ON A CENTRAL LIMIT THEOREM IN RENEWAL THEORY

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Abstract. Serfozo (2009, Theorem 2.65) gives a useful central limit theorem for processes with regenerative increments. Unfortunately, there is a gap in the proof. We fill this gap, and at the same time we weaken the assumptions. Furthermore, we give conditions for moment convergence in this setting. We give also further results complementing results in Serfozo (2009) on the law of large numbers and estimates for the mean; in particular, we show that there is a gap between conditions for the weak and strong laws of large numbers.

1. Introduction

The standard setting in renewal theory is that we have a stochastic process (in continuous or discrete time) such that some event occurs at random times $0 < T_1 < T_2 < \dots$, and the process "starts again" at each such event. Formally this means that the times between renewals $\xi_n := T_n - T_{n-1}$, $n \geqslant 1$, (where we define $T_0 := 0$) are i.i.d. (independent and identically distributed) random variables. We thus have

$$T_n := \sum_{i=1}^{n} \xi_i, \qquad n \geqslant 0. \tag{1.1}$$

with $(\xi_i)_1^\infty$ i.i.d., where ξ_1 may be any (strictly) positive random variable. We then define, for $t \geqslant 0$,

$$N(t) := \max\{n : T_n \leqslant t\} = \sum_{n=1}^{\infty} \mathbf{1}\{T_n \leqslant t\}, \tag{1.2}$$

$$\tau(t) := \min\{n : T_n > t\} = N(t) + 1. \tag{1.3}$$

It is well-known that $T_n \to \infty$ a.s. as $n \to \infty$, and thus $N(t)$ and $\tau(t)$ are well defined for any $t \geqslant 0$. Note that by the definitions,

$$T_{N(t)} \leqslant t < T_{N(t)+1} = T_{\tau(t)}. \tag{1.4}$$

In applications it is common to study the values of another stochastic process at the renewal times T_n. One common version of this is to let $(\eta_i)_1^\infty$ be another sequence of random variables such that the random vectors (ξ_i, η_i), $i \geqslant 1$, are i.i.d., and define their partial sums

$$V_n := \sum_{i=1}^{n} \eta_i; \tag{1.5}$$

we then may consider $V_{N(t)}$ or $V_{\tau(t)}$, which can be interpreted as the values of a stochastic process at the time $T_{N(t)}$ or $T_{\tau(t)}$. Asymptotic results such as a law of large numbers and a central limit theorem for $V_{N(t)}$ or $V_{\tau(t)}$ are well known, see e.g. [2, Section 4.2, including Remark 4.2.10].

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A version of this is that we are given another (real-valued) stochastic process $Z(t)$ defined for all times $t \geqslant 0$ such that $Z(t)$ also "starts again" at each renewal time T_n. We are interested in asymptotic results for $Z(t)$ as $t \to \infty$, and by (1.4), we may under suitable assumptions approximate $Z(t)$ by $V_{N(t)}$ or $V_{\tau(t)}$ (with η_n given below) and obtain results for $Z(t)$ from results of the type just mentioned. To make this formal, Serfozo [4, Definition 2.52, Section 2.10] makes the following definition:

Definition 1.1. The process $Z(t)$ has *regenerative increments over the times* T_n if $Z(0) = 0$ and the increments

$$\zeta_n := \big(\xi_n, \{Z(t + T_{n-1}) - Z(T_{n-1}) : 0 \leqslant t \leqslant \xi_n\}\big), \qquad n \geqslant 1, \tag{1.6}$$

are i.i.d.

Remark 1.2. Note that the second component of ζ_n is a stochastic process, defined over the random interval $[0, \xi_n]$; we may for the purpose of this definition regard it as a process on $[0, \infty)$ stopped at ξ_n, i.e., equal to $Z(\xi_n + T_{n-1}) - Z(T_{n-1})$ for all $t \geqslant \xi_n$. In general we may regard this process as an element of the product space $\mathbb{R}^{[0,\infty)}$ but we will also need some regularity property. We assume, for convenience, that $Z(t)$ is càdlàg (right-continuous with left limits, also written $Z \in D[0, \infty)$); then the second component of ζ_n is also càdlàg. △

Remark 1.3. The definition given in [4, Definition 2.52] actually differs slightly from the one above, using only the interval $0 \leqslant t < \xi_n$ instead of $0 \leqslant t \leqslant \xi_n$ in (1.6). This is obviously a typo, and should be interpreted as in (1.6), since otherwise the definition would be trivially satisfied by the process $Z(t) := X_{N(t)}$ for *any* random sequence $(X_n)_0^\infty$ with $X_0 = 0$; this evidently cannot imply any limit results. △

Suppose that $Z(t)$ has regenerative increments over T_n, and define

$$\eta_n := Z(T_n) - Z(T_{n-1}), \qquad n \geqslant 1. \tag{1.7}$$

It follows from the definition above, taking $t = \xi_n$ in (1.6), that the sequence of pairs (ξ_n, η_n), $n \geqslant 1$, is i.i.d. Hence, we may define V_n by (1.5), and note that (1.7) yields

$$V_n = Z(T_n). \tag{1.8}$$

Consequently, using (1.4), we have for any $t \geqslant 0$

$$Z(t) = V_{N(t)} + \big(Z(t' + T_{N(t)}) - Z(T_{N(t)})\big), \tag{1.9}$$

where $t' := t - T_{N(t)} \in [0, \xi_{N(t)+1})$. We define, following [4],

$$M_n := \sup_{T_{n-1} \leqslant t \leqslant T_n} |Z(t) - Z(T_{n-1})|, \qquad n \geqslant 1. \tag{1.10}$$

In particular,

$$M_1 := \sup_{0 \leqslant t \leqslant T_1} |Z(t)|. \tag{1.11}$$

Note that it follows from Definition 1.1 that the random variables M_n are i.i.d.

It is clear from (1.9) that with suitable conditions on M_n, asymptotic results for $Z(T)$ follow from results for $V_{N(t)}$. In particular, Serfozo [4] gives the following central limit theorem, which is well suited for applications.

