{2}$ is given by (5.28) below.

It is interesting to see that changing 3 to other remainders results in drastically different periodic functions; compare also [27, Example 3.7]. Let $\Lambda_{2,2}[f_j] = \mathbf{1}_{n \bmod 4 \equiv j}$ for $j = 0, 1, 2, 3$ with $f_j(1) = 0$. Then

$$f_j(n) = \begin{cases} n^2 P_0(\log_2 n) - \frac{1}{3} + \frac{1}{4} \cdot \mathbf{1}_{n \text{ odd}}, & \text{if } j = 0, \\ n^2 P_j(\log_2 n), & \text{if } j = 1, 3, \\ n^2 P_2(\log_2 n) - \frac{1}{4} \cdot \mathbf{1}_{n \text{ odd}}, & \text{if } j = 2, \end{cases} \tag{5.24}$$

where $P_0(t)$ is continuously differentiable on $[0, 1]$ (not the same as $P_0(t)$ from Section 2), $P_2(t) = \frac{1}{4}$ is *a constant*, and $P_1(t)$ has many visible cusps in the unit interval (see Figure 7). More explicit expressions are given by

$$P_0(t) = \frac{3}{4} - 2^{1-\{t\}} + \frac{1}{3}4^{1-\{t\}}, \tag{5.25}$$

$$P_1(t) = \frac{3}{4\log 2} - \frac{\pi}{8\log 2} - \frac{1}{2} - \frac{1}{4\log 2} \sum_{k \neq 0} \frac{\zeta(1 + \chi_k, \frac{1}{4}) - \zeta(1 + \chi_k, \frac{3}{4}) - 6}{(1 + \chi_k)(2 + \chi_k)} e^{2k\pi it}, \tag{5.26}$$

$$P_2(t) = \frac{1}{4}, \tag{5.27}$$

$$P_3(t) = \frac{\pi}{8\log 2} - \frac{1}{2} + \frac{1}{4\log 2} \sum_{k \neq 0} \frac{\zeta(1 + \chi_k, \frac{1}{4}) - \zeta(1 + \chi_k, \frac{3}{4})}{(1 + \chi_k)(2 + \chi_k)} e^{2k\pi it}; \tag{5.28}$$

see Figure 7 for an illustration. The difference $\zeta(s, \frac{1}{4}) - \zeta(s, \frac{3}{4})$ is also known as the Dirichlet beta function; it appears also in several other formulas above and below. Note that $f_2(n) =$

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$\left\lfloor \frac{1}{4}n^2 \right\rfloor$, which equals the quarter-squares A002620, and the sum $f_1 + f_2 + f_3 + f_4$ equals A063915 in Example 5.1 and Table 8. Also the mean values of these periodic functions are given by

$$\left(-\frac{1}{2\log 2} + \frac{3}{4}, \frac{3}{4\log 2} - \frac{\pi}{8\log 2} - \frac{1}{2}, \frac{1}{4}, \frac{\pi}{8\log 2} - \frac{1}{2} \right) \approx \left(0.02865, 0.01548, 0.25, 0.06655 \right), \quad (5.29)$$

respectively, and we see that the mean value of P_2 is much larger than those of other three.

Similarly, by comparing with Table 8, the sequence A256249 for a sum for the Josephus problem has $f_{\text{A256249}} = f_0 + f_2$. The sequence A266540 for another sum in the Josephus problem has $f(1) = 0$ and $\Lambda_{2,2}[f] = 1 + \cos\left(\frac{1}{2}n\pi\right)$ (which has the periodic pattern $(2, 1, 0, 1)$ for $n \geqslant 0$); hence $f_{\text{A266540}} = 2f_0 + f_1 + f_3$. It is written on the OEIS page that "It appears that this sequence has a fractal (or like-fractal) behaviour." This is untrue because from the generating function given there

$$\sum_{n \geqslant 1} f(n)z^n = \frac{z^3(1 + z^2)}{(1 - z^2)(1 - z)^2} - \frac{z}{(1 - z)^2} \sum_{k \geqslant 2} 2^{k-1} z^{2^k}, \quad (5.30)$$

or from our approach, we can derive the identity $f(n) = n^2 P(\log_2 n) - \frac{2}{3} + \frac{1}{2}\mathbf{1}_{n \text{ is odd}}$ with $P(t) = \frac{1}{2} - 2^{-\{t\}} + \frac{2}{3}4^{-\{t\}}$.

Changing the recurrence to $\Lambda_{2,2}[f] = 1 - \cos\left(\frac{1}{2}n\pi\right)$ does not alter the smooth nature of the periodic function because $f(n) = n^2 P(\log_2 n) - \frac{1}{2}\mathbf{1}_{n \text{ is odd}}$, where $P(t) = -\frac{1}{2} + 3 \cdot 2^{-t} - 2 \cdot 4^{-t}$. However, switching cosine function to sine function does change the nature of the periodic oscillation because we then have the solution $f(n) = n^2 P(\log_2 n) - \frac{1}{3}$ when (*i*) $g(n) = 1 + \sin\left(\frac{1}{2}n\pi\right)$, where

$$P(t) = \frac{1}{\log 2} - \frac{\pi}{4\log 2} - \frac{1}{2\log 2} \sum_{k \neq 0} \frac{\zeta\left(1 + \chi_k, \frac{1}{4}\right) - \zeta\left(1 + \chi_k, \frac{3}{4}\right) - 4}{(1 + \chi_k)(2 + \chi_k)} e^{2k\pi it}, \quad (5.31)$$

and (*ii*) $g(n) = 1 - \sin\left(\frac{1}{2}n\pi\right)$, where

$$P(t) = \frac{\pi}{4\log 2} - \frac{1}{2\log 2} + \frac{1}{2\log 2} \sum_{k \neq 0} \frac{\zeta\left(1 + \chi_k, \frac{1}{4}\right) - \zeta\left(1 + \chi_k, \frac{3}{4}\right) - 2}{(1 + \chi_k)(2 + \chi_k)} e^{2k\pi it}. \quad (5.32)$$

$\square$

Consider finally the partial sum of A229762:

$$f(n) := \sum_{0 \leqslant k < n} \left((k \text{ XOR } \lfloor \tfrac{k}{2} \rfloor) \text{ AND } \lfloor \tfrac{k}{2} \rfloor \right). \quad (5.33)$$

Then $\Lambda_{2,2}[f] = \lfloor \frac{n+1}{4} \rfloor$. Let $\bar{f}(n) := f(n) + \frac{n}{4} - \frac{1}{8}\mathbf{1}_{n \text{ odd}}$. Then $\bar{f}(1) = \frac{1}{8}$ and for $n \geqslant 2$

$$\Lambda_{2,2}[\bar{f}] = \frac{((n \bmod 4) - 1)^2 - 1}{8} = \begin{cases} -\frac{1}{8}, & \text{if } n \bmod 4 \equiv 1, \\ \frac{3}{8}, & \text{if } n \bmod 4 \equiv 3, \\ 0, & \text{otherwise.} \end{cases} \quad (5.34)$$

We then deduce that $f(n) = n^2 P(\log_2 n) - \frac{n}{4} + \frac{1}{8}\mathbf{1}_{n \text{ odd}}$, where we have, using (3.29) again, $P(t) = \frac{1}{2}P_3(t) + \frac{1}{8}$ with $P_3(t)$ given in (5.28).

Similarly, the partial sum of the sequence A229763 satisfies $\Lambda_{2,2}[f] = \frac{1}{2}(1 - (-1)^{\lfloor \frac{1}{2}n \rfloor})$ and can be dealt with by the same procedure.
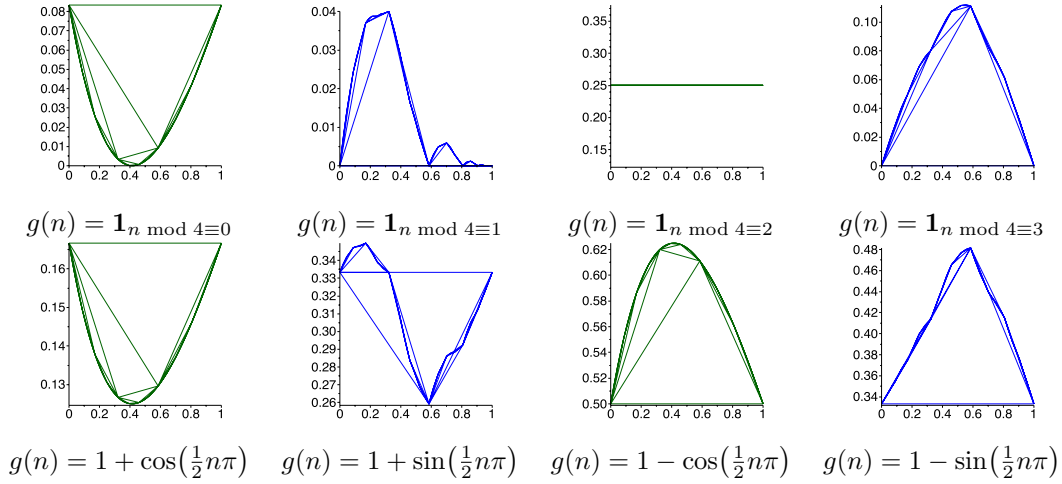
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$g(n) = \mathbf{1}_{n \bmod 4 \equiv 0}$ $g(n) = \mathbf{1}_{n \bmod 4 \equiv 1}$ $g(n) = \mathbf{1}_{n \bmod 4 \equiv 2}$ $g(n) = \mathbf{1}_{n \bmod 4 \equiv 3}$

$g(n) = 1 + \cos\left(\tfrac{1}{2}n\pi\right)$ $g(n) = 1 + \sin\left(\tfrac{1}{2}n\pi\right)$ $g(n) = 1 - \cos\left(\tfrac{1}{2}n\pi\right)$ $g(n) = 1 - \sin\left(\tfrac{1}{2}n\pi\right)$

Figure 7: Sensitive dependence of the periodic functions on $g(n)$.

5.3 $(\alpha, \beta) = (4, 4)$

Similarly to the $(2, 2)$ case, sequences in OEIS of this category include digital sums connected to 2-D arrays and double sums involving bitwise logic operators. Note that a degenerate case occurs when $f(1) = 0$ and $g(n) = 3n$ when n is even and $g(n) = 0$ otherwise; in this case, $f(n) = n^3 - n$.

Example 5.8 (Different sequences, same periodic oscillations)**.** We consider the following eight OEIS sequences in which the first four have $g(n)$ involving $\sin \frac{1}{2} n\pi$ (of period 4) while the last four have $g(n)$ depending simply on the parity of n.

OEIS id.	Context	$g(n)$	$f(1)$	$f(n)$
A163242	2-D Gray code	$\frac{3}{2}\left(n - \sin \frac{1}{2}n\pi\right)$	0	$n^3 P(\log_2 n) - \frac{1}{2}n$
A163365	Hilbert curve	$\frac{1}{2}\left(3 + \cos n\pi\right)n - \sin \frac{1}{2}n\pi$	0	$n^3 P(\log_2 n) - \frac{2}{3}n$
A163477	Hilbert curve	$\frac{1}{8}\left(3 + \cos n\pi\right)n - \frac{1}{4}\sin \frac{1}{2}n\pi$	0	$n^3 P(\log_2 n) - \frac{1}{6}n$
A163478	2-D Gray code	$\frac{1}{2}\left(n - \sin \frac{1}{2}n\pi\right)$	0	$n^3 P(\log_2 n) - \frac{1}{6}n$
A224923	$\sum\limits_{1 \leqslant i,j < n} (i\ \mathsf{XOR}\ j)$	$\left\lfloor \frac{1}{2}n^2 \right\rfloor$	0	$n^3 P(\log_2 n) - \frac{1}{2}n^2$
A224924	$\sum\limits_{1 \leqslant i,j < n} (i\ \mathsf{AND}\ j)$	$\begin{cases} \frac{1}{4}n^2 \\ \frac{1}{4}(n-1)(n-5) \end{cases}$	0	$n^3 P(\log_2 n) - \frac{1}{4}n^2$
A241522	game of Nim	$\begin{cases} 0 \\ -3(2n-1) \end{cases}$	1	$n^3 P(\log_2 n)$
A258438	$\sum\limits_{1 \leqslant i,j < n} (i\ \mathsf{OR}\ j)$	$\begin{cases} \frac{1}{4}n(7n-12) \\ \frac{1}{4}(n-1)(7n-11) \end{cases}$	0	$n^3 P(\log_2 n) - \frac{7}{4}n^2 + n$

Table 10: Sequences satisfying $\Lambda_{4,4}[f] = g$ (Example 5.8).

