UNCOVERING A GRAPH

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Abstract. Uncover the vertices of a given graph, deterministic or random, in random order; we consider both a discrete-time and a continuous-time version. We study the evolution of the number of visible edges, and show convergence after normalization to a Gaussian process. This problem was studied by Hackl, Panholzer, and Wagner for the case when the graph is a random labelled tree; we generalize their result to more general graphs, including both other classes of random and non-random trees, and denser graphs. The results are similar in all cases, but some differences can be seen depending on the size of the average degree and of the variance of the vertex degrees.

1. Introduction

Let G be a (finite) graph, deterministic or random, and uncover its vertices one by one, in uniformly random order; we say that a vertex becomes *visible* when it is uncovered. This yields a growing sequence of (random) induced subgraphs of G, and we are interested in the evolution of this sequence. In particular, we study in this paper the evolution of the number of edges in these subgraphs, regarded as a stochastic process. More precisely, we consider a sequence of graphs G_n with order $|G_n| = n$, and study the asymptotic behaviour of this stochastic process as $n \to \infty$, under suitable conditions. (See Section 2 for more details, and for definitions of notation used below.) The methods extend to the number of other small subgraphs, see Section 9.

This question (among others) was studied by Hackl, Panholzer and Wagner [12] for the case when G is a random labelled tree. They showed that the stochastic process given by the number of visible edges, after suitable rescaling, converges to a continuous Gaussian process, which resembles a Brownian bridge but with a somewhat different distribution; see Example 4.1. Our main result is that this extends to a wide class of deterministic and random trees and graphs, see Section 3.

Remark 1.1. If G is a random graph with vertex set $[n] = \{1, \dots, n\}$, we may alternatively consider uncovering the vertices in the given order. (Actually, this is the formulation used in [12].) For a random graph with a distribution that is invariant under permutations of the vertices (and in particular for the random tree in [12]), this is obviously equivalent to taking the vertices in random order, and we will for convenience use only the formulation above. △

We will consider two versions of the problem. In the first, the vertices are uncovered at fixed times (as in [12]); in the second, they are uncovered at random times which are independent for different vertices. The two versions are related by a simple

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random change of time. We find it interesting to give results for both versions, and see their similarities and differences under different conditions on the graphs G_n.

The second version means that at every given time t, each vertex is uncovered with some probability $p = p(t)$ independently of all other vertices; in other words, this is site percolation on the graph G, regarded as a stochastic process where p increases from 0 to 1. There is a large literature on site percolation on various finite and infinite graphs, see e.g. Grimmett [10]; much of it concerns global properties, but we do not know any references studying the local properties studied here.

Our method of analysis is based on the second version, with random times. (The method in [12] for random labelled trees is very different, and is based on a remarkable exact formula for a multivariate generating function.) The main part of our proofs are done for the case when the graph G is deterministic and the uncovering times are random. By standard methods we then randomize and get results for random G, and also derandomize and get results for fixed uncovering times.

Our method is a variant of methods used since a long time for the study of Erdős–Rényi random graphs. Recall that Erdős and Rényi in their seminal papers [8; 9] on random graphs considered the sequence of graphs obtained by uncovering the edges of the complete graph K_n in random order; the problem studied here is thus the "dual" vertex analogue (for an arbitrary graph G). As is well known, it is often easier to consider the random time version of the Erdős–Rényi random graph process, where edges are added (or uncovered) at independent, uniform random times. (This process was introduced by Stepanov [30, 31], although there with exponential times.) See further e.g. [23, p. 4]. We will here use a vertex version of a method used for these random graph processes in [15; 16]; the method is based on a martingale limit theorem for continuous-time martingales by Jacod and Shiryaev [14].

Notation and some other preliminaries are given in Section 2. The main theorems are stated in Section 3. A number of examples are given in Section 4, both for their own sake and to illustrate various features of the results. Proofs are given in Sections 5–7; Section 5 contain further preliminaries: Section 6 contains the basic technical work including a decomposition of the continuous-time process using some martingales that are defined and studied there. The proofs are then completed in a rather straightforward manner in Section 7. Section 8 gives, as a corollary, for the case of trees a result on the number of components in the visible subgraph. Section 9 briefly discusses extensions to the number of other small subgraphs.

The appendices give some background results used in the main part of the paper, for which we have not found any references; the results in the appendices are stated in rather general forms for future reference. Appendix A shows how results for the discrete-time version can be obtained from continuous-time result. Appendix B shows results on vertex degrees for some random trees that are used in examples in Section 4.

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2. Notation and preliminaries

2.1. **General (mainly standard) notation.** We denote the size (number of elements) of a (finite) set A by $|A|$.

If H and G are graphs, then $\hom(H, G)$ denotes the number of homomorphisms $H \to G$, i.e., the number of (labelled, not necessarily induced) copies of H in G.

C_n is a cycle with n vertices, P_n is a path with n vertices, $\mathsf{K}_{\ell,m}$ is a complete bipartite graph with $\ell + m$ vertices.

C denotes unspecified constants that may vary from one occurrence to the next.

Unspecified limits are as $n \to \infty$.

We use standard O and o notation; furthermore, $a_n \ll b_n$ means the same as $a_n = o(b_n)$.

We use $\overset{\mathrm{p}}{\longrightarrow}$ for convergence in probability, and $\overset{\mathrm{d}}{\longrightarrow}$ for convergence in distribution of random variables. Moreover, if X_n is a sequence of random variables and a_n is a sequence of positive numbers, then $X_n = o_{\mathrm{p}}(a_n)$ means that $X_n/a_n \overset{\mathrm{p}}{\longrightarrow} 0$ as $n \to \infty$, and $X_n = O_{\mathrm{p}}(a_n)$ means that for every $\varepsilon > 0$, there exists $C < \infty$ such that $\mathbb{P}(|X_n| > Ca_n) < \varepsilon$ for all n. (This is called that X_n/a_n is *bounded in probability* or *tight.*) For stochastic processes $(X_n(t))_{t\in J}$ defined on some interval J, we write $X_n(t) = o^*_{\mathrm{p}}(a_n)$ and $X_n(t) = O^*_{\mathrm{p}}(a_n)$ when $\sup_{t\in J} |X_n(t)| = o_{\mathrm{p}}(a_n)$ and $\sup_{t\in J} |X_n(t)| = O_{\mathrm{p}}(a_n)$, respectively.

$\mathcal{L}(X)$ denotes the distribution of a random variable X.

$W^\circ(t)$ denotes a Brownian bridge, i.e., a continuous Gaussian process on $[0, 1]$ with mean 0 and covariance function

$$\mathrm{Cov}\big(W^\circ(s), W^\circ(t)\big) = s(1 - t), \qquad 0 \leqslant s \leqslant t \leqslant 1. \tag{2.1}$$

For typographical reasons, we write vectors as row vectors. The transpose of v is denoted v'. The covariance matrix $\mathrm{Cov}(X)$ of a random vector X, for simplicity assumed centred, is thus $\mathbb{E}\,(XX')$, and similarly $\mathrm{Cov}(X, Y) = \mathbb{E}\,(XY')$ for two centred random vectors X and Y.

2.2. **Notation for our problem.** Let G be a deterministic or random graph with vertex set $V(G) = [n] = \{1, \dots, n\}$ and edge set $E = E(G)$. (Thus the number of vertices is n and the number of edges is $|E|$.) As usual, we denote (potential) edges by ij, where $i, j \in [n]$ (with $i \neq j$) are the endpoints. We sometimes write $i \sim j$ instead of $ij \in E$.

In the first version (discrete-time) of our problem, we uncover the vertices in uniformly random order as $v_1, \dots, v_n$; we say that vertex v_k becomes visible at time k, and we let $\dot{L}_k$ be the number of edges visible at time k (meaning that both endpoints are visible),[1] i.e.,

$$\dot{L}_k := \big|\big\{(i, j) : 1 \leqslant i < j \leqslant k \text{ and } v_iv_j \in E\big\}\big|, \qquad 0 \leqslant k \leqslant n. \tag{2.2}$$

In the second version (random times), we instead give each vertex i a random time T_i when it becomes visible; we assume that $T_1, \dots, T_n$ are independent and have the uniform distribution $\mathsf{U}(0, 1)$. We let $L(t)$ be the number of edges visible at time t, i.e.,

$$L(t) := \big|\big\{ij \in E : T_i \leqslant t \text{ and } T_j \leqslant t\big\}\big|, \qquad 0 \leqslant t \leqslant 1. \tag{2.3}$$

In the case when G is a random graph, we assume that the random permutation $v_1, \dots, v_n$ and the random times $T_1, \dots, T_n$ are independent of G.

We note that there is a natural coupling of the two versions. The random times $T_1, \dots, T_n$ are a.s. distinct, and we will tacitly assume in the sequel that this is the case. We may then let v_k be the kth vertex that becomes visible; this yields

[1] To distinguish the two versions, we use a dot in our notation $\dot{L}$ for the discrete-time process, and also later for continuous-time limits $\dot{Z}$ of such processes, and other quantities related to $\dot{L}$.

a uniformly random permutation of the vertices as required above. We will always assume that we have coupled the two versions in this way.

To express this coupling in formulas, let

$$N(t) := \sum_{i=1}^{n} \mathbf{1}\{T_i \leqslant t\}, \qquad 0 \leqslant t \leqslant 1, \tag{2.4}$$

i.e., the number of vertices visible at time t. Furthermore, let

$$\tau_k := \inf\{t : N(t) \geqslant k\}, \qquad k = 1, \dots, n, \tag{2.5}$$

i.e., the time when the kth vertex becomes visible. Then

$$\dot{L}_k = L(\tau_k). \tag{2.6}$$

We will do most of the analysis for $L(t)$, and then use (2.6) to derive corresponding results for $\dot{L}_k$.

We introduce some further notation. We denote the degree of vertex i by d_i, and let as usual

$$\Delta := \max_{1 \leqslant i \leqslant n} d_i. \tag{2.7}$$

(Although we will for emphasis also write $\max_i d_i$ sometimes.) We further define

$$\bar{d} := \frac{1}{n} \sum_{i=1}^{n} d_i, \tag{2.8}$$

$$\chi := \frac{1}{n} \sum_{i=1}^{n} d_i^2, \tag{2.9}$$

i.e., the first and second moments of the degree of a randomly chosen vertex in G.

In particular, we note that

$$|E| = \frac{1}{2} \sum_{i=1}^{n} d_i = \frac{n\bar{d}}{2}. \tag{2.10}$$

Our theorems are stated as limits for a sequence $G^{(n)}$ of graphs as above, with $V(G^{(n)}) = [n]$. We then add a superscript $^{(n)}$ to the notation for all variables relating to $G^{(n)}$; however, this may be omitted when it is clear from the context.

2.3. **The Skorohod topology.** We state our main results as convergence of (continuous-time) stochastic processes, defined on $[0, 1]$. In the proofs we will also show auxiliary results with convergence of stochastic processes defined on the half-open interval $[0, 1)$. All our continuous-time stochastic processes will be right-continuous with left limits everywhere; such functions are often called *càdlàg*.

We denote left limits by $f(t-) := \lim_{s \nearrow t} f(s)$, and jump sizes by $\Delta f(t) := f(t) - f(t-)$.

In general, for any interval $J \subseteq \mathbb{R}$, let $D(J)$ be the space of càdlàg functions $f : J \to \mathbb{R}$. We equip $D(J)$, as usual, with the *Skorohod topology*; a general definition is given in [16, §2] but is a bit technical, and for our purposes it suffices to note that the topology is Polish (i.e., can be defined by a separable and complete metric), and that if $f_n, f \in D(J)$ ($n \in \mathbb{N}$) and f is continuous, then $f_n \to f$ in $D(J)$ (i.e., in the Skorohod topology) if and only if $f_n \to f$ uniformly on every compact subset of J. (All limits considered below will be continuous; thus the Skorohod topology can be seen as a substitute for the uniform topology, which is non-separable and has technical problems with measurability, see [4, §18].) See also e.g. [4; 14; 24] for details. (These references treat only $J = [0,1]$ or $J = [0,\infty)$; the latter is equivalent to $[0,1)$ by a change of time).

More generally, we may also define the space $D(J)$ for vector-valued functions; this enables us to talk about joint convergence in $D(J)$ of several processes.

Note that convergence in $D[0,1]$ is substantially stronger than convergence in $D[0,1)$. We will use both. When nothing is said explicitly, we mean convergence in $D[0,1]$.

3. Main results

We state our main results in this section. Proofs are given in Section 7. We use the notation in Section 2; in particular, recall that $E^{(n)} := E(G^{(n)})$. We state the results in three different theorems, with different conditions on the vertex degrees. Actually, the first two theorems (Theorems 3.1 and 3.5) are special cases of the third theorem (Theorem 3.6), but we have chosen to present (and prove) them separately, in order to illustrate different features of the results (and proofs); this also gives slightly simpler statements of the first two theorems.

We begin with the sparse case, with $|E^{(n)}| = O(n)$. This includes the random labelled tree studied in [12]. Moreover, the sparse case is some sense the most interesting case, where (as we will see below) different contributions to the result turn out to be of the same order, and therefore interact.

Theorem 3.1. *Let $G^{(n)}$ be a sequence of deterministic or random graphs with $V(G^{(n)}) = [n]$. Assume also that for some (non-random) constants $d_*, \chi_* \in [0,\infty)$, we have, as $n \to \infty$,*

$$\bar{d}^{(n)} := \frac{1}{n}\sum_{i=1}^{n} d_i^{(n)} = \frac{2|E^{(n)}|}{n} \overset{\mathrm{p}}{\longrightarrow} d_*, \tag{3.1}$$

$$\chi^{(n)} := \frac{1}{n}\sum_{i=1}^{n} \bigl(d_i^{(n)}\bigr)^2 \overset{\mathrm{p}}{\longrightarrow} \chi_*, \tag{3.2}$$

$$n^{-1/2} \max_i d_i^{(n)} \overset{\mathrm{p}}{\longrightarrow} 0. \tag{3.3}$$

(i) *Then, in $D[0,1]$,*

$$n^{-1/2}\bigl(\dot{L}^{(n)}_{\lfloor nt\rfloor} - t^2|E^{(n)}|\bigr) \overset{\mathrm{d}}{\longrightarrow} \dot{Z}(t), \tag{3.4}$$

where $\dot{Z}(t)$ is a continuous Gaussian process on $[0,1]$ with $\mathbb{E}\,\dot{Z}(t) = 0$ and covariance function, for $0 \leqslant s \leqslant t \leqslant 1$,

$$\operatorname{Cov}\bigl(\dot{Z}(s), \dot{Z}(t)\bigr) = \dot{\sigma}(s,t) := \frac{d_*}{2} s^2(1-t)^2 + \gamma_* s^2 t(1-t), \tag{3.5}$$

where $\gamma_* := \chi_* - d_*^2$.

(ii) *Similarly, in $D[0,1]$,*

$$n^{-1/2}\bigl(L^{(n)}(t) - t^2|E^{(n)}|\bigr) \overset{\mathrm{d}}{\longrightarrow} Z(t), \tag{3.6}$$

where $Z(t)$ is a continuous Gaussian process on $[0,1]$ with $\mathbb{E}\,Z(t) = 0$ and covariance function, for $0 \leqslant s \leqslant t \leqslant 1$,

$$\operatorname{Cov}\bigl(Z(s), Z(t)\bigr) = \sigma(s,t) := \frac{d_*}{2} s^2(1-t)^2 + \chi_* s^2 t(1-t). \tag{3.7}$$

The condition (3.3) on the maximum degree is necessary, see Example 4.9.

Remark 3.2. Note that the left-hand sides of (3.1) and (3.2) are the first and second moments of the degree distribution in $G^{(n)}$; thus our assumptions say that these moments are asymptotically d_* and χ_*, respectively; as a consequence the constant γ_* in (3.5) is the asymptotic variance of the degree distribution. △

A special case is when $G^{(n)}$ is a tree (as in [12]). Then $|E^{(n)}| = n-1$, and thus (3.1) always holds with $d_* = 2$, so we only have to verify (3.2) and (3.3); we state this as a corollary.

Corollary 3.3. *Let $G^{(n)}$ be a sequence of deterministic or random trees with $V(G^{(n)}) = [n]$. Assume also that* (3.2) *and* (3.3) *hold. Then* (3.4)–(3.5) *and* (3.6)–(3.7) *hold, with $d_* = 2$, $\gamma_* = \chi_* - 4$, and $|E^{(n)}| = n-1$ (which may be replaced by n).*

Remark 3.4. We see from (3.5), and in more detail from the proof in Section 7, that the limit process $\dot{Z}(t)$ in (3.4) can be regarded as consisting of two components: the second term in $\dot{\sigma}(s,t)$ comes from the randomness of the degrees of the vertices that are visible and first term comes from additional randomness in the structure of the visible subgraph. In continuous time, the last term in $\sigma(s,t)$ in (3.7) includes also a term $d_*^2 s^2 t(1-t)$ coming from the randomness of the number of visible vertices, which contributes a third component to the limit. More precisely, the proofs show that for finite n, we can decompose the processes $\dot{L}^{(n)}_{\lfloor nt \rfloor} - t^2|E^{(n)}|$ and $L^{(n)}(t) - t^2|E^{(n)}|$ into two or three components (+ smaller error terms) with the origins just described. Note that in the sparse case these three contributions to the processes are of the same order, unlike in other cases discussed below; this makes the sparse case more complicated, and therefore is a sense more interesting, than more dense cases. △

We next consider regular graphs. The statement below includes both the sparse case and denser cases. We state the regular case separately, since the case of regular graphs is special, and somewhat simpler than others, because there is no randomness in the degrees of the visible vertices, and thus one of the three contributions discussed in Remark 3.4 disappears. The sparse case (with the degree $d^{(n)}$ bounded, which is essentially equivalent to a constant degree $d^{(n)}$) is a special case of Theorem 3.1, but the conclusions are written in a somewhat different (but equivalent) form. In denser cases, with $d^{(n)} \to \infty$, note that the normalizing factors in (3.8) and (3.10) are of different orders. This is because in the continuous-time version, the third contribution discussed in Remark 3.4 (which does not appear for the discrete-time version) is of larger order than the others, and thus dominates the limit. This also means that in the dense case, the limit for the continuous-time version (Theorem 3.5(iii)) is rather uninteresting and determined solely by the number of visible vertices.

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Theorem 3.5. *Let $G^{(n)}$ be a sequence of deterministic or random graphs with $V(G^{(n)}) = [n]$. Assume also that each $G^{(n)}$ is regular, with (non-random) degree $d^{(n)} \geqslant 1$, and that $d^{(n)} = o(n)$.*

(i) *Then, in $D[0,1]$,*

$$(nd^{(n)})^{-1/2}\bigl(\dot{L}^{(n)}_{\lfloor nt \rfloor} - t^2|E^{(n)}|\bigr) \overset{\mathrm{d}}{\longrightarrow} \dot{Z}(t), \tag{3.8}$$

where $\dot{Z}(t)$ *is a continuous Gaussian process on* $[0,1]$ *with* $\mathbb{E}\,\dot{Z}(t) = 0$ *and covariance function, for* $0 \leqslant s \leqslant t \leqslant 1$,

$$\operatorname{Cov}\bigl(\dot{Z}(s), \dot{Z}(t)\bigr) = \dot{\sigma}(s,t) := \frac{1}{2}s^2(1-t)^2. \tag{3.9}$$

(ii) *If furthermore* $d^{(n)} \to d_* \leqslant \infty$, *then, in* $D[0,1]$,

$$(n^{1/2}d^{(n)})^{-1}\bigl(L^{(n)}(t) - t^2|E^{(n)}|\bigr) \xrightarrow{\mathrm{d}} Z(t), \tag{3.10}$$

where $Z(t)$ *is a continuous Gaussian process on* $[0,1]$ *with* $\mathbb{E}\,Z(t) = 0$ *and covariance function, for* $0 \leqslant s \leqslant t \leqslant 1$,

$$\operatorname{Cov}\bigl(Z(s), Z(t)\bigr) = \sigma(s,t) := \frac{1}{2d_*}s^2(1-t)^2 + s^2t(1-t). \tag{3.11}$$

(iii) *In particular, if* $d^{(n)} \to \infty$, *then* (3.10) *holds with* $Z(t) = tW^\circ(t)$ *for a Brownian bridge* $W^\circ(t)$, *and thus, for* $0 \leqslant s \leqslant t \leqslant 1$,

$$\operatorname{Cov}\bigl(Z(s), Z(t)\bigr) = s^2t(1-t). \tag{3.12}$$

The condition $d^{(n)} = o(n)$ is necessary in Theorem 3.5, at least for part (i), see Example 4.9; see also Example 4.11 where this condition is violated and a non-normal limit appears.