Theorem 1.4 ([4, Theorem 2.65])**.** *Suppose $Z(t)$ is a stochastic process with regenerative increments over T_n such that $\mu := \mathbb{E}\,[T_1]$, $a := \mathbb{E}\,[Z(T_1)]/\mu$, and $\sigma^2 :=$*

$\operatorname{Var}[Z(T_1) - aT_1]$ *are finite. In addition, let* M_1 *be defined by* (1.11)*, and assume* $M_1 < \infty$ *a.s. Then*

$$\frac{Z(t) - at}{\sqrt{t}} \xrightarrow{\mathrm{d}} N(0, \sigma^2/\mu), \qquad \text{as } t \to \infty. \tag{1.12}$$

We include the case $\sigma^2 = 0$, letting $N(0,0)$ denote (the distribution of) 0.

Remark 1.5. The theorem stated in [4] also assumes $\mathbb{E}\, M_1 < \infty$, but the proof below shows that it suffices to assume $M_1 < \infty$ a.s., as done here. △

Unfortunately, there is a gap in the proof given in [4], see Remark 2.1, so we give a proof filling that gap (under our, weaker, conditions) in Section 2.

In Section 3, we give conditions for moment convergence in Theorem 1.4. Furthermore, we give in Section 4 a weak law of large numbers, complementing the strong law in [4]; we show that the weak law holds under weaker conditions than the strong law. Finally, Section 5 gives further estimates for the mean under various moment conditions.

Remark 1.6. We let throughout the paper the time parameter $t \in [0, \infty)$ be a continuous variable. Results for a discrete time parameter $t \in \mathbb{N}$ follow immediately, by assuming that the times T_n are integer-valued and then considering only $t \in \mathbb{N}$. △

1.1. **Some notation.** We use the notation introduced above throughout the paper. In particular, μ, a, and σ^2 have the same meanings as in Theorem 1.4. We also use the *renewal function*

$$U(t) := \sum_{n=0}^{\infty} \mathbb{P}\bigl(T_n \leqslant t\bigr) = \mathbb{E}\, N(t) + 1. \tag{1.13}$$

We let $\xrightarrow{\mathrm{d}}$, $\xrightarrow{\mathrm{p}}$, and $\xrightarrow{\mathrm{a.s.}}$ denote convergence in distribution, probability, and almost surely, respectively. Unspecified limits are as $t \to \infty$.

For real x, y, we let $x \wedge y := \min\{x, y\}$ and $x \vee y := \max\{x, y\}$.

"Decreasing" is interpreted in the weak sense.

2. Proof of Theorem 1.4

We basically follow the proof in [4, pp. 136–137]. We define as there

$$Z'(t) := \frac{Z(T_{N(t)}) - aT_{N(t)}}{\sqrt{t}} = t^{-1/2} \sum_{i=1}^{N(t)} \bigl(\eta_i - a\xi_i\bigr), \tag{2.1}$$

where we used (1.7) and (1.1), and note that $X_i := \eta_i - a\xi_i$ are i.i.d. random variables with $\mathbb{E}\, X_i = \mathbb{E}\, X_1 = \mathbb{E}\,[Z(T_1)] - a\,\mathbb{E}\,[T_1] = 0$ and $\operatorname{Var} X_i = \sigma^2$. Hence it follows from Anscombe's theorem [4, Theorem 2.64] (see also [2, Section 1.3]; alternatively, use instead Donsker's theorem [3, Theorem 7.7.13]), together with the (weak) law of large numbers $N(t)/t \xrightarrow{\mathrm{p}} 1/\mu$, [4, Corollary 2.11] that

$$Z'(t) \xrightarrow{\mathrm{d}} N\bigl(0, \sigma^2/\mu\bigr), \qquad \text{as } t \to \infty. \tag{2.2}$$

(The case $\sigma^2 = 0$ is trivial since then $X_i = 0$ and thus $Z'(t) = 0$ a.s.)

Hence, by Cramér–Slutsky's theorem [3, Theorem 5.11.4], it suffices to show that

$$\frac{Z(t) - at}{\sqrt{t}} - Z'(t) \xrightarrow{\mathrm{p}} 0. \tag{2.3}$$

We have, by (2.1),

$$\frac{Z(t)-at}{\sqrt{t}} - Z'(t) = \frac{Z(t) - Z_{N(t)} - a(t - T_{N(t)})}{\sqrt{t}} \tag{2.4}$$

and thus, recalling (1.4), (1.10), and (1.1),

$$\left|\frac{Z(t)-at}{\sqrt{t}} - Z'(t)\right| \leqslant \frac{M_{N(t)+1} + |a|\xi_{N(t)+1}}{\sqrt{t}} =: \frac{Y_{N(t)+1}}{\sqrt{t}}, \tag{2.5}$$

where we let

$$Y_n := M_n + |a|\xi_n. \tag{2.6}$$

Consequently, to show (2.3), and thus Theorem 1.4, it suffices to show that

$$\frac{Y_{N(t)+1}}{t^{1/2}} \overset{\mathrm{p}}{\longrightarrow} 0, \tag{2.7}$$

or, equivalently,

$$\frac{Y_{N(t)+1}}{N(t)^{1/2}} \overset{\mathrm{p}}{\longrightarrow} 0, \tag{2.8}$$

By (2.6), (1.10) and Definition 1.1, the random vectors (ξ_n, Y_n) are i.i.d., and thus (2.7) follows from Lemma 2.2 below (with $\delta(t) := t^{-1/2}$), which completes the proof of Theorem 1.4. ☐

Remark 2.1. The gap in the proof in [4] is the claim $n^{-1/2}Y_n \overset{\mathrm{d}}{=} n^{-1/2}Y_1 \overset{\mathrm{a.s.}}{\longrightarrow} 0$ as $n \to \infty$ made there; although $Y_n \overset{\mathrm{d}}{=} Y_1$ and $n^{-1/2}Y_1 \overset{\mathrm{a.s.}}{\longrightarrow} 0$ as $n \to \infty$, this only shows $Y_n/n^{1/2} \overset{\mathrm{p}}{\longrightarrow} 0$, which in general is not enough to imply (2.8). In fact, it is a well-known consequence of the Borel–Cantelli lemmas (see [3, Proposition 6.1.1]) that, for any i.i.d. sequence Y_n, we have $Y_n/n^{1/2} \overset{\mathrm{a.s.}}{\longrightarrow} 0$ as $n \to \infty$ (which does imply (2.8)) if and only if $\mathbb{E}\, Y_1^2 < \infty$, which requires the stronger assumption $\mathbb{E}\, M_1^2 < \infty$ (and also $\mathbb{E}\, T_1^2 < \infty$ unless $a = 0$). △

We used in the proof the following lemma. (For related results under moment assumptions, see [2, Theorem 1.8.1].) Recall that a family $(X_\alpha)_{\alpha\in\mathcal{A}}$ of random variables is *tight* (a.k.a. *stochastically bounded*) if for every $\varepsilon > 0$, there exists $c > 0$ such that $\mathbb{P}(|X_\alpha| > c) < \varepsilon$ for all $\alpha \in \mathcal{A}$. Recall also that (the distribution of) ξ_1 is *arithmetic* if there exists $d > 0$ such that $\xi_1 \in d\mathbb{N} = \{d, 2d, \dots\}$ a.s.; then the largest such d is called the *span* of ξ_1.