While it is visible that $f_{\text{A163242}}(n) = 3f_{\text{A163478}}(n)$ and $f_{\text{A163365}}(n) = 4f_{\text{A163477}}(n)$, it is less transparent but can be proved by induction that

$$
\begin{aligned}
f_{\text{A163477}}(n) &= \tfrac{1}{12}n(n^2 - 1) + \tfrac{1}{6}f_{\text{A163242}}(n) \\
f_{\text{A224924}}(n) &= \tfrac{1}{2}n^2(n - 1) - \tfrac{1}{2}f_{\text{A224923}}(n) \\
f_{\text{A241522}}(n) &= n^2(2n - 1) - 2f_{\text{A224923}}(n) \\
f_{\text{A258348}}(n) &= \tfrac{1}{2}n(n - 1)(n - 2) + \tfrac{1}{2}f_{\text{A224923}}(n),
\end{aligned}
\tag{5.35}
$$

implying that these eight sequences lead indeed to only two periodically different ones: the first four and the last four.

Also the Fourier expansions are given by

$$
P_{\text{A163242}}(t) = \frac{\zeta\left(2, \tfrac{1}{4}\right) - \zeta\left(2, \tfrac{3}{4}\right)}{32 \log 2} + \frac{3}{16 \log 2} \sum_{k \neq 0} \frac{\zeta\left(2 + \chi_k, \tfrac{1}{4}\right) - \zeta\left(2 + \chi_k, \tfrac{3}{4}\right)}{(2 + \chi_k)(3 + \chi_k)} e^{2k\pi i t}
\tag{5.36}
$$

$$
P_{\text{A224923}}(t) = \frac{\pi^2}{24 \log 2} + \frac{3}{2 \log 2} \sum_{k \neq 0} \frac{\zeta(2 + \chi_k)}{(2 + \chi_k)(3 + \chi_k)} e^{2k\pi i t},
\tag{5.37}
$$

respectively. The two series are absolutely convergent. $\qquad\square$
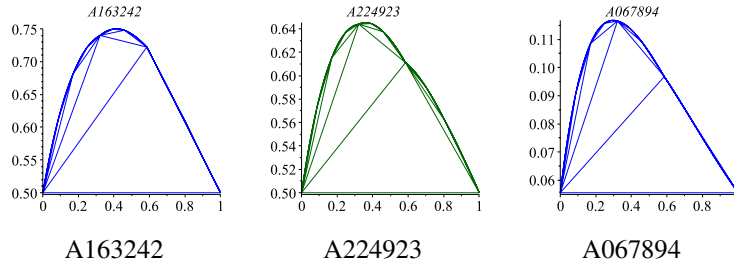

A163242   A224923   A067894

Figure 8: The periodic functions in the two representative $(4, 4)$ *cases (Example 5.8) and A067894 (Example 5.9) as approximated by* $n^{-\varrho}(f(n) + Q(n))$ *for* $n = 2^k + j$ *with* $k = 0, 1, \ldots$ *and* $0 \leqslant j \leqslant 2^k$.

5.4 $(\alpha, \beta) = (10, 10)$

Example 5.9 (A067894). Write $0, \ldots, n-1$ in binary and add as if they were decimal numbers. Then $f(1) = 0$ and

$$
\Lambda_{10,10}[f] = \lfloor \tfrac{n}{2} \rfloor.
\tag{5.38}
$$

The solution is

$$
f(n) = n^{\log_2(20)} P(\log_2 n) - \tfrac{1}{18}n,
\tag{5.39}
$$

In particular, we can derive the Fourier series expansion (as in our previous paper) for $P_{\text{A067894}}(t)$:

$$
P_{\text{A067894}}(t) = \frac{4}{5 \log 2} \sum_{k \in \mathbb{Z}} \frac{\zeta(\varrho - 1 + \chi_k)}{(\varrho - 1 + \chi_k)(\varrho + \chi_k)} e^{2k\pi i t},
\tag{5.40}
$$

where $\varrho = \log_2 20$.

An extension by replacing 10 by other values is discussed in Example 5.10. $\qquad\square$

5.5 Partial sums of $\Lambda_{\alpha,0}[f] = g$

One simple way to generate sequences satisfying $\Lambda_{\alpha,\alpha}[f] = g$ with $\alpha = \beta$ is to consider the partial sum $f(n) := \sum_{1 \leqslant k < n} h(k)$, where $\Lambda_{\alpha,0}[h] = \xi$; then

$$\Lambda_{\alpha,\alpha}[f] = h(1) + \sum_{2 \leqslant k < n} \xi(k), \tag{5.41}$$

for $n \geqslant 2$. Such a sum (after normalising by n) gives the average order and more smooth asymptotics than the original sequence (which leads almost always to functions with discontinuities; see Section 6.1 and [27] for more details). For example, the partial sum of the sequence A006520 (satisfying $\Lambda_{2,0}[h] = \lceil \frac{1}{2}n \rceil$) gives A022560 discussed in Example 5.6 with $g(n) = \lfloor \frac{1}{4}n^2 \rfloor$.

In a similar manner, if we define $f(n) := \sum_{1 \leqslant k \leqslant n} h(k)$, where $\Lambda_{0,\beta}[h] = \xi$; then

$$\Lambda_{\beta,\beta}[f] = (1 - \beta)h(1) + \sum_{2 \leqslant k \leqslant n} \xi(k), \tag{5.42}$$

for $n \geqslant 2$. Note that the equations $\Lambda_{\alpha,0}[h] = \xi$ and $\Lambda_{0,\alpha}[h] = \xi$ are equivalent up to a shift of both h and ξ; thus it suffices to consider only one of them.

The number of such sequences on OEIS exceeds several hundred after removing sequences whose generating functions are rational (with all singularities on the unit circle). So far we discussed only OEIS sequences leading to $(1, 1)$, $(2, 2)$, $(4, 4)$ and $(10, 10)$, but in such partial sum constructions, sequences with different values of α are frequently found; see Table 11.

Example 5.10 (Partial sum of Moser-de Bruijn sequences: $\alpha \geqslant 0$)**.** The Moser-de Bruijn sequence A000695 consists of the integers whose digits in base 4 are in $\{0, 1\}$; equivalently, A000695(n) is obtained by reading the binary representation of n in base 4. This sequence $h(n)$ satisfies $\Lambda_{4,0}[h] = \mathbf{1}_{n \text{ odd}}$. Hence, (5.41) shows that the partial sum $f(n) := \sum_{k < n} h(n)$ satisfies

$$\Lambda_{4,4}[f](n) = \lfloor \tfrac{n}{2} \rfloor, \qquad n \geqslant 2. \tag{5.43}$$

We can here replace 4 by any base $\alpha > 1$ (possibly non-integer); more generally, we can take any real α and define, for any integer $n = \sum_{j \geqslant 0} b_j 2^j$ with $b_j \in \{0, 1\}$,

$$h\left(\sum_{j \geqslant 0} b_j 2^j\right) := \sum_{j \geqslant 0} b_j \alpha^j. \tag{5.44}$$

(One might also take complex α, but we leave that case to the adventurous reader.) We then have the generating function

$$\sum_{n \geqslant 1} h(n) z^n = \frac{1}{1 - z} \sum_{k \geqslant 0} \frac{\alpha^k z^{2^k}}{1 + z^{2^k}}. \tag{5.45}$$

Table 11 lists many of such "α-Moser-de Bruijn sequences" that we found on OEIS. Note that $\alpha = 1$ gives A000120$(n) = \nu(n)$, the number of 1s in the binary expansion of n (see Example 4.4 and (4.9)), and that $\alpha = 2$ gives the trivial case A000027$(n) = n$.

Taking partial sums as above gives us a sequence $f(n) := \sum_{k < n} h(n)$ with

$$\Lambda_{\alpha,\alpha}[f](n) = \lfloor \tfrac{n}{2} \rfloor, \qquad n \geqslant 2. \tag{5.46}$$

The resulting sequence f is found in OEIS in the cases $\alpha = 1$, 2, 10, and -1, which give A000788, A000217 ($\binom{n}{2}$), A067894 (Example 5.9), and A005536 [27, Example 7.1], respectively.

α	1	2	3	4	5	6
OEIS	A000120	A000027	A005836	A000695	A033042	A033043
α	7	8	9	10	11	12
OEIS	A033044	A033045	A033046	A007088	A033047	A033048
α	13	14	15	16	17	18
OEIS	A033049	A033050	A033051	A033052	A197351	A197352
α	19	20	64	100	-1	-2
OEIS	A197353	A063012	A135124	A063010	A065359	A053985

Table 11: α-Moser-de Bruijn sequences in OEIS (Example 5.10).

$\boldsymbol{\alpha > \tfrac{1}{2}}$. For any $\alpha > 1$, we have, by (5.46), (2.29) and (2.28),

$$f(n) = n^{1+\log_2 \alpha} P(\log_2 n) - \frac{n}{2(\alpha - 1)}. \tag{5.47}$$

The same result holds for $\frac{1}{2} < \alpha < 1$ by considering $f_1(n) := f(n) + \frac{1}{2(\alpha-1)}n$, which satisfies $\Lambda_{\alpha,\alpha}[f_1](n) = g_1(n) := -\{\frac{n}{2}\}$.

$\boldsymbol{\alpha \leqslant \tfrac{1}{2}}$. If $0 < \alpha \leqslant \frac{1}{2}$, then $\varrho \leqslant 0$, and even if we consider $f_1(n)$ as above, the sum (2.28) does not converge uniformly since the individual terms do not converge uniformly to 0. Hence, Theorem 2.10 shows that (5.47) cannot hold with a continuous $P(t)$. In fact, if $s(x)$ is the sawtooth function that is defined by $s(n) := \mathbf{1}_{n \text{ is odd}}$ for $n \in \mathbb{Z}$, with linear interpolation between the integers, then $g(x) = \frac{1}{2}x - \frac{1}{2}s(x)$ for all $x \geqslant 0$, and thus (2.21) implies that, in view of $f(1) = 0$,

$$f(x) = \frac{x}{2(1-\alpha)} - \frac{1}{2}\sum_{k \geqslant 0}(2\alpha)^k s(2^{-k}x), \qquad x \geqslant 0. \tag{5.48}$$

(This actually holds as soon as $|\alpha| < 1$.) If $0 < \alpha < \frac{1}{2}$, then the sum in (5.48) converges uniformly for all $x \geqslant 0$ to a bounded continuous function on $[0, \infty)$. Note that this function is dominated by the first few terms in the sum, which are periodic with small periods $(2, 4, \dots)$. Hence $f(n) - \frac{n}{2(1-\alpha)}$ can be approximated arbitrarily well by a periodic function, without any scaling; such behaviour is very different from the case $\alpha > \frac{1}{2}$ when we instead have a periodic function of $\log_2 n$, scaled by n^ϱ; see also Remark 2.18.

If $\alpha = \frac{1}{2}$, the sum in (5.48) is $O(\log x)$ (for $x \geqslant 2$), but again, there is no smooth asymptotic behaviour. For example, with $f_1(n) := f(n) - n$, we have $f_1(\frac{1}{3}(2^{2\ell} - 2^{2k})) = -\frac{2}{3}(\ell - k) + O(1)$ for $0 \leqslant k < \ell$; taking $0 \leqslant k \leqslant \frac{1}{2}\ell$, say, shows a rather large variation on a relatively small interval of length $O(n^{1/2})$. Note also $f_1(2^k) = -1$ for all $k \geqslant 0$.

The case $\alpha = 0$ is trivial, with $f(n) = g(n) = \lfloor \frac{n}{2} \rfloor$. The case $\alpha < 0$ will be discussed in Example 6.5.

Fourier expansion. We have, by (3.28) with $g(n) = \lfloor \frac{n}{2} \rfloor$, when $\Re s$ is large enough,

$$D(s) = \sum_{n \geqslant 1}(\mathbf{1}_{n \text{ is odd}} - \mathbf{1}_{n \text{ is even}})n^{-s} = (1 - 2^{1-s})\zeta(s). \tag{5.49}$$

It follows by Corollary 3.8 that if $\alpha > \frac{1}{2}$ with $\alpha \neq 1, 2$, then the periodic function $P(t)$ in (5.47) has the Fourier expansion

$$P(t) = \frac{\alpha - 2}{\alpha \log 2} \sum_{k \in \mathbb{Z}} \frac{\zeta(\varrho - 1 + \chi_k)}{(\varrho - 1 + \chi_k)(\varrho + \chi_k)} e^{2k\pi it}. \tag{5.50}$$

Alternatively, by (A.4), $f(1) = 0$, (3.32) and (5.49), we derive the Mellin transform of $f(x)$:

$$f^*(s) = \frac{(1 - 2^{s+2})\zeta(-s-1)}{(1 - \alpha 2^{s+1})s(s+1)}, \qquad \Re s < \min(-1, -\varrho). \tag{5.51}$$

This extends to all real α, with $\varrho := \log(2|\alpha|)$ as in Example 6.5 when $\alpha \leqslant 0$. Hence, Mellin inversion yields (after a change of variable) the integral formula

$$f(n) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{(1 - 2^{1-s})\zeta(s)n^{s+1}}{s(s+1)(1 - \alpha 2^{-s})} \, ds, \tag{5.52}$$

for any α and any $\sigma > \max(\varrho - 1, 0)$. This provides by standard methods an alternative proof of several of the results above.