Finally, we give a more general version. It is easily seen that Theorems 3.1 and 3.5 are special cases, with $\beta_n = n^{1/2}$ and $\beta_n = (nd^{(n)})^{1/2}$, respectively. We see also that the sizes of the first two contributions discussed in Remark 3.4 are governed by λ_1 and λ_2 in (3.13)–(3.14); any of these may vanish (see Example 4.8), and then only the other contributes to the limit for the discrete-time version. Similarly, for the continuous-time version, the third contribution is governed by α. We will see in Example 4.8 that more or less arbitrary combinations of λ_1, λ_2, and α may occur. (However, see Remark 3.7 below.) Hence, different combinations of the three components discussed in Remark 3.4 may dominate in different examples. In particular, in dense cases, for the continuous-time version we typically have $\alpha = \infty$, and then (Theorem 3.6(ii)b) we have, as in the regular case, a rather uninteresting limit determined by the number of visible vertices, which dominates the contributions coming from the structure of the visible subgraph. Here, however, the condition for this is a little more complicated.

Theorem 3.6. *Let* $G^{(n)}$ *be a sequence of deterministic or random graphs with* $V(G^{(n)}) = [n]$, *and let* β_n *be a sequence of positive constants with* $\beta_n = o(n)$. *Assume also that for some (non-random) constants* $\lambda_1, \lambda_2 \in [0,\infty)$, *we have, as* $n \to \infty$,

$$\frac{2|E^{(n)}|}{\beta_n^2} = \frac{n\bar{d}^{(n)}}{\beta_n^2} \xrightarrow{\mathrm{p}} \lambda_1, \tag{3.13}$$

$$\frac{1}{\beta_n^2}\sum_{i=1}^{n}\bigl(d_i^{(n)} - \bar{d}^{(n)}\bigr)^2 \xrightarrow{\mathrm{p}} \lambda_2, \tag{3.14}$$

$$\beta_n^{-1}\max_i d_i^{(n)} \xrightarrow{\mathrm{p}} 0. \tag{3.15}$$

(i) *Then, in* $D[0,1]$,

$$\beta_n^{-1}\bigl(\dot{L}^{(n)}_{\lfloor nt\rfloor} - t^2|E^{(n)}|\bigr) \xrightarrow{\mathrm{d}} \dot{Z}(t), \tag{3.16}$$

where $\dot Z(t)$ *is a continuous Gaussian process on* $[0,1]$ *with* $\mathbb{E}\,\dot Z(t) = 0$ *and covariance function, for* $0 \leqslant s \leqslant t \leqslant 1$,

$$\operatorname{Cov}\bigl(\dot Z(s), \dot Z(t)\bigr) = \dot\sigma(s,t) := \frac{\lambda_1}{2}s^2(1-t)^2 + \lambda_2 s^2 t(1-t). \tag{3.17}$$

(ii) *Suppose further that, for some non-random constant* $\alpha \in [0,\infty]$,

$$n^{1/2}\bar d^{(n)}/\beta_n \xrightarrow{\text{p}} \alpha. \tag{3.18}$$

(a) *If* $0 \leqslant \alpha < \infty$, *then, in* $D[0,1]$,

$$\beta_n^{-1}\bigl(L^{(n)}(t) - t^2|E^{(n)}|\bigr) \xrightarrow{\text{d}} Z(t), \tag{3.19}$$

where $Z(t)$ *is a continuous Gaussian process on* $[0,1]$ *with* $\mathbb{E}\,Z(t) = 0$ *and covariance function, for* $0 \leqslant s \leqslant t \leqslant 1$,

$$\operatorname{Cov}\bigl(Z(s), Z(t)\bigr) = \sigma(s,t) := \frac{\lambda_1}{2}s^2(1-t)^2 + (\lambda_2+\alpha^2)s^2 t(1-t). \tag{3.20}$$

(b) *If* $\alpha = \infty$, *then, in* $D[0,1]$,

$$(n^{1/2}\bar d^{(n)})^{-1}\bigl(L^{(n)}(t) - t^2|E^{(n)}|\bigr) \xrightarrow{\text{d}} Z(t), \tag{3.21}$$

where $Z(t) = tW^\circ(t)$ *for a Brownian bridge* $W^\circ(t)$, *and thus, for* $0 \leqslant s \leqslant t \leqslant 1$,

$$\operatorname{Cov}\bigl(Z(s), Z(t)\bigr) = \sigma(s,t) := s^2 t(1-t). \tag{3.22}$$

The condition (3.15) on the maximum degree is necessary for Theorem 3.6(i) and (ii)a, see Example 4.9.

Remark 3.7. The conditions (3.13) and (3.18) imply

$$\bar d^{(n)} \xrightarrow{\text{p}} \alpha^2/\lambda_1 \tag{3.23}$$

unless $\alpha = \lambda_1 = 0$. Hence, we can have $\lambda_1 > 0$ and $\alpha = 0$ only when $\bar d^{(n)} \xrightarrow{\text{p}} 0$, which is a rather extreme (and perhaps less interesting) case when necessarily most vertices are isolated.

Furthermore, the condition (3.14) may also be written

$$\frac{1}{\beta_n^2}\Bigl(\sum_{i=1}^n (d_i^{(n)})^2 - n(\bar d^{(n)})^2\Bigr) \xrightarrow{\text{p}} \lambda_2. \tag{3.24}$$

In particular, when $\bar d^{(n)} \xrightarrow{\text{p}} 0$, and assuming (3.13), (3.14) is equivalent to

$$\frac{1}{\beta_n^2}\sum_{i=1}^n (d_i^{(n)})^2 \xrightarrow{\text{p}} \lambda_2. \tag{3.25}$$

It follows that in the case $\bar d^{(n)} \xrightarrow{\text{p}} 0$, we must have $\lambda_2 \geqslant \lambda_1$. In particular, if $\lambda_1 > 0$ and $\alpha = 0$, then $\lambda_2 \geqslant \lambda_1 > 0$. △

Remark 3.8. The limit processes $\dot Z(t)$ and $Z(t)$ in the theorems above all are centred Gaussian, with covariance functions of the type $\operatorname{Cov}(Z(s), Z(t)) = s^2\varphi(t)$ $(0 \leqslant s \leqslant t \leqslant 1)$, where $\varphi(t) = a(1-t)^2 + bt(1-t)$ for some $a, b \geqslant 0$. As noted in a special case in [12], this means that they can be represented as

$$\varphi(t)W\bigl(t^2/\varphi(t)\bigr), \qquad 0 \leqslant t < 1, \tag{3.26}$$

where $W(t)$ is a standard Brownian motion (Wiener process). △

Remark 3.9. For the discrete-time version $\dot{L}_k$, we thus extend it to continuous time by considering $\dot{L}_{\lfloor t\rfloor}$, $t \in [0,n]$; we then scale time to $[0,1]$ and consider $\dot{L}_{\lfloor nt\rfloor}$ in our theorems. As is well known from many other problems, an alternative would be to extend $\dot{L}_k$ to $[0,n]$ by linear interpolation to a continuous process $\dot{L}_t$ as in [12]; it follows immediately that the limit results above hold also if we replace $\dot{L}_{\lfloor nt\rfloor}$ by $\dot{L}_{nt}$; moreover, then the results could be stated as convergence (after rescaling) of $\dot{L}_{nt}$ in the space $C[0,1]$ of continuous functions on $[0,1]$, since for continuous processes, convergence in $C[0,1]$ is equivalent to the convergence in $D[0,1]$ considered in the present paper. (We have chosen to use $D[0,1]$ and the formulations above with $\dot{L}_{\lfloor nt\rfloor}$, at least partly because it is convenient to use discontinuous processes in our proofs.) △

4. Examples

In our first examples, $G^{(n)}$ is a tree, so we can use Corollary 3.3.

Example 4.1. Let us first revisit the case studied by Hackl, Panholzer and Wagner [12], where $G^{(n)}$ is a random labelled tree. It is well known that the asymptotic degree distribution is $1+\mathrm{Po}(1)$, and it is easily seen that also all moments converge (given $G^{(n)}$, in probability), see Remark B.4. In particular, (3.2) holds with $\chi_* = \mathbb{E}\,(\xi+1)^2 = 5$, and (3.3) holds as a consequence of the convergence of the third moment of the degree distribution, or by the argument in Section B.2. (In fact, more strongly, $\Delta^{(n)} = o_{\mathrm{p}}(\log n)$, and a very precise result is known, see [26] and [7, Remark 3.14].) See also Example 4.2 for a generalization. Consequently, Corollary 3.3 shows that (3.4)–(3.7) hold, with $\gamma_* = \chi_* - 4 = 1$; thus the limits $Z(t)$ and $\dot{Z}(t)$ are centred Gaussian processes with covariance functions

$$\operatorname{Cov}\bigl(\dot{Z}(s),\dot{Z}(t)\bigr) = s^2(1-t)^2 + s^2t(1-t) = s^2(1-t), \tag{4.1}$$

$$\operatorname{Cov}\bigl(Z(s),Z(t)\bigr) = s^2(1-t)^2 + 5s^2t(1-t). \tag{4.2}$$

The limit $\dot{Z}(t)$ with covariance function (4.1) was found by [12, Theorem 3], which inspired the present work. △

Example 4.2. More generally, let $G^{(n)}$ be a conditioned Galton–Watson tree with n vertices, defined by conditioning a Galton–Watson tree $\mathcal{T}$ with offspring distributed as a random variable ξ with values in $\{0,1,\dots\}$, see Appendix B.2 and e.g. [20].

Assume that $\mathbb{E}\,\xi = 1$ and $0 < \operatorname{Var}\xi < \infty$. Then the asymptotic outdegree distribution is given by ξ [20, Theorem 7.11], and the asymptotic degree distribution is thus $1+\xi$. We will verify in Appendix B.2 that (3.2) and (3.3) hold, with

$$\chi_* = \mathbb{E}\,(\xi+1)^2 = \operatorname{Var}\xi + 4, \tag{4.3}$$

Corollary 3.3 thus applies and yields (3.4)–(3.7), with

$$\gamma_* = \chi_* - 4 = \operatorname{Var}\xi. \tag{4.4}$$

Some well known examples of conditioned Galton–Watson trees are

(i) The random labelled tree in Example 4.1, with $\xi \sim \mathrm{Po}(1)$ and $\gamma_* = 1$.
(ii) The random binary tree, with $\xi \sim \mathrm{Bin}(2,\frac12)$ and $\gamma_* = \frac12$.
(iii) The random ordered (plane) tree, with $\xi \sim \mathrm{Ge}(1/2)$ and $\gamma_* = 2$.

See e.g. [2], [6], [7], and [20, Section 10], where also further examples are given, △

Example 4.3. Let $G^{(n)}$ be a random binary search tree. All outdegrees are 0, 1, or 2, and thus all degrees are 1, 2, or 3. The proportion of vertices of each type tends in probability to 1/3, see e.g. [5, Theorem 2], [3, Section 3.3], and [13, Example 6.2]. Since there is only a finite number of possible vertex degrees, this implies immediately that (3.2) holds with $\chi_* = (1+4+9)/3 = 14/3$. Furthermore, (3.3) is trivial. Consequently, Corollary 3.3 shows that (3.4)–(3.7) hold, with $\gamma_* = \chi_* - 4 = 2/3$. △

Example 4.4. Let $G^{(n)}$ be a random recursive tree. The asymptotic outdegree distribution is geometric Ge(1/2), just as for random ordered trees in Example 4.2, see e.g. [25], [3, Section 3.2], [18, Theorem 1], [7, Theorem 6.8], and [13, Example 6.1]. Furthermore, if $\xi \in \mathrm{Ge}(1/2)$, then (3.2) and (3.3) hold, with $\chi_* = \mathbb{E}\,(\xi+1)^2 = \operatorname{Var}\xi + 4 = 6$; see Appendix B.3 for a detailed verification. (In fact, $\Delta^{(n)} = O_{\mathrm{p}}(\log n)$, see [7, Theorem 6.12] for a precise result.) Consequently, Corollary 3.3 applies, with $\gamma_* = \operatorname{Var}\xi = 2$, and yields (3.4)–(3.7) with exactly the same limits as for the random ordered tree in Example 4.2. (This coincidence is thus because the two types of random trees have the same asymptotic degree distribution. In other respects, the trees are quite different.) △

Example 4.5. Let $G^{(n)} = \mathsf{P}_n$, a path of length n. (Thus $G^{(n)}$ is non-random.) This case was studied in [19], in the analysis of a problem by [1] which we briefly discuss in Section 8. It is obvious that (3.1)–(3.3) hold, with $d_* = 2$ and $\chi_* = 4$. Hence, Theorem 3.1 (or Corollary 3.3) applies and shows (3.4)–(3.7), with $\gamma_* = \chi_* - d_*^2 = 0$.

Alternatively, let $G^{(n)} = \mathsf{C}_n$, a cycle of length n. This differs from P_n only in a single edge, and thus $\dot{L}_k$ and $L(t)$ differ by at most 1 between the two graphs; consequently, we have the same limit results for P_n and C_n. Indeed, Theorem 3.5 applies to C_n with $d_* = 2$, which gives the same results as just obtained for P_n (although written with somewhat different normalizations). △

Example 4.6. Let $G^{(n)}$ be the random graph $G(n, m_n)$, where m_n are given with $1 \ll m_n \ll \binom{n}{2}$. In this case, our problem is essentially trivial, since by definition, $G(n, m_n)$ has exactly m_n edges, which form a uniformly random subset of size m_n in the set of all $\binom{n}{2}$ possible edges. By symmetry, we may as well uncover the vertices in order $1, 2, \dots$, and then $\dot{L}_k^{(n)}$ is the number of edges seen in the first $\binom{k}{2}$ possible positions (in the order the possible edge positions are uncovered). This number thus has a hypergeometric distribution. Moreover, the functional limit theorem for sampling from a finite population [4, Theorem 24.1] implies easily that, with $p_n := m_n/\binom{n}{2} \to 0$, in $D[0,1]$,

$$\frac{1}{\sqrt{\binom{n}{2} p_n (1-p_n)}} \Bigl(\dot{L}_{\lfloor nt \rfloor}^{(n)} - \frac{\lfloor nt \rfloor (\lfloor nt \rfloor - 1)}{n(n-1)} m_n \Bigr) \xrightarrow{\mathrm{d}} W^{\circ}(t^2) \tag{4.5}$$

and thus

$$\frac{1}{\sqrt{m_n}} \bigl(\dot{L}_{\lfloor nt \rfloor}^{(n)} - t^2 m_n \bigr) \xrightarrow{\mathrm{d}} W^{\circ}(t^2). \tag{4.6}$$

The limit process is thus a time-changed Brownian bridge.

As an illustration of our results, we show how this also follows from the theorems above, more precisely Theorem 3.6. (If $m_n = \Theta(n)$, we may also use Theorem 3.1.) We take $\beta_n := \sqrt{m_n}$. We have $\bar{d}^{(n)} = 2m_n/n$, and thus (3.13) holds trivially with $\lambda_1 = 2$. We see also that (3.18) holds with $\alpha = 2\lambda^{1/2}$ if $m_n/n \to \lambda \in [0, \infty]$. We claim that (3.14) holds with $\lambda_2 = 2$, and that (3.15) holds. Then Theorem 3.6 applies, and (3.17) yields

$$\dot\sigma(s,t) = s^2(1-t)^2 + 2s^2t(1-t) = s^2(1-t^2), \qquad 0 \leqslant s \leqslant t \leqslant 1, \tag{4.7}$$

which shows that $\dot Z(t) \overset{\mathrm{d}}{=} W^\circ(t^2)$ (as processes). Hence, (3.16) yields (4.6). The corresponding result for $L^{(n)}(t)$ is given by (3.19) or (3.21), depending on α.

To verify the claims, let I_{ij} be the indicator that there is an edge ij in $G^{(n)}$. Then

$$\sum_{i=1}^n d_i^2 = \sum_{i=1}^n d_i + \sum_{i=1}^n d_i(d_i-1) = 2m_n + \sum_{i,j,k}^{*} I_{ij}I_{ik}, \tag{4.8}$$

where $\sum^*$ denotes the sum over distinct i,j,k. With our choice $\beta_n = \sqrt{m_n}$, the condition (3.14), in the form (3.24), with $\lambda_2 = 2$ is thus equivalent to

$$m_n^{-1}\sum_{i,j,k}^{*} I_{ij}I_{ik} - \frac{n}{m_n}(\bar d^{(n)})^2 = m_n^{-1}\sum_{i,j,k}^{*} I_{ij}I_{ik} - \frac{4m_n}{n} \overset{\mathrm{p}}{\longrightarrow} 0. \tag{4.9}$$

Denote the triple sum in (4.8) and (4.9) by V. Note that V is twice the number of copies of P_3 in $G(n,m_n)$; the results we need are thus closely related to result on subgraph counts (for the special case P_3) in $G(n,m_n)$, and it seems possible (at least for some ranges of m_n) to derive what we need from known (and more advanced) such results, see e.g. [9, Theorems 2a–2b], [28] (for $G(n,p)$) and [16, Theorem 19], but we find it easier to show it directly by calculating moments. We have (writing $m = m_n$)

$$\begin{aligned}\mathbb{E}\,V &= n(n-1)(n-2)\,\mathbb{P}(I_{12}I_{13}=1) = n(n-1)(n-2)\frac{m}{\binom{n}{2}}\frac{m-1}{\binom{n}{2}-1}\\ &= \frac{4m(m-1)}{n+1} = \frac{4m^2}{n} + o(m).\end{aligned} \tag{4.10}$$

A straightforward calculation, which we omit, shows that $\operatorname{Var} V = o(m^2)$. Hence,

$$V = \mathbb{E}\,V + o_{\mathrm{p}}(m) = \frac{4m^2}{n} + o_{\mathrm{p}}(m), \tag{4.11}$$

which shows (4.9) and thus (3.14). Finally, a similar calculation yields

$$\mathbb{E}\sum_{i=1}^n\Bigl(d_i(d_i-1) - \frac{\mathbb{E}\,V}{n}\Bigr)^2 = \sum_{i=1}^n \operatorname{Var}\Bigl(\sum_{j,k}^{*} I_{ij}I_{ik}\Bigr) = o(m^2). \tag{4.12}$$

Hence, using also (4.10) and our assumption $m \ll \binom{n}{2}$,

$$\Delta(\Delta-1) \leqslant \frac{\mathbb{E}\,V}{n} + \biggl(\sum_{i=1}^n\Bigl(d_i(d_i-1) - \frac{\mathbb{E}\,V}{n}\Bigr)^2\biggr)^{1/2} = o(m) + o_{\mathrm{p}}(m), \tag{4.13}$$

which yields $\Delta = o_{\mathrm{p}}(m^{1/2})$ and thus (3.15), completing the verification that Theorem 3.6 applies and yields (4.6). △

Example 4.7. Let $G^{(n)}$ be the random graph $G(n,p_n)$, where $p_n \in (0,1)$ and we assume $n^2p_n \to \infty$ and $p_n \to 0$. Thus, each possible edge appears with probability p_n, independently of each other.

This example is, as the closely related preceding example, essentially trivial. When we have uncovered k vertices, the visible subgraph is a random graph $G(k,p_n)$.

Hence, $\dot{L}_k^{(n)} \sim \operatorname{Bin}\bigl(\binom{k}{2}, p_n\bigr)$, and, under our conditions on p_n, it follows from a version of Donsker's theorem for triangular arrays (easily proved by Proposition 5.4 below) that

$$\bigl(\tbinom{n}{2}p_n\bigr)^{-1/2}\Bigl(\dot{L}_{\lfloor nt\rfloor}^{(n)} - t^2\binom{n}{2}p_n\Bigr) \overset{\mathrm{d}}{\longrightarrow} W(t^2) \qquad \text{in } D[0,1], \tag{4.14}$$

where $W(t)$ is a Brownian motion on $[0,1]$. Since $E^{(n)} = \dot{L}_n^{(n)}$, it follows that

$$\bigl(\tbinom{n}{2}p_n\bigr)^{-1/2}\Bigl(\dot{L}_{\lfloor nt\rfloor}^{(n)} - t^2|E^{(n)}|\Bigr) \overset{\mathrm{d}}{\longrightarrow} W(t^2) - t^2W(1) = W^\circ(t^2). \tag{4.15}$$

This follows also from (4.6) by conditioning on $|E^{(n)}|$.

As in Example 4.6, we can also see this as an example of Theorem 3.6. If we choose $\beta_n := \bigl(\binom{n}{2}p_n\bigr)^{1/2}$, then (3.13)–(3.15) hold with $\lambda_1 = \lambda_2 = 2$; this follows from the same result for $G(n, m_n)$ in Example 4.6 by conditioning on $|E^{(n)}|$, or by similar calculations in $G(n, p_n)$. Hence, Theorem 3.6 yields (4.15). In the special case $p_n = \lambda/n$, with $\lambda \in (0,\infty)$, we may also use Theorem 3.1. △

Example 4.8. We may construct different examples of Theorem 3.6 by choosing vertex degrees $d_1^{(n)}, \dots, d_n^{(n)}$ and then, for example, taking $G^{(n)}$ to be a random graph with the given degrees. (As is well known, $G^{(n)}$ can be constructed by the configuration model, conditioned to be simple. Of course, we have to choose the degrees such that a simple graph $G^{(n)}$ exists; in particular we need $\sum_i d_i^{(n)}$ to be even.)