Lemma 2.2. *With the notations above, let T_n be a sequence of renewal times with $\mathbb{E}\, T_1 = \mathbb{E}\,\xi_1 < \infty$, and let Y_n, $n \geqslant 1$, be another sequence of random variables such that the random vectors (ξ_n, Y_n) are i.i.d. Then the family of random variables $\{Y_{N(t)+1} : t \geqslant 0\}$ is tight. In particular, if $\delta(t)$ is any positive function such that $\delta(t) \to 0$ as $t \to \infty$, then*

$$\delta(t) Y_{N(t)+1} \overset{\mathrm{p}}{\longrightarrow} 0 \qquad \text{as } t \to \infty. \tag{2.9}$$

Proof. By replacing Y_n with $|Y_n|$, we may for convenience assume that $Y_n \geqslant 0$. Moreover, if ξ_1 is arithmetic, with span $d > 0$, then it suffices to consider $t \in d\mathbb{N}$.

Let $c > 0$. Then, since (ξ_{n+1}, Y_{n+1}) is independent of T_n and further $(\xi_{n+1}, Y_{n+1}) \overset{\mathrm{d}}{=} (\xi_1, Y_1)$,

$$\begin{aligned}\mathbb{P}\bigl(Y_{N(t)+1} > c\bigr) &= \mathbb{E}\sum_{n=0}^{\infty} \mathbf{1}\{T_n \leqslant t < T_{n+1},\ Y_{n+1} > c\} \\ &= \mathbb{E}\sum_{n=0}^{\infty} \mathbb{P}\bigl(T_n \leqslant t < T_{n+1},\ Y_{n+1} > c \mid T_n\bigr) \\ &= \mathbb{E}\sum_{n=0}^{\infty} \mathbb{P}\bigl(0 \leqslant t - T_n < \xi_{n+1},\ Y_{n+1} > c \mid T_n\bigr) \\ &= \mathbb{E}\sum_{n=0}^{\infty} h_c(t - T_n), \end{aligned} \tag{2.10}$$

where

$$h_c(s) := \mathbf{1}\{s \geqslant 0\}\,\mathbb{P}\bigl(\xi_1 > s,\ Y_1 > c\bigr) = \mathbf{1}\{s \geqslant 0\}\,\mathbb{P}\bigl(T_1 > s,\ Y_1 > c\bigr). \tag{2.11}$$

Using the renewal function $U(t)$ defined in (1.13), we can write (2.10) as

$$\mathbb{P}\bigl(Y_{N(t)+1} > c\bigr) = U * h_c(t) := \int_0^\infty h(t-u)\,\mathrm{d}U(u). \tag{2.12}$$

Note that $h_c(s) \geqslant 0$ and that $h_c(s)$ is decreasing on $[0, \infty)$ with

$$\int_0^\infty h_c(s)\,\mathrm{d}s \leqslant \int_0^\infty \mathbb{P}(\xi_1 > s)\,\mathrm{d}s = \mathbb{E}\,\xi_1 = \mu < \infty. \tag{2.13}$$

Hence, $h_c(s)$ is directly Riemann integrable, and thus the key renewal theorem (see [4, Theorems 2.35–37] or [2, Theorem 2.4.3]) yields (in both the arithmetic and non-arithmetic cases)

$$\lim_{t\to\infty} \mathbb{P}\bigl(Y_{N(t)+1} > c\bigr) = \frac{1}{\mu}\int_0^\infty h_c(s)\,\mathrm{d}s = \frac{1}{\mu}\int_0^\infty \mathbb{P}\bigl(\xi_1 > s,\ Y_1 > c\bigr)\,\mathrm{d}s =: \lambda_c. \tag{2.14}$$

Since $\mathbb{P}\bigl(\xi_1 > s,\ Y_1 > c\bigr) \leqslant \mathbb{P}\bigl(\xi_1 > s\bigr)$ and

$$\int_0^\infty \mathbb{P}\bigl(\xi_1 > s\bigr)\,\mathrm{d}s = \mathbb{E}\,\xi_1 < \infty, \tag{2.15}$$

dominated convergence yields

$$\lim_{c\to\infty} \lambda_c = \lim_{c\to\infty} \frac{1}{\mu}\int_0^\infty \mathbb{P}\bigl(\xi_1 > s,\ Y_1 > c\bigr)\,\mathrm{d}s = \frac{1}{\mu}\int_0^\infty \lim_{c\to\infty} \mathbb{P}\bigl(\xi_1 > s,\ Y_1 > c\bigr)\,\mathrm{d}s = 0. \tag{2.16}$$

Let $\varepsilon > 0$. Then (2.16) shows that we may choose $c > 0$ such that $\lambda_c < \varepsilon$, and then (2.14) shows that for all sufficiently large t, we have

$$\mathbb{P}\bigl(Y_{N(t)+1} > c\bigr) < \varepsilon. \tag{2.17}$$

This is enough to imply (2.9), since for large t we also have $c\delta(t) < \varepsilon$, and consequently

$$\mathbb{P}\bigl(\delta(t)Y_{N(t)+1} > \varepsilon\bigr) \leqslant \mathbb{P}\bigl(\delta(t)Y_{N(t)+1} > \delta(t)c\bigr) = \mathbb{P}\bigl(Y_{N(t)+1} > c\bigr) < \varepsilon. \tag{2.18}$$

Moreover, we have shown that for every $\varepsilon > 0$, there exists c and t_0 such that (2.17) holds for $t \geqslant t_0$. There exists n_0 such that $\mathbb{P}(N(t_0)+1 > n_0) < \varepsilon/2$, and we may increase c so that $\mathbb{P}(Y_n > c) < \varepsilon/(2n_0)$ for all $n \leqslant n_0$. Then, for every $t \leqslant t_0$,

$$\mathbb{P}(Y_{N(t)+1} > c) \leqslant \mathbb{P}\bigl(N(t)+1 > n_0\bigr) + \sum_{n=1}^{n_0} \mathbb{P}\bigl(Y_n > c\bigr) < \varepsilon. \tag{2.19}$$

Hence, there exists c such that (2.17) holds for all $t \geqslant 0$, which proves that the family $\{Y_{N(t)+1} : t \geqslant 0\}$ is tight. □

3. Moment convergence

If we assume further moment conditions, we also have convergence of moments in Theorem 1.4.