In particular, when $\alpha = 2$, we have $f(n) = \binom{n}{2}$ and $P(t) = \frac{1}{2}$. When $\alpha = 1$, we can use Theorem A.2 and conclude that

$$f(n) = \frac{1}{2}n \log_2 n + nP(\log_2 n), \tag{5.53}$$

where $P(t)$ is the Trollope–Delange fractal function (see [6, 12]):

$$P(t) = \frac{1}{2} \log_2 \pi - \frac{1}{4} - \frac{1}{2\log 2} - \frac{1}{\log 2} \sum_{l \neq 0} \frac{\zeta(\chi_k)}{\chi_k(\chi_k + 1)} e^{2k\pi it}. \tag{5.54}$$

Convergence of Fourier series. For a fixed real σ, and all real t with $|t| \geqslant 2$, it is known that

$$|\zeta(\sigma + it)| = \begin{cases} \Theta(|t|^{\frac{1}{2} - \sigma}), & \sigma < 0, \\ O(|t|^{\frac{1}{2}} \log |t|), & \sigma = 0, \\ O(|t|^{\frac{1}{2} - \frac{\sigma}{2}}), & 0 < \sigma < 1, \\ O(\log |t|), & \sigma = 1, \\ \Theta(1), & \sigma > 1, \end{cases} \tag{5.55}$$

see, e.g., [39, Chapter 5] or [29, Section 1.5]. (Stronger results are known for $0 \leqslant \sigma \leqslant 1$; the best exponents are still not known for $0 < \sigma < 1$.) Hence, the Fourier series in (5.50) and (5.54) are absolutely convergent if and only if $\varrho > \frac{1}{2}$, i.e., if $\alpha > \frac{\sqrt{2}}{2}$. Thus, if $\frac{1}{2} < \alpha < \frac{\sqrt{2}}{2}$, then the Fourier series is *not* absolutely convergent. (As a check, we note that (5.55) verifies that the Fourier coefficients are in ℓ^2 for every $\alpha > \frac{1}{2}$ ($\varrho > 0$), which is obvious from Parseval's formula. On the other hand, for $0 < \alpha \leqslant \frac{1}{2}$, (5.50) is not the Fourier series of any L^2 function.) $\qquad \square$

6 Extension to nonpositive α or β

We have so far considered the recurrence (1.1), or, equivalently (2.1), with $\alpha, \beta > 0$. Here we discuss rather briefly α and β with other sign combinations. The case $\alpha = \beta = 0$ is trivial, with (1.1) reduced to $f(n) = g(n)$, and is ignored in the sequel.

6.1 Recurrences with $\alpha = 0$ or $\beta = 0$

The situation when α or β equals zero is very similar to the special cases $(\alpha, \beta) = (2, 0)$ and $(0, 2)$ discussed in our previous paper [27], and we give only some brief comments. Such sequences abound in OEIS; see Table 11 for a few examples of one kind.

For any $\alpha \neq 0$, (2.4) is solved by $\varphi_{\alpha,0}(t) = 0$, $t \in [0, 1)$, and thus (2.2) yields $f(x) = f(\lfloor x \rfloor)$. Similarly, for $\beta \neq 0$, $\varphi_{0,\beta}(t) = 1$, $t \in (0, 1]$, and thus $f(x) = f(\lceil x \rceil)$. The main difference from the case $\alpha, \beta > 0$ is that now φ is discontinuous (at an endpoint), and thus $f(x)$ is discontinuous except in trivial cases, In the cases $(\alpha, 0)$ with $\alpha > 0$ and $(0, \beta)$ with $\beta > 0$, it is easily verified that Theorem 2.10 holds with modifications similar to the special cases in [27, Theorems 4 and 5]; note that the periodic function $P(t)$ now is discontinuous except in trivial cases. If $\alpha < 0$ or $\beta < 0$, this holds with further modifications as in Section 6.2 below. We omit the details.

6.2 Recurrences with both α and β negative

In this section, we consider the case when α and β both are negative; see Table 12 for a few examples from OEIS (discussed below). We thus assume the recurrence (1.1), or, equivalently, (2.1) with $\alpha, \beta < 0$. In this case we define

$$\varrho := \log_2(|\alpha| + |\beta|). \tag{6.1}$$

Thus $\alpha + \beta = -2^\varrho$. As above, we define $g(1) := 0$. We also extend the definition of $\varphi_{\alpha,\beta}(t)$ to negative values of α, β by $\varphi_{\alpha,\beta}(t) := \varphi_{|\alpha|,|\beta|}(t)$, and note that then (2.4) holds. The proof of Lemma 2.1 applies, *mutatis mutandis*, and shows that (2.3) holds in this case too. It follows that the theory developed in Sections 2–3 extends to this case, but with an important modification.

We say that a function $P(t)$ on $\mathbb{R}$ is 1-*antiperiodic* if $P(t + 1) = -P(t)$ for all $t \in \mathbb{R}$. In other words, $P(t)$ is 1-antiperiodic if and only if $e^{\pi i t} P(t)$ is 1-periodic.

Note that every 1-antiperiodic function is 2-periodic. Moreover, a 1-antiperiodic function that is integrable on $[0, 1]$ (and thus on any compact interval) has a Fourier series that can be written

$$P(t) \sim \sum_{k \in \mathbb{Z}} \widehat{P}(k + \tfrac{1}{2}) e^{(2k+1)\pi i t}. \tag{6.2}$$

We collect the most important results in the following theorem, leaving the others in Sections 2–3 to the reader.

Theorem 6.1. *Suppose that $\alpha, \beta < 0$, and let $\varrho := \log_2(|\alpha| + \beta|)$. Then Theorem 2.10 holds with the modifications that 1-periodic is replaced by 1-antiperiodic, and that* (2.28) *and* (2.30) *are replaced by*

$$Q(x) := \sum_{m \geqslant 1} (-2^{-\varrho})^m g(2^m x) = \sum_{m \geqslant 1} (\alpha + \beta)^{-m} g(2^m x), \tag{6.3}$$

and

$$P(t) = \sum_{m \in \mathbb{Z}} (-1)^m 2^{-\varrho(m+t)} g(2^{m+t}) + f(1) P_0(t), \qquad t \in \mathbb{R}, \tag{6.4}$$

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where now

$$P_0(t) := (-1)^{\lfloor t \rfloor}\big(1 + (\alpha + \beta - 1)\varphi\big(2^{\{t\}} - 1\big)\big)2^{-\varrho\{t\}}. \tag{6.5}$$

Moreover, Theorem 3.6 also holds, with the condition $|\alpha| + |\beta| > 1$ *in part* (iii)*, and with* (3.25) *replaced by*

$$\widehat{P}(k + \tfrac{1}{2}) = \frac{1}{\log 2}\int_1^\infty \frac{g(u)}{u^{\varrho + \chi_k' + 1}}\,\mathrm{d}u + \frac{f(1)}{\log 2}\int_0^1 \frac{1 + (\alpha + \beta - 1)\varphi(u)}{(1 + u)^{\varrho + \chi_k' + 1}}\,\mathrm{d}u, \tag{6.6}$$

for $k \in \mathbb{Z}$, *where*

$$\chi_k' := \frac{(2k + 1)\pi}{\log 2}i, \qquad k \in \mathbb{Z}. \tag{6.7}$$

If $\alpha = \beta$*, then, in analogy with Corollary 3.8,* (6.6) *simplifies to*

$$\widehat{P}(k + \tfrac{1}{2}) = \frac{1}{(\varrho + \chi_k')(\varrho - 1 + \chi_k')\log 2}\Big(D(\varrho - 1 + \chi_k') + \frac{(2\alpha - 1)(\alpha - 1)}{\alpha}f(1)\Big), \tag{6.8}$$

where $D(s)$ *is defined in* (3.27)*.*

Proof. The proof follows by the same arguments used in Sections 2–3, with minor modifications. In particular, Lemma 2.7 and Propositions 2.11–2.12 hold with P_0 and Q as above, 1-periodic replaced by 1-antiperiodic, and extra factors $(-1)^k$ or $(-1)^m$ in the sums in (2.21), (2.32), (2.33) and in the definition of $G_m(x)$. Note that (2.35) now becomes

$$h\big(2^m y\big) = (-1)^m 2^{m\varrho}G_m(y) = (\alpha + \beta)^m G_m(y). \tag{6.9}$$

For the proof of (6.8), note that (3.31) holds with χ_k replaced by χ_k', since now $2^{\varrho + \chi_k'} = -2^\varrho = 2\alpha$. $\qquad\square$

Remark 6.2. Thus $P(\log_2 n)$ in our formulas now is an 1-antiperiodic function of $\log_2 n$, which implies that it is a 2-periodic function of $\log_2 n$, or, equivalently, a 1-periodic function of $\frac{1}{2}\log_2 n = \log_4 n$. $\qquad\square$

Remark 6.3. Most of the formulas in Sections 2–3 hold verbatim for $\alpha, \beta < 0$ if we instead replace ϱ by the complex logarithm $\varrho_{\mathsf{c}} := \log_2(\alpha + \beta) = \varrho + \pi i/\log 2$. However, this seems less convenient for applications. $\qquad\square$

Example 6.4. Consider the basic case $g(n) = 0$, $f(1) = 1$ as in Example 2.15; we again denote the solution $f(n)$ by $S_{\alpha,\beta}(n)$. Theorem 6.1 shows that

$$S_{\alpha,\beta}(n) = n^\varrho P_0(\log_2 n), \qquad n \geqslant 1, \tag{6.10}$$

where now $P_0(t)$ is given by (6.5). It follows that (2.45) still holds, which also follows because for fixed n, $S_{\alpha,\beta}(n)$ is a polynomial in $\alpha, \beta \in \mathbb{R}$, as is $\varphi\big(2^{\{\log_2 n\} - 1}\big)(\alpha + \beta)^{\lfloor \log_2 n \rfloor}$ by (2.11).

In the special case $\alpha = \beta < 0$, (6.8) shows that, in analogy to (3.33),

$$\widehat{P}_0(k + \tfrac{1}{2}) = \frac{(2\alpha - 1)(\alpha - 1)}{\alpha \log 2}\cdot \frac{1}{(\varrho + \chi_k')(\varrho - 1 + \chi_k')}, \qquad k \in \mathbb{Z}, \tag{6.11}$$

and, thus (see (6.2)), $P_0(t)$ has the absolutely convergent Fourier series

$$P_0(t) = \frac{(2\alpha - 1)(\alpha - 1)}{\alpha \log 2}\sum_{k \in \mathbb{Z}} \frac{e^{(2k+1)\pi i t}}{(\varrho + \chi_k')(\varrho - 1 + \chi_k')}. \tag{6.12}$$

$\qquad\square$

OEIS	(α, β)	$g(n)$	Initials
A005536	$(-1, -1)$	$\lfloor \frac{1}{2}n \rfloor$	$f(1) = 0$
A079947	$(-1, -1)$	$n - 1$	$f(1) = 0$
A079954	$(-1, -1)$	$n - 2$	$f(1) = 0$
A094120	$(-2, -2)$	$\lfloor \frac{1}{4}n^2 \rfloor$	$f(1) = 0$

Table 12: Sequences in OEIS satisfying $\Lambda_{\alpha,\beta}[f] = g$ with $\alpha, \beta < 0$.

Example 6.5 (Partial sum of Moser-de Bruijn sequences: $\alpha < 0$). Let $\alpha < 0$. Consider the digital sum $f(n)$ defined in Example 5.10 as the partial sum of the α-Moser-de Bruijn sequence. Then f satisfies (5.46) with $f(1) = 0$. Recall that $\alpha = -1$ gives A005536 treated in [27, Example 7.1], and that $\alpha = -2$ gives the partial sums of A053985.

For $\alpha < -\frac{1}{2}$, we obtain by Theorem 6.1, directly if $\alpha < -1$ and otherwise by considering $f_1(n) := f(n) + \frac{1}{2(\alpha-1)}n$,

$$f(n) = n^{1+\log_2 |\alpha|} P(\log_2 n) - \frac{n}{2(\alpha - 1)}, \qquad (6.13)$$

for a continuous 1-antiperiodic function $P(t)$ with the Fourier expansion (using (5.49); cf. (5.50))

$$P(t) = \frac{\alpha - 2}{\alpha \log 2} \sum_{k \in \mathbb{Z}} \frac{\zeta(\varrho - 1 + \chi'_k)}{(\varrho - 1 + \chi'_k)(\varrho + \chi'_k)} e^{(2k+1)\pi i t}. \qquad (6.14)$$

As in Example 5.10, the Fourier series is absolutely convergent if and only if $\varrho > \frac{1}{2}$, i.e., if and only if $|\alpha| > \frac{\sqrt{2}}{2}$.