Note that examples of Theorem 3.6 with $\lambda_1 > 0$, $\lambda_2 = 0$, and either $0 < \alpha < \infty$ or $\alpha = \infty$ (see (3.23)) are provided by regular graphs as in Theorem 3.5, and that examples with $\lambda_1 > 0$, $\lambda_2 > 0$, and $0 < \alpha < \infty$ are provided by many instances of Theorem 3.1, see the examples above.

Another interesting case is to choose the degrees such that the average degree is roughly constant, more precisely $\bar{d}^{(n)} \to d_*$ for some $d_* \in (0,\infty)$, but the variance of the degrees

$$\gamma^{(n)} := \frac{1}{n}\sum_{i=1}^n \bigl(d_i^{(n)} - \bar{d}^{(n)}\bigr)^2 = \chi^{(n)} - (\bar{d}^{(n)})^2 \to \infty. \tag{4.16}$$

For example, let $\delta_n \to 0$ with $n\delta_n \to \infty$, and let all degrees $d_i^{(n)}$ be 2 except the first $n\delta_n + O(1)$ which are $\lfloor 1/\delta_n \rfloor$. Then, choosing $\beta_n := \sqrt{n\gamma^{(n)}}$, Theorem 3.6 holds with $\lambda_1 = 0$, $\lambda_2 = 1$, and $\alpha = 0$. Hence, in this case the second component in Remark 3.4 dominates both the others. As a result, we have the same covariance in (3.17) and (3.20), and thus the same limit in (3.16) and (3.19).

We may also construct an example with $\lambda_1 > 0$ and $\alpha = 0$, which by Remark 3.7 implies $\bar{d}^{(n)} \overset{\mathrm{p}}{\longrightarrow} 0$ and $\lambda_2 \geqslant \lambda_1 > 0$. We may simply let G_n by the cycle C_{m_n} plus $n - m_n$ isolated vertices, where $m_n \to \infty$ with $m_n = o(n)$, and take $\beta_n := \sqrt{m_n}$; we omit the simple verifications.

Finally, for given sequences a_n and b_n of positive integers with $b_n \leqslant a_n$, let n be even and let $n/2$ vertices have degree $a_n + b_n$ and the other $n/2$ degree $a_n - b_n$. Choose $\beta_n := \sqrt{n}b_n$; thus (3.14) holds with $\lambda_2 = 1$. Let b_n be any sequence such that $1 \ll b_n \ll \sqrt{n}$; it is the easy to verify that in the following three cases, the conditions (3.13)–(3.15) and (3.18) hold with the stated parameters:

(i) If $a_n = b_n^2$, then $\lambda_1 = 1$, $\lambda_2 = 1$, and $\alpha = \infty$.

(ii) If $a_n = b_n$, then $\lambda_1 = 0$, $\lambda_2 = 1$, and $\alpha = 1$.
(iii) If $b_n \ll a_n \ll b_n^2$, then $\lambda_1 = 0$, $\lambda_2 = 1$, and $\alpha = \infty$.

△

Example 4.9. It is easy to see that condition (3.3) on the maximum degree is necessary in Theorem 3.1. In fact, suppose that for some fixed $\delta > 0$, there is a vertex, say $i = 1$, with $d_1 \geqslant \delta n^{1/2}$. Since vertex 1 and its neighbours become visible in uniformly random order, $\Delta L(T_1)$ is uniformly distributed on $\{0, \dots, d_1\}$; hence, with probability $\geqslant 1/2$, we have $\Delta L(T_1) \geqslant d_1/2 \geqslant (\delta/2)n^{1/2}$. Consequently, $n^{-1/2}L(t)$ has with large probability a macroscopic jump (at T_1), and thus it cannot converge in distribution to a continuous process. The same applies to $n^{-1/2}\dot{L}_k$.

The same argument shows that $d^{(n)} = o(n)$ is necessary for (3.8) in Theorem 3.5, and that (3.15) is necessary for (3.16) and (3.19) in Theorem 3.6. △

Remark 4.10. In spite of Example 4.9, it is for some graphs $G^{(n)}$ possible to obtain a continuous limit in (3.8) or (3.16) even if $d^{(n)} = o(n)$ or $\Delta^{(n)} = o(\beta_n)$ fails, provided we use linear interpolation of $\dot{L}_k$ as in Remark 3.9 instead of $\dot{L}_{\lfloor nt \rfloor}$. We give one example in Example 4.11, and note that the limit obtained there is *not* Gaussian. We have not investigated this possibility any further, but it seems that it reguires $G^{(n)}$ to be regular or almost regular (in a suitable sense); moreover, we conjecture that limits always will be as in Example 4.11, and thus not normal. △

Example 4.11. Let $G^{(n)} = \mathsf{K}_{n/2,n/2}$, the symmetric complete bipartite graph, where we assume that n is even. Then $G^{(n)}$ is regular with degree $d^{(n)} = n/2$. Note that this example does *not* satisfy the condition $d^{(n)} = o(n)$ in Theorem 3.5. We will show that, indeed, Theorem 3.6 does not hold, and that we in this example have non-normal limit distributions.

Colour the vertices of the two parts white and black, respectively; each edge has thus one white and one black endpoint.

Let $\dot{W}_k$ be the number of uncovered white vertices among the first k uncovered vertices, and let $\dot{Y}_k := \dot{W}_k - k/2$. Thus, at time k there are $\dot{W}_k = k/2 + \dot{Y}_k$ visible white vertices and $\dot{B}_k = k - \dot{W}_k = k/2 - \dot{Y}_k$ visible black vertices; consequently,

$$\dot{L}_k = (\tfrac{k}{2} + \dot{Y}_k)(\tfrac{k}{2} - \dot{Y}_k) = \tfrac{k^2}{4} - \tilde{Y}_k^2. \tag{4.17}$$

The random variable $\dot{W}_k$ has a hypergeometric distribution. Moreover, it is well known (e.g. by [4, Theorem 24.1]) that, in $D[0,1]$, as $n \to \infty$,

$$n^{-1/2}(\dot{W}_{\lfloor nt \rfloor} - \dot{B}_{\lfloor nt \rfloor}) = 2n^{-1/2}\dot{Y}_{\lfloor nt \rfloor} \overset{\mathrm{d}}{\longrightarrow} W^\circ(t), \tag{4.18}$$

where $W^\circ(t)$ is a Brownian bridge, see Section 2.1. Consequently, we see from (4.17) and (4.18) that, in $D[0,1]$,

$$n^{-1}\Bigl(\dot{L}^{(n)}_{\lfloor nt \rfloor} - \frac{\lfloor nt \rfloor^2}{4}\Bigr) = -n^{-1}(\dot{Y}^{(n)}_{\lfloor nt \rfloor})^2 \overset{\mathrm{d}}{\longrightarrow} -\frac{1}{4}W^\circ(t)^2. \tag{4.19}$$

In this case, thus the limit process is a (negative) square of a Gaussian process, and for a fixed $t \in (0,1)$, the distribution of $\dot{L}_{\lfloor nt \rfloor}$ is, after normalization and change of sign, a χ^2-distribution $\chi^2(1)$. In particular, the limit distribution is not normal.

Note also that we in (4.19) cannot replace $\lfloor nt \rfloor^2/4$ by $(nt)^2/4 = t^2|E^{(n)}|$, as we have in (3.8); the reason is that the jumps in $\dot{L}_k^{(n)}$ and in $\lfloor nt \rfloor^2$ are of order n, and do not disappear asymptotically with the normalization in (4.19); in (4.19) the jumps of the two terms cancel asymptotically, but we cannot replace one of the terms with a continuous version. However, if we as in Remark 3.9 define $\dot{L}_t^{(n)}$ for real $t \in [0, n]$ by linear interpolation between integers, and thus $\dot{L}_t^{(n)}$ is a continuous stochastic process, then it follows easily from (4.19) that

$$n^{-1}\bigl(\dot{L}_{nt}^{(n)} - t^2|E^{(n)}|\bigr) = n^{-1}\Bigl(\dot{L}_{nt}^{(n)} - \frac{n^2t^2}{4}\Bigr) \xrightarrow{\mathrm{d}} -\frac{1}{4}W^\circ(t)^2 \tag{4.20}$$

in $D[0, 1]$ (and in $C[0, 1]$, since here all processes are continuous). △

5. Preliminaries

5.1. **Addition in the Skorohod topology.** Addition is not continuous in $D(J)$ in general, but if $f_n, f, g_n, g \in D(J)$ with $f_n \to f$ and $g_n \to g$, and furthermore f and g are continuous, then $f_n + g_n \to f + g$. (This follows immediately from the description in Section 2.3.) As a consequence, we have the following results, which often will be used without comment.

Lemma 5.1. *Let X, Y, X_n and Y_n ($n \geqslant 1$) be stochastic processes on an interval J, with X and Y continuous a.s. If $(X_n, Y_n) \xrightarrow{\mathrm{d}} (X, Y)$ in $D(J)$, then $X_n + Y_n \xrightarrow{\mathrm{d}} X + Y$ in $D(J)$.*

Proof. By the comment above and [4, Corollary 1, p. 31]. □

Lemma 5.2. *Let X, X_n and Y_n ($n \geqslant 1$) be stochastic processes on an interval J, with X continuous a.s., and suppose that $X_n \xrightarrow{\mathrm{d}} X$ in $D(J)$ and $Y_n = o_{\mathrm{p}}^*(1)$. Then $X_n + Y_n \xrightarrow{\mathrm{d}} X$ in $D(J)$.*

Proof. Recall that $Y_n = o_{\mathrm{p}}^*(1)$ means $\sup_{t\in J} |Y_n(t)| \xrightarrow{\mathrm{p}} 0$. Hence, $Y_n \xrightarrow{\mathrm{p}} 0$ in $D(J)$, and thus $(X_n, Y_n) \xrightarrow{\mathrm{d}} (X, 0)$ [4, Theorem 4.4]. Consequently, the result follows from Lemma 5.1. □

Remark 5.3. Lemma 5.1 actually holds assuming only that either X or Y is continuous a.s.; similarly, continuity of X is not needed in Lemma 5.2. We omit the proofs, since we need only the cases above. △

5.2. **Quadratic variation of martingales.** Let M_t be a continuous-time martingale, defined for $t \in J$ where J is some interval $[0, b]$ or $[0, b)$ with $0 < b \leqslant \infty$. Assume for convenience $M(0) = 0$.

Let $\Delta M(t) := M(t) - M(t-)$ be the size of the jump (if any) at t. (For $t \in J$, where we for completeness define $\Delta M(0) := 0$.)

We will use the *quadratic variation* $[M, M]_t$ of a continuous-time martingale M, and its bilinear version, the *quadratic covariation* $[M, M_1]_t$ of two martingales M and M_1. For a general definition (which further extends beyond martingales to semimartingales) see e.g. [14, §I.4e] or [24, p. 519], but we only need a simple case: If the martingale $M(t)$ a.s. has finite variation over every compact subinterval of J (this holds trivially for the martingales defined in (6.11)–(6.13) below, since they are piecewise smooth), then its quadratic variation is given by, see e.g. [24, Theorem 26.6(viii)],

$$[M, M]_t := \sum_{0\leqslant s\leqslant t} (\Delta M(s))^2, \tag{5.1}$$

and similarly, for any martingale M_1 on J,

$$[M, M_1]_t := \sum_{0 \leqslant s \leqslant t} \Delta M(s) \Delta M_1(s). \tag{5.2}$$

(The sums are formally uncountable, but there is only a countable number of non-zero terms, since $M(s)$ has at most countably many jumps.)

We recall the basic identity [27, p. 73, Corollary 3]

$$\mathbb{E}\,[M, M]_t = \mathbb{E}\,|M(t)|^2 \tag{5.3}$$

and, more generally (by polarization), provided $\mathbb{E}\,|M(t)|^2 < \infty$ and $\mathbb{E}\,|M_1(t)|^2 < \infty$,

$$\mathbb{E}\,[M, M_1]_t := \mathbb{E}\,\big(M(t)M_1(t)\big). \tag{5.4}$$

5.3. **A martingale convergence theorem.** Our proofs are based on the following convergence theorem for martingales; it is a special case of a more general theorem by Jacod and Shiryaev [14, Theorem VIII.3.12], and the present formulation is taken from [15, Theorem 0] and [16, Proposition 2.6], where proofs are given. (See also [17, Proposition 9.1] for a similar version with somewhat weaker assumptions.)

Proposition 5.4. *Let J be an interval $[0, b]$ or $[0, b)$, $0 < b \leqslant \infty$. Assume that for each n, $M^{(n)}(t) = (M_i^{(n)}(t))_{i=1}^q$ is a q-dimensional martingale on J with $M^{(n)}(0) = 0$, and that $\Sigma(t) = (\sigma_{ij}(t))_{i,j=1}^q$, $t \in J$, is a (non-random) continuous matrix-valued function such that for every fixed $t \in J$ and $1 \leqslant i, j \leqslant q$, we have*

$$\mathbb{E}\,\big[M_i^{(n)}, M_j^{(n)}\big]_t \to \sigma_{ij}(t), \tag{5.5}$$

$$\operatorname{Var}\,\big[M_i^{(n)}, M_j^{(n)}\big]_t \to 0. \tag{5.6}$$

Then $M^{(n)} \xrightarrow{\text{d}} M$ in $D(J)$ as $n \to \infty$, where M is a continuous q-dimensional Gaussian process with $\mathbb{E}\,M(t) = 0$ and covariance function

$$\operatorname{Cov}\big(M_i(s), M_j(t)\big) = \mathbb{E}\,\big(M_i(s)M_j(t)\big) = \sigma_{ij}(s), \qquad s, t \in J \text{ and } s \leqslant t. \tag{5.7}$$

(Furthermore, the limit process $M(t)$ is a martingale, but we have no use for this extra property in the present paper.) Note that by (5.4), (5.5) can equivalently be written as $\mathbb{E}\,\big(M_i^{(n)}(t)M_j^{(n)}(t)\big) \to \sigma_{ij}(t)$.

In our applications the martingales will blow up at $t = 1$, making it impossible to use Proposition 5.4 directly on the closed interval $[0, 1]$; instead we will use Proposition 5.4 on $[0, 1)$, and then obtain convergence on $[0, 1]$ using the following lemma.

Lemma 5.5. *Suppose that $M^{(n)}(t)$, $n \geqslant 1$, are martingales on $[0, 1)$ such that $M^{(n)}(t) \xrightarrow{\text{d}} M(t)$ in $D[0, 1)$ for some continuous stochastic process $M(t)$. Suppose furthermore that*

$$\mathbb{E}\,\big|M^{(n)}(t)\big|^2 \leqslant C(1 - t)^{-a} \tag{5.8}$$

for some $a \geqslant 0$, uniformly in $n \geqslant 1$ and $t \in [0, 1)$. Let $b > a/2$ and define $\widetilde{M}^{(n)}(t) := (1 - t)^b M^{(n)}(t)$ and $\widetilde{M}(t) := (1 - t)^b M(t)$ for $t \in [0, 1)$, and $\widetilde{M}^{(n)}(1) = \widetilde{M}(1) := 0$. Then, a.s., $\widetilde{M}^{(n)}(1-) = \widetilde{M}(1-) = 0$, and thus a.s. $\widetilde{M}^{(n)} \in D[0, 1]$ and $\widetilde{M}$ is continuous on $[0, 1]$; moreover,

$$\widetilde{M}^{(n)}(t) \xrightarrow{\text{d}} \widetilde{M}(t) \tag{5.9}$$

in $D[0, 1]$.

Proof. First, for any $N \geqslant 1$, Doob's inequality [24, Proposition 7.16] and (5.8) imply

$$\begin{aligned}\mathbb{E} \sup_{1-2^{-N}\leqslant t\leqslant 1-2^{-N-1}} |\widetilde{M}^{(n)}(t)|^2 &\leqslant 2^{-2bN}\,\mathbb{E} \sup_{0\leqslant t\leqslant 1-2^{-N-1}} |M^{(n)}(t)|^2 \\ &\leqslant 2^{-2bN} 4\,\mathbb{E}\,|M^{(n)}(1-2^{-N-1})|^2 \\ &\leqslant C2^{aN-2bN} = C2^{-(2b-a)N}, \qquad (5.10)\end{aligned}$$

and thus

$$\mathbb{E} \sup_{1-2^{-N}\leqslant t<1} |\widetilde{M}^{(n)}(t)|^2 \leqslant C \sum_{\ell=N}^{\infty} 2^{-(2b-a)\ell} \leqslant C2^{-(2b-a)N}. \qquad (5.11)$$

Letting $N \to \infty$ yields, by Fatou's lemma,

$$\begin{aligned}\mathbb{E} \limsup_{t\nearrow 1} |\widetilde{M}^{(n)}(t)|^2 &= \mathbb{E} \lim_{N\to\infty} \sup_{1-2^{-N}\leqslant t<1} |\widetilde{M}^{(n)}(t)|^2 \leqslant C \lim_{N\to\infty} 2^{-(2b-a)N} \\ &= 0, \qquad (5.12)\end{aligned}$$

and thus $\widetilde{M}^{(n)}(1-) := \lim_{t\nearrow 1} \widetilde{M}^{(n)}(t) = 0$ a.s. as asserted. Moreover, since (5.10) holds uniformly in n, it follows (by Fatou's lemma again) that it holds for $\widetilde{M}(t)$ too, and thus the same argument shows that a.s. $\widetilde{M}(1-) = 0$, and thus $\widetilde{M}$ is continuous.

Finally, (5.11) and Markov's inequality imply that, for any $\varepsilon > 0$ and $u \in [0, 1)$,

$$\sup_n \mathbb{P}\bigl(\sup_{u\leqslant t<1} |\widetilde{M}^{(n)}(t)| > \varepsilon\bigr) \leqslant C\varepsilon^{-2}(1-u)^{2b-a}, \qquad (5.13)$$

which tends to 0 as $u \nearrow 1$; hence [16, Proposition 2.4] applies and yields $\widetilde{M}^{(n)} \xrightarrow{\mathrm{d}} \widetilde{M}$ in $D[0, 1]$. ☐

Remark 5.6. Lemma 5.5 extends to vector-valued martingales $(M_i^{(n)}(t))_{i=1}^q$, where we may have separate exponents a_i and b_i (with $b_i > a_i/2 \geqslant 0$) for different components $M_i^{(n)}$, $i = 1, \dots, q$. (Thus, $\widetilde{M}_i(t) := (1-t)^{b_i} M_i(t)$.) To see this, note that the proof above shows (5.13) for each component $\widetilde{M}_i^{(n)}$, and it follows that (5.13) holds for $\widetilde{M}^{(n)}$, with $2b - a$ replaced by $\min_i(2b_i - a_i) > 0$. This and the convergence $\widetilde{M}^{(n)}(t) \xrightarrow{\mathrm{d}} M^{(n)}(t)$ in $D[0, 1)$ then implies convergence in $D[0, 1]$ just as in the 1-dimensional case in [16, Proposition 2.4]. (For a proof, use e.g. the Skorohod coupling theorem [24, Theorem 4.30]; we omit the details.) △

6. A decomposition into martingales

The main idea of the proofs is to decompose $L(t)$ as a linear combination (with coefficients that are deterministic functions of t) of some martingales defined below, and then to show (joint) convergence of these martingales.

In this section we consider a fixed n and construct the martingales and the decomposition that we use; we also calculate the quadratic (co)variations of the martingales and make some estimates of them. In Section 7, we then let $n \to \infty$ and show the desired convergence.

We thus assume throughout this section that G is a fixed, deterministic graph on $[n]$. Thus $E = E(G)$ is a fixed set, and the vertex degrees d_i are deterministic.

As said in the introduction, we use a vertex version of the method in [15; 16]. We define, for $i \in [n]$, the random function

$$I_i(t) := \mathbf{1}\{T_i \leqslant t\}, \qquad 0 \leqslant t \leqslant 1. \qquad (6.1)$$

Thus $I_i(t)$ is the indicator of the event that vertex i is visible at time t. We define further

$$\widetilde{I}_i(t) := I_i(t) - \mathbb{E}\, I_i(t) = I_i(t) - t, \qquad 0 \leqslant t \leqslant 1, \tag{6.2}$$

$$\breve{I}_i(t) := (1-t)^{-1}\widetilde{I}_i(t) = \frac{I_i(t) - t}{1-t}, \qquad 0 \leqslant t < 1. \tag{6.3}$$

Note that $\widetilde{I}_i(0) = \widetilde{I}_i(1) = 0$, and that

$$\mathbb{E}\, \breve{I}_i(t) = \mathbb{E}\, \widetilde{I}_i(t) = 0. \tag{6.4}$$

Furthermore,

$$\mathbb{E}\, \breve{I}_i(t)^2 = (1-t)^{-2}\, \mathbb{E}\, \widetilde{I}_i(t)^2 = (1-t)^{-2} \operatorname{Var} I_i(t) = \frac{t}{1-t}, \tag{6.5}$$

and, when $i \neq j$, by independence,

$$\mathbb{E}\big(\breve{I}_i(t)\breve{I}_j(t)\big) = \mathbb{E}\, \breve{I}_i(t)\, \mathbb{E}\, \breve{I}_j(t) = 0. \tag{6.6}$$

Let $\mathcal{F}_t$ be the σ-field generated by $\{I_i(s) : i \in [n] \text{ and } s \leqslant t\}$. The martingales below are martingales with respect to the filtration $(\mathcal{F}_t)_t$.