Theorem 3.1. *Let $r \geqslant 2$, and assume in addition to the assumptions of Theorem 1.4 that $\mathbb{E}\,[T_1^r] < \infty$ and $\mathbb{E}\,[M_1^r] < \infty$. Then the family of random variables*

$$\left\{ \left| \frac{Z(t) - at}{\sqrt{t}} \right|^r : t \geqslant 1 \right\} \tag{3.1}$$

is uniformly integrable, and consequently (1.12) *holds with convergence of all moments (absolute and ordinary) of orders $\leqslant r$.*

Proof. It will be convenient to use $\tau(t) = N(t)+1$ instead of $N(t)$, since $\tau(t)$ is a stopping time. We thus define, similarly to (2.1),

$$Z''(t) := \frac{Z(T_{\tau(t)}) - aT_{\tau(t)}}{\sqrt{t}} = t^{-1/2} \sum_{i=1}^{\tau(t)} \bigl(\eta_i - a\xi_i\bigr). \tag{3.2}$$

Since $|\eta_n| \leqslant M_n$ by (1.7) and (1.10), we have, using also (2.1) and (2.6),

$$|Z''(t) - Z'(t)| = \left| \frac{\eta_{\tau(t)} - a\xi_{\tau(t)}}{\sqrt{t}} \right| \leqslant \frac{M_{\tau(t)} + |a|\xi_{\tau(t)}}{\sqrt{t}} = \frac{Y_{\tau(t)}}{\sqrt{t}}, \tag{3.3}$$

Thus (2.5) yields

$$\left| \frac{Z(t) - at}{\sqrt{t}} - Z''(t) \right| \leqslant \frac{2Y_{\tau(t)}}{\sqrt{t}} \tag{3.4}$$

and consequently

$$\left| \frac{Z(t) - at}{\sqrt{t}} \right| \leqslant \bigl|Z''(t)\bigr| + \frac{2Y_{\tau(t)}}{\sqrt{t}}. \tag{3.5}$$

Since $Z(T_{\tau(t)}) = V_{\tau(t)}$ by (1.8), the uniform integrability of $\bigl\{|Z''(t)|^r : t \geqslant 1\bigr\}$ follows by [2, Theorem 4.2.3(ii)] applied to $\sum_{i=1}^{\tau(t)} \widehat{\eta}_i$ with $\widehat{\eta}_i := \eta_i - a\xi_i$; note that $\mathbb{E}\,|\eta_1|^r \leqslant \mathbb{E}\,[M_1^r] < \infty$ and thus also $\mathbb{E}\,|\widehat{\eta}_1|^r < \infty$.

Furthermore, recalling $\xi_1 = T_1$, the assumptions and (2.6) yield $\mathbb{E}\,|Y_1|^r < \infty$. The family $\{\tau(t)/t : t \geqslant 1\}$ is uniformly integrable by [2, (2.5.6), see also the more general Theorem 3.7.1], and $\tau(t)$ are stopping times, and thus [2, Theorem 1.8.1] shows that the family $\bigl\{|Y_{\tau(t)}|^r/t : t \geqslant 1\bigr\}$ is uniformly integrable. Since $r \geqslant 2$, this implies that also the family $\bigl\{\bigl(|Y_{\tau(t)}|/\sqrt{t}\bigr)^r : t \geqslant 1\bigr\}$ is uniformly integrable.

The uniform integrability of (3.1) now follows by (3.5). As is well known, this implies moment convergence in (1.12), see e.g. [3, Theorem 5.5.9]. □

Corollary 3.2. *Suppose $Z(t)$ is a stochastic process with regenerative increments over T_n such that $\mathbb{E}[T_1^2]$ and $\mathbb{E}[M_1^2]$ are finite, and* $\mathrm{Var}\,[Z(T_1) - \frac{\mathbb{E}\,Z(T_1)}{\mathbb{E}\,T_1}T_1] > 0$. *Then*

$$\frac{Z(t)-\mathbb{E}\,[Z(t)]}{\sqrt{\mathrm{Var}[Z(t)]}} \xrightarrow{\mathrm{d}} N(0,1), \qquad \text{as } t\to\infty. \tag{3.6}$$

Proof. Note that $\big|Z(T_1)\big| \leqslant M_1$ by (1.11), and thus $\mathbb{E}\,[Z(T_1)^2] < \infty$. It follows that the assumptions of Theorem 1.4 hold, and so do the assumptions of Theorem 3.1 with $r = 2$.

Hence, (1.12) holds, with

$$\mathbb{E}\,[Z(t)] = at + o\big(\sqrt{t}\big), \tag{3.7}$$

$$\mathrm{Var}\,[Z(t)] = \big(\sigma^2/\mu + o(1)\big)\,t. \tag{3.8}$$

The result (3.6) follows from (1.12) and (3.7)–(3.8) by Cramér–Slutsky's theorem. □

4. Law of large numbers

As a complement to the results on asymptotic normality, we give both weak and strong laws of large numbers for processes with regenerative increments. The strong law is given in [4], but repeated here for completeness.

Theorem 4.1. *Suppose $Z(t)$ is a stochastic process with regenerative increments over T_n such that $\mu := \mathbb{E}\,[T_1]$ and $a := \mathbb{E}\,[Z(T_1)]/\mu$ are finite. In addition, let M_1 be defined by* (1.11).