The case $-\frac{1}{2} \leqslant \alpha < 0$ is similar to the case $0 < \alpha \leqslant \frac{1}{2}$ discussed in Example 5.10. (For $\alpha = -\frac{1}{2}$, consider the example $n = \frac{1}{15}(16^\ell - 16^k)$.) Also note that (5.52) holds for all $\alpha < 0$. $\square$

Example 6.6 ($\alpha = \beta = -1$ and thus $\varrho = 1$). The function $h(n) := (-1)^{L_n}$ obviously satisfies $\Lambda_{-1,0}[h](n) = 0$, with $h(1) = 1$; thus $h(n) = S_{-1,0}(n)$. By (5.41), the partial sums $f(n) := \sum_{1 \leqslant k < n}(-1)^{L_k}$ satisfy

$$\Lambda_{-1,-1}[f](n) = 1, \qquad n \geqslant 2, \qquad (6.15)$$

with $f(1) = 0$. It follows that

$$f(n) = \tfrac{1}{3} - \tfrac{1}{3}S_{-1,-1}(n), \qquad (6.16)$$

which is Example 5.1 and (5.1) with $\alpha = -1$; cf. also Example 6.4. By induction, we have the explicit formula $f(n) = (-1)^{L_n}n - \frac{4}{3}(-2)^{L_n} + \frac{1}{3}$, which implies that $f(n) = nP(\log_2 n) + \frac{1}{3}$ with the 1-antiperiodic function

$$P(t) = (-1)^{\lfloor t \rfloor}\left(1 - \frac{2^{2-\{t\}}}{3}\right) = \frac{2}{\log 2} \sum_{k \in \mathbb{Z}} \frac{e^{(2k+1)\pi i t}}{\chi'_k(1 + \chi'_k)}, \qquad (6.17)$$

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where the Fourier coefficients follow from (6.16) and (6.12). This sequence is not in OEIS, but the following two periodically equivalent variants are.

The sequence A079947 is $f(n) := \frac{1}{2}\sum_{1\leqslant k<n}(1 + (-1)^{L_k})$ (partial sums of A030300), which, by (5.41) or (6.15), satisfies $\Lambda_{-1,-1}[f] = n - 1$ with $f(1) = 0$. We then have, by (6.16),

$$f_{\text{A079947}}(n) = \frac{n-1}{2} + \frac{1}{6} - \frac{1}{6}S_{-1,-1}(n) = \frac{n}{2} - \frac{1}{6}S_{-1,-1}(n) - \frac{1}{3}. \tag{6.18}$$

Explicitly, $f(n) = \frac{1}{2}\big(1 + (-1)^{L_n}\big)n - \frac{2}{3}(-2)^{L_n} - \frac{1}{3}$, so that $f(n) = nP(\log_2 n) - \frac{1}{3}$, where $P(t)$ is 2-periodic with

$$P(t) = \frac{1}{2} + (-1)^{\lfloor t \rfloor}\left(\frac{1}{2} - \frac{2^{1-\{t\}}}{3}\right) = \frac{1}{2} + \frac{1}{\log 2}\sum_{k\in\mathbb{Z}}\frac{e^{(2k+1)\pi it}}{\chi'_k(1 + \chi'_k)}. \tag{6.19}$$

Similarly, A079954 is $f(n) := \frac{1}{2}\sum_{1\leqslant k<n}(1 - (-1)^{L_k})$ (partial sums of A030301), which satisfies $\Lambda_{-1,-1}[f] = n - 2$ with $f(1) = 0$. We have $f_{\text{A079954}}(n) = n - 1 - f_{\text{A079947}}(n)$. Thus, by (6.16),

$$f_{\text{A079954}}(n) = \frac{n-1}{2} - \frac{1}{6} + \frac{1}{6}S_{-1,-1}(n) = \frac{n}{2} + \frac{1}{6}S_{-1,-1}(n) - \frac{2}{3}. \tag{6.20}$$

Explicitly, $f(n) = \frac{1}{2}\big(1 - (-1)^{L_n}\big)n + \frac{2}{3}(-2)^{L_n} - \frac{2}{3}$, so that $f(n) = nP(\log_2 n) - \frac{2}{3}$, where $P(t)$ is 2-periodic with

$$P(t) = \frac{1}{2} - (-1)^{\lfloor t \rfloor}\left(\frac{1}{2} - \frac{2^{1-\{t\}}}{3}\right) = \frac{1}{2} - \frac{1}{\log 2}\sum_{k\in\mathbb{Z}}\frac{e^{(2k+1)\pi it}}{\chi'_k(1 + \chi'_k)}. \tag{6.21}$$

$\square$

Example 6.7 ($\alpha = \beta = -2$ and thus $\varrho = 2$). Consider the sequence A094120 given by $f(n) := \sum_{1\leqslant k<n}\sum_{1\leqslant j\leqslant k}(-2)^{v_2(j)} = \sum_{1\leqslant j<n}(-2)^{v_2(j)}(n - j)$, where $v_2(j)$ is the dyadic valuation of j; see Example 5.6 for definition. Note that this is the analogue of A022560 in Example 5.6 with $2^{v_2(j)}$ replaced by $(-2)^{v_2(j)}$.

By definition, $f(n)$ is the partial sum of $h(k) := \sum_{1\leqslant j\leqslant k}(-2)^{v_2(j)}$; it is easily seen that $\Lambda_{-2,0}[h](n) = \lceil\frac{n}{2}\rceil$, and thus (5.41) yields $g(n) := \Lambda_{-2,-2}[f](n) = \lfloor\frac{1}{4}n^2\rfloor$. Theorem 6.1 does not directly apply because $g(n)$ grows too rapidly, but we can use a standard trick and subtract a multiple of n^2. We have $\Lambda_{-2,-2}[n^2] = 2n^2 + \mathbf{1}_{n \text{ is odd}}$, and thus $f_1(n) := f(n) - \frac{1}{8}n^2$ yields $\Lambda_{-2,-2}[f_1](n) = -\frac{3}{8}\mathbf{1}_{n \text{ is odd}}$ with $f_1(1) = -\frac{1}{8}$. Theorem 6.1 applies to $f_1(n)$ and implies that $f_1(n) = n^2P_1(\log_2 n)$ with $P_1(t)$ 1-antiperiodic, and consequently $f(n) = n^2P(\log_2 n)$, where $P(t) = P_1(t) + \frac{1}{8}$ is 2-periodic with Fourier expansion given by, from (6.8) applied to f_1,

$$P(t) = \frac{1}{8} + \frac{3}{2\log 2}\sum_{k\in\mathbb{Z}}\frac{\zeta(1 + \chi'_k)}{(1 + \chi'_k)(2 + \chi'_k)}e^{(2k+1)\pi it}. \tag{6.22}$$

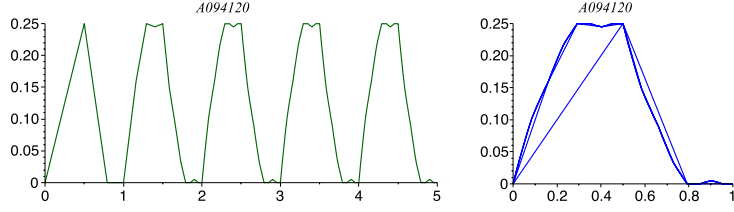See Figure 9. Note the difference from A022560 in Example 5.6 where a logarithmic term appears in (5.22). $\square$

Figure 9: Periodic fluctuations of A094120$(n)/n^2$ (Example 6.7).

6.3 Recurrences with either α or β negative

There are many examples in OEIS of sequences satisfying $\Lambda_{\alpha,\beta}[f] = g$ with $\alpha\beta < 0$, i.e., one of α and β is positive and the other is negative. In this case, we can still define $\varphi(t)$ for dyadic rationals $t \in [0, 1]$ by (2.11) (and it has to have this value), but since now $|\alpha+\beta| < |\alpha|, |\beta|$, this function is unbounded in every interval, and thus cannot be extended to a continuous function (or any other reasonable function) on $[0, 1]$. Hence our methods break down, and we have no general theorem in this case. Moreover, we do not expect any simple asymptotics in general, which is illustrated by the following example.

Example 6.8. Consider again $f(n) = S_{\alpha,\beta}(n)$ defined by $\Lambda_{\alpha,\beta}[f](n) = 0$ and $f(1) = 1$. In particular, we have

$$f(2n) = (\alpha + \beta)f(n), \qquad n \geqslant 1. \tag{6.23}$$

Moreover, it is easily seen by induction, or by (4.19) and (4.16), that, for any α and β,

$$f(n + 1) - f(n) = (\alpha + \beta - 1)\alpha^{\nu(n)-1}\beta^{\nu_0(n)}. \tag{6.24}$$

By (6.23) and (6.24), we have, for example,

$$f(2^k) = (\alpha + \beta)^k, \qquad\qquad k \geqslant 0, \tag{6.25}$$
$$f(2^k - 1) = (\alpha + \beta)^k - (\alpha + \beta - 1)\alpha^{k-1}, \qquad k \geqslant 1, \tag{6.26}$$
$$f(2^k + 1) = (\alpha + \beta)^k + (\alpha + \beta - 1)\beta^k, \qquad k \geqslant 0. \tag{6.27}$$

Consider, for definiteness, the case $\alpha > 0 > \beta$ with $|\alpha| \geqslant |\beta|$. Then $0 \leqslant \alpha + \beta < \alpha$. Assume also $\alpha + \beta \neq 1$. (Otherwise, $S_{\alpha,\beta}(n) = 1$ for all n.) We see from (6.26) that $|f(n)|$ may be of the order $\alpha^{\log_2 n} = n^{\log_2 \alpha}$, although (6.25) shows that $|f(n)|$ also may be much smaller. In fact, it is easily shown by induction using (6.23)–(6.24) that $|f(n)| \leqslant C\alpha^{L_n}$ for some constant C (depending on α and β) and all n. Hence, $\widetilde{f}(n) := n^{-\log_2 \alpha}f(n)$ is bounded. However, $\widetilde{f}(n)$ does not seem to have any simple asymptotic approximations, as the following arguments show.

First, (6.25) and (6.26) show that $\widetilde{f}(n)$ have infinitely many jumps with size of order 1. More precisely, it can be seen from (6.24) that for every $\varepsilon > 0$, there exists $\delta > 0$ such that every interval $[N, (1 + \varepsilon)N]$ with $N \geqslant 1$ contains some n such that the jump $|\widetilde{f}(n + 1) - \widetilde{f}(n)| > \delta$.

Secondly, we cannot have $|f(n)| = n^{\log_2 \alpha}\big(P(\log_2 n) + o(1)\big)$ for any 1-periodic function $P(t)$, because then substituting $2n$ would give $|f(2n)| = \alpha n^{\log_2 \alpha}\big(P(\log_2 n) + o(1)\big)$, while (6.23) would imply $|f(2n)| = (\alpha + \beta)n^{\log_2 \alpha}\big(P(\log_2 n) + o(1)\big)$, and together these imply $P(\log_2 n) = o(1)$, so we would have $f(n) = o(n^{\log_2 \alpha})$, which contradicts (6.26). $\qquad\square$

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There are many OEIS sequences in this category too, and most of them have $\alpha + \beta = 0$. We do not discuss these examples further since we have nothing new to add by our methods, but just mention a prototype sequence A115384, the partial sum of Thue-Morse sequence (A010060, the parity of the dyadic valuation), which satisfies $\Lambda_{-1,1}[f] = \lfloor \frac{n}{2} \rfloor$ with $f(1) = 0$. The exact solution is given by $f(n) = \lfloor \frac{n}{2} \rfloor + \frac{1}{4}(1 - (-1)^n)(1 + (-1)^{\nu(n-1)})$, where the last term indicates why there is no simple smooth function providing good asymptotic approximation to $f(n) - \frac{n}{2}$. See also other related sequences A076826, A159481, A173209 and A245710, which have a very similar behaviour.

7 Extension from binary to q-ary

We consider briefly the more general recurrence

$$f(n) = \sum_{0 \leqslant j < q} \alpha_j f\left(\left\lfloor \tfrac{n+j}{q} \right\rfloor\right) + g(n) \qquad (n \geqslant q), \tag{7.1}$$

for some integer $q \geqslant 2$ and q given constants $\alpha_0, \ldots, \alpha_{q-1}$; note that the case $q = 2$ corresponds to (1.1). We now require $g(n)$ for $n \geqslant q$ and the initial values $\{f(1), \ldots, f(q-1)\}$.

Just as the recurrence (1.1), for example, occurs naturally in many combinatorial and algorithmic contexts where a problem is split into two halves, the generalisation (7.1) occurs typically in divide-and-conquer context or recursive structures where we instead divide the source problem into q subproblems of sizes as evenly as possible.

The case $\alpha_0 = \cdots = \alpha_{q-1} = 1$ was discussed in [27] with several examples from the literature and OEIS. We can similarly extend the method of Section 2 to treat the general recursion (7.1) under suitable conditions. We assume that α_j are real with

$$\max_{0 \leqslant j < q} |\alpha_j| < A := \sum_{0 \leqslant j < q} \alpha_j. \tag{7.2}$$

Note that (7.2) holds in the standard case when all $\alpha_j > 0$; it also holds, more generally, if $\alpha_j \geqslant 0$ for all j and at least two α_j are non-zero.