Lemma 6.1. *$\breve{I}_i(t)$ is a martingale for $t \in [0,1)$, for every $i \in [n]$.*

More generally, for any sequence $1 \leqslant i_1 < \cdots < i_r \leqslant n$, the product $\prod_{j=1}^r \breve{I}_{i_j}(t)$ is a martingale on $[0,1)$.

Proof. (After [16, Lemma 2.1]; see also [15, Lemma 2.1].) It is easy to see that $\breve{I}_i$ is a Markov process with $\mathbb{E}\big(\breve{I}_i(t) \mid \mathcal{F}_s\big) = \breve{I}_i(s)$ when $0 \leqslant s \leqslant t < 1$, which implies that $I_i(t)$ is a martingale.

The final sentence follows because the collections of random variable $\{\breve{I}_i(t)\}_{t\in[0,1]}$, $i \in [n]$, are independent of each other. ∎

We have by (2.3) and (6.1)–(6.2) (summing over unordered pairs ij)

$$\begin{aligned} L(t) &= \sum_{ij\in E} I_i(t)I_j(t) = \sum_{ij\in E} \big(\widetilde{I}_i(t) + t\big)\big(\widetilde{I}_j(t) + t\big) \\ &= \sum_{ij\in E} \widetilde{I}_i(t)\widetilde{I}_j(t) + \sum_{i=1}^n \sum_{j:ij\in E} \widetilde{I}_i(t)t + \sum_{ij\in E} t^2 \\ &= \sum_{ij\in E} \widetilde{I}_i(t)\widetilde{I}_j(t) + t\sum_{i=1}^n d_i\widetilde{I}_i(t) + t^2|E|. \end{aligned} \tag{6.7}$$

We define, for $t \in [0,1]$,

$$\widetilde{Q}(t) := \sum_{ij\in E} \widetilde{I}_i(t)\widetilde{I}_j(t), \tag{6.8}$$

$$\widetilde{S}(t) := \sum_{i=1}^n d_i\widetilde{I}_i(t), \tag{6.9}$$

$$\widetilde{N}(t) := \sum_{i=1}^n \widetilde{I}_i(t) = \sum_{i=1}^n I_i(t) - nt = N(t) - nt, \tag{6.10}$$

and further, for $t \in [0,1)$, recalling (6.3),

$$\breve{Q}(t) := \sum_{ij\in E} \breve{I}_i(t)\breve{I}_j(t) = (1-t)^{-2}\widetilde{Q}(t), \tag{6.11}$$

$$\breve{S}(t) := \sum_{i=1}^n d_i\breve{I}_i(t) = (1-t)^{-1}\widetilde{S}(t), \tag{6.12}$$

$$\breve{N}(t) := \sum_{i=1}^n \breve{I}_i(t) = (1-t)^{-1}\widetilde{N}(t). \tag{6.13}$$

By Lemma 6.1, $\breve{Q}(t)$, $\breve{S}(t)$ and $\breve{N}(t)$ are martingales on $[0,1)$.

We can now rewrite (6.7) as

$$\begin{aligned} L(t) &= \widetilde{Q}(t) + t\widetilde{S}(t) + t^2|E| & (6.14)\\ &= (1-t)^2\breve{Q}(t) + t(1-t)\breve{S}(t) + t^2|E|, & (6.15)\end{aligned}$$

where the second line is meaningful only for $t \in [0,1)$.

The main idea in our proofs is to use the decomposition (6.14)–(6.15) together with limit theorems for $\widetilde{Q}$ and $\widetilde{Z}$ (and $\widetilde{N}$, for reasons that will be seen later), or (essentially equivalently) for the martingales $\breve{Q}$, $\breve{S}$ and $\breve{N}$, which we obtain from Proposition 5.4 and calculations of quadratic (co)variations.

6.1. **Quadratic variations.** To find the quadratic (co)variations, we note that $I_i(t)$ and $\widetilde{I}_i(t)$ have jumps $+1$ at $t = T_i$; hence $\breve{I}_i(t)$ has a jump $(1-T_i)^{-1}$ at T_i (and no other jump). It follows from (6.11)–(6.13) that $\breve{Q}(t)$, $\breve{S}(t)$ and $\breve{N}(t)$ have jumps only at the points T_i, $i = 1,\dots,n$, and that

$$\Delta\breve{Q}(T_i) = (1-T_i)^{-1}\sum_{j\sim i}\breve{I}_j(T_i), \tag{6.16}$$

$$\Delta\breve{S}(T_i) = d_i(1-T_i)^{-1}, \tag{6.17}$$

$$\Delta\breve{N}(T_i) = (1-T_i)^{-1}. \tag{6.18}$$

Hence, (5.1)–(5.2) yield

$$[\breve{Q},\breve{Q}]_t = \sum_{i=1}^n \mathbf{1}\{T_i \leqslant t\}(1-T_i)^{-2}\Bigl(\sum_{j\sim i}\breve{I}_j(T_i)\Bigr)^2, \tag{6.19}$$

$$[\breve{S},\breve{S}]_t = \sum_{i=1}^n \mathbf{1}\{T_i \leqslant t\}d_i^2(1-T_i)^{-2}, \tag{6.20}$$

$$[\breve{N},\breve{N}]_t = \sum_{i=1}^n \mathbf{1}\{T_i \leqslant t\}(1-T_i)^{-2}. \tag{6.21}$$

$$[\breve{Q},\breve{S}]_t = \sum_{i=1}^n \mathbf{1}\{T_i \leqslant t\}(1-T_i)^{-2}d_i\sum_{j\sim i}\breve{I}_j(T_i), \tag{6.22}$$

$$[\breve{Q},\breve{N}]_t = \sum_{i=1}^n \mathbf{1}\{T_i \leqslant t\}(1-T_i)^{-2}\sum_{j\sim i}\breve{I}_j(T_i), \tag{6.23}$$

$$[\breve{S},\breve{N}]_t = \sum_{i=1}^n \mathbf{1}\{T_i \leqslant t\}d_i(1-T_i)^{-2}. \tag{6.24}$$

We easily calculate the expectations. Since the $T_i \in \mathsf{U}(0,1)$ are independent, we obtain from (6.19)–(6.24), recalling (6.4)–(6.6),

$$\begin{aligned}
\mathbb{E}\,[\check{Q},\check{Q}]_t &= \sum_{i=1}^n \int_0^t (1-s)^{-2}\,\mathbb{E}\Bigl(\sum_{j\sim i}\check{I}_j(s)\Bigr)^2\,\mathrm{d}s = \sum_{i=1}^n \int_0^t (1-s)^{-2}\sum_{j\sim i}\mathbb{E}\,\check{I}_j(s)^2\,\mathrm{d}s \\
&= \sum_{i=1}^n \int_0^t d_i\frac{s}{(1-s)^3}\,\mathrm{d}s = \sum_{i=1}^n d_i\cdot\frac{1}{2}\frac{t^2}{(1-t)^2} = |E|\frac{t^2}{(1-t)^2}. && (6.25)\\
\mathbb{E}\,[\check{S},\check{S}]_t &= \sum_{i=1}^n \int_0^t d_i^2(1-s)^{-2}\,\mathrm{d}s = \sum_{i=1}^n d_i^2\cdot\frac{t}{1-t}, && (6.26)\\
\mathbb{E}\,[\check{N},\check{N}]_t &= \sum_{i=1}^n \int_0^t (1-s)^{-2}\,\mathrm{d}s = n\frac{t}{1-t}, && (6.27)\\
\mathbb{E}\,[\check{Q},\check{S}]_t &= \sum_{i=1}^n \int_0^t (1-s)^{-2}d_i\,\mathbb{E}\sum_{j\sim i}\check{I}_j(s)\,\mathrm{d}s = 0, && (6.28)\\
\mathbb{E}\,[\check{Q},\check{N}]_t &= \sum_{i=1}^n \int_0^t (1-s)^{-2}\,\mathbb{E}\sum_{j\sim i}\check{I}_j(s)\,\mathrm{d}s = 0, && (6.29)\\
\mathbb{E}\,[\check{S},\check{N}]_t &= \sum_{i=1}^n \int_0^t d_i(1-s)^{-2}\,\mathrm{d}s = \sum_{i=1}^n d_i\cdot\frac{t}{1-t} = 2|E|\frac{t}{1-t}. && (6.30)
\end{aligned}$$

We also need estimates of the variances of the quadratic (co)variations. (We do not bother to calculate the variances exactly, although this clearly can be done.) For simplicity, we consider a fixed t. (It is easily seen that the constants below can be taken bounded for $t \in [0,t_0]$ for any $t_0 < 1$, but that they blow up as $t \to 1$.)

Recall that $\hom(\mathsf{C}_4, G)$ is the number of (labelled) copies of C_4 in G.

Lemma 6.2. *For each fixed $t \in [0,1)$, we have, with constants C that depend on t,*

$$\operatorname{Var}\,[\check{Q},\check{Q}]_t \leqslant C\sum_{i=1}^n d_i^3. \qquad (6.31)$$

$$\operatorname{Var}\,[\check{S},\check{S}]_t \leqslant C\sum_{i=1}^n d_i^4, \qquad (6.32)$$

$$\operatorname{Var}\,[\check{N},\check{N}]_t \leqslant Cn, \qquad (6.33)$$

$$\operatorname{Var}\,[\check{Q},\check{S}]_t \leqslant C\sum_{i=1}^n d_i^4, \qquad (6.34)$$

$$\operatorname{Var}\,[\check{Q},\check{N}]_t \leqslant C\sum_{i=1}^n d_i^2, \qquad (6.35)$$

$$\operatorname{Var}\,[\check{S},\check{N}]_t \leqslant C\sum_{i=1}^n d_i^2. \qquad (6.36)$$

Proof. To begin with the simplest case, (6.21) shows that $[\check{N},\check{N}]_t$ is the sum of n independent random variables, each bounded by $(1-t)^{-2}$. Hence each term has variance $\leqslant (1-t)^{-4}$, and thus $\operatorname{Var}\,[\check{N},\check{N}]_t \leqslant n(1-t)^{-4}$, which we simplify to (6.33).

The same argument also gives (6.32) and (6.36).

Next, we write (6.19) as

$$[\check{Q},\check{Q}]_t=\sum_{i=1}^n\sum_{j\sim i}A_{ij}+\sum_{i,j,k:j\sim i\sim k}B_{ijk}, \tag{6.37}$$

where in the second sum we assume $j\neq k$, and we let

$$A_{ij}:=\mathbf{1}\{T_i\leqslant t\}(1-T_i)^{-2}\check{I}_j(T_i)^2, \tag{6.38}$$

$$B_{ijk}:=\mathbf{1}\{T_i\leqslant t\}(1-T_i)^{-2}\check{I}_j(T_i)\check{I}_k(T_i)\mathbf{1}\{j\neq k\}. \tag{6.39}$$

We estimate the variances of the two sums separately. Note that (for a fixed t) all A_{ij} and B_{ijk} are uniformly bounded. Two variables A_{ij} and $A_{i'j'}$ are independent unless they have at least one common index. Hence, using symmetry,

$$\begin{aligned}\operatorname{Var}\Bigl(\sum_{i,j:i\sim j}A_{ij}\Bigr)&\leqslant C\bigl|\{(i,j,i',j')\in[n]^4:i\sim j,\ i'\sim j',\ \{i,j\}\cap\{i',j'\}\geqslant 1\}\bigr|\\&\leqslant C\bigl|\{(i,j,j')\in[n]^3:i\sim j,\ i\sim j'\}\bigr|\\&=C\sum_{i=1}^n d_i^2.\end{aligned} \tag{6.40}$$

For B_{ijk}, we first note that $\mathbb{E}\,(B_{ijk}\mid T_i,T_j)=0$ since $\mathbb{E}\,(\check{I}_k(T_i)\mid T_i)=0$ by (6.4); thus $\mathbb{E}\,B_{ijk}=0$. Similarly, if, say, $k\notin\{i',j',k'\}$, then, by conditioning on T_ℓ for all $\ell\neq k$, we have $\mathbb{E}\,(B_{ijk}B_{i'j'k'})=0$. Hence, if $\mathbb{E}\,(B_{ijk}B_{i'j'k'})\neq 0$, then each of j,k,j',k' equals one of the other five indices. Since we assume that i,j,k are distinct, as well as i',j',k', this is possible only if either $\{i,j,k\}=\{i',j',k'\}$ or $\bigl|\{i,j,k\}\cap\{i',j',k'\}\bigr|=2$, and in the latter case furthermore $\{i,j,k\}\cap\{i',j',k'\}=\{j,k\}=\{j',k'\}$, and thus j,i,k,i' form a cycle C_4 in G. In the first case, there are at most 6 choices of i',j',k' for each (i,j,k). Consequently, we obtain, using (6.40),

$$\begin{aligned}\operatorname{Var}\Bigl(\sum_{i,j,k:k\sim i\sim j}B_{ijk}\Bigr)&\leqslant C\bigl|\{(i,j,k)\in[n]^3:k\sim i\sim j\}\bigr|+C\hom(\mathsf{C}_4,G)\\&=C\sum_{i=1}^n d_i^2+C\hom(\mathsf{C}_4,G).\end{aligned} \tag{6.41}$$

It follows from (6.37), (6.40) and (6.41) that

$$\operatorname{Var}\,[\check{Q},\check{Q}]_t\leqslant C\sum_{i=1}^n d_i^2+C\hom(\mathsf{C}_4,G). \tag{6.42}$$

The inequality (6.31) then follows because by an inequality by Sidorenko [29] (see also [22]),

$$\hom(\mathsf{C}_4,G)\leqslant\hom(\mathsf{P}_4,G)\leqslant\hom(\mathsf{K}_{1,3},G)\leqslant\sum_{i=1}^n d_i^3. \tag{6.43}$$

Alternatively, this follows because we have

$$\begin{aligned}\hom(\mathsf{C}_4,G)\leqslant\hom(\mathsf{P}_4,G)&\leqslant\sum_{i,j:\,i\sim j}d_id_j\leqslant\sum_{i,j:\,i\sim j}\tfrac12\bigl(d_i^2+d_j^2\bigr)=\sum_{i,j:\,i\sim j}d_i^2\\&=\sum_{i=1}^n d_i^3.\end{aligned} \tag{6.44}$$

Similarly, we write (6.23) as

$$[\breve{Q},\breve{N}]_t=\sum_{i=1}^n\sum_{j\sim i}D_{ij}, \tag{6.45}$$

where

$$D_{ij}:=\mathbf{1}\{T_i\leqslant t\}(1-T_i)^{-2}\breve{I}_j(T_i), \tag{6.46}$$

noting that (for a given t), the random variables D_{ij} are uniformly bounded; furthermore, D_{ij} and $D_{i'j'}$ are independent unless they have at least one common index. Consequently, as in (6.40),

$$\operatorname{Var}[\breve{Q},\breve{N}]_t=\operatorname{Var}\Big(\sum_{i,j:i\sim j}D_{ij}\Big)\leqslant C\sum_{i=1}^n d_i^2. \tag{6.47}$$

Finally, we have, with D_{ij} as in (6.46),

$$[\breve{Q},\breve{S}]_t=\sum_{i=1}^n\sum_{j\sim i}d_iD_{ij}, \tag{6.48}$$

and it follows similarly, using symmetry as in (6.40),

$$\begin{aligned}\operatorname{Var}[\breve{Q},\breve{S}]_t&\leqslant C\sum_{i\sim j,\,i'\sim j',\,\{i,j\}\cap\{i',j'\}\geqslant 1}d_id_{i'}\\&\leqslant C\sum_{i,j,j':\,i\sim j,\,i\sim j'}(d_i+d_j)(d_i+d_{j'}).\end{aligned} \tag{6.49}$$

It is easily seen that this sum can be estimated by $C\sum_{|E(H)|\leqslant 4}\hom(H,G)$, summing over connected graphs H with at most 4 edges. Consequently, using again Sidorenko's inequality [29; 22] (or a more complicated version of (6.44), which we leave to the reader)

$$\operatorname{Var}[\breve{Q},\breve{S}]_t\leqslant C\sum_{|E(H)|\leqslant 4}\hom(H,G)\leqslant C\hom(\mathsf{K}_{1,4},G)=C\sum_{i=1}^n d_i^4. \tag{6.50}$$

This completes the proof of (6.31)–(6.36). □

We will also use a version where we "normalize" $\widetilde{S}(t)$ and $\breve{S}(t)$ by subtracting the average degree $\bar{d}$ from the degrees in the definition. We define, cf. (6.9) and (6.12),

$$\widetilde{R}(t):=\sum_{i=1}^n(d_i-\bar{d})\widetilde{I}_i(t)=\widetilde{S}(t)-\bar{d}\widetilde{N}(t), \tag{6.51}$$

$$\breve{R}(t):=\sum_{i=1}^n(d_i-\bar{d})\breve{I}_i(t)=(1-t)^{-1}\widetilde{R}(t)=\breve{S}(t)-\bar{d}\breve{N}(t). \tag{6.52}$$

Then $\breve{R}(t)$ is a martingale on $[0,1)$, with quadratic (co)variation, similarly to (6.20)–(6.23),

$$[\breve{R},\breve{R}]_t=\sum_{i=1}^n\mathbf{1}\{T_i\leqslant t\}(d_i-\bar{d})^2(1-T_i)^{-2}, \tag{6.53}$$

$$[\breve{Q},\breve{R}]_t=\sum_{i=1}^n\mathbf{1}\{T_i\leqslant t\}(1-T_i)^{-2}(d_i-\bar{d})\sum_{j\sim i}\breve{I}_j(T_i), \tag{6.54}$$

$$[\breve{R},\breve{N}]_t=\sum_{i=1}^n \mathbf{1}\{T_i\leqslant t\}(1-T_i)^{-2}(d_i-\bar{d}). \tag{6.55}$$

Hence, or by (6.52) and (6.26)–(6.30), recalling (2.8),

$$\mathbb{E}\,[\breve{R},\breve{R}]_t=\sum_{i=1}^n (d_i-\bar{d})^2\cdot\frac{t}{1-t}, \tag{6.56}$$

$$\mathbb{E}\,[\breve{Q},\breve{R}]_t=0, \tag{6.57}$$

$$\mathbb{E}\,[\breve{R},\breve{N}]_t=\sum_{i=1}^n (d_i-\bar{d})\cdot\frac{t}{1-t}=0. \tag{6.58}$$

Lemma 6.3. *For each fixed $t\in[0,1)$, we have, with constants C that depend on t,*

$$\operatorname{Var}\,[\breve{R},\breve{R}]_t\leqslant C\sum_{i=1}^n (d_i-\bar{d})^4, \tag{6.59}$$

$$\operatorname{Var}\,[\breve{Q},\breve{R}]_t\leqslant C\Delta^2\sum_{i=1}^n (d_i-\bar{d})^2, \tag{6.60}$$

$$\operatorname{Var}\,[\breve{R},\breve{N}]_t\leqslant C\sum_{i=1}^n (d_i-\bar{d})^2. \tag{6.61}$$

Proof. First, (6.59) and (6.61) follow from (6.53) and (6.55) since the terms in the sums are independent, similarly to (6.32) and (6.33) in Lemma 6.2.