(i) *(Weak LLN.) If $M_1 < \infty$ a.s., then $Z(t)/t \xrightarrow{\mathrm{p}} a$ as $t\to\infty$.*
(ii) *(Strong LLN [4, Theorem 2.54].) If $\mathbb{E}\,M_1 < \infty$, then $Z(t)/t \xrightarrow{\mathrm{a.s.}} a$ as $t\to\infty$.*

Proof. By [2, Theorem 4.2.1], we have the strong law of large numbers

$$\frac{Z(T_{\tau(t)})}{t} = \frac{V_{\tau(t)}}{t} \xrightarrow{\mathrm{a.s.}} \frac{\mathbb{E}\,[Z(T_1)]}{\mathbb{E}\,[T_1]} = a. \tag{4.1}$$

Moreover, $\tau(t)/t \xrightarrow{\mathrm{a.s.}} 1/\mu$ by [2, Theorem 2.5.1(i)], and thus [2, Theorem 1.2.3(i)] (with $r = 1$) shows that

$$\frac{\eta_{\tau(t)}}{t} \xrightarrow{\mathrm{a.s.}} 0. \tag{4.2}$$

Consequently,

$$\frac{Z(T_{N(t)})}{t} = \frac{Z(T_{\tau(t)}) - \eta_{\tau(t)}}{t} \xrightarrow{\mathrm{a.s.}} a. \tag{4.3}$$

Hence, the weak and strong laws in (i) and (ii) are equivalent to, respectively,

$$\frac{Z(t)-Z(T_{N(t)})}{t} \xrightarrow{\mathrm{p}} 0, \tag{4.4}$$

$$\frac{Z(t)-Z(T_{N(t)})}{t} \xrightarrow{\mathrm{a.s.}} 0. \tag{4.5}$$

The proof is completed as follows.

(i): We have, by (1.10) and (1.4),

$$M_{N(t)+1} = \sup_{T_{N(t)}\leqslant s\leqslant T_{N(t)+1}} \big|Z(s)-Z(T_{N(t)})\big| \geqslant \big|Z(t)-Z(T_{N(t)})\big|. \tag{4.6}$$

Lemma 2.2 with $Y_n := M_n$ shows that $M_{N(t)+1}/t \xrightarrow{\text{p}} 0$, and thus (4.4) follows from (4.6).

(ii): Since $N(t)/t \xrightarrow{\text{a.s.}} 1/\mu > 0$ [2, Theorem 2.5.1(i)], (4.5) is equivalent to

$$\frac{Z(t) - Z(T_{N(t)})}{N(t)} \xrightarrow{\text{a.s.}} 0, \qquad \text{as } t \to \infty, \tag{4.7}$$

and thus to

$$\sup_{T_n \leqslant t < T_{n+1}} \frac{|Z(t) - Z(T_n)|}{n} \xrightarrow{\text{a.s.}} 0, \qquad \text{as } n \to \infty. \tag{4.8}$$

Define

$$M'_n := \sup_{T_{n-1} \leqslant t < T_n} |Z(t) - Z(T_{n-1})|, \qquad n \geqslant 1; \tag{4.9}$$

cf. (1.10) and note that

$$M_n = M'_n \vee |Z(T_n) - Z(T_{n-1})|. \tag{4.10}$$

We can write (4.8) as

$$\frac{M'_{n+1}}{n} \xrightarrow{\text{a.s.}} 0, \qquad \text{as } n \to \infty. \tag{4.11}$$

Since the sequence $\{M'_n\}$ is i.i.d., (4.11) is equivalent to

$$\mathbb{E}\, M'_1 < \infty, \tag{4.12}$$

see [3, Proposition 6.1.1]. Furthermore, $\mathbb{E}\,|Z(T_1)| < \infty$ by assumption, and thus (4.10) shows that (4.12) is equivalent to

$$\mathbb{E}\, M_1 = \mathbb{E}\,[M'_1 \vee |Z(T_1)|] < \infty. \tag{4.13}$$

The chain of equivalences above shows that (4.5) is equivalent to (4.13). □

Remark 4.2. The proof shows that, under the assumption that $\mathbb{E}\,[T_1]$ and $\mathbb{E}\,[Z(T_1)]$ are finite, the strong law of large numbers $Z(t)/t \xrightarrow{\text{a.s.}} a$ holds if and only if $\mathbb{E}\,[M_1] < \infty$. The gap between the conditions in (i) and (ii) is thus not an artefact of the proof. △

5. The mean

We add also some further results for the mean. First, we note the estimate (3.7) obtained above when T_1 and M_1 have finite second moments can be improved. In fact, [4, Theorem 2.85] shows the following. (The assumptions in [4] are slightly more general. Also, [4] states only the non-arithmetic case, but the arithmetic case is similar.) For completeness, we give a proof later.

Theorem 5.1 (Essentially [4, Theorem 2.85])**.** *Suppose $Z(t)$ is a stochastic process with regenerative increments over T_n such that $\mathbb{E}\,[T_1^2]$, $\mathbb{E}\,[M_1]$, and $\mathbb{E}\,[T_1 M_1]$ are finite. (In particular, this holds if $\mathbb{E}\,[T_1^2]$ and $\mathbb{E}\,[M_1^2]$ are finite.) Then*

$$\mathbb{E}\,[Z(t)] = at + O(1). \tag{5.1}$$

More precisely,

(i) *If the distribution of T_1 is non-arithmetic, then, as $t \to \infty$,*

$$\mathbb{E}\,[Z(t)] = at + a\frac{\mathbb{E}\,[T_1^2]}{2\mu} + \frac{1}{\mu}\,\mathbb{E}\left[\int_0^{T_1} Z(s)\,\mathrm{d}s - T_1 Z(T_1)\right] + o(1). \tag{5.2}$$

(ii) *If the distribution of T_1 is arithmetic with span d, then*

$$\mathbb{E}\,[Z(t)] = at + a\frac{\mathbb{E}\,[T_1^2]}{2\mu} + \frac{ad}{2} + \frac{1}{\mu}\mathbb{E}\left[d\sum_{k=1}^{T_1/d-1} Z(kd) - T_1 Z(T_1)\right] + o(1)$$

$$= at + a\frac{\mathbb{E}\,[T_1^2]}{2\mu} - \frac{ad}{2} + \frac{1}{\mu}\mathbb{E}\left[d\sum_{k=1}^{T_1/d} Z(kd) - T_1 Z(T_1)\right] + o(1). \tag{5.3}$$

as $t \to \infty$ with $t \in d\mathbb{N}$.

Remark 5.2. In the special case $Z(t) := V_{N(t)}$ for some sequence V_n as in (1.5), i.e., in the case $Z(t) = Z(T_n)$ for $T_n \leqslant t < T_{n+1}$, (5.1) is a special case of [2, Theorem 4.2.4(i) with Remark 4.2.10], and the proof given there shows also (5.2) and (5.3).

The even more special case $Z(t) := N(t)$ is classical, see [2, Theorem 2.5.2] and [4, Proposition 2.84]. △

Under weaker moment assumptions (where asymptotic normality does not necessarily hold), we have the following results. (Proofs are given below.) First, we assume only finite expectations.

