ROUNDING OF DISCRETE VARIABLES

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Abstract. Let X be a random variable that takes its values in $\frac{1}{q}\mathbb{Z}$, for some integer $q \geqslant 2$, and consider X rounded to an integer, either downwards or upwards or to the nearest integer. We give general formulas for the characteristic function and moments of the rounded variable. These formulas complement the related but different formulas in the case that X has a continuous distribution, which was studied by Janson (2006).

1. Introduction

Let X be a random variable and consider also X rounded to an integer; more precisely we may consider, for example, $\lfloor X \rfloor$ (rounding downwards) or $\langle X \rangle$ (rounding to the nearest integer), see Section 2 for precise definitions.

We gave in [2] general formulas for the characteristic function and moments (in particular, the first and second moments) of the rounded variable in the important case when X has a continuous distribution; this was motivated by several examples that had appeared as subsequence limits of integer-valued random variables in different problems.

In the present paper we consider instead the case of X with a discrete distribution. More precisely, we suppose that there exists an integer $q \geqslant 2$ such that X takes its values in $\frac{1}{q}\mathbb{Z}$, or equivalently that $qX \in \mathbb{Z}$ a.s. (For completeness, we allow also $q = 1$ below, but the results are trivial in this case.) Again, we give general formulas for the characteristic function and moments.

The results can be compared to those given in [2] for the case of a continuous distribution; the results are similar but different.

Remark 1.1. Our setting is obviously equivalent to rounding an integer-valued random variable to multiples of q, and the results can easily be translated to that case. △

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2. Notation

The characteristic function of a random variable X is denoted by $\varphi_X(t) := \mathbb{E}\, e^{\mathrm{i}tX}$.

The indicator function of an event $\mathcal{E}$ is denoted $\mathbf{1}\{\mathcal{E}\}$.

For two integers j and k, $j \mid k$ means that j is a divisor