Next, by (6.52), (6.45), and (6.48),

$$[\breve{Q},\breve{R}]_t=[\breve{Q},\breve{S}]_t-\bar{d}[\breve{Q},\breve{N}]_t=\sum_{i=1}^n\sum_{j\sim i}(d_i-\bar{d})D_{ij}, \tag{6.62}$$

with D_{ij} defined in (6.46). Recall that for a fixed t, the random variables D_{ij} are uniformly bounded, and that D_{ij} and $D_{i'j'}$ are independent unless they have at least one common index. Hence, similarly to (6.49), it follows that

$$\begin{aligned}\operatorname{Var}\,[\breve{Q},\breve{R}]_t&\leqslant C\sum_{i\sim j,\,i'\sim j',\,\{i,j\}\cap\{i',j'\}\geqslant 1}|d_i-\bar{d}|\,|d_{i'}-\bar{d}|\\&\leqslant C\sum_{i,j,j'\colon i\sim j,\,i\sim j'}\bigl(|d_i-\bar{d}|+|d_j-\bar{d}|\bigr)\bigl(|d_i-\bar{d}|+|d_{j'}-\bar{d}|\bigr)\\&=C\sum_i d_i^2|d_i-\bar{d}|^2+C\sum_{i,j:i\sim j}d_i|d_i-\bar{d}||d_j-\bar{d}|+\sum_i\Bigl(\sum_{j\sim i}|d_j-\bar{d}|\Bigr)^2.\end{aligned} \tag{6.63}$$

The first sum on the right-hand side of (6.63) is clearly at most $\Delta^2\sum_{i=1}^n(d_i-\bar{d})^2$. The second sum is

$$\begin{aligned}&\leqslant\Delta\sum_{i,j:i\sim j}\bigl(|d_i-\bar{d}|^2+|d_j-\bar{d}|^2\bigr)\\&=\Delta\sum_i d_i|d_i-\bar{d}|^2+\Delta\sum_j d_j|d_j-\bar{d}|^2\\&\leqslant 2\Delta^2\sum_i|d_i-\bar{d}|^2.\end{aligned} \tag{6.64}$$

Finally, the third sum is, by the Cauchy–Schwarz inequality,

$$\leqslant \sum_i d_i \sum_{j\sim i} |d_j - \bar{d}|^2 \leqslant \Delta \sum_j d_j |d_j - \bar{d}|^2 \leqslant \Delta^2 \sum_j |d_j - \bar{d}|^2. \tag{6.65}$$

Combining these estimates, we obtain (6.60). □

7. Proofs of convergence and main theorems

We are now prepared to show convergence of the martingales defined in Section 6, and then to prove the theorems in Section 3. Although Theorems 3.1 and 3.5 can be proved as special cases of Theorem 3.6, we have chosen to give separate proofs of the three theorems, using the same general method but with some variations; this illustrates the differences between the cases. We begin with the sparse case in Theorem 3.1, which shows the main ideas without unnecessary complications. We first state a lemma, where we assume that the graphs $G^{(n)}$ are non-random. (Thus the assumptions (3.1)–(3.3) are replaced by their non-random counterparts (7.1)–(7.3).)

In this section we use convergence in both $D[0,1)$ and $D[0,1]$. For processes defined on $[0,1]$, we will always use convergence in $D[0,1]$, even when this is not explicitly said.

Lemma 7.1. *Assume that $G^{(n)}$ is a sequence of non-random graphs with $V(G^{(n)}) = [n]$ such that, as in Theorem 3.1,*

$$\frac{1}{n}\sum_{i=1}^n d_i^{(n)} = \frac{2|E^{(n)}|}{n} \to d_* \in [0,\infty), \tag{7.1}$$

$$\frac{1}{n}\sum_{i=1}^n (d_i^{(n)})^2 \to \chi_* \in [0,\infty), \tag{7.2}$$

$$\Delta^{(n)} := \max_i d_i^{(n)} = o(n^{1/2}). \tag{7.3}$$

(i) *Then, in $D[0,1)$,*

$$n^{-1/2}\big(\breve{Q}^{(n)}(t), \breve{S}^{(n)}(t), \breve{N}^{(n)}(t)\big) \overset{\mathrm{d}}{\longrightarrow} \breve{Z}(t) = \big(\breve{Z}_{\mathsf{Q}}(t), \breve{Z}_{\mathsf{S}}(t), \breve{Z}_{\mathsf{N}}(t)\big), \tag{7.4}$$

where $\breve{Z}(t)$ is a continuous Gaussian process on $[0,1)$ with $\mathbb{E}\,\breve{Z}(t) = 0$ and covariance function, for $0 \leqslant s \leqslant t < 1$,

$$\operatorname{Cov}\big(\breve{Z}(s), \breve{Z}(t)\big) = \mathbb{E}\left(\breve{Z}(s)\breve{Z}(t)'\right) = \breve{\Sigma}(s) := \begin{pmatrix} \frac{d_*}{2}\frac{s^2}{(1-s)^2} & 0 & 0 \\ 0 & \chi_*\frac{s}{1-s} & d_*\frac{s}{1-s} \\ 0 & d_*\frac{s}{1-s} & \frac{s}{1-s} \end{pmatrix}. \tag{7.5}$$

(ii) *Similarly, in $D[0,1]$,*

$$n^{-1/2}\big(\widetilde{Q}^{(n)}(t), \widetilde{S}^{(n)}(t), \widetilde{N}^{(n)}(t)\big) \overset{\mathrm{d}}{\longrightarrow} \widetilde{Z}(t) = \big(\widetilde{Z}_{\mathsf{Q}}(t), \widetilde{Z}_{\mathsf{S}}(t), \widetilde{Z}_{\mathsf{N}}(t)\big), \tag{7.6}$$

where $\widetilde{Z}(t)$ is a continuous Gaussian process on $[0,1]$ with $\mathbb{E}\,\widetilde{Z}(t) = 0$ and covariance function, for $0 \leqslant s \leqslant t \leqslant 1$,

$$\operatorname{Cov}\big(\widetilde{Z}(s), \widetilde{Z}(t)\big) = \widetilde{\Sigma}(s,t) := \begin{pmatrix} \frac{d_*}{2}s^2(1-t)^2 & 0 & 0 \\ 0 & \chi_* s(1-t) & d_* s(1-t) \\ 0 & d_* s(1-t) & s(1-t) \end{pmatrix}. \tag{7.7}$$

Proof. (i): Consider the vector-valued martingale on $[0,1)$ defined by

$$M^{(n)}(t)=\bigl(M^{(n)}_{\mathsf{Q}}(t),M^{(n)}_{\mathsf{S}}(t),M^{(n)}_{\mathsf{N}}(t)\bigr):=n^{-1/2}\bigl(\breve{Q}^{(n)}(t),\breve{S}^{(n)}(t),\breve{N}^{(n)}(t)\bigr). \tag{7.8}$$

For any fixed $t\in[0,1)$, (6.25)–(6.30) together with (7.1)–(7.2) and (7.5) show that the matrix of quadratic covariations has expectation

$$\mathbb{E}\,[M^{(n)},(M^{(n)})']_t\to\breve{\Sigma}(t), \tag{7.9}$$

while Lemma 6.2 shows that for each $i,j\in\{\mathsf{Q},\mathsf{S},\mathsf{N}\}$ we have, for the corresponding $\breve{X},\breve{Y}\in\{\breve{Q},\breve{S},\breve{N}\}$, using (7.2)–(7.3),

$$\begin{aligned}\operatorname{Var}\bigl[M^{(n)}_i,M^{(n)}_j\bigr]_t&=\operatorname{Var}\bigl(n^{-1}[\breve{X}^{(n)},\breve{Y}^{(n)}]_t\bigr)\leqslant Cn^{-2}\Bigl(\sum_{i=1}^n\bigl(d^{(n)}_i\bigr)^4+n\Bigr)\\&\leqslant Cn^{-2}(\Delta^{(n)})^2\sum_{i=1}^n\bigl(d^{(n)}_i\bigr)^2+Cn^{-1}\to0.\end{aligned} \tag{7.10}$$

Hence, Proposition 5.4 applies and yields the result.

(ii): Convergence in $D[0,1)$ follows immediately from (i) and the relations (6.11)–(6.13), defining $\widetilde{Z}_{\mathsf{Q}}(t):=(1-t)^2\breve{Z}_{\mathsf{Q}}(t)$, $\widetilde{Z}_{\mathsf{S}}(t):=(1-t)\breve{Z}_{\mathsf{S}}(t)$, and $\widetilde{Z}_{\mathsf{N}}(t):=(1-t)\breve{Z}_{\mathsf{N}}(t)$ for $t\in[0,1)$. We define further $\widetilde{Z}_{\mathsf{Q}}(1):=\widetilde{Z}_{\mathsf{S}}(1):=\widetilde{Z}_{\mathsf{N}}(1):=0$; continuity of $\widetilde{Z}(t)$ at $t=1$ and convergence in $D[0,1]$ then follows by Lemma 5.5 and Remark 5.6, taking $a_{\mathsf{Q}}=b_{\mathsf{Q}}=2$ and $a_{\mathsf{S}}=b_{\mathsf{S}}=a_{\mathsf{N}}=b_{\mathsf{N}}=1$ and recalling (5.3) and (6.25)–(6.27). ☐

Remark 7.2. In particular, $\widetilde{Z}_{\mathsf{N}}(t)$ has covariance function $s(1-t)$ for $0\leqslant s\leqslant t\leqslant1$, and is thus a Brownian bridge $W^\circ(t)$, see (2.1). The convergence of $n^{-1/2}\widetilde{N}^{(n)}(t)=n^{-1/2}(N^{(n)}(t)-nt)$ to a Brownian bridge is a well-known fact, since by (2.4), $n^{-1}N^{(n)}(t)$ is the empirical distribution function of the i.i.d. uniformly distributed random variables T_i, $i\in[n]$; see e.g. [4, Theorems 16.4 and 13.1]. This holds for any graphs $G^{(n)}$ (without any conditions on the degrees), since the edges do not affect $N^{(n)}(t)$, and therefore this result for $\widetilde{N}^{(n)}(t)$ will return in other proofs below. △

Proof of Theorem 3.1. Assume first that the graphs $G^{(n)}$ are non-random. In particular, the variables in (3.1)–(3.3) are non-random, and the limits there are thus usual limits of real numbers. Hence, Lemma 7.1 applies. We prove first the continuous-time result (ii), and then use it to derive the discrete-time result (i).

$G^{(n)}$ *non-random*, (ii): By (6.14) and Lemma 7.1(ii), using also Lemma 5.1,

$$n^{-1/2}\bigl(L^{(n)}(t)-t^2|E^{(n)}|\bigr)=n^{-1/2}\bigl(\widetilde{Q}^{(n)}(t)+t\widetilde{S}^{(n)}(t)\bigr)\overset{\mathrm{d}}{\longrightarrow}Z(t):=\widetilde{Z}_{\mathsf{Q}}(t)+t\widetilde{Z}_{\mathsf{S}}(t) \tag{7.11}$$

in $D[0,1]$. The process $Z(t)$ is clearly continuous and Gaussian, with mean $\mathbb{E}\,Z(t)=0$ and covariance function, using (7.6)–(7.7),

$$\begin{aligned}\operatorname{Cov}\bigl(Z(s),Z(t)\bigr)&=\operatorname{Cov}\bigl(\widetilde{Z}_{\mathsf{Q}}(s),\widetilde{Z}_{\mathsf{Q}}(t)\bigr)+st\operatorname{Cov}\bigl(\widetilde{Z}_{\mathsf{S}}(s),\widetilde{Z}_{\mathsf{S}}(t)\bigr)\\&=\frac{d_*}{2}s^2(1-t)^2+\chi_*s^2t(1-t).\end{aligned} \tag{7.12}$$

This proves (3.6)–(3.7).

$G^{(n)}$ *non-random*, (i): The proof just given for (ii) shows that the result (7.11) holds jointly with $n^{-1/2}\widetilde{N}(t)\overset{\mathrm{d}}{\longrightarrow}\widetilde{Z}_{\mathsf{N}}(t)$. Hence, we can apply Theorem A.1, with $\dot{X}^{(n)}_k:=\dot{L}_k(t)$, $X^{(n)}(t):=L^{(n)}(t)$, $W(t)=\widetilde{Z}_{\mathsf{N}}(t)$, $a_n=n^{-1/2}$, $b_n=2|E^{(n)}|$, $f(t)=$

$t^2/2$, and thus $c = \lim_{n\to\infty} 2|E^{(n)}|/n = d_*$. This yields (3.4) with $\dot{Z}(t) := Z(t) - d_* t\widetilde{Z}_{\mathsf{N}}(t)$, and (A.5) yields the covariance function (3.5).

$G^{(n)}$ random: Finally, consider the general case when $G^{(n)}$ may be random. By the Skorohod coupling theorem [24, Theorem 4.30], we may for convenience assume that the limits in (3.1)–(3.3) hold a.s. We then condition on the sequence $(G^{(n)})_1^\infty$, and note that thus a.s. the deterministic case just proved applies to the sequence. Hence, a.s. the conclusions (3.4) and (3.6) hold conditionally on $(G^{(n)})_1^\infty$. Since the limits have distributions determined by (3.5) and (3.7) which do not depend on the sequence $(G^{(n)})_1^\infty$, it follows that (3.4) and (3.6) hold unconditionally too. ☐

We turn to the regular case, again beginning with a lemma for non-random $G^{(n)}$.

Lemma 7.3. *Assume that $G^{(n)}$ is a sequence of non-random graphs with $V(G^{(n)}) = [n]$ such that, as in Theorem 3.5, $G^{(n)}$ is regular with degree $1 \leqslant d^{(n)} = o(n)$. Then $\widetilde{S}^{(n)}(t) = d^{(n)}\widetilde{N}^{(n)}(t)$ and $\breve{S}^{(n)}(t) = d^{(n)}\breve{N}^{(n)}(t)$. Furthermore:*

(i) *We have, in $D[0,1)$,*

$$\big((nd^{(n)})^{-1/2}\breve{Q}^{(n)}(t), n^{-1/2}\breve{N}^{(n)}(t)\big) \xrightarrow{\mathrm{d}} \breve{Z}(t) = \big(\breve{Z}_{\mathsf{Q}}(t), \breve{Z}_{\mathsf{N}}(t)\big), \tag{7.13}$$

where $\breve{Z}_{\mathsf{Q}}(t)$ and $\breve{Z}_{\mathsf{N}}(t)$ are independent continuous Gaussian processs on $[0,1)$ with means 0 and covariance functions, for $0 \leqslant s \leqslant t < 1$,

$$\operatorname{Cov}\Big(\breve{Z}_{\mathsf{Q}}(s), \breve{Z}_{\mathsf{Q}}(t)\Big) = \frac{1}{2}\frac{s^2}{(1-s)^2}, \tag{7.14}$$

$$\operatorname{Cov}\Big(\breve{Z}_{\mathsf{N}}(s), \breve{Z}_{\mathsf{N}}(t)\Big) = \frac{s}{1-s}. \tag{7.15}$$

(ii) *Similarly, in $D[0,1]$,*

$$\big((nd^{(n)})^{-1/2}\widetilde{Q}^{(n)}(t), n^{-1/2}\widetilde{N}^{(n)}(t)\big) \xrightarrow{\mathrm{d}} \widetilde{Z}(t) = \big(\widetilde{Z}_{\mathsf{Q}}(t), \widetilde{Z}_{\mathsf{N}}(t)\big), \tag{7.16}$$

where $\widetilde{Z}_{\mathsf{Q}}(t)$ and $\widetilde{Z}_{\mathsf{N}}(t)$ are independent continuous Gaussian processs on $[0,1)$ with means 0 and covariance functions, for $0 \leqslant s \leqslant t \leqslant 1$,

$$\operatorname{Cov}\Big(\widetilde{Z}_{\mathsf{Q}}(s), \widetilde{Z}_{\mathsf{Q}}(t)\Big) = \frac{1}{2}s^2(1-t)^2, \tag{7.17}$$

$$\operatorname{Cov}\Big(\widetilde{Z}_{\mathsf{N}}(s), \widetilde{Z}_{\mathsf{N}}(t)\Big) = s(1-t). \tag{7.18}$$

Proof. Since $G^{(n)}$ is regular with $d_i^{(n)} = d^{(n)}$ for every $i \in [n]$, it follows from (6.9)–(6.10) and (6.12)–(6.13) that $\widetilde{S}^{(n)}(t) = d^{(n)}\widetilde{N}^{(n)}(t)$ and $\breve{S}^{(n)}(t) = d^{(n)}\breve{N}^{(n)}(t)$.

The rest of the proof is similar to the proof of Lemma 7.1. We ignore $\breve{S}^{(n)}$ and define now

$$M^{(n)}(t) = \big(M_{\mathsf{Q}}^{(n)}(t), M_{\mathsf{N}}^{(n)}(t)\big) := \big((nd^{(n)})^{-1/2}\breve{Q}^{(n)}(t), n^{-1/2}\breve{N}^{(n)}(t)\big). \tag{7.19}$$

For any fixed $t \in [0,1)$, (6.25), (6.27) and (6.29) together with $|E^{(n)}| = nd^{(n)}/2$ show that the matrix of quadratic covariations has expectation

$$\mathbb{E}\,[M^{(n)}, (M^{(n)})']_t = \Sigma(t) := \begin{pmatrix} \frac{1}{2}\frac{t^2}{(1-t)^2} & 0 \\ 0 & \frac{t}{1-t} \end{pmatrix} \tag{7.20}$$

(in this case an identity for all n), while Lemma 6.2 shows that

$$\operatorname{Var}\,[M_{\mathsf{Q}}^{(n)}, M_{\mathsf{Q}}^{(n)}]_t = \operatorname{Var}\big((nd^{(n)})^{-1}[\breve{Q}^{(n)}, \breve{Q}^{(n)}]_t\big) \leqslant (nd^{(n)})^{-2}Cn(d^{(n)})^3$$

$$= Cd^{(n)}/n = o(1), \tag{7.21}$$

$$\operatorname{Var}[M^{(n)}_{\mathsf{Q}}, M^{(n)}_{\mathsf{N}}]_t = \operatorname{Var}\big((n^2 d^{(n)})^{-1/2}[\breve{Q}^{(n)}, \breve{N}^{(n)}]_t\big) \leqslant (n^2 d^{(n)})^{-1} Cn (d^{(n)})^2$$
$$= Cd^{(n)}/n = o(1), \tag{7.22}$$

$$\operatorname{Var}[M^{(n)}_{\mathsf{N}}, M^{(n)}_{\mathsf{N}}]_t = \operatorname{Var}\big(n^{-1}[\breve{N}^{(n)}, \breve{N}^{(n)}]_t\big) \leqslant n^{-2} Cn = Cn^{-1} = o(1), \tag{7.23}$$

Hence, Proposition 5.4 applies and yields (i), with $\operatorname{Cov}(\breve{Z}(t)) = \Sigma(t)$ in (7.20). The independence of $\breve{Z}_{\mathsf{Q}}$ and $\breve{Z}_{\mathsf{N}}$ follows from the fact that the matrix $\Sigma(t)$ is diagonal, and thus all covariances $\operatorname{Cov}\big(\breve{Z}_{\mathsf{Q}}(s), \breve{Z}_{\mathsf{N}}(t)\big) = \operatorname{Var}\breve{Z}_{\mathsf{Q}}(s)$ $(0 \leqslant s \leqslant t)$ vanish.

Finally, (ii) follows as in the proof of Lemma 7.1 by Lemma 5.5 and Remark 5.6, using (6.25) and (6.27). □

Proof of Theorem 3.5. As in the proof of Theorem 3.1, we may by conditioning assume that each $G^{(n)}$ is non-random. Then Lemma 7.3 applies. As in the proof of Theorem 3.1, we first consider continuous time, but this time we cannot derive (i) from (ii), since the normalizing factors are different; hence we derive instead first the intermediate continuous-time result (7.25) below.

(i): The decomposition (6.14) yields, noting that $\widetilde{S}^{(n)}(t) = d^{(n)}\widetilde{N}^{(n)}(t)$ by Lemma 7.3 and $|E^{(n)}| = nd^{(n)}/2$ by (2.10), and using also (6.10),

$$L^{(n)}(t) - \frac{d^{(n)}}{2n} N^{(n)}(t)^2 = \widetilde{Q}^{(n)}(t) + td^{(n)}\widetilde{N}^{(n)}(t) + t^2\frac{nd^{(n)}}{2} - \frac{d^{(n)}}{2n}\big(\widetilde{N}^{(n)}(t) + nt\big)^2$$
$$= \widetilde{Q}^{(n)}(t) - \frac{d^{(n)}}{2n}\widetilde{N}^{(n)}(t)^2. \tag{7.24}$$

As a consequence of (7.16), we have $\sup_t |\widetilde{N}^{(n)}(t)| = O_{\mathrm{p}}\big(n^{1/2}\big)$; recall that we write this as $\widetilde{N}^{(n)}(t) = O^*_{\mathrm{p}}\big(n^{1/2}\big)$. Hence, (7.24) implies, using $d^{(n)} = o(n)$, (7.16), and Lemma 5.2,

$$(nd^{(n)})^{-1/2}\Big(L^{(n)}(t) - \frac{d^{(n)}}{2n}N^{(n)}(t)^2\Big) = (nd^{(n)})^{-1/2}\widetilde{Q}^{(n)}(t) + O^*_{\mathrm{p}}((d^{(n)})^{1/2} n^{-1/2})$$
$$= (nd^{(n)})^{-1/2}\widetilde{Q}^{(n)}(t) + o^*_{\mathrm{p}}(1)$$
$$\overset{\mathrm{d}}{\longrightarrow} \widetilde{Z}_{\mathsf{Q}}(t). \tag{7.25}$$

Furthermore, (7.16) implies that (7.25) holds jointly with $n^{-1/2}(N^{(n)}(t) - nt) \overset{\mathrm{d}}{\longrightarrow} \widetilde{Z}_{\mathsf{N}}(t)$. Thus by Theorem A.1 (with $b_n = f(t) = c = 0$), or in this simple case directly by substituting $\tau_{\lfloor nt\rfloor}$ for t in (7.25),

$$(nd^{(n)})^{-1/2}\Big(\dot{L}_{\lfloor nt\rfloor} - \frac{d^{(n)}}{2n}\lfloor nt\rfloor^2\Big) \overset{\mathrm{d}}{\longrightarrow} \widetilde{Z}_{\mathsf{Q}}(t). \tag{7.26}$$

Furthermore, we have

$$\frac{d^{(n)}}{2n}\lfloor nt\rfloor^2 - t^2|E^{(n)}| = \frac{d^{(n)}}{2n}\big(\lfloor nt\rfloor^2 - (nt)^2\big) = O(d^{(n)}) = o\big((nd^{(n)})^{1/2}\big), \tag{7.27}$$

and thus (7.26) implies (3.8) with $\dot{Z}(t) = \widetilde{Z}_{\mathsf{Q}}(t)$. Hence, (3.9) holds by (7.17).