Lemma 7.1. *Assume* (7.2). *Then there exists a unique continuous function $\varphi(t)$ on $[0, 1]$ such that $\varphi(0) = 0$, $\varphi(1) = 1$, and for $j = 0, \ldots, q - 1$,*

$$\varphi(t) = \frac{\alpha_{q-1-j}}{A} \varphi(qt - j) + \frac{\sum_{q-j \leqslant i < q} \alpha_i}{A}, \qquad if \quad \frac{j}{q} \leqslant t \leqslant \frac{j+1}{q}. \tag{7.3}$$

Moreover, if $\alpha_j \geqslant 0$ for all j, then φ is strictly increasing.

Proof. This follows with only notational changes as in our third proof of Lemma 2.3 based on the recursive construction (2.16); we obtain by an analogue of (2.16) a sequence of continuous functions φ_k that, using (7.2), converge uniformly to a function $\varphi(t)$ satisfying (7.3). $\qquad \square$

We then extend $f(n)$ and $g(n)$ to functions of a real variable $x \geqslant 1$ by (2.2) as before, and it is easily verified that (2.3) generalises to

$$f(x) = A f\left(\frac{x}{q}\right) + g(x), \qquad x \geqslant q. \tag{7.4}$$

We may now argue as in Section 2 and prove extensions of Theorem 2.10 and its corollaries for the recursion (7.1). We now define

$$\varrho := \log_q A = \log\Big(\sum_{0 \leqslant j < q} \alpha_j\Big). \tag{7.5}$$

The simplest situation is when $g(n) = O(n^{\varrho - \varepsilon})$ for some $\varepsilon > 0$; then

$$f(x) = x^\varrho P(\log_q x) - Q(x), \qquad x \geqslant 1, \tag{7.6}$$

where $P(t)$ is a continuous 1-periodic function, and

$$Q(x) := \sum_{m \geqslant 1} q^{-\varrho m} g(q^m x) = o(x^\varrho). \tag{7.7}$$

We leave further details to the reader and content ourselves with the discussion of two classes of examples.

7.1 Binomial coefficients not divisible by a prime q

Let $f(n)$ denote the number of binomial coefficients $\binom{m}{k}$, $0 \leqslant k \leqslant m < n$, that are not divisible by a given prime q. This sequence has a long history, at least dating back to Fine's [17] observation that almost all binomial coefficients are even; see, e.g., [8] and the references therein. It equals A006046 (see Example 4.9) when $q = 2$. The case $q = 3$ corresponds to A006048, while $q = 5$ gives A194458. We then deduce the recurrence

$$f(n) = \sum_{0 \leqslant j < q} (q - j) f\big(\lfloor \tfrac{n+j}{q} \rfloor\big), \tag{7.8}$$

with $f(0) = 0$ and $f(1) = 1$. This is (7.1) with $\alpha_j = q - j$ and $g(n) = 0$; furthermore, in this example (7.8) holds for all $n \geqslant 0$. We have $A = \binom{q+1}{2}$ and $\varrho := \log_q A$. Stein [37] proved that

$$\frac{1}{A} \leqslant \frac{f(n)}{n^\varrho} \leqslant 1, \tag{7.9}$$

and extended $f(n)$ to a continuous function $f(x)$; see [21, 40, 42] for finer lower bounds. Our general approach yields the same continuous extension $f(x)$ as in [37]; we obtain (by (7.6))

$$\frac{f(n)}{n^\varrho} = P(\log_q n) \qquad (n \geqslant 1), \tag{7.10}$$

where $P(t) := A^{-\{t\}} f(q^{\{t\}})$ is a continuous 1-periodic function. Moreover, since (7.8) holds for all $n \geqslant 1$, it is easily verified (using (7.3)) that $f(x) = A\varphi(x/q)$ for $x \in [1, q]$, and thus

$$P(t) = A^{1-\{t\}} \varphi(q^{\{t\}-1}). \tag{7.11}$$

(Cf. Remark 2.9 for a similar simplification, for related reasons.) Here $\varphi(t)$ satisfies $\varphi(0) = 0$, $\varphi(1) = 1$, and for $0 \leqslant j < q$,

$$\varphi(t) = \frac{j+1}{A} \varphi(\{qt\}) + \frac{\binom{j+1}{2}}{A}, \qquad \text{if} \quad \frac{j}{q} \leqslant t \leqslant \frac{j+1}{q}. \tag{7.12}$$

$$\varphi_3 \qquad \varphi_5 \qquad \varphi_7 \qquad \varphi_{11}$$
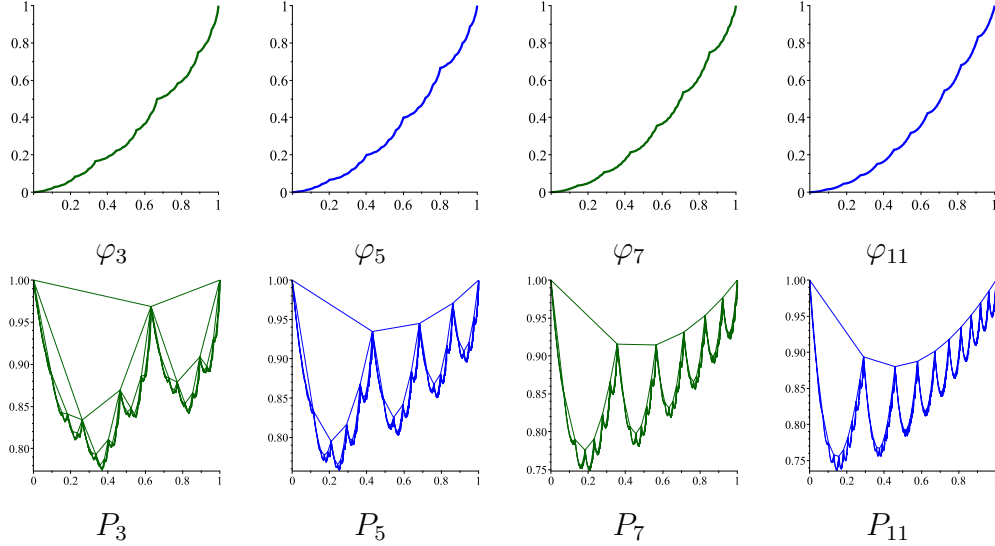
$$P_3 \qquad P_5 \qquad P_7 \qquad P_{11}$$

Figure 10: The function φ (Section 7.1) when $q = 3, 5, 7, 11$ (upper half) and the periodic function P for the same set of values of q (lower half).

By (7.9) or (7.10), *almost all binomial coefficients are divisible by any given prime q* because

$$\varrho = \log_q(q+1) - \log_q 2 + 1 < 2. \tag{7.13}$$

Alternatively, it is known [8] that

$$f(n) = \frac{1}{2} \sum_{0 \leqslant j \leqslant s} \binom{p+1}{2}^j b_j \prod_{j \leqslant i \leqslant s} (b_i + 1), \tag{7.14}$$

when $n = b_0 + b_1 q + \cdots b_s q^s$ with $0 \leqslant b_j < q$ and $s = \lfloor \log_q n \rfloor$, from which we can obtain an alternative representation for the periodic function P. Also the generating function [37]

$$\sum_{n \geqslant 0} f(n) z^n = \frac{z}{1-z} \prod_{k \geqslant 0} \sum_{0 \leqslant j < q} (j+1) z^{j \cdot q^k}, \tag{7.15}$$

is helpful in applying the Mellin transform approach; see [20, 24, 25].

The more general problem of the number of multinomial coefficients $\binom{m}{j_1, \ldots, j_d}$ not divisible by a prime q, for $0 \leqslant m < n$ and $j_1 + \cdots + j_d = m$, $j_1, \ldots, j_d \geqslant 0$, where $d \geqslant 1$ is given (see [7, 8, 40]) can be similarly dealt with. This number satisfies the recurrence

$$f(n) = \sum_{0 \leqslant j < q} \binom{q-j+d-2}{d-1} f\left(\left\lfloor \frac{n+j}{q} \right\rfloor\right); \tag{7.16}$$

we then deduce, by (7.6), the identity $f(n) = n^\varrho P(\log_q n)$, $n \geqslant 1$, for some continuous periodic function P with $\varrho = \log_q \binom{q+d-1}{d}$.

7.2 Generating polynomial of Gray codes

Gray codes of integers are strings of binary words in which neighboring code words differ by one bit only; we already discussed some properties of the binary reflected Gray codes in

Examples 5.6 and 5.8. Here, we consider a simple extended version of binary Gray codes to q-ary ones (non-reflected); see [9, 36]. The construction is as follows. If

$$n = \sum_{0 \leqslant j \leqslant s} \kappa_j q^j, \qquad (0 \leqslant \kappa_j < q), \tag{7.17}$$

then the Gray code of n is given by $(\kappa'_s, \ldots, \kappa'_0)$, where $\kappa'_s = \kappa_s$ and

$$\kappa'_j := (\kappa_j - \kappa_{j+1}) \bmod q \qquad (0 \leqslant j < s). \tag{7.18}$$

For simplicity, we consider the number of nonzero digits $\gamma(n) := \sum_{0 \leqslant j \leqslant s} \mathbf{1}_{\kappa'_j > 0}$ in this Gray code representation of n; other quantities such as the sum of digits $\sum_{0 \leqslant j \leqslant s} \kappa'_j$ can be considered similarly (a sketch given below). Then by the recurrence

$$\gamma(qk + j) = \gamma(k) + \begin{cases} 0, & \text{if } k \bmod q \equiv j; \\ 1, & \text{otherwise}, \end{cases} \tag{7.19}$$

for $0 \leqslant j < q$, we deduce that the generating polynomial $f(n) := \sum_{0 \leqslant k < n} \alpha^{\gamma(k)}$ of $\gamma(n)$ satisfies

$$f(n) = f\left(\left\lfloor \tfrac{n}{q} \right\rfloor\right) + \alpha \sum_{1 \leqslant j < q} f\left(\left\lfloor \tfrac{n+j}{q} \right\rfloor\right) + g(n), \tag{7.20}$$

where g can be expressed as

$$g(q^2 k + qr + j) = (1 - \alpha)\alpha^{\gamma(qk+r)+1} \qquad (k \geqslant 0), \tag{7.21}$$

for $0 \leqslant r \leqslant q - 2$ and $r + 1 \leqslant j \leqslant q - 1$, and $g(n) = 0$ for all other values of n. Alternatively, in terms of q-ary expansion, the nonzero $g(n)$, $0 \leqslant n < q^2$, occurs when $n = (\kappa_1, \kappa_0)_q$ is of the form:

$$(0, 1)_q, (0, 2)_q, (0, 3)_q, (0, 4)_q, \ldots, (0, q - 1)_q$$
$$(1, 2)_q, (1, 3)_q, (1, 3)_q, \ldots, (1, q - 1)_q$$
$$\vdots$$
$$(q - 3, q - 2)_q, (q - 3, q - 1)_q$$
$$(q - 2, q - 1)_q$$

We then deduce from (7.6) the exact and asymptotic expansion (since $Q(n) = 0$ by (7.7))

$$f(n) = P(\log_q n) n^{\log_q(1 + (q-1)\alpha)} \qquad (n \geqslant 1), \tag{7.22}$$

whenever $\alpha > 0$, for some continuous periodic function $P = P_\alpha$; note that then $g(n) = O\big((\alpha \vee 1)^{\log_q n}\big) = O\big(n^{0 \vee \log_q \alpha}\big)$ with $0 \vee \log_q \alpha < \varrho = \log_q(1 + (q-1)\alpha)$.