Theorem 5.3. *Suppose $Z(t)$ is a stochastic process with regenerative increments over T_n such that $\mathbb{E}\,[T_1]$ and $\mathbb{E}\,[M_1]$ are finite. Then*

$$\mathbb{E}\,[Z(t)] = at + o(t), \qquad \text{as } t \to \infty. \tag{5.4}$$

More generally, we have the following theorem that "interpolates" between Theorems 5.1 and 5.3. Note that the case $r = 1$ is Theorem 5.3 (which we have stated separately for emphasis), and that Theorem 5.1 is a substitute for the excluded case $r = 0$.

Theorem 5.4. *Suppose $Z(t)$ is a stochastic process with regenerative increments over T_n. Let $0 < r \leqslant 1$.*

(i) *If $\mathbb{E}\,[T_1^{2-r}]$, $\mathbb{E}\,[M_1]$, and $\mathbb{E}\,[T_1^{1-r}M_1]$ are finite, then*

$$\mathbb{E}\,[Z(t)] = at + o(t^r), \qquad \text{as } t \to \infty. \tag{5.5}$$

(ii) *In particular, (5.5) holds if there exist $p \geqslant 2 - r$ and $q \geqslant 1$ such that $\mathbb{E}\,[T_1^p]$ and $\mathbb{E}\,[M_1^q]$ are finite, and*

$$p\Bigl(1 - \frac{1}{q}\Bigr) \geqslant 1 - r. \tag{5.6}$$

Remark 5.5. In the special case $Z(t) := V_{N(t)}$ as in Remark 5.2, (5.4) follows from [2, Theorem 4.2.1 with Remark 4.2.10]. Moreover, in this case and for $r \in (\frac{1}{2}, 1)$, under somewhat stronger moment assumptions, (5.5) follows from [2, Theorem 4.2.2 with Remark 4.2.10]. These theorems in [2] yield also estimates for higher moments of $Z(t) - at$; we leave it to the reader to extend those results to general processes with regenerative increments.

The special case $Z(t) = N(t)$ of (5.4) is the elementary renewal theorem. Furthermore, in this special case, (5.5) was proved in [5]. △

Proof of Theorem 5.3. Consider $\widehat{Z}(t) := Z(t) - at$, which also is a process with regenerative increments over T_n; we have $\mathbb{E}\,[\widehat{Z}(T_1)] = \mathbb{E}\,[Z(T_1)] - a\,\mathbb{E}\,[T_1] = 0$ and

$$\widehat{M}_1 := \sup_{0 \leqslant t \leqslant T_1} |\widehat{Z}(t)| \leqslant M_1 + |a|T_1. \tag{5.7}$$

Hence, by replacing $Z(t)$ with $\widehat{Z}(t)$, it follows that we may without loss of generality assume $\mathbb{E}\left[Z(T_1)\right] = 0$ and thus $a = 0$. (We could have done so also in earlier proofs, but we preferred to stay close to [4].)

Thus assume $a = 0$. Recall that $\tau(t)$ is a stopping time, and note that $\mathbb{E}\left[\tau(t)\right] < \infty$ for every $t \geqslant 0$, see e.g. [2, Theorem 2.3.1(ii) or Theorem 2.4.1]. Hence Wald's equation [2, Theorem 1.5.3(i)], [4, Proposition 2.53] applies to $Z(T_{\tau(t)}) = V_{\tau(t)}$, which yields

$$\mathbb{E}\left[Z(T_{\tau(t)})\right] = \mathbb{E}\left[\tau(t)\right] \cdot \mathbb{E}\left[Z(T_1)\right] = 0. \tag{5.8}$$

Furthermore, it follows from (1.4), (1.10), and $\tau(t) = N(t) + 1$ that

$$\left|Z(t) - Z(T_{\tau(t)})\right| \leqslant \left|Z(t) - Z(T_{N(t)})\right| + \left|Z(T_{\tau(t)}) - Z(T_{N(t)})\right| \leqslant 2M_{\tau(t)}. \tag{5.9}$$

Consequently,

$$\left|\mathbb{E}\left[Z(t)\right]\right| = \left|\mathbb{E}\left[Z(t) - Z(T_{\tau(t)})\right]\right| \leqslant \mathbb{E}\left|Z(t) - Z(T_{\tau(t)})\right| \leqslant 2\,\mathbb{E}\, M_{\tau(t)}. \tag{5.10}$$

Moreover, $\tau(t)$ are stopping times, with $\tau(t)/t \to 1/\mu$ a.s. as $t \to \infty$, and the random variables $\{\tau(t)/t : t \geqslant 1\}$ are uniformly integrable, see [2, Theorem 2.5.1 and (2.5.6) (or Theorem 3.7.1)]. Hence, [2, Theorem 1.8.1] shows that the assumption $\mathbb{E}\left[M_1\right] < \infty$ implies

$$\mathbb{E}\left[M_{\tau(t)}\right] = o(t), \tag{5.11}$$

and the result follows from (5.10) □

Proof of Theorem 5.4. We note first that (ii) follows from (i) and Hölder's inequality. In fact, suppose that the assumptions of (ii) hold. If $q > 1$, let q' be the conjugate exponent defined by $1/q' = 1 - 1/q$; then

$$\mathbb{E}\left[T_1^{1-r} M_1\right] \leqslant \mathbb{E}\left[T_1^{q'(1-r)}\right]^{1/q'} \mathbb{E}\left[M_1^q\right]^{1/q} < \infty, \tag{5.12}$$

since (5.6) says $p/q' \geqslant 1 - r$ and thus $p \geqslant q'(1 - r)$. Hence, the assumptions of (i) hold. The case $q = 1$ occurs by (5.6) only for $r = 1$, and then the result immediately follows from (i).

It thus suffices to prove (i). As in the proof of Theorem 5.3, we may assume $a = 0$. Then (5.10) holds, and the result follows from the following lemma. □

Lemma 5.6. *As in Lemma 2.2, let T_n be a sequence of renewal times and let Y_n, $n \geqslant 1$, be another sequence of random variables such that the random vectors (ξ_n, Y_n) are i.i.d. Let $0 < r \leqslant 1$ and assume that $\mathbb{E}\left[T_1\right]$, $\mathbb{E}\left[Y_1\right]$ and $\mathbb{E}\left[T_1^{1-r} Y_1\right]$ are finite. Then*

$$\mathbb{E}\, Y_{\tau(t)} = o(t^r) \qquad \textit{as } t \to \infty. \tag{5.13}$$

Proof. The case $r = 1$ follows by [2, Theorem 1.8.1] as for (5.11) in the proof of Theorem 5.3. Hence, we may assume $0 < r < 1$. We may also assume that $Y_n \geqslant 0$, by otherwise replacing Y_n by $|Y_n|$.