(ii) and (iii): It follows from (2.8), (6.10) and (7.16) that

$$\frac{1}{n^{1/2}d^{(n)}}\Big(\frac{d^{(n)}}{2n}N^{(n)}(t)^2 - t^2|E^{(n)}|\Big) = \frac{1}{2n^{3/2}}\big(N^{(n)}(t)^2 - n^2t^2\big)$$

$$= \frac{1}{2n^{3/2}}\big(\widetilde{N}^{(n)}(t)^2 + 2nt\widetilde{N}^{(n)}(t)\big)$$
$$\xrightarrow{\mathrm{d}} t\widetilde{Z}_{\mathsf{N}}(t). \tag{7.28}$$

Furthermore, this holds jointly with (7.25) and its consequence

$$(n^{1/2}d^{(n)})^{-1}\Big(L^{(n)}(t) - \frac{d^{(n)}}{2n}N^{(n)}(t)^2\Big) \xrightarrow{\mathrm{d}} d_*^{-1/2}\widetilde{Z}_{\mathsf{Q}}(t). \tag{7.29}$$

Combining (7.29) and (7.28) yields

$$\frac{1}{n^{1/2}d^{(n)}}\big(L^{(n)}(t) - t^2|E^{(n)}|\big) \xrightarrow{\mathrm{d}} Z(t) := d_*^{-1/2}\widetilde{Z}_{\mathsf{Q}}(t) + t\widetilde{Z}_{\mathsf{N}}(t). \tag{7.30}$$

This proves (3.10), and (7.17)–(7.18) imply that the covariance function is given by (3.11); when $d_* = \infty$ this simplifies to (3.12), and we see also directly from (7.30) that in this case $Z(t) = t\widetilde{Z}_{\mathsf{N}}(t) = tW^\circ(t)$, see Remark 7.2. □

Finally, we treat the general case in Theorem 3.6, again beginning with a lemma.

Lemma 7.4. *Assume that $G^{(n)}$ is a sequence of non-random graphs with $V(G^{(n)}) = [n]$ and that β_n is a sequence of positive constants such that, as in Theorem 3.6, for some constants $\lambda_1, \lambda_2 \in [0,\infty)$, we have, as $n \to \infty$,*

$$\beta_n = o(n), \tag{7.31}$$
$$\frac{2|E^{(n)}|}{\beta_n^2} = \frac{n\bar{d}^{(n)}}{\beta_n^2} \to \lambda_1, \tag{7.32}$$
$$\frac{1}{\beta_n^2}\sum_{i=1}^{n}\big(d_i^{(n)} - \bar{d}^{(n)}\big)^2 \to \lambda_2, \tag{7.33}$$
$$\Delta^{(n)} = o(\beta_n). \tag{7.34}$$

(i) *Then, in $D[0,1)$,*

$$\big(\beta_n^{-1}\breve{Q}^{(n)}(t), \beta_n^{-1}\breve{R}^{(n)}(t), n^{-1/2}\breve{N}^{(n)}(t)\big) \xrightarrow{\mathrm{d}} \breve{Z}(t) = \big(\breve{Z}_{\mathsf{Q}}(t), \breve{Z}_{\mathsf{R}}(t), \breve{Z}_{\mathsf{N}}(t)\big), \tag{7.35}$$

where $\breve{Z}_{\mathsf{Q}}(t)$, $\breve{Z}_{\mathsf{R}}(t)$, and $\breve{Z}_{\mathsf{N}}(t)$ are independent continuous Gaussian processes on $[0,1)$ with means 0 and covariance functions, for $0 \leqslant s \leqslant t < 1$,

$$\operatorname{Cov}\Big(\breve{Z}_{\mathsf{Q}}(s), \breve{Z}_{\mathsf{Q}}(t)\Big) = \frac{\lambda_1}{2}\frac{s^2}{(1-s)^2}, \tag{7.36}$$
$$\operatorname{Cov}\Big(\breve{Z}_{\mathsf{R}}(s), \breve{Z}_{\mathsf{R}}(t)\Big) = \lambda_2\frac{s}{1-s}, \tag{7.37}$$
$$\operatorname{Cov}\Big(\breve{Z}_{\mathsf{N}}(s), \breve{Z}_{\mathsf{N}}(t)\Big) = \frac{s}{1-s}. \tag{7.38}$$

(ii) *Similarly, in $D[0,1]$,*

$$\big(\beta_n^{-1}\widetilde{Q}^{(n)}(t), \beta_n^{-1}\widetilde{R}^{(n)}(t), n^{-1/2}\widetilde{N}^{(n)}(t)\big) \xrightarrow{\mathrm{d}} \widetilde{Z}(t) = \big(\widetilde{Z}_{\mathsf{Q}}(t), \widetilde{Z}_{\mathsf{R}}(t), \widetilde{Z}_{\mathsf{N}}(t)\big), \tag{7.39}$$

where $\widetilde{Z}_{\mathsf{Q}}(t)$, $\widetilde{Z}_{\mathsf{R}}(t)$, and $\widetilde{Z}_{\mathsf{N}}(t)$ are independent continuous Gaussian processes on $[0,1]$ with means 0 and covariance functions, for $0 \leqslant s \leqslant t \leqslant 1$,

$$\operatorname{Cov}\Big(\widetilde{Z}_{\mathsf{Q}}(s), \widetilde{Z}_{\mathsf{Q}}(t)\Big) = \frac{\lambda_1}{2}s^2(1-t)^2, \tag{7.40}$$
$$\operatorname{Cov}\Big(\widetilde{Z}_{\mathsf{R}}(s), \widetilde{Z}_{\mathsf{R}}(t)\Big) = \lambda_2 s(1-t), \tag{7.41}$$

$$\operatorname{Cov}\Bigl(\widetilde{Z}_{\mathsf{N}}(s),\widetilde{Z}_{\mathsf{N}}(t)\Bigr)=s(1-t). \tag{7.42}$$

Proof. This is similar to the proofs of Lemmas 7.1 and 7.3. We now define the martingale

$$M^{(n)}(t)=\bigl(M^{(n)}_{\mathsf{Q}}(t),M^{(n)}_{\mathsf{R}}(t),M^{(n)}_{\mathsf{N}}(t)\bigr):=\bigl(\beta_n^{-1}\breve{Q}^{(n)}(t),\beta_n^{-1}\breve{R}^{(n)}(t),n^{-1/2}\breve{N}^{(n)}(t)\bigr). \tag{7.43}$$

For any fixed $t\in[0,1)$, (6.25), (6.27), (6.29) and (6.56)–(6.58) together with (7.32)–(7.33) show that the matrix of quadratic covariations has expectation

$$\mathbb{E}\,[M^{(n)},(M^{(n)})']_t\to\breve{\Sigma}(t):=\begin{pmatrix}\frac{\lambda_1}{2}\frac{t^2}{(1-t)^2} & 0 & 0\\ 0 & \lambda_2\frac{t}{1-t} & 0\\ 0 & 0 & \frac{t}{1-t}\end{pmatrix}. \tag{7.44}$$

Similarly, Lemmas 6.2 and 6.3 together with (7.31)–(7.34) show that (omitting superscripts $^{(n)}$ for convenience), for any fixed $t\in[0,1)$,

$$\begin{aligned}\operatorname{Var}\,[M_{\mathsf{Q}},M_{\mathsf{Q}}]_t&=\beta_n^{-4}\operatorname{Var}\,[\breve{Q},\breve{Q}]_t\leqslant C\beta_n^{-4}\sum_{i=1}^n d_i^3\\&\leqslant C\beta_n^{-4}\Delta^2\sum_{i=1}^n d_i=C\Bigl(\frac{\Delta}{\beta_n}\Bigr)^2\cdot\frac{2|E|}{\beta_n^2}\to0,\end{aligned} \tag{7.45}$$

$$\begin{aligned}\operatorname{Var}\,[M_{\mathsf{R}},M_{\mathsf{R}}]_t&=\beta_n^{-4}\operatorname{Var}\,[\breve{R},\breve{R}]_t\leqslant C\beta_n^{-4}\sum_{i=1}^n(d_i-\bar{d})^4\\&\leqslant C\beta_n^{-4}\Delta^2\sum_{i=1}^n(d_i-\bar{d})^2=C\Bigl(\frac{\Delta}{\beta_n}\Bigr)^2\cdot\frac{1}{\beta_n^2}\sum_{i=1}^n(d_i-\bar{d})^2\to0,\end{aligned} \tag{7.46}$$

$$\operatorname{Var}\,[M_{\mathsf{N}},M_{\mathsf{N}}]_t=n^{-2}\operatorname{Var}\,[\breve{N},\breve{N}]_t\leqslant Cn^{-1}\to0, \tag{7.47}$$

$$\operatorname{Var}\,[M_{\mathsf{Q}},M_{\mathsf{R}}]_t=\beta_n^{-4}\operatorname{Var}\,[\breve{Q},\breve{R}]_t\leqslant C\beta_n^{-4}\Delta^2\sum_{i=1}^n(d_i-\bar{d})^2\to0, \tag{7.48}$$

$$\begin{aligned}\operatorname{Var}\,[M_{\mathsf{Q}},M_{\mathsf{N}}]_t&=\beta_n^{-2}n^{-1}\operatorname{Var}\,[\breve{Q},\breve{N}]_t\leqslant C\beta_n^{-2}\frac{1}{n}\sum_{i=1}^n d_i^2\\&=C\beta_n^{-2}\Bigl(\frac{1}{n}\sum_{i=1}^n(d_i-\bar{d})^2+\bar{d}^2\Bigr)\leqslant\frac{C}{n}+C\left(\frac{\beta_n}{n}\cdot\frac{n\bar{d}}{\beta_n^2}\right)^2\to0,\end{aligned} \tag{7.49}$$

$$\operatorname{Var}\,[M_{\mathsf{R}},M_{\mathsf{N}}]_t=\beta_n^{-2}n^{-1}\operatorname{Var}\,[\breve{R},\breve{N}]_t\leqslant Cn^{-1}\beta_n^{-2}\sum_{i=1}^n(d_i-\bar{d})^2\leqslant Cn^{-1}\to0. \tag{7.50}$$

Hence, Proposition 5.4 applies and yields the result; the three components of the limit process are independent since the matrix $\breve{\Sigma}(t)$ in (7.44) is diagonal.

Finally, (ii) follows as in the proof of Lemma 7.1 by Lemma 5.5 and Remark 5.6, using (6.25), (6.27), and (6.56) together with (7.32)–(7.33). ∎

Proof of Theorem 3.6. As in the proof of Theorems 3.1 and 3.5, we may by conditioning assume that $G^{(n)}$ are non-random. Then Lemma 7.4 applies. As in the proof of Theorem 3.5, we first consider continuous time and derive an intermediate result.

(i): We now use (6.51) and (2.10) to write the decomposition (6.14) as

$$L^{(n)}(t) = \widetilde{Q}^{(n)}(t) + t\widetilde{R}^{(n)}(t) + t\bar{d}^{(n)}\widetilde{N}^{(n)}(t) + t^2\frac{n\bar{d}^{(n)}}{2}. \tag{7.51}$$

Hence, using also (6.10), cf. the regular case (7.24) where $\widetilde{R}^{(n)}(t) = 0$,

$$\begin{aligned} L^{(n)}(t) - \frac{\bar{d}^{(n)}}{2n}N^{(n)}(t)^2 &= L^{(n)}(t) - \frac{\bar{d}^{(n)}}{2n}\big(\widetilde{N}^{(n)}(t) + nt\big)^2 \\ &= \widetilde{Q}^{(n)}(t) + t\widetilde{R}^{(n)}(t) - \frac{\bar{d}^{(n)}}{2n}\widetilde{N}^{(n)}(t)^2. \end{aligned} \tag{7.52}$$

As a consequence of (7.39), we have $\widetilde{N}^{(n)}(t) = O_{\mathrm{p}}^{*}\big(n^{1/2}\big)$. Hence, (7.52) implies, using $\bar{d}^{(n)} \leqslant \Delta^{(n)} = o(\beta_n)$, (7.39), and Lemma 5.2,

$$\begin{aligned} \beta_n^{-1}\Big(L^{(n)}(t) - \frac{\bar{d}^{(n)}}{2n}N^{(n)}(t)^2\Big) &= \beta_n^{-1}\widetilde{Q}^{(n)}(t) + \beta_n^{-1}t\widetilde{R}^{(n)}(t) + O_{\mathrm{p}}^{*}(\bar{d}^{(n)}/\beta_n) \\ &= \beta_n^{-1}\widetilde{Q}^{(n)}(t) + t\beta_n^{-1}\widetilde{R}^{(n)}(t) + o_{\mathrm{p}}^{*}(1) \\ &\overset{\mathrm{d}}{\longrightarrow} \widetilde{Z}_{\mathsf{Q}}(t) + t\widetilde{Z}_{\mathsf{R}}(t). \end{aligned} \tag{7.53}$$

Furthermore, (7.39) implies that (7.53) holds jointly with $n^{-1/2}(N^{(n)}(t) - nt) \overset{\mathrm{d}}{\longrightarrow} \widetilde{Z}_{\mathsf{N}}(t)$. Thus by Theorem A.1 (with $b_n = f(t) = c = 0$), or directly by substituting $\tau_{\lfloor nt\rfloor}$ for t in (7.53),

$$\beta_n^{-1}\Big(\dot{L}_{\lfloor nt\rfloor} - \frac{\bar{d}^{(n)}}{2n}\lfloor nt\rfloor^2\Big) \overset{\mathrm{d}}{\longrightarrow} \dot{Z}(t) := \widetilde{Z}_{\mathsf{Q}}(t) + t\widetilde{Z}_{\mathsf{R}}(t). \tag{7.54}$$

Furthermore, we have

$$\frac{\bar{d}^{(n)}}{2n}\lfloor nt\rfloor^2 - t^2|E^{(n)}| = \frac{\bar{d}^{(n)}}{2n}\big(\lfloor nt\rfloor^2 - (nt)^2\big) = O(\bar{d}^{(n)}) = O(\Delta^{(n)}) = o\big(\beta_n\big), \tag{7.55}$$

and thus (7.54) implies (3.16). We have (3.17) by (7.40)–(7.41).

(ii): By (2.8), (6.10) and (7.39), we have, as in (7.28),

$$\begin{aligned} \frac{1}{n^{1/2}\bar{d}^{(n)}}\Big(\frac{\bar{d}^{(n)}}{2n}N^{(n)}(t)^2 - t^2|E^{(n)}|\Big) &= \frac{1}{2n^{3/2}}\big(\widetilde{N}^{(n)}(t)^2 + 2nt\widetilde{N}^{(n)}(t)\big) \\ &\overset{\mathrm{d}}{\longrightarrow} t\widetilde{Z}_{\mathsf{N}}(t). \end{aligned} \tag{7.56}$$

(The case $\bar{d}^{(n)} = 0$ is trivial and may be excluded.) Furthermore, this holds jointly with (7.53).

(ii)a: If $\alpha < \infty$, then (3.18) and (7.56) imply

$$\frac{1}{\beta_n}\Big(\frac{\bar{d}^{(n)}}{2n}N^{(n)}(t)^2 - t^2|E^{(n)}|\Big) \overset{\mathrm{d}}{\longrightarrow} \alpha t\widetilde{Z}_{\mathsf{N}}(t), \tag{7.57}$$

jointly with (7.53). Consequently, recalling also (2.8), (3.19) holds with

$$Z(t) := \dot{Z}(t) + \alpha t\widetilde{Z}_{\mathsf{N}} = \widetilde{Z}_{\mathsf{Q}}(t) + t\widetilde{Z}_{\mathsf{R}}(t) + \alpha t\widetilde{Z}_{\mathsf{N}}. \tag{7.58}$$

The covariance function (3.20) follows from (7.40)–(7.42).

(ii)b: If $\alpha = \infty$, then $\beta_n = o_{\mathrm{p}}(n^{1/2}\bar{d}^{(n)})$. In this case, $L^{(n)}(t)$ is dominated by the contribution from (7.56). More precisely, (7.53) now implies

$$\frac{1}{n^{1/2}\bar{d}^{(n)}}\Big(L^{(n)}(t) - \frac{\bar{d}^{(n)}}{2n}N^{(n)}(t)^2\Big) \overset{\mathrm{d}}{\longrightarrow} 0, \tag{7.59}$$

which together with (7.56) yields (3.21) with $Z(t) := t\widetilde{Z}_{\mathsf{N}}(t)$. By Remark 7.2, $\widetilde{Z}_{\mathsf{N}}(t)$ is a Brownian bridge $W^\circ(t)$, and the result follows. □

8. Number of components

Consider the case when G is a tree; in this case, the visible part of the graph is a forest. Let $\dot{K}_k^{(n)}$ and $K^{(n)}(t)$ be the number of components in the visible forest at time k or t, repectively, for the discrete-time and continuous-time versions.

Theorem 8.1. *Assume that $G^{(n)}$ is a sequence of deterministic or random trees with $V(G^{(n)}) = [n]$, and that (3.2) and (3.3) hold. Let $\gamma_* := \chi_* - 4$. For the number of components in the visible forest, we then have, in $D[0,1]$,*

$$n^{-1/2}\bigl(\dot{K}_{\lfloor nt\rfloor}^{(n)} - t(1-t)n\bigr) \xrightarrow{\mathrm{d}} \dot{Z}(t), \tag{8.1}$$

and

$$n^{-1/2}\bigl(K^{(n)}(t) - t(1-t)n\bigr) \xrightarrow{\mathrm{d}} Z(t), \tag{8.2}$$

where $\dot{Z}(t)$ and $Z(t)$ are continuous Gaussian processes with $\mathbb{E}\,\dot{Z}(t) = \mathbb{E}\,Z(t) = 0$ and covariance functions given by, for $0 \leqslant s \leqslant t \leqslant 1$,

$$\operatorname{Cov}\bigl(\dot{Z}(s), \dot{Z}(t)\bigr) = s^2(1-t)^2 + \gamma_* s^2 t(1-t), \tag{8.3}$$

as in (3.5), and

$$\operatorname{Cov}\bigl(Z(s), Z(t)\bigr) = s^2(1-t)^2 + \gamma_* s^2 t(1-t) + s(1-2s)(1-t)(1-2t). \tag{8.4}$$

Proof. In the discrete-time version, at time k the visible forest has k vertices and $\dot{L}_k^{(n)}$ edges, and thus

$$\dot{K}_k^{(n)} = k - \dot{L}_k^{(n)}. \tag{8.5}$$

Hence,

$$\dot{K}_{\lfloor nt\rfloor}^{(n)} - t(1-t)n = -\bigl(\dot{L}_{\lfloor nt\rfloor}^{(n)} - t^2 n\bigr) + O(1). \tag{8.6}$$

Corollary 3.3 applies and thus (3.4)–(3.7) hold, with $d_* = 2$ and thus $\gamma_* = \chi_* - 4$. Hence, (8.1) follows from (8.6) and (3.4), together with the fact that $-\dot{Z}(t) \overset{\mathrm{d}}{=} \dot{Z}(t)$ (as processes); the covariances (8.3) are given in (3.5).