Similarly, the generating polynomial of the sum-of-digits function in such q-ary Gray codes $f(n) = \sum_{0 \leqslant k < n} \alpha^{\sigma(k)}$, $\sigma(k) := \sum_{0 \leqslant j \leqslant s} \kappa'_j$ when $k = \sum_{0 \leqslant j \leqslant s} \kappa_j q^j$, satisfies the recurrence

$$f(n) = \sum_{0 \leqslant j < q} \alpha^j f\left(\left\lfloor \tfrac{n+j}{q} \right\rfloor\right) + g(n), \tag{7.23}$$

where g can be expressed as

$$g(q^2 k + qr + j) = \begin{cases} \alpha^{\sigma(qk+r)} \dfrac{(1 - \alpha^j)(\alpha^{q-r} - \alpha^{q-j})}{1 - \alpha} & (0 \leqslant j < r), \\ \alpha^{\sigma(qk+r)} \dfrac{(1 - \alpha^{j-r})(1 - \alpha^{q-j})}{1 - \alpha} & (r \leqslant j < q). \end{cases} \tag{7.24}$$

$$q = 3, \alpha = \frac{\sqrt{5} \pm 1}{2} \qquad q = 3, \alpha = 1 \pm 0.01 \qquad q = 4, \alpha = \frac{\sqrt{5} \pm 1}{2} \qquad q = 4, \alpha = 1 \pm 0.01$$
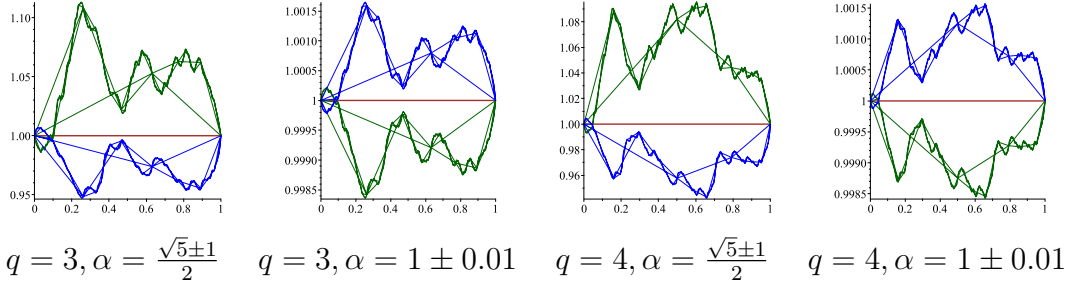
Figure 11: Periodic fluctuations of $f(n)n^{-\log_q(1+(q-1)\alpha)}$ in the cases of nonzero digits of q-ary Gray codes for $q = 3, 4$ and different α (Section 7.2).

for $k \geqslant 0$, $0 \leqslant r \leqslant q - 1$, and $0 \leqslant j \leqslant q - 1$. This is derived by the relation

$$\sigma(qk + j) = \sigma(k) + (j - k) \bmod q, \tag{7.25}$$

which in turn follows from (7.18). We then deduce that

$$f(n) = P(\log_q n) n^{\log_q(1 + \alpha + \cdots + \alpha^{q-1})}, \tag{7.26}$$

for $n \geqslant 1$.

A Mellin transforms

Mellin transforms are another useful techniques in analysing divide-and-conquer recurrences; see [18, 19, 20, 25, 28] and the references therein for more information. Up to now most of the tools we adopt to solve (1.1) are direct and elementary in nature; it is however possible to apply Mellin transforms for a more effective characterisation of the underlying periodic oscillations, notably calculations of the Fourier coefficients, as already observed before in the literature (although an analytic approach often requires stronger conditions).

Let again $\alpha, \beta > 0$, and $\varrho := \log_2(\alpha + \beta)$, and assume that $f(n)$ and $g(n)$ satisfy the recursion (1.1). Extend again f and g to $[1, \infty)$ by (2.2), with $g(1) := 0$, and define $f(x) := g(x) := 0$ for $x \in [0, 1)$. Denote the Mellin transform of $f(x)$ by $f^*(s)$:

$$f^*(s) := \int_0^\infty f(x)x^{s-1}\,\mathrm{d}x = \int_1^\infty f(x)x^{s-1}\,\mathrm{d}x, \tag{A.1}$$

for all complex s such that the integral is absolutely convergent, and similarly for $g^*(s)$. If $f(n) = O(n^c)$ for large n and some real c, then $f^*(s)$ exists at least in the half-plane $\Re s < -c$, and is analytic there. If $f^*(s)$ or $g^*(s)$ extends meromorphically to a larger domain, we use the same notation there.

Assume that $f^*(s)$ and $g^*(s)$ exist. Then (2.3) implies that

$$\begin{aligned}
f^*(s) &= \int_1^2 f(x)x^{s-1}\,\mathrm{d}x + \int_2^\infty \left(2^\varrho f\left(\frac{x}{2}\right) + g(x)\right)x^{s-1}\,\mathrm{d}x \\
&= \int_1^2 \left(f(x) - g(x)\right)x^{s-1}\,\mathrm{d}x + 2^{\varrho+s} f^*(s) + g^*(s).
\end{aligned} \tag{A.2}$$

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Furthermore, (2.21) or (2.23) shows that if $1 \leqslant x < 2$, then $f(x) - g(x) = f(1)P_0(\log_2 x)x^\varrho$. Hence, recalling the definition (2.22),

$$\int_1^2 \left(f(x) - g(x)\right)x^{s-1}\,\mathrm{d}x = f(1)\int_1^2 P_0(\log_2 x)x^{\varrho+s-1}\,\mathrm{d}x$$

$$= f(1)\int_1^2 \left(1 + (2^\varrho - 1)\varphi\,(x-1)\right)x^{s-1}\,\mathrm{d}x. \qquad (A.3)$$

Substituting this into (A.2) yields

$$\left(1 - 2^{\varrho+s}\right)f^*(s) = g^*(s) + f(1)\int_0^1 \left(1 + (2^\varrho - 1)\varphi(u)\right)(1+u)^{s-1}\,\mathrm{d}u. \qquad (A.4)$$

Note that the right-hand side of (3.25) equals, apart from a factor $\frac{1}{\log 2}$, the right-hand side of (A.4) at $s = -\varrho - \chi_k$. On the other hand, for such s, the left-hand side factor $1 - 2^{\varrho+s}$ equals zero. Hence, combining (A.4) and Theorem 3.6(ii) yields the following (see [19] for more information).

Lemma A.1. *If $g(n) = O(n^{\varrho-\varepsilon})$ for some $\varepsilon > 0$, then $g^*(s)$ is analytic in (at least) the half-plane $\Re s < -\varrho + \varepsilon$, and $f^*(s)$ is meromorphic in the same half-plane with only simple poles, which are at $-\varrho - \chi_k = -\varrho - \frac{2k\pi i}{\log 2}$ for some $k \in \mathbb{Z}$. Furthermore, Theorem 3.6(ii) applies, and*

$$\widehat{P}(k) = -\operatorname{Res}\left[f^*(s); s = -\varrho - \chi_k\right]. \qquad (A.5)$$

We do not claim that every $-\varrho - \chi_k$ actually is a pole. In fact, by (A.5), $-\varrho - \chi_k$ is a pole of $f^*(s)$ if and only if $\widehat{P}(k) \neq 0$.

For a better demonstration of the approach, we study the case when $g(n) \sim n^\varrho$, and thus Theorem 2.10 does not directly apply (although readily amenable) due to an extra logarithmic leading term in the asymptotic approximation of $f(n)$. For a meromorphic function $F(z)$, let $F(s)_{\mathrm{fin}}$ denote *finite value* (or the constant term) in the Laurent series expansion at $z = s$; thus $F(s)_{\mathrm{fin}} = F(s)$ when the latter is finite.

Theorem A.2. *Assume that $g(n) = n^\varrho + g_0(n)$, where $g_0(n) = O(n^{\varrho-\varepsilon})$ for some $\varepsilon > 0$. Then*

$$f(n) = n^\varrho \log_2 n + n^\varrho P(\log_2 n) - Q(n), \qquad n \geqslant 1, \qquad (A.6)$$

where (as in Theorem 2.10) $P(t)$ is a continuous 1-periodic function and $Q(n) = o\,(n^\varrho)$ as $n \to \infty$. The Fourier coefficients of $P(t)$ are given by

$$\widehat{P}(k) = \frac{f(1)}{\log 2}\int_0^1 \frac{1 + (\alpha + \beta - 1)\varphi(u)}{(1+u)^{\varrho+\chi_k+1}}\,\mathrm{d}u + \begin{cases} \frac{1}{\log 2}g^*(-\varrho - \chi_k), & k \neq 0, \\ \frac{1}{\log 2}g^*(-\varrho)_{\mathrm{fin}} + \frac{1}{2}, & k = 0, \end{cases} \qquad (A.7)$$

where $g^*(s)$ is meromorphic in $\Re s < -\varrho + (\varepsilon \wedge 1)$ with a sole simple pole at $s = -\varrho$. In particular, (3.25) holds for $k \neq 0$.

If, moreover, $g(n) = n^\varrho$ for even $n \geqslant 2$ (i.e., $g_0(2m) = 0$), then $Q(n) = 0$ for $n \geqslant 1$.

Proof. Assume, without loss of generality, $\varepsilon < 1$. The assumption and (2.2) then imply that we have, for $n \geqslant 1$ and $t \in [0,1]$,

$$|g(n+t) - g(n)| \leqslant |g(n+1) - g(n)| = |(n+1)^\varrho - n^\varrho + g_0(n+1) - g_0(n)|$$
$$= O\bigl(n^{\varrho-\varepsilon}\bigr), \qquad (A.8)$$

which together with a Taylor expansion of $(n+t)^{s-1}$ yields, for $\Re s < -\varrho$,

$$g^*(s) = \sum_{n \geqslant 1} \int_0^1 g(n+t)(n+t)^{s-1} \, \mathrm{d}t = \sum_{n \geqslant 1} \left(g(n)n^{s-1} + O\left(n^{\varrho+\Re s-1-\varepsilon}\right) \right)$$

$$= \sum_{n \geqslant 1} n^{\varrho+s-1} + \sum_{\geqslant 1} O\left(n^{\varrho+\Re s-1-\varepsilon}\right)$$

$$= \zeta(1 - \varrho - s) + \sum_{n \geqslant 1} O\left(n^{\varrho+\Re s-1-\varepsilon}\right). \tag{A.9}$$

Moreover, each term in the final sum is an entire function in s, and the O is uniform for s in any compact set; thus the sum converges to an analytic function in H_ε. Hence, $g^*(s)$ extends to a meromorphic function in H_ε, with a single simple pole at $s = -\varrho$, as asserted.

Let $f_1(n) := n^\varrho \log_2 n$, and $f_2(n) := f(n) - f_1(n)$, and let $g_j := \Lambda_{\alpha,\beta}[f_j]$, $j = 1, 2$. We have

$$\Lambda_{\alpha,\beta}[f_1](2n) = (2n)^\varrho \log_2(2n) - (\alpha + \beta)n^\varrho \log_2(n) = (2n)^\varrho, \tag{A.10}$$

and, with $\psi(x) := x^\varrho \log_2(x)$, using the mean-value theorem.

$$\Lambda_{\alpha,\beta}[f_1](2n+1) = (2n+1)^\varrho + 2^\varrho \psi\left(n + \tfrac{1}{2}\right) - \alpha\psi(n) - \beta\psi(n+1)$$

$$= (2n+1)^\varrho + O\left(n^{\varrho-1} \log(n+1)\right). \tag{A.11}$$

In other words, for all $n \geqslant 2$,

$$g_1(n) := \Lambda_{\alpha,\beta}[f_1](n) = n^\varrho + O\left(n^{\varrho-1} \log n\right) = n^\varrho + O\left(n^{\varrho-\varepsilon}\right), \tag{A.12}$$

and thus

$$g_2(n) := g(n) - g_1(n) = g_0(n) + O\left(n^{\varrho-\varepsilon}\right) = O\left(n^{\varrho-\varepsilon}\right). \tag{A.13}$$

Consequently, $g_2^*(s)$ is analytic in the half-plane $H_\varepsilon := \{s : \Re s < -\varrho + \varepsilon\}$, and, by Corollary 2.14,

$$f_2(x) = x^\varrho P_2(\log_2 x) - Q_2(x), \qquad x \geqslant 1, \tag{A.14}$$

where $Q_2(x) = \sum_{m \geqslant 1} 2^{-\varrho m} g_2(2^m x)$, and $P_2(t)$ is a periodic continuous function. Since $f(n) = f_1(n) + f_2(n)$, this shows (A.6) with $P(t) := P_2(t)$ and $Q(t) := Q_2(t)$.