With these assumptions, we argue similarly to the proof of Lemma 2.2. We have

$$\begin{aligned}
\mathbb{E}\left[Y_{\tau(t)}\right] &= \mathbb{E} \sum_{n=0}^{\infty} \mathbf{1}\{T_n \leqslant t < T_{n+1}\} \cdot Y_{n+1} \\
&= \mathbb{E} \sum_{n=0}^{\infty} \mathbb{E}\left[Y_{n+1} \cdot \mathbf{1}\{T_n \leqslant t < T_{n+1}\} \mid T_n\right]
\end{aligned}$$

$$= \mathbb{E} \sum_{n=0}^{\infty} \mathbb{E}\big[Y_{n+1} \cdot \mathbf{1}\{0 \leqslant t - T_n < \xi_{n+1}\} \mid T_n\big]$$

$$= \mathbb{E} \sum_{n=0}^{\infty} h(t - T_n), \tag{5.14}$$

where now

$$h(s) := \mathbf{1}\{s \geqslant 0\} \,\mathbb{E}\big[Y_1 \mathbf{1}\{\xi_1 > s\}\big] = \mathbf{1}\{s \geqslant 0\} \,\mathbb{E}\big[Y_1 \mathbf{1}\{T_1 > s\}\big]. \tag{5.15}$$

Define, for $r, \varepsilon > 0$,

$$h_{r,\varepsilon}(s) := (s^{-r} \wedge \varepsilon) h(s). \tag{5.16}$$

If $t \geqslant t_\varepsilon := \varepsilon^{-1/r}$, then $t^{-r} \leqslant s^{-r} \wedge \varepsilon$ for every $s \in [0, t]$, and thus (5.14) implies

$$t^{-r} \,\mathbb{E}\big[Y_{\tau(t)}\big] = \mathbb{E} \sum_{n=0}^{\infty} t^{-r} h(t - T_n) \leqslant \mathbb{E} \sum_{n=0}^{\infty} h_{r,\varepsilon}(t - T_n)$$

$$= \int_0^{\infty} h_{r,\varepsilon}(t - u) \,\mathrm{d}U(u). \tag{5.17}$$

By (5.15)–(5.16), $h_{r,\varepsilon}(s)$ is decreasing on $[0, \infty)$ with, using Fubini's theorem,

$$\int_0^{\infty} h_{r,\varepsilon}(s) \,\mathrm{d}s \leqslant \int_0^{\infty} s^{-r} h(s) \,\mathrm{d}s = \int_0^{\infty} \mathbb{E}\big[Y_1 \mathbf{1}\{T_1 > s\} s^{-r}\big] \,\mathrm{d}s$$

$$= \mathbb{E} \int_0^{\infty} Y_1 \mathbf{1}\{T_1 > s\} s^{-r} \,\mathrm{d}s = \frac{1}{1-r} \,\mathbb{E}\big[Y_1 T_1^{1-r}\big] < \infty. \tag{5.18}$$

Hence, $h_{r,\varepsilon}(s)$ is directly Riemann integrable, and thus (5.17) and the key renewal theorem yield, in the non-arithmetic case, for every $\varepsilon > 0$,

$$\limsup_{t\to\infty} t^{-r} \,\mathbb{E}\big[Y_{\tau(t)}\big] \leqslant \lim_{t\to\infty} \int_0^{\infty} h_{r,\varepsilon}(t - u) \,\mathrm{d}U(u) = \frac{1}{\mu} \int_0^{\infty} h_{r,\varepsilon}(s) \,\mathrm{d}s =: \lambda_{r,\varepsilon}. \tag{5.19}$$

Moreover, $h_{r,\varepsilon}(s) \to 0$ as $\varepsilon \to 0$ for every fixed s by (5.16), and the inequality $h_{r,\varepsilon}(s) \leqslant s^{-r} h(s)$ together with (5.18) allows us to use the dominated convergence theorem and conclude that

$$\lim_{\varepsilon\to 0} \lambda_{r,\varepsilon} = \frac{1}{\mu} \lim_{\varepsilon\to 0} \int_0^{\infty} h_{r,\varepsilon}(s) \,\mathrm{d}s = 0. \tag{5.20}$$

Hence, (5.13) follows from (5.19).

The case when T_1 is arithmetic is similar; if d is the span of T_1, then it suffices to consider $n \in d\mathbb{N}$, and we then have (5.19) with

$$\lambda_{r,\varepsilon} := \frac{d}{\mu} \sum_{n=0}^{\infty} h_{r,\varepsilon}(nd). \tag{5.21}$$

Again, $\lambda_{r,\varepsilon} \to 0$ as $\varepsilon \to 0$ by dominated convergence, and (5.13) follows. □

The following example shows that the condition (5.6) is best possible.

Example 5.7. Let $\alpha > 1$ and $\beta > 0$. Let the renewal times T_n be given by (1.1) where ξ_i are i.i.d. and have the Pareto distribution with $\mathbb{P}[\xi_n > t] = t^{-\alpha}$ for $t \geqslant 1$. (Thus $\mathbb{E}\, T_1 = \mathbb{E}\, \xi_1 < \infty$, since $\alpha > 1$.) Let $Z(t)$ be the process

$$Z(t) := \xi_{\tau(t)}^{\beta} \mathbf{1}\{t \geqslant T_{N(t)} + 1\}. \tag{5.22}$$

Recall (1.4), and note that $Z(t)$ is càdlàg and that $Z(T_n) = 0$ for all n; consequently $\mathbb{E}[Z(T_1)] = 0$ and $a = 0$. Then, for any $t \geqslant 1$,

$$\begin{aligned}\mathbb{P}[Z(t) > t^\beta] &= \sum_{n=0}^{\infty} \mathbb{P}\bigl[T_n + 1 \leqslant t < T_{n+1},\ \xi_{n+1} > t\bigr] \\ &= \sum_{n=0}^{\infty} \mathbb{P}[T_n + 1 \leqslant t]\, \mathbb{P}[\xi_{n+1} > t] \\ &= \sum_{n=0}^{\infty} \mathbb{P}[T_n \leqslant t-1] t^{-\alpha} \\ &= U(t-1)t^{-\alpha}. \end{aligned} \tag{5.23}$$

Since $U(t)/t \to 1/\mu > 0$ as $t \to \infty$, and $U(t) \geqslant U(0) = 1$, we have $U(t-1) > ct$ for some $c > 0$ and all $t \geqslant 1$. Hence, for $t \geqslant 1$,

$$\mathbb{P}[Z(t) > t^\beta] > ct^{1-\alpha} \tag{5.24}$$

and thus

$$\mathbb{E}[Z(t)] > ct^{1+\beta-\alpha}. \tag{5.25}$$

Consequently, (5.4) does *not* hold for $r := 1 + \beta - \alpha$.

If we are given $p, q \geqslant 1$ and $0 < r < 1$ such that (5.6) does not hold, i.e.,

$$p\Bigl(1 - \frac{1}{q}\Bigr) < 1 - r, \tag{5.26}$$

choose $\beta := p/q$ and $\alpha := \beta + 1 - r$. Note that $\alpha > 1$ since (5.26) yields

$$p - \beta < 1 - r \tag{5.27}$$

and thus

$$\alpha > p \geqslant 1. \tag{5.28}$$

This also implies $\mathbb{E}[T_1^p] < \infty$. Furthermore, $M_1 \leqslant \xi_1^\beta = T_1^\beta$ and thus $\mathbb{E}[M_1^q] = \mathbb{E}[T_1^p] < \infty$. We have seen that (5.4) does not hold. △

Finally, we give a proof of Theorem 5.1, since the statement and proof in [4] do not explicitly include the arithmetic case. (We find it illustrative to include both cases in our proof. The ideas are similar to the proof in [4], although the details differ.)

Proof of Theorem 5.1. First, the two expressions in (5.3) are equal, since their difference is $ad - \frac{1}{\mu}\mathbb{E}[dZ(T_1)] = ad - da = 0$.

Next, note that in the case $Z(t) = at$, simple calculations show that (5.2) and (5.3) hold (without the remainder term). Hence, we may again replace $Z(t)$ by $Z(t) - at$ and thus assume that $a = 0$.

We then have $\mathbb{E}[Z(T_{\tau(t)})] = 0$ by (5.8). We argue as in (5.14) and obtain, recalling that ζ_{n+1} in (1.6) is independent of T_n and noting that absolute convergence holds by (5.9) and (5.14) (with $Y_n := M_n$),

$$\mathbb{E}[Z(t)] = \mathbb{E}\bigl[Z(t) - Z(T_{\tau(t)})\bigr] = \mathbb{E}\sum_{n=0}^{\infty} \mathbf{1}\{T_n \leqslant t < T_{n+1}\} \cdot \bigl(Z(t) - Z(T_{n+1})\bigr)$$

$$
\begin{aligned}
&= \mathbb{E} \sum_{n=0}^{\infty} \mathbb{E}\big[\big(Z(t) - Z(T_{n+1})\big) \cdot \mathbf{1}\{T_n \leqslant t < T_{n+1}\} \mid T_n\big] \\
&= \mathbb{E} \sum_{n=0}^{\infty} \mathbb{E}\big[\big(Z(t) - Z(T_{n+1})\big) \cdot \mathbf{1}\{0 \leqslant t - T_n < \xi_{n+1}\} \mid T_n\big] \\
&= \mathbb{E} \sum_{n=0}^{\infty} g(t - T_n), \qquad (5.29)
\end{aligned}
$$

where

$$
g(s) := \mathbf{1}\{s \geqslant 0\} \, \mathbb{E}\big[\big(Z(s) - Z(T_1)\big)\mathbf{1}\{T_1 > s\}\big]. \qquad (5.30)
$$

We have by (5.9), for $s \geqslant 0$,

$$
|g(s)| \leqslant \mathbb{E}\big[\big(Z(s) - Z(T_1)\big)\mathbf{1}\{T_1 > s\}\big] \leqslant \mathbb{E}\big[2M_1\mathbf{1}\{T_1 > s\}\big] = 2h(s), \qquad (5.31)
$$

where we let $h(s)$ be as in (5.15) with $Y_1 := M_1$. Then $h(s)$ is decreasing on $[0, \infty)$ and

$$
\int_0^{\infty} h(s) \, \mathrm{d}s = \mathbb{E}\,[M_1 T_1] < \infty \qquad (5.32)
$$

by the calculation in (5.18) with $r = 0$, and thus $h(s)$ is directly Riemann integrable. Furthermore, since $Z(s)$ is assumed to be càdlàg, it follows from (5.30), using (5.9) and dominated convergence, that $g(s)$ also is càdlàg, and in particular a.e. continuous. Hence, using (5.31), $g(s)$ too is directly Riemann integrable, see [4, Proposition 2.88(c)]. In the non-arithmetic case, (5.29)–(5.30) and the key renewal theorem now yield, using Fubini's theorem justified by (5.31) and (5.32),

$$
\begin{aligned}
\mathbb{E}\,[Z(t)] \to \frac{1}{\mu}\int_0^{\infty} g(s)\,\mathrm{d}s &= \frac{1}{\mu}\int_0^{\infty} \mathbb{E}\big[\big(Z(s) - Z(T_1)\big)\mathbf{1}\{T_1 > s\}\big]\,\mathrm{d}s \\
&= \frac{1}{\mu}\,\mathbb{E}\left[\int_0^{T_1} Z(s)\,\mathrm{d}s - T_1 Z(T_1)\right]. \qquad (5.33)
\end{aligned}
$$

(This also follows by [4, Theorem 2.45], applied to $X(t) := Z(t) - Z(T_{\tau(t)})$.) In the arithmetic case, with span d, we obtain instead, as $t \to \infty$ with $t \in d\mathbb{N}$,

$$
\begin{aligned}
\mathbb{E}\,[Z(t)] \to \frac{d}{\mu}\sum_{k=0}^{\infty} g(kd) &= \frac{d}{\mu}\sum_{k=0}^{\infty} \mathbb{E}\big[\big(Z(kd) - Z(T_1)\big)\mathbf{1}\{T_1 > kd\}\big] \\
&= \frac{1}{\mu}\,\mathbb{E}\left[d \sum_{k=0}^{T_1/d-1} Z(kd) - T_1 Z(T_1)\right]. \qquad (5.34)
\end{aligned}
$$

Since we have assumed $a = 0$, these results (5.33) and (5.34) show (5.2) and (5.3), which completes the proof. ☐

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