Similarly, recalling (6.10),

$$\begin{aligned} K^{(n)}(t) &= N^{(n)}(t) - L^{(n)}(t) \\ &= nt(1-t) + \widetilde{N}^{(n)}(t) - \bigl(L^{(n)}(t) - t^2|E^{(n)}|\bigr) + O(1). \end{aligned} \tag{8.7}$$

Hence, (8.2) follows from (7.6) and (7.11) (which hold jointly), with

$$Z(t) := \widetilde{Z}_{\mathsf{N}}(t) - \bigl(\widetilde{Z}_{\mathsf{Q}}(t) + t\widetilde{Z}_{\mathsf{S}}(t)\bigr) = -\widetilde{Z}_{\mathsf{Q}}(t) - t\widetilde{Z}_{\mathsf{S}}(t) + \widetilde{Z}_{\mathsf{N}}(t), \tag{8.8}$$

and the covariances (8.4) follow from (8.8) and (7.7). Alternatively, (8.2) and (8.4) follow from Theorem A.4 with $a_n = n^{-1/2}$, $b_n = n$, $c = 1$ and $f(t) = t(1-t)$. □

Remark 8.2. The case $G = \mathsf{P}_n$ was studied in [19], where the results in Theorem 8.1 where proved for this case (using the same method as here), which solved a problem from [1]. The main results in [19] concern asymptotics of the maximum $\max_k \dot{K}_k^{(n)}$ and the difference $\max_k \dot{K}_k^{(n)} - \dot{K}_{\lceil n/2\rceil}^{(n)}$. (Note that the maximum is attained for k

close to $n/2$.) The proofs of these results in [19] are easily modified to the present more general case, using (8.1) or (8.2); we leave the details to the reader. △

Theorem 8.1 extends easily to forests $G^{(n)}$; again the visible part is always a forest. It would be interesting to have similar results for general graphs G, but this seems to require different methods.

Problem 8.3. Study the number of components in the visible part of G when G is not a forest.

9. Other small subgraphs

We have in this paper studied the evolution of the number of visible edges as a given graph is uncovered randomly; this number equals the number of visible copies of K_2. The methods in Sections 6–7 above can be used to show similar results for the number of visible copies of other small graphs. This does not seem to involve any new ideas, but the calculations become long and tedious, with more cases to treat, and since the paper already is long enough, we give only a brief sketch for one instance, the number of K_3 (triangles).

As in Section 6, we let G be a given non-random graph on $[n]$ and consider the continuous-time version of the uncovering process. Let $T(t)$ be the number of visible triangles in G at time $t \in [0,1]$. Similarly to (6.7) we have, using (6.1)–(6.2) and symmetry, and summing over ordered triples of distinct indices $i,j,k \in [n]$ satisfying the indicated conditions,

$$\begin{aligned}
6T(t) &= \sum_{ij,ik,jk\in E} I_i(t)I_j(t)I_k(t) = \sum_{ij,ik,jk\in E} \big(\widetilde{I}_i(t)+t\big)\big(\widetilde{I}_j(t)+t\big)\big(\widetilde{I}_k(t)+t\big)\\
&= \sum_{ij,ik,jk\in E} \widetilde{I}_i(t)\widetilde{I}_j(t)\widetilde{I}_k(t) + 3t\sum_{ij,ik,jk\in E} \widetilde{I}_i(t)\widetilde{I}_j(t) + 3t^2\sum_{ij,ik,jk\in E} \widetilde{I}_i(t) + 6t^3T(1)\\
&=: \widetilde{T}_1(t) + 3t\widetilde{T}_2(t) + 3t^2\widetilde{T}_3 + 6t^3T(1), \qquad (9.1)
\end{aligned}$$

where $T(1)$ is the (non-random) number of triangles in G. We define

$$\breve{T}_1(t) := (1-t)^{-3}\widetilde{T}_1(t) = \sum_{i,j,k:ij,ik,jk\in E} \breve{I}_i(t)\breve{I}_j(t)\breve{I}_k(t), \qquad (9.2)$$

$$\breve{T}_2(t) := (1-t)^{-2}\widetilde{T}_2(t) = \sum_{i,j:ij\in E} \delta_{ij}\breve{I}_i(t)\breve{I}_j(t) \qquad (9.3)$$

$$\breve{T}_3(t) := (1-t)^{-1}\widetilde{T}_3(t) = \sum_{i=1}^{n} 2\varepsilon_i\breve{I}_i(t), \qquad (9.4)$$

where δ_{ij} is the number of common neighbours of i and j, and ε_i is the number of triangles in G that contain i.

Similarly to Section 6, $\breve{T}_\ell$ ($\ell = 1,2,3$) are martingales on $[0,1)$, and (9.1) together with (9.2)–(9.4) yields a decomposition of $T(t)$ into them. The quadratic (co)variations and their expectations are found as in Section 6; we have for example

$$[\breve{T}_1,\breve{T}_1]_t = \sum_{i=1}^{n} \mathbf{1}\{T_i \leqslant t\}(1-T_i)^{-2}\Bigg(3\sum_{j,k:ij,ik,jk\in E} \breve{I}_j(t)\breve{I}_k(t)\Bigg)^2 \qquad (9.5)$$

and as a consequence

$$\begin{aligned}\mathbb{E}\,[\breve{T}_1,\breve{T}_1]_t &= \sum_{i=1}^{n}\int_0^t (1-s)^{-2}\cdot 36\varepsilon_i\,\mathbb{E}\left(\breve{I}_1(s)\breve{I}_2(s)\right)^2\,\mathrm{d}s\\ &= 36\sum_{i=1}^{n}\varepsilon_i\int_0^t \frac{s^2}{(1-s)^4}\,\mathrm{d}s\\ &= 12\sum_{i=1}^{n}\varepsilon_i\cdot\frac{t^3}{(1-t)^3} = 36T(1)\frac{t^3}{(1-t)^3}. \end{aligned}\tag{9.6}$$

Furthermore, it seems clear that it is possible to estimate the variances of the quadratic (co)variations by arguments similar to the ones in the proof of Lemma 6.2. Then, for a sequence of graphs $G^{(n)}$ with suitable hypotheses on vertex degrees and on other small structures in $G^{(n)}$ (in particular the number of triangles and the numbers of pairs of triangles sharing one or two vertices), Proposition 5.4 and Lemma 5.5 would apply and show joint convergence, after suitable normalization, of $\breve{T}_\ell(t)$ in $D[0,1)$ and of $\widetilde{T}_\ell(t)$ i $D[0,1]$, which by the decomposition (9.1) would yield convergence of $T(t)-t^3T(1)$ to a Gaussian process. Furthermore, corresponding results for $\dot{T}_{\lfloor nt\rfloor}$ would follow from Theorem A.1.

We have, however, not checked the details, nor found a precise set of conditions; we leave this to vigorous readers to explore further.

Appendix A. Derandomizing time

We consider in this appendix the problem of recovering results for a discrete-time stochastic process such as $\dot{L}_k$ from results for the corresponding continuous-time process $L(t)$. We adapt the method from e.g. [15; 16] to the present situation.

We state the result generally. We assume that (for each $n \geqslant 1$) we have a given set of n elements, which we may identify with $[n]$, and that we draw its elements one by one in random order (i.e., uniformly at random and without replacement); we assume also that we have a discrete-time stochastic process $(\dot{X}_k)_{k=0}^n = (\dot{X}_k^{(n)})_{k=0}^n$, where $\dot{X}_k$ is the value of some variable when we have drawn k objects. In the corresponding continuous-time model, we give, as in Section 2, each element $i \in [n]$ a random variable $T_i \in \mathsf{U}(0,1)$ representing the time when i is drawn; we assume that $T_1,\dots,T_n$ are independent. Then $X(t) = X^{(n)}(t)$ is the value of our variable at time $t \in [0,1]$. (The variables $\dot{X}_k$ and $X(t)$ may depend on other underlying random variables too; in that case, these variables are assumed to be independent of the order the elements are drawn and of $(T_i)_1^n$.)

We define $N(t) = N^{(n)}(t)$ and $\tau_k = \tau_k^{(n)}$ by (2.4) and (2.5), and note that, as in (2.6), we may for each n couple the discrete-time and continuous-time processes such that

$$X(t) = \dot{X}_{N(t)} \qquad \text{and conversely} \qquad \dot{X}_k = X(\tau_k), \tag{A.1}$$

for all $t \in [0,1]$ and $k = 0,\dots,n$, respectively. Note that the process $(N(t))_{t\in[0,1]}$, which describes the collection of times $\{T_i\}_1^n$, is stochastically independent of the order of these times, and thus of the process $(\dot{X}_k)_k$. Moreover, $n^{-1}N(t)$ is the empirical distribution of $\{T_i\}_1^n$, and thus, as noted in Remark 7.2, it is well known, see e.g. [4, Theorems 16.4 and 13.1], that, as $n \to \infty$,

$$n^{-1/2}\big(N^{(n)}(t) - nt\big) \xrightarrow{\mathrm{d}} W^\circ(t) \qquad \text{in } D[0,1], \tag{A.2}$$

where W° is a Brownian bridge, see Section 2.1 and in particular (2.1). (In fact, we have proved (A.2) as part of Lemmas 7.1, 7.3, and 7.4.)

We say that a function $f(t)$ is continuously differentiable on $[0,1]$ if it is continuously differentiable in $(0,1)$ and $f'(t)$ extends continuously to $[0,1]$.

Theorem A.1. *Suppose that, for each n, $(\dot X_k^{(n)})_{k=0}^n$ and $(X^{(n)}(t))_{t\in[0,1]}$ are stochastic processes as above; in particular we assume that* (A.1) *holds. Suppose also that $(a_n)_1^\infty$ and $(b_n)_1^\infty$ are sequences of positive numbers, that $f(t)$ is a continuously differentiable function on $[0,1]$, and that $\big(Z(t), W(t)\big)_{t\in[0,1]}$ is a continuous 2-dimensional Gaussian process on $[0,1]$ such that, as $n \to \infty$, in $D[0,1]$,*

$$\big(a_n(X^{(n)}(t) - b_n f(t)), n^{-1/2}(N^{(n)}(t) - nt)\big) \xrightarrow{\mathrm{d}} \big(Z(t), W(t)\big). \tag{A.3}$$

Suppose further that $n^{-1/2}a_n b_n \to c \in [0,\infty)$. Then,

$$a_n\big(\dot X_{\lfloor nt\rfloor}^{(n)} - b_n f(t)\big) \xrightarrow{\mathrm{d}} \dot Z(t) := Z(t) - cf'(t)W(t) \qquad \textit{in } D[0,1]. \tag{A.4}$$

Moreover, $\dot Z(t)$ is also a continuous Gaussian process on $[0,1]$, it has mean $\mathbb{E}\,\dot Z(t) = \mathbb{E}\,Z(t)$, and covariance function

$$\operatorname{Cov}(\dot Z(s), \dot Z(t)) = \operatorname{Cov}(Z(s), Z(t)) - c^2 s(1-t)f'(s)f'(t), \qquad 0 \leqslant s \leqslant t \leqslant 1. \tag{A.5}$$

Remark A.2. By (A.2), (A.3) implies that $(W(t))_{t\in[0,1]} \overset{\mathrm{d}}{=} (W^\circ(t))_{t\in[0,1]}$, so $W(t)$ is just a Brownian bridge. Nevertheless, we keep the notation $W(t)$ since in general $Z(t)$ and $W(t)$ are dependent, and their joint distribution is important in (A.4). We may also note that (A.3) is equivalent to the limits (A.2) and

$$a_n\big(X^{(n)}(t) - b_n f(t)\big) \xrightarrow{\mathrm{d}} Z(t) \tag{A.6}$$

holding jointly, for some particular coupling of $Z(t)$ and $W^\circ(t)$ (which we then denote by $W(t)$). △

Proof. By replacing $\dot X_k^{(n)}$, $X^{(n)}(t)$, and b_n by $a_n\dot X_k^{(n)}$, $a_nX^{(n)}(t)$, and a_nb_n, respectively, we may for convenience assume that $a_n = 1$ for all n.

By the Skorohod coupling theorem [24, Theorem 4.30], we may assume that the limit in (A.3) holds a.s. Since convergence in $D[0,1]$ to a continuous limit is equivalent to uniform convergence, this means that a.s., as $n \to \infty$,

$$X^{(n)}(t) = b_n f(t) + Z(t) + o(1), \tag{A.7}$$

$$N^{(n)}(t) = nt + n^{1/2}W(t) + o\big(n^{1/2}\big), \tag{A.8}$$

uniformly for $t \in [0,1]$. In particular, substituting $t = \tau_k^{(n)}$ in (A.8), we obtain, a.s.,

$$k = N^{(n)}(\tau_k^{(n)}) = n\tau_k^{(n)} + n^{1/2}W(\tau_k^{(n)}) + o\big(n^{1/2}\big), \tag{A.9}$$

and thus

$$\tau_k^{(n)} = \frac{k}{n} - n^{-1/2}W(\tau_k^{(n)}) + o\big(n^{-1/2}\big), \tag{A.10}$$

uniformly for $0 \leqslant k \leqslant n < \infty$. Since $W(t)$ is a continuous function of t, it is bounded (with a random bound), and thus (A.10) implies in particular that a.s.

$$\tau_k^{(n)} = \frac{k}{n} + o(1) \qquad \text{uniformly in } k = 0, \dots, n. \tag{A.11}$$

Furthermore, $W(t)$ is uniformly continuous on the compact interval $[0,1]$, and thus (A.11) implies $W(\tau_k^{(n)}) - W(k/n) = o(1)$, uniformly in k. Consequently, using (A.10) again, a.s.

$$\tau_k^{(n)} = \frac{k}{n} - n^{-1/2}W(k/n) + o\bigl(n^{-1/2}\bigr), \tag{A.12}$$

and similarly, uniformly for $t \in [0,1]$,

$$\tau_{\lfloor nt \rfloor}^{(n)} = t - n^{-1/2}W_t + o\bigl(n^{-1/2}\bigr) = t + o(1). \tag{A.13}$$

We substitute this in (A.7) and obtain a.s., recalling (A.1) and the fact that $Z(t)$ is uniformly continuous, and using a Taylor expansion of f,

$$\begin{aligned} \dot{X}_{\lfloor nt \rfloor}^{(n)} &= X^{(n)}(\tau_{\lfloor nt \rfloor}^{(n)}) = b_n f\bigl(\tau_{\lfloor nt \rfloor}^{(n)}\bigr) + Z(\tau_{\lfloor nt \rfloor}^{(n)}) + o(1) \\ &= b_n f\bigl(t - n^{-1/2}W(t) + o\bigl(n^{-1/2}\bigr)\bigr) + Z(t) + o(1) \\ &= b_n f(t) - b_n f'(t) n^{-1/2} W(t) + o\bigl(b_n n^{-1/2}\bigr) + Z(t) + o(1), \end{aligned} \tag{A.14}$$

uniformly for $t \in [0,1]$. The result (A.4) follows, since $b_n n^{-1/2} = a_n b_n n^{-1/2} \to c$, and in particular $b_n n^{-1/2} = O(1)$.

Finally, we recall that, as noted above, $(\dot{X}_k^{(n)})_k$ and $(N^{(n)}(t))_t$ are independent for each n. Since the proof shows that the limits (A.3) and (A.4) hold jointly, we conclude that the limits $(\dot{Z}(t))$ and $(W(t))$ are independent. Since $Z(t) = \dot{Z}(t) + cf'(t)W(t)$ by (A.4), it follows that

$$\operatorname{Cov}(Z(s), Z(t)) = \operatorname{Cov}(\dot{Z}(s), \dot{Z}(t)) + c^2 f'(s) f'(t) s(1-t), \qquad 0 \leqslant s \leqslant t \leqslant 1, \tag{A.15}$$

and thus (A.5) holds. ∎

Remark A.3. As just noted, the proof shows that the limit (A.4) holds jointly with (A.3), and furthermore that $(\dot{Z}(t))$ is independent of $(W(t))$. In particular, for all $s, t \in [0,1]$, we have $\operatorname{Cov}(\dot{Z}(s), W(t)) = 0$, and thus by (A.4) and (2.1) necessarily

$$\operatorname{Cov}(Z(s), W(t)) = cf'(s) \operatorname{Cov}(W(s), W(t)) = \begin{cases} cf'(s)s(1-t), & 0 \leqslant s \leqslant t \leqslant 1, \\ cf'(s)(1-s)t, & 0 \leqslant t \leqslant s \leqslant 1. \end{cases} \tag{A.16}$$

△

We note also that a converse to Theorem A.1 holds.

Theorem A.4. *Suppose that (as in Theorem A.1) $(\dot{X}_k^{(n)})_0^n$ and $(X^{(n)}(t))_{t \in [0,1]}$ are stochastic processes as above, that $(a_n)_1^\infty$ and $(b_n)_1^\infty$ are positive numbers, and that $f(t)$ is a continuously differentiable function on $[0,1]$. Suppose further that $(\dot{Z}(t))_{t \in [0,1]}$ is a continuous Gaussian process such that*

$$a_n\bigl(\dot{X}_{\lfloor nt \rfloor}^{(n)} - b_n f(t)\bigr) \xrightarrow{\text{d}} \dot{Z}(t) \qquad \text{in } D[0,1]. \tag{A.17}$$

Suppose also that $a_n b_n/\sqrt{n} \to c \in [0,\infty)$*. Then*

$$a_n\bigl(X^{(n)}(t) - b_n f(t)\bigr) \xrightarrow{\mathrm{d}} Z(t) := \dot{Z}(t) + cf'(t)W^\circ(t) \qquad \text{in } D[0,1], \tag{A.18}$$

where $(W^\circ(t))_{t\in[0,1]}$ *is a Brownian bridge independent of* $(\dot{Z}(t))_{t\in[0,1]}$.

Moreover, $Z(t)$ *is also a continuous Gaussian process on* $[0,1]$*, it has mean* $\mathbb{E}\, Z(t) = \mathbb{E}\,\dot{Z}(t)$*, and covariance function*

$$\operatorname{Cov}(Z(s),Z(t)) = \operatorname{Cov}(\dot{Z}(s),\dot{Z}(t)) + c^2 s(1-t)f'(s)f'(t), \qquad 0 \leqslant s \leqslant t \leqslant 1. \tag{A.19}$$

Proof. We argue as in the proof of Theorem A.1, and we may again assume $a_n = 1$. First, as noted above, $(\dot{X}_k^{(n)})_k$ and $(N^{(n)}(t))_t$ are independent for each n; hence, the limits (A.17) and (A.2) hold jointly, with independent limits $(\dot{Z}(t))_t$ and $(W^\circ(t))_t$. Consequently, we may by the Skorohod coupling theorem assume that both (A.17) and (A.2) hold a.s., and thus

$$\dot{X}^{(n)}_{\lfloor nt\rfloor} = b_n f(t) + \dot{Z}(t) + o(1), \tag{A.20}$$

$$N^{(n)}(t) = nt + n^{1/2}W^\circ(t) + o\bigl(n^{1/2}\bigr), \tag{A.21}$$

uniformly for $t \in [0,1]$. Consequently, a.s.,

$$n^{-1}N^{(n)}(t) = t + n^{-1/2}W^\circ(t) + o\bigl(n^{-1/2}\bigr), \tag{A.22}$$

and, using (A.1), the uniform continuity of $\dot{Z}(t)$, and the continuous differentiability of f,

$$\begin{aligned} X^{(n)}(t) = \dot{X}^{(n)}_{N^{(n)}(t)} &= b_n f\bigl(n^{-1}N^{(n)}(t)\bigr) + \dot{Z}\bigl(n^{-1}N^{(n)}(t)\bigr) + o(1) \\ &= b_n f(t) + b_n f'(t)n^{-1/2}W^\circ(t) + o\bigl(b_n n^{-1/2}\bigr) + \dot{Z}\bigl(t\bigr) + o(1) \end{aligned} \tag{A.23}$$

uniformly for $t \in [0,1]$, which yields (A.18).

The covariance formula (A.19) follows from (2.1) and the independence of $(\dot{Z}(t))_t$ and $(W^\circ(t))_t$. ☐

Remark A.5. Theorems A.1 and A.4 extend to vector-valued $\dot{X}_k$ and $X(t)$, *mutatis mutandis*; we may assume that all $\dot{X}_k$ and $X(t)$ take their values in $\mathbb{R}^q$ for some fixed $q \geqslant 1$ (not depending on k or n), and that then also $f(t)$ and $Z(t)$ or $\dot{Z}(t)$ take their values in $\mathbb{R}^q$. △

Remark A.6. We have assumed in Theorem A.1 that $(Z(t),W(t))$ is a Gaussian process, since this is the case we use. The theorem is valid (with the same proof) for any continuous stochastic process $(Z(t),W(t))$, except that (of course) $\dot{Z}(t)$ then is not necessarily Gaussian, and that (A.5) requires that the processes have finite variances.

Similarly, Theorem A.4 holds for any continuous stochastic process $\dot{Z}(t)$, with corresponding modifications. △

Appendix B. On degree distributions in some random trees

We apply our main results to several classes of random trees in examples in Section 4. In order to do so, we have to verify the condition (3.2), which says that the average of the squared degrees of the vertices converges to χ_* in probability. While this is closely related to known results on the degree distribution in the random trees, we do not know any references stating precisely this result. However, in this appendix we show that for the random trees considered in Section 4, (3.2) easily follows from known results. We also show that the condition (3.3) on the maximum degree holds for these trees.

Throughout this section, we assume that $G^{(n)}$ $(n \geqslant 1)$ is some sequence of random trees with $V(G^{(n)}) = [n]$. Let D_n be the degree of a random vertex of $G^{(n)}$. Then (3.2) can be written

$$\mathbb{E}\left(D_n^2 \mid G^{(n)}\right) \xrightarrow{\text{p}} \chi_*. \tag{B.1}$$

Our uncovering problem and the results for it in Section 3 are stated for unrooted graphs and trees, but they can of course be applied also to rooted trees by regarding them as unrooted, forgetting the choice of root. Indeed, most of our examples of random trees in Section 4 consider rooted trees. For rooted trees, it is usually more convenient to consider outdegrees. We assume (without loss of generality) that the root is vertex 1, and we denote the outdegree of vertex i by $\hat{d}_i$. Thus

$$d_i = \hat{d}_i + \mathbf{1}\{i \neq 1\}. \tag{B.2}$$

Similarly, we let $\widehat{D}_n$ denote the outdegree of a random vertex of $G^{(n)}$. We then have the following simple reformulations of (3.2) and (B.1).

Lemma B.1. *Suppose that $G^{(n)}$ is a sequence of rooted trees with $V(G^{(n)}) = [n]$, and suppose that* (3.3) *holds. Then* (3.2) *is equivalent to*

$$\widehat{\chi}^{(n)} := \frac{1}{n}\sum_{i=1}^{n}\bigl(\hat{d}_i^{(n)}\bigr)^2 \xrightarrow{\text{p}} \widehat{\chi}_* := \chi_* - 3, \tag{B.3}$$

and thus also to

$$\mathbb{E}\left(\widehat{D}_n^2 \mid G^{(n)}\right) \xrightarrow{\text{p}} \widehat{\chi}_*. \tag{B.4}$$

Proof. By (B.2), we have, using $\sum_{i=1}^n \hat{d}_i^{(n)} = n-1$ and (3.3),

$$\begin{aligned}\sum_{i=1}^{n}\bigl(d_i^{(n)}\bigr)^2 &= \sum_{i=1}^{n}\Bigl(\bigl(\hat{d}_i^{(n)}\bigr)^2 + 2\hat{d}_i^{(n)} + 1\Bigr) - 2\hat{d}_1^{(n)} - 1\\ &= \sum_{i=1}^{n}\bigl(\hat{d}_i^{(n)}\bigr)^2 + 2(n-1) + n - 2\hat{d}_1^{(n)} - 1\\ &= \sum_{i=1}^{n}\bigl(\hat{d}_i^{(n)}\bigr)^2 + 3n + o(n).\end{aligned} \tag{B.5}$$

The result follows by dividing by n. □

B.1. **Preliminaries.** We let $\mathcal{P}(\mathbb{N})$ be the space of probability distributions on $\mathbb{N} = \{0, 1, \dots\}$. We give $\mathcal{P}(\mathbb{N})$ the standard weak topology (for the space of probability measures on any metric space), see e.g. [4]. Thus, since $\mathbb{N}$ is discrete, if X_n $(n \geqslant 1)$ and X are random variables with values in $\mathbb{N}$, then convergence in $\mathcal{P}(\mathbb{N})$ of the distributions

$$\mathcal{L}(X_n) \to \mathcal{L}(X) \tag{B.6}$$

means

$$\mathbb{E} f(X_n) \to \mathbb{E} f(X) \qquad \text{for every bounded function } f : \mathbb{N} \to \mathbb{R}. \tag{B.7}$$

It is well known [11, Theorem 5.6.4] that, again since $\mathbb{N}$ is discrete, this is equivalent both to convergence of point probabilities

$$\mathbb{P}(X_n = d) \to \mathbb{P}(X = d) \qquad \text{for every } d \in \mathbb{N}, \tag{B.8}$$

and also to convergence in total variation:

$$\sum_{d=0}^{\infty} \bigl|\mathbb{P}(X_n = d) - \mathbb{P}(X = d)\bigr| \to 0. \tag{B.9}$$

Consider now a sequence $G^{(n)}$ of random trees, as usual with $V(G^{(n)}) = [n]$. Then the conditional distribution $\mathcal{L}(D_n \mid G^{(n)})$ is a random distribution on $\mathbb{N}$, i.e. a random element of $\mathcal{P}(\mathbb{N})$. The equivalence of (B.6)–(B.9) above transfers to convergence in probability of random distributions in $\mathcal{P}(\mathbb{N})$, and in particular, in our situation, we have the following (note that the right-hand side of (B.10) is a constant element of $\mathcal{P}(\mathbb{N})$):

Lemma B.2. *For any random variable $\zeta \in \mathbb{N}$, the following are equivalent:*

$$\mathcal{L}\bigl(D_n \mid G^{(n)}\bigr) \xrightarrow{\text{p}} \mathcal{L}(\zeta), \tag{B.10}$$

$$\mathbb{E}\bigl(f(D_n) \mid G_n\bigr) \xrightarrow{\text{p}} \mathbb{E} f(\zeta) \quad \textit{for every bounded } f : \mathbb{N} \to \mathbb{R}, \tag{B.11}$$

$$\mathbb{P}\bigl(D_n = d \mid G^{(n)}\bigr) \xrightarrow{\text{p}} \mathbb{P}(\zeta = d) \quad \textit{for every fixed } d \in \mathbb{N}, \tag{B.12}$$

$$\sum_{d=0}^{\infty} \bigl|\mathbb{P}(D_n = d \mid G^{(n)}) - \mathbb{P}(\zeta = d)\bigr| \xrightarrow{\text{p}} 0. \tag{B.13}$$

If $G^{(n)}$ are rooted trees, we also have the same equivalences with $\widehat{D}_n$ instead of D_n.

Proof. Using the fact that (in any metric space) a sequence converging in probability has a subsequence converging a.s., this follows easily from the equivalence of (B.6)–(B.9); we omit the details. ☐

We state a simple lemma in a general form; recall that $\Delta^{(n)}$ denotes the maximum degree in $G^{(n)}$. We are mainly interested in the case $f(x) = x^2$; note that in this case, (B.16) and (B.17) are equivalent to (3.2) and (3.3), with $\chi_* = \mathbb{E} f(\zeta) = \mathbb{E}\,\zeta^2$.

Lemma B.3. *Let ζ be a random variable with values in $\mathbb{N}$ and let $f : \mathbb{N} \to [0, \infty)$ be any function such that $\mathbb{E} f(\zeta) < \infty$. If, as $n \to \infty$,*

$$\mathcal{L}(D_n \mid G^{(n)}) \xrightarrow{\text{p}} \mathcal{L}(\zeta) \tag{B.14}$$

and

$$\mathbb{E} f(D_n) \to \mathbb{E} f(\zeta), \tag{B.15}$$

then

$$\mathbb{E}\,(f(D_n) \mid G^{(n)}) \xrightarrow{\text{p}} \mathbb{E} f(\zeta), \tag{B.16}$$

$$f(\Delta^{(n)})/n \xrightarrow{\text{p}} 0. \tag{B.17}$$

If $G^{(n)}$ are rooted trees, we also have the same equivalences with $\widehat{D}_n$ instead of D_n and $\widehat{\Delta}^{(n)} := \max_i \hat{d}_i^{(n)} \geqslant \Delta^{(n)} - 1$ instead of $\Delta^{(n)}$.

Proof. We use truncations. For $M \geqslant 0$, let $f_M(x) := f(x) \wedge M$. Then each f_M is bounded, and thus by (B.14) and Lemma B.2,

$$\mathbb{E}\left(f_M(D_n) \mid G^{(n)}\right) \xrightarrow{\mathrm{p}} \mathbb{E} f_M(\zeta). \tag{B.18}$$

Furthermore, taking the expectation in (B.18), we obtain by dominated convergence (for convergence in probability, see e.g. [11, Theorem 5.5.4]), again because f_M is bounded,

$$\mathbb{E} f_M(D_n) \to \mathbb{E} f_M(\zeta). \tag{B.19}$$

We have $f_M(D_n) \leqslant f(D_n)$ and thus, using Markov's inequality, for any $\varepsilon > 0$.

$$\begin{aligned}\mathbb{P}\bigl[\mathbb{E}\left(f(D_n) \mid G^{(n)}\right) - \mathbb{E}\left(f_M(D_n) \mid G^{(n)}\right) > \varepsilon\bigr]\\ &= \mathbb{P}\bigl[\mathbb{E}\left(f(D_n) - f_M(D_n) \mid G^{(n)}\right) > \varepsilon\bigr]\\ &\leqslant \varepsilon^{-1}\,\mathbb{E}\bigl[\mathbb{E}\left(f(D_n) - f_M(D_n) \mid G^{(n)}\right)\bigr]\\ &= \varepsilon^{-1}\,\mathbb{E}\bigl[f(D_n) - f_M(D_n)\bigr]. \end{aligned} \tag{B.20}$$

Hence, by (B.15) and (B.19),

$$\begin{aligned}\limsup_{n\to\infty} \mathbb{P}\bigl[\mathbb{E}\left(f(D_n) \mid G^{(n)}\right) - \mathbb{E}\left(f_M(D_n) \mid G^{(n)}\right) > \varepsilon\bigr]\\ \leqslant \varepsilon^{-1} \limsup_{n\to\infty} \mathbb{E}\bigl[f(D_n) - f_M(D_n)\bigr] = \varepsilon^{-1}\bigl(\mathbb{E} f(\zeta) - \mathbb{E} f_M(\zeta)\bigr). \end{aligned} \tag{B.21}$$

By monotone convergence,

$$\mathbb{E} f_M(\zeta) \to \mathbb{E} f(\zeta) \qquad \text{as } M \to \infty, \tag{B.22}$$

and thus

$$\lim_{M\to\infty} \limsup_{n\to\infty} \mathbb{P}\bigl[\mathbb{E}\left(f(D_n) \mid G^{(n)}\right) - \mathbb{E}\left(f_M(D_n) \mid G^{(n)}\right) > \varepsilon\bigr] = 0. \tag{B.23}$$

We have, for any $\varepsilon > 0$,

$$\begin{aligned}\mathbb{P}\bigl[\bigl|\mathbb{E}\left(f(D_n) \mid G^{(n)}\right) - \mathbb{E} f(\zeta)\bigr| > 3\varepsilon\bigr]\\ &\leqslant \mathbb{P}\bigl[\bigl|\mathbb{E}\left(f(D_n) \mid G^{(n)}\right) - \mathbb{E}\left(f_M(D_n) \mid G^{(n)}\right)\bigr| > \varepsilon\bigr]\\ &\qquad + \mathbb{P}\bigl[\bigl|\mathbb{E}\left(f_M(D_n) \mid G^{(n)}\right) - \mathbb{E} f_M(\zeta)\bigr| > \varepsilon\bigr]\\ &\qquad + \mathbb{P}\bigl[\bigl|\mathbb{E} f_M(\zeta) - \mathbb{E} f(\zeta)\bigr| > \varepsilon\bigr] \end{aligned} \tag{B.24}$$

Taking first the limsup as $n \to \infty$, and then letting $M \to \infty$, we see from (B.23), (B.18), and (B.22) that the right-hand side tends to 0, which yields (B.16). (Alternatively, since convergence in probability to a constant is the same as convergence in distribution to the same constant, [4, Theorem 4.2] shows that (B.22) and (B.23) enable us to let $M \to \infty$ in (B.18), which yields (B.16).)

Moreover, $\mathbb{P}\bigl(D_n = \Delta^{(n)} \mid G^{(n)}\bigr) \geqslant 1/n$, and thus, for any $M \geqslant 0$,

$$f(\Delta^{(n)}) \leqslant f(\Delta^{(n)}) - f_M(\Delta^{(n)}) + M \leqslant n\,\mathbb{E}\left(f(D_n) - f_M(D_n) \mid G^{(n)}\right) + M. \tag{B.25}$$

Hence, for any $\varepsilon > 0$ and $M \geqslant 0$, we have for $n > 2M/\varepsilon$,

$$\mathbb{P}[f(\Delta^{(n)})/n > \varepsilon] \leqslant \mathbb{P}\bigl[\mathbb{E}\left(f(D_n) - f_M(D_n) \mid G^{(n)}\right) > \varepsilon/2\bigr], \tag{B.26}$$

and consequently,

$$\limsup_{n\to\infty} \mathbb{P}[f(\Delta^{(n)})/n > \varepsilon] \leqslant \limsup_{n\to\infty} \mathbb{P}\bigl[\mathbb{E}\left(f(D_n) - f_M(D_n) \mid G^{(n)}\right) > \varepsilon/2\bigr]. \tag{B.27}$$

This holds for every $M \geqslant 0$, and thus (B.23) shows that

$$\lim_{n\to\infty} \mathbb{P}\big(f(\Delta^{(n)})/n > \varepsilon\big) = 0 \tag{B.28}$$

for every $\varepsilon > 0$, which is (B.17).

The proof for $\widehat{D}^{(n)}$ is the same. ☐

B.2. **Conditioned Galton–Watson trees.** Let, as in Example 4.2, $G^{(n)}$ be a conditioned Galton–Watson tree with n vertices defined with offspring distributed as a random variable $\xi \in \mathbb{N}$ with $\mathbb{E}\,\xi = 1$ and $0 < \operatorname{Var}\xi < \infty$. Recall that this means that $G^{(n)}$ is obtained by conditioning a Galton–Watson tree $\mathcal{T}$ on having exactly n vertices, where in $\mathcal{T}$, every vertex has a number of children distributed as independent copies of ξ; see e.g. [20]. We denote the distribution of ξ by $\mathcal{L}(\xi) \in \mathcal{P}(\mathbb{N})$; this is known as the offspring distribution.

We regard the trees $G^{(n)}$ and $\mathcal{T}$ as rooted and ordered.

It is well known that the asymptotic degree distribution of $G^{(n)}$ is the offspring distribution $\mathcal{L}(\xi)$, i.e., $\widehat{D}_n \xrightarrow{\mathrm{d}} \xi$ as $n \to \infty$; moreover, this holds also (in probability) conditioned on $G^{(n)}$, i.e.,

$$\mathcal{L}(\widehat{D}_n \mid G^{(n)}) \xrightarrow{\mathrm{p}} \mathcal{L}(\xi), \tag{B.29}$$

see [20, Theorem 7.11(ii)]. (The notation is different there, but [20, Theorem 7.11(ii)] is equivalent to $\mathbb{P}(\widehat{D}_n = d \mid G^{(n)}) \xrightarrow{\mathrm{p}} \mathbb{P}(\xi = d)$, which is equivalent to (B.29) by Lemma B.2.) Furthermore, let $f(x) := \binom{x}{2} = x(x-1)/2$. Then,

$$\mathbb{E}\left(f(\widehat{D}_n) \mid G^{(n)}\right) = \frac{1}{n}\sum_{i=1}^{n} \frac{\hat{d}_i(\hat{d}_i - 1)}{2}. \tag{B.30}$$

Denote the sum in (B.30) by Υ_n, and note that if $\mathbf{t}$ is the rooted tree consisting of a root with two children, then Υ_n is the number of subtrees of $G^{(n)}$ that are isomorphic to $\mathbf{t}$, where we consider general subtrees, and consider each subtree as rooted with the same parent-child relations as in $G^{(n)}$. This number Υ_n is treated (for general rooted and ordered trees $\mathbf{t}$) in [21, Theorem 1.1], which shows, in particular, that under our conditions $\mathbb{E}\,\xi = 1$ and $\mathbb{E}\,\xi^2 < \infty$, we have

$$\frac{\Upsilon_n}{n} \xrightarrow{\mathrm{p}} \mathbb{E}\,\frac{\xi(\xi-1)}{2} = \mathbb{E}\,f(\xi), \tag{B.31}$$

and

$$\mathbb{E}\,\frac{\Upsilon_n}{n} \longrightarrow \mathbb{E}\,\frac{\xi(\xi-1)}{2} = \mathbb{E}\,f(\xi). \tag{B.32}$$

By (B.30), these are precisely (B.16) and (B.15), for $\widehat{D}_n$, with $\zeta = \xi$. Since trivially $\mathbb{E}\,D_n = (n-1)/n \to 1 = \mathbb{E}\,\xi$, (B.15) is trivial for $\widehat{D}_n$ and $f(x) = x$. Hence, still taking $\zeta = \xi$, it follows by linearity that (B.15) holds also for $f(x) = x^2$, and thus, recalling (B.29), Lemma B.3 shows (B.16) and (B.17) for $\widehat{D}_n$ and $f(x) = x^2$. By linearity again, we may also take $f(x) = (x+1)^2$ in (B.16), which (using (B.17)) easily is seen to be equivalent to (B.16) for D_n with $f(x) = x^2$ and $\zeta = \xi + 1$. Consequently, we have (3.2) and (3.3), with $\chi_* = \mathbb{E}\,(\xi+1)^2$. Since $d_* = 2 = \mathbb{E}\,(\xi+1)$, it follows that

$$\gamma_* := \chi_* - d_*^2 = \operatorname{Var}(\xi+1) = \operatorname{Var}\xi. \tag{B.33}$$

Remark B.4. It follows similarly from [21] (by taking $\mathbf{t}$ as a star with a root of degree $k \leqslant r$) that for any integer $r \geqslant 2$, if $\mathbb{E}\,\xi^r < \infty$, then

$$\frac{1}{n}\sum_{i=1}^{n} \hat{d}_i^r = \mathbb{E}\left(\widehat{D}_n^r \mid G^{(n)}\right) \xrightarrow{\mathrm{p}} \mathbb{E}\,\xi^r, \tag{B.34}$$

$$\frac{1}{n}\sum_{i=1}^{n} d_i^r = \mathbb{E}\left(D_n^r \mid G^{(n)}\right) \xrightarrow{\mathrm{p}} \mathbb{E}\,(\xi+1)^r. \tag{B.35}$$

In other words, the moments of the (out)degree distribution converge to the moments of ξ or $\xi+1$, provided the latter moments are finite. △

B.3. **The random recursive tree.** Let now $G^{(n)}$ be a random recursive tree. It is well known that the asymptotic outdegree distribution is geometric Ge(1/2); moreover, this holds conditioned on $G^{(n)}$ in the sense (B.12), see e.g. [25], [3, Section 3.2], [18, Theorem 1], [7, Theorem 6.8], and [13, Example 6.1]. Hence, Lemma B.2 shows that (B.10)–(B.13) hold for $\widehat{D}_n$ with $\zeta = \xi \sim \mathrm{Ge}(1/2)$; consequently they also hold for the total degree D_n with $\zeta = \xi + 1$.

We next verify (B.15), again first taking $f(x) := \binom{x}{2} = x(x-1)/2$. Since $G^{(n)}$ is constructed with vertices added in increasing order, $f(\hat{d}_i)$ is the number of pairs (j,k) with $i < j < k \leqslant n$ such that ij and ik are edges. Hence, since $\mathbb{P}(j \sim i) = 1/(j-1)$ for $j > i$, and these events for different j are independent,

$$\begin{aligned} n\,\mathbb{E}\,f(\widehat{D}_n) &= \sum_{i=1}^{n} \mathbb{E}\,f(\hat{d}_i) = \sum_{1\leqslant i<j<k\leqslant n} \mathbb{P}(j \sim i \text{ and } k \sim i) = \sum_{1\leqslant i<j<k\leqslant n} \frac{1}{(j-1)(k-1)} \\ &= \sum_{2\leqslant j<k\leqslant n} \frac{1}{k-1} = \sum_{3\leqslant k\leqslant n} \frac{k-2}{k-1} = n + O(\log n). \end{aligned} \tag{B.36}$$

Consequently, $\mathbb{E}\,f(\widehat{D}_n) \to 1$, which verifies (B.15) since $\mathbb{E}\,f(\xi) = \frac{1}{2}(\mathbb{E}\,\xi^2 - \mathbb{E}\,\xi) = \frac{1}{2}(\mathrm{Var}\,(\xi) + 1 - 1) = 1$.

We have shown (B.15) with $f = x(x-1)/2$, and, as in Section B.2, it follows by linearity that (B.15) holds also for $f(x) = x^2$, and furthermore that (B.15)–(B.17) hold for D_n and $f(x) = x^2$. Hence, (3.2) and (3.3) hold, with $\chi_* = \mathbb{E}\,(\xi+1)^2$.

Remark B.5. A similar calculation shows that (B.15) holds for $\widehat{D}_n$ and $f(x) := \binom{x}{r}$ for any integer $r \geqslant 2$; it follows that (B.15) holds also for $f(x) = x^r$, and thus Lemma B.3 shows that all moments of the outdegree distribution (given $G^{(n)}$) converge in probability to the corresponding moments of $\xi \sim \mathrm{Ge}(1/2)$. Hence, all moments of the degree distribution D_n converge in probability to the corresponding moments of $\xi + 1$. △

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