Furthermore, by Lemma A.1,

$$\widehat{P}(k) = \widehat{P}_2(k) = -\operatorname{Res}\left[f_2^*(s); s = -\varrho - \chi_k\right]. \tag{A.15}$$

By (A.4), $f_2^*(s)$ is meromorphic in H_ε, with poles only at $-\varrho - \chi_k$. Moreover, similarly to (A.8), it follows from (2.2) that $|f_1(n+t) - f_1(n)| = O\left(n^{\varrho-\varepsilon}\right)$ for $n \geqslant 1$ and $t \in [0,1]$, which in turn, similarly to (A.9), yields

$$f_1^*(s) = \sum_{n \geqslant 1} \int_0^1 f_1(n+t)(n+t)^{s-1} \, \mathrm{d}t = \sum_{n \geqslant 1} n^{\varrho+s-1} \log_2 n + \sum_{n \geqslant 1} O\left(n^{\varrho+\Re s-1-\varepsilon}\right)$$

$$= -\frac{1}{\log 2}\zeta'(1 - \varrho - s) + \sum_{n \geqslant 1} O\left(n^{\varrho+\Re s-1-\varepsilon}\right). \tag{A.16}$$

Again, each term in the final sum is an entire function in s, and the O is uniform for s in any compact set; thus the sum converges to an analytic function in H_ε. Hence, $f_1^*(s)$ extends to a meromorphic function in H_ε, with a single (double) pole at $s = -\varrho$. Furthermore, the residue $\operatorname{Res}\left[f_1^*(s); s = -\varrho\right] = \frac{-1}{\log 2} \operatorname{Res}\left[\zeta'(1 - \varrho - s); s = -\varrho\right] = 0$, since ζ' is a derivative of a meromorphic function. Consequently, $f^*(s) = f_1^*(s) + f_2^*(s)$ is meromorphic in H_ε, with poles only at $-\varrho - \chi_k$, and

$$\operatorname{Res}\left[f^*(s); s = -\varrho - \chi_k\right] = \operatorname{Res}\left[f_2^*(s); s = -\varrho - \chi_k\right], \qquad k \in \mathbb{Z}. \tag{A.17}$$

Thus, by (A.15) and (A.17),

$$\widehat{P}(k) = -\operatorname{Res}\left[f^*(s); s = -\varrho - \chi_k\right]. \tag{A.18}$$

Consequently, for $k \neq 0$, (A.4) yields, with $s = -\varrho - \chi_k$,

$$\widehat{P}(k) \log 2 = g^*(-\varrho - \chi_k) + f(1) \int_0^1 \left(1 + (2^\varrho - 1)\varphi(u)\right)(1 + u)^{-\varrho - \chi_k - 1} \, \mathrm{d}u, \tag{A.19}$$

which is (A.7) in this case. For $k = 0$, by the expansions $\zeta(z) = (z - 1)^{-1} + O(1)$ and $1 - 2^z = -(\log 2)z - \frac{(\log 2)^2}{2} z^2 + O(z^3)$ for small $|z|$, and the relations (A.16) and (A.18), we have

$$(1 - 2^z)f^*(-\varrho + z) = (1 - 2^z)\left(\frac{1}{\log 2} z^{-2} - \widehat{P}(0)z^{-1} + O(1)\right)$$

$$= -z^{-1} - \frac{\log 2}{2} + \widehat{P}(0)\log 2 + O(z). \tag{A.20}$$

Hence,

$$\widehat{P}(0) = \frac{1}{2} + \frac{1}{\log 2}\left((1 - 2^{\varrho+s})f^*(s)\big|_{s=-\varrho}\right)_{\mathrm{fin}}, \tag{A.21}$$

and (A.7) for $k = 0$ follows from (A.4).

Finally, (A.10) shows that $g_1(n) = n^\varrho$ for even n. Hence, if $g(n) = n^\varrho$ for even n, then $g_2(2m) = 0$ for $m \geqslant 1$, and then $Q(n) = Q_2(n) = 0$ for $n \geqslant 1$ by Example 2.16, which proves the final claim of the theorem. $\qquad\square$

Example A.3. Let $\alpha = \beta = 2$ (so $\varrho = 2$), and let $g(n) = n^2$, $n \geqslant 2$. Then Theorem A.2 applies, with $g_0(n) = 0$. Hence, $Q(n) = 0$, and (A.6) yields

$$f(n) = n^\varrho \log_2 n + n^\varrho P(\log_2 n), \qquad n \geqslant 1. \tag{A.22}$$

Furthermore, (3.32) shows that, at least for $\Re s < -2$,

$$g^*(s) = \frac{D(-s - 1)}{s(s + 1)}, \tag{A.23}$$

where (3.28) yields, recalling $g(0) = g(1) = 0$,

$$D(s) = 4 + 2^{-s} + \sum_{n \geqslant 3} 2n^{-s} = 2\zeta(s) + 2 - 2^{-s}, \qquad \Re s > 1. \tag{A.24}$$

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Accordingly,

$$g^*(s) = \frac{1}{s(s+1)}\Big(2\zeta(-s-1) + 2 - 2^{s+1}\Big). \tag{A.25}$$

This shows that $g^*(s)$ is meromorphic in $\mathbb{C}$, with poles at $0, -1$ and -2. The finite value $g^*(-2)_{\text{fin}} = \gamma - \frac{3}{4}$, where γ is Euler's constant. Hence, if for simplicity $f(1) = 0$, the Fourier series of $P(t)$ is, by (A.7) and (A.25),

$$P(t) = \frac{4\gamma - 3}{4\log 2} + \frac{1}{2} + \frac{1}{2\log 2} \sum_{k \neq 0} \frac{4\zeta(1 + \chi_k) + 3}{(1 + \chi_k)(2 + \chi_k)}\, e^{2k\pi i t}. \tag{A.26}$$

This sequence $f(n)$ with $g(n) = n^2$ and $f(1) = 0$ is not in OEIS, but the sequence $f_{\text{A022560}}(n) = \text{A022560}(n - 1)$ discussed in Example 5.6 satisfies the recurrence $\Lambda_{2,2}[f] = \lfloor \frac{1}{4}n^2 \rfloor$ with $f_{\text{A022560}}(1) = 0$. Hence, $4f_{\text{A022560}}(n) = f(n) + n^2 - S_{2,2}(n)$ with $f(n)$ as above. The formulas (5.22)–(5.23) now follow from (A.22) and (A.26) together with (5.15). $\qquad\square$

B A series representation for $\hat{P}(k)$

We prove (3.34) in Remark 3.12. For notational simplicity we consider the case $k = 0$; the general case is the same, with ϱ replaced by $\varrho + \chi_k$ below. (Note that $2^{\varrho+\chi_k} = 2^\varrho = \alpha + \beta$ for all $k \in \mathbb{Z}$.) Consider the integral

$$J := \int_0^1 \frac{1 + (\alpha + \beta - 1)\varphi(t)}{(1 + t)^{\varrho+1}}\, \mathrm{d}t = \frac{1 - 2^{-\varrho}}{\varrho} + (\alpha + \beta - 1)J(1) \tag{B.1}$$

where

$$J(m) := \int_0^1 \frac{\varphi(t)}{(m + t)^{\varrho+1}}\, \mathrm{d}t. \tag{B.2}$$

We now express $J(1)$ in a series form as follows. First, by applying the recursive definition (2.4) of φ:

$$\varphi(t) = \begin{cases} \frac{\beta}{\alpha+\beta}\, \varphi(2t), & \text{if } 0 \leqslant t \leqslant \frac{1}{2}; \\ \frac{\alpha}{\alpha+\beta}\, \varphi(2t - 1) + \frac{\beta}{\alpha+\beta}, & \text{if } \frac{1}{2} \leqslant t \leqslant 1, \end{cases} \tag{B.3}$$

we obtain, for any $m \geqslant 1$,

$$\begin{aligned} J(m) &:= \int_0^1 \frac{\varphi(t)}{(m + t)^{\varrho+1}}\, \mathrm{d}t \\ &= \beta \int_0^1 \frac{\varphi(t)}{(2m + t)^{\varrho+1}}\, \mathrm{d}t + \alpha \int_0^1 \frac{\varphi(t)}{(2m + 1 + t)^{\varrho+1}}\, \mathrm{d}t + \frac{\beta}{\alpha + \beta} \int_{\frac{1}{2}}^1 \frac{1}{(m + t)^{\varrho+1}}\, \mathrm{d}t \\ &=: \beta J(2m) + \alpha J(2m + 1) + K(m), \end{aligned} \tag{B.4}$$

say. Iterating this gives, for any $N \geqslant 0$,

$$J(1) = \sum_{0 \leqslant m < N}\ \sum_{0 \leqslant j < 2^m} \alpha^{\nu(j)} \beta^{m-\nu(j)} K(2^m + j) + \sum_{0 \leqslant j < 2^N} \alpha^{\nu(j)} \beta^{N-\nu(j)} J(2^N + j). \tag{B.5}$$

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where $\nu(j)$ denotes the number of 1's in j's binary expansion. Since $|J(m)|, |K(m)| = O(m^{-\varrho-1})$, the last sum in (2.38) is $O\big((\alpha + \beta)^N 2^{-(\varrho+1)N}\big) = O\big(2^{-N}\big)$ and similarly the inner sum in the double sum is $O(2^{-m})$; hence we can let $N \to \infty$, which yields

$$J(1) = \sum_{m \geqslant 0} \sum_{0 \leqslant j < 2^m} \alpha^{\nu(j)} \beta^{m-\nu(j)} K(2^m + j). \tag{B.6}$$

Now

$$K(m) = \frac{\beta}{\alpha + \beta} \int_{\frac{1}{2}}^{1} \frac{1}{(m+t)^{\varrho+1}} \, \mathrm{d}t = \frac{\beta}{\varrho(\alpha + \beta)} \Big(\frac{1}{(m + \frac{1}{2})^\varrho} - \frac{1}{(m+1)^\varrho} \Big). \tag{B.7}$$

Thus

$$
\int_0^1 \frac{1 + (\alpha + \beta - 1)\varphi(t)}{(1+t)^{\varrho+1}} \, \mathrm{d}t
$$
$$
= \frac{1 - 2^{-\varrho}}{\varrho} + \frac{\beta(\alpha + \beta - 1)}{\varrho(\alpha + \beta)} \sum_{m \geqslant 0} \sum_{0 \leqslant j < 2^m} \alpha^{\nu(j)} \beta^{m-\nu(j)} \Big(\frac{1}{(2^m + j + \frac{1}{2})^\varrho} - \frac{1}{(2^m + j + 1)^\varrho} \Big)
$$
$$
= \frac{\alpha + \beta - 1}{\varrho(\alpha + \beta)} \Big(1 + \beta \sum_{m \geqslant 0} \sum_{0 \leqslant j < 2^m} \alpha^{\nu(j)} \beta^{m-\nu(j)} \Big(\frac{1}{(2^m + j + \frac{1}{2})^\varrho} - \frac{1}{(2^m + j + 1)^\varrho} \Big) \Big). \tag{B.8}
$$

The double sum can be converted into a single one as follows.

$$
\sum_{m \geqslant 0} \sum_{0 \leqslant j < 2^m} \alpha^{\nu(j)} \beta^{m+1-\nu(j)} \Big(\frac{1}{(2^m + j + \frac{1}{2})^\varrho} - \frac{1}{(2^m + j + 1)^\varrho} \Big)
$$
$$
= 2^\varrho \sum_{m \geqslant 0} \sum_{0 \leqslant j < 2^m} \alpha^{\nu(j)} \beta^{m+1-\nu(j)} \Big(\frac{1}{(2^{m+1} + 2j + 1)^\varrho} - \frac{1}{(2^{m+1} + 2j + 2)^\varrho} \Big)
$$
$$
= 2^\varrho \sum_{m \geqslant 1} \sum_{0 \leqslant j < 2^{m-1}} \alpha^{\nu(j)} \beta^{m-\nu(j)} \Big(\frac{1}{(2^m + 2j + 1)^\varrho} - \frac{1}{(2^m + 2j + 2)^\varrho} \Big)
$$
$$
= 2^\varrho \sum_{k \geqslant 2} \frac{(-1)^k}{(k+1)^\varrho} \alpha^{\nu(\lfloor 2^{L_k-1}\{k/2^{L_k}\} \rfloor)} \beta^{L_k - \nu(\lfloor 2^{L_k-1}\{k/2^{L_k}\} \rfloor)}. \tag{B.9}
$$

C Recurrences with minimisation or maximisation

Consider the class of sequences satisfying recurrences of the form

$$u(n) = \min_{1 \leqslant k \leqslant \lfloor \frac{n}{2} \rfloor} \{ \alpha u(k) + \beta u(n-k) \} \qquad (n \geqslant 2), \tag{C.1}$$

with $u(1) = 1$.

This was studied in [5], where it was shown that if α and β are positive integers with $\beta \geqslant \alpha$, then the minimum in (C.1) is reached at $k = \lfloor \frac{n}{2} \rfloor$ and the solution is given by

$$u(n) = 1 + (\alpha + \beta - 1) \sum_{1 \leqslant j < n} w(j), \tag{C.2}$$

where
$$w(j) := \alpha^{L_j - \nu_0(j)} \beta^{\nu_0(j)}, \tag{C.3}$$

and $\nu_0(j)$ denotes the number of zeros in the binary expansion of j. We will extend this result and the arguments in [5], and prove the following.

Proposition C.1. *Let $\alpha, \beta > 0$ be real numbers such that either*

(i) $\beta \geqslant \alpha$ *and* $\beta \geqslant 1$, *or*

(ii) $\alpha \geqslant \beta$ *and* $\alpha + \beta \leqslant 1$.

Then the minimum in (C.1) *is reached at* $k = \left\lfloor \frac{n}{2} \right\rfloor$. *Hence,* (C.1) *reduces to* $\Lambda_{\alpha,\beta}[u] = 0$, *and thus* (C.1) *is solved by* $u(n) = S_{\alpha,\beta}(n)$.

Proof. We note first that $S_{\alpha,\beta}(n)$ is given by the formula in (C.2) for any α, β. This follows by (2.44), (2.22), and (2.14) in Section 2, or by the proof in [5].

It thus remains to show that if $u(n)$ is defined by (C.2), then

$$\alpha u(k) + \beta u(n-k) \geqslant \alpha u\left(\left\lfloor \tfrac{n}{2} \right\rfloor\right) + \beta u\left(\left\lceil \tfrac{n}{2} \right\rceil\right) \tag{C.4}$$

for $1 \leqslant k < \left\lfloor \frac{n}{2} \right\rfloor$. By (C.2), the difference between the two sides of (C.4) is

$$(\alpha + \beta - 1)\left(\beta \sum_{\lceil \frac{n}{2} \rceil \leqslant j < n-k} w(j) - \alpha \sum_{k \leqslant j < \lfloor \frac{n}{2} \rfloor} w(j) \right). \tag{C.5}$$

To prove that this is non-negative, we will show that

$$\beta \sum_{\lceil \frac{n}{2} \rceil \leqslant j < n-k} w(j) \geqslant \alpha \sum_{k \leqslant j < \lfloor \frac{n}{2} \rfloor} w(j) \tag{C.6}$$

if (i) holds, and that (C.6) holds with the inequality reversed if (ii) holds. The key is the following claim:

(a) *If* $\beta \geqslant \alpha$ *and* $\beta \geqslant 1$, *then*

$$\beta w(n + 2^j) \geqslant \alpha w(n), \quad \text{for all } n \geqslant 1 \text{ and } j \geqslant 0. \tag{C.7}$$

(b) *If* $\alpha \geqslant \beta$ *and* $\beta \leqslant 1$, *then*

$$\beta w(n + 2^j) \leqslant \alpha w(n), \quad \text{for all } n \geqslant 1 \text{ and } j \geqslant 0. \tag{C.8}$$

Proof of the claim. From (C.3), we have

$$\frac{\beta w(n + 2^j)}{\alpha w(n)} = \left(\frac{\beta}{\alpha}\right)^{1 + \nu_0(n+2^j) - \nu_0(n)} \alpha^{L_{n+2^j} - L_n}$$

$$= \left(\frac{\beta}{\alpha}\right)^{1 + \nu_0(n+2^j) - \nu_0(n) - (L_{n+2^j} - L_n)} \beta^{L_{n+2^j} - L_n}. \tag{C.9}$$

It is easily seen that

$$1 + \nu_0(n + 2^j) - \nu_0(n) \geqslant L_{n+2^j} - L_n \geqslant 0. \tag{C.10}$$

Both parts of the claim thus follow from (C.9). $\qquad\square$

To show (C.6), or its converse in case (ii), which will then complete the proof of the proposition, we combine the claim above with a pairing between the sets, with $m := \left\lfloor \frac{n}{2} \right\rfloor - k$,

$$\left\{ \left\lfloor \frac{n}{2} \right\rfloor - m, \cdots, \left\lfloor \frac{n}{2} \right\rfloor - 1 \right\} \quad \text{and} \quad \left\{ \left\lceil \frac{n}{2} \right\rceil, \cdots, \left\lceil \frac{n}{2} \right\rceil + m - 1 \right\}, \tag{C.11}$$

such that the difference between the elements of each pair is a power of 2. In other words, the proof is completed by applying the following lemma (with a translation). $\qquad\square$

Lemma C.2. *Let $n \geqslant 1$ and*

$$A := \{1, 2, \cdots, n\}, \tag{C.12}$$
$$C_1 := \{n + 1, n + 2, \cdots, 2n\}, \tag{C.13}$$
$$C_2 := \{n + 2, n + 3, \cdots, 2n + 1\}. \tag{C.14}$$

There exist one-to-one mappings

$$h_1 : A \to C_1 \quad \text{and} \quad h_2 : A \to C_2, \tag{C.15}$$

such that for each $k \in A$ there exist j_1, j_2 with

$$h_1(k) = k + 2^{j_1}, \quad \text{for some } j_1 \geqslant 0, \tag{C.16}$$
$$h_2(k) = k + 2^{j_1}, \quad \text{for some } j_2 \geqslant 0. \tag{C.17}$$

Proof. We prove the existence of h_1 by induction. Write $n = 2^\ell + j$ for some $\ell \geqslant 0$ and $0 \leqslant j < 2^\ell$. (Thus, $\ell = L_n$.) We want to show that there exist a one-to-one mapping

$$h_1 : \{1, \cdots, 2^\ell + j\} \to \{2^\ell + j + 1, \cdots, 2^{\ell+1} + 2j\}, \tag{C.18}$$

such that for each $1 \leqslant k \leqslant 2^\ell + j$, (C.16) holds.

If $j = 0$, we simply define $h_1(k) := k + 2^\ell$.

If $j \geqslant 1$, we first define $h_1(k)$ for $k \leqslant 2j$ by $h_1(k) := k + 2^{\ell+1}$. This gives a mapping from $\{1, 2, \cdots, 2j\}$ to $\{2^{\ell+1} + 1, \cdots, 2^{\ell+1} + 2j\}$. We remove these two blocks, and it remains to define a one-to-one mapping satisfying (C.16) between $\{2j + 1, \cdots, 2^\ell + j\}$ and $\{2^\ell + j + 1, \cdots, 2^{\ell+1}\}$. By subtracting $2j$ from each term, it is equivalent to showing that there exist such a mapping from $\{1, \cdots, m\}$ to $\{m + 1, \cdots, 2m\}$, where $m = 2^\ell - j$; this is true by the induction hypothesis.

The proof of the existence of h_2 is similar (but with $n+1 = 2^\ell + j$); we omit the details. $\qquad\square$

If we replace $\min$ by $\max$ in (C.1), we obtain a similar result; see also [32] for $\alpha > \beta = 1$.

Proposition C.3. *Let $\alpha, \beta > 0$ be real numbers such that $\alpha \geqslant \beta$, $\beta \leqslant 1$, and $\alpha + \beta \geqslant 1$. Then the maximum in the recursion*

$$u(n) = \max_{1 \leqslant k \leqslant \lfloor \frac{n}{2} \rfloor} \{\alpha u(k) + \beta u(n - k)\} \qquad (n \geqslant 2), \tag{C.19}$$

with $u(1) = 1$, is attained at $k = \left\lfloor \frac{n}{2} \right\rfloor$. Hence, (C.19) reduces to $\Lambda_{\alpha,\beta}[u] = 0$, and thus (C.19) is solved by $u(n) = S_{\alpha,\beta}(n)$.

Proof. The proof above shows that under these conditions, (C.8) holds, and hence (C.6) holds in the opposite direction. Thus the difference in (C.5) is $\leqslant 0$. (We do not obtain a second case with $\alpha + \beta < 1$; we then would need (C.7), but the condition $\beta \geqslant 1$ in (a) above is incompatible with $\alpha + \beta < 1$.) $\qquad\square$

The results above can be extended to the recurrences of the form

$$u(n) = \min_{1 \leqslant k \leqslant \lfloor \frac{n}{2} \rfloor} \{\alpha u(k) + \beta u(n-k)\} + c \qquad (n \geqslant 2), \tag{C.20}$$

with the same conditions as above on α and β; see [22, 28] for more general versions. The solution for the recurrence

$$u(n) = \alpha u\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + \beta u\left(\left\lceil \frac{n}{2} \right\rceil\right) + c \tag{C.21}$$

is, with $w(j)$ as in (C.2)–(C.3) above (see also (4.3))

$$u(n) = u(1) + \big((\alpha + \beta - 1)u(1) + c\big) \sum_{1 \leqslant j < n} w(j). \tag{C.22}$$

D   Nowhere differentiability of $P_{\mathbf{A006581}}(t)$

We prove in this appendix the fractal nature of the periodic function $P(t)$ arising from A006581 (discussed in Example 5.6 with Fourier expansion given in (5.20)), namely, $\Lambda_{2,2}[f] = g$ where $g(n) := \{\frac{n}{2}\}(n-1)$:

$$f(n) = n^2 P(\log_2 n), \tag{D.1}$$

where, with $\bar{g}(x) := g(x)/x^2$,

$$P(t) = \sum_{m \in \mathbb{Z}} 4^{-m-t} g(2^{m+t}) = \sum_{m \in \mathbb{Z}} \bar{g}(2^{m+t}). \tag{D.2}$$

Here $g(x)$ is extended from $g(n)$ as in (2.2) with $\varphi(t) = t$:

$$\bar{g}(x) = \frac{1}{2x^2} \times \begin{cases} \{x\}\lfloor x \rfloor, & \text{if } \lfloor x \rfloor \text{ is even;} \\ (1 - \{x\})(\lfloor x \rfloor - 1), & \text{if } \lfloor x \rfloor \text{ is odd.} \end{cases} \tag{D.3}$$

Since $\bar{g}(2^{m+t}) = 0$ for $t \in [0, 1]$ and $m \leqslant 0$, we have

$$P(t) = \sum_{m \geqslant 1} \bar{g}(2^{m+t}) \qquad (0 \leqslant t \leqslant 1). \tag{D.4}$$

The method of proof used here to prove the nowhere differentiability of P is standard and similar to that for the Takagi function given in the survey paper [1].

Let $t \in [0, 1)$, and define

$$\tau_n := \log_2 \frac{\lfloor 2^{n+t} \rfloor}{2^n} \quad \text{and} \quad \tau'_n := \log_2 \frac{\lfloor 2^{n+t} \rfloor + 1}{2^n}. \tag{D.5}$$

To prove that P does not have a finite derivative at t, it suffices to show that the sequence

$$\frac{P(\tau_n) - P(\tau'_n)}{\tau_n - \tau'_n} = \sum_{1 \leqslant m \leqslant n} \frac{\bar{g}(2^{m+\tau_n}) - \bar{g}(2^{m+\tau'_n})}{\tau_n - \tau'_n} \tag{D.6}$$

does not converge to a finite limit. Here we used the relation $\bar{g}(2^{m+\tau_n}) = \bar{g}(2^{m+\tau'_n}) = 0$ for $m > n$. Now for $\theta \in [0, 1)$

$$\begin{cases} \bar{g}(2k + \theta) = \dfrac{k\theta}{(2k+\theta)^2} \\ \bar{g}(2k + 1 + \theta) = \dfrac{k(1-\theta)}{(2k+1+\theta)^2} \end{cases} \tag{D.7}$$

so that, taking the right derivative at integer points here and below,

$$\begin{cases} \bar{g}'(2k + \theta) = \dfrac{k(2k-\theta)}{2(2k+\theta)^3} \\ \bar{g}'(2k + 1 + \theta) = -\dfrac{k(2k+3-\theta)}{(2k+1+\theta)^3}. \end{cases} \tag{D.8}$$

If $1 \leqslant m \leqslant n$, then

$$\lfloor 2^{m+\tau_n} \rfloor \leqslant 2^{m+\tau_n} \leqslant 2^{m+t} < 2^{m+\tau'_n} \leqslant \lfloor 2^{m+\tau_n} \rfloor + 1. \tag{D.9}$$

It follows that $h(x) := \bar{g}(2^x)$ is infinitely differentiable on $[m + \tau_n, m + \tau'_n]$, and it is easily seen from (D.8) that $h''(x) = O(1)$ (uniformly in m and n). We have, for some $\tau''_n \in (\tau_n, \tau'_n)$,

$$\begin{aligned} \frac{\bar{g}(2^{m+\tau_n}) - \bar{g}(2^{m+\tau'_n})}{\tau_n - \tau'_n} &= \frac{h(m+\tau_n) - h(m+\tau'_n)}{\tau_n - \tau'_n} = h'(m + \tau''_n) \\ &= h'(m+t) + O(|\tau''_n - t|) = h'(m+t) + O(|\tau'_n - \tau_n|) \\ &= h'(m+t) + O(2^{-n}). \end{aligned} \tag{D.10}$$

Hence, (D.6) implies that

$$\frac{P(\tau_n) - P(\tau'_n)}{\tau_n - \tau'_n} = \sum_{1 \leqslant m \leqslant n} h'(m+t) + O(n2^{-n}), \tag{D.11}$$

and thus, if P is differentiable at t, then the sum

$$\sum_{m \geqslant 1} h'(m+t) \tag{D.12}$$

converges (and equals $P'(t)$).

On the other hand, it follows easily from (D.8) that if $x \geqslant 2$, then $x\bar{g}'(x) \geqslant \frac{1}{9}$ for even $\lfloor x \rfloor$ and $x\bar{g}'(x) \leqslant -\frac{1}{4}$ for odd $\lfloor x \rfloor$. Hence, $|h'(m+t)| = |2^{m+t}\bar{g}'(2^{m+t})\log 2| \geqslant \frac{\log 2}{9}$ for all $m \geqslant 1$. Consequently, the sum (D.12) diverges for any t, and thus P is nowhere differentiable.

Moreover, we note that if 2^t is a dyadic rational, then for all large m, 2^{m+t} is an even integer, and thus $h'(m+t) \geqslant \frac{\log 2}{9}$. Hence, in this case the sum (D.12) diverges to $+\infty$, and thus so does $\left(P(\tau_n) - P(\tau'_n)\right)/(\tau_n - \tau'_n)$ in (D.6); consequently, P is not Lipschitz.

We do not know whether P is Hölder continuous, and leave that as an open problem. Note that Lemma 3.4 does not apply since (3.10) does not hold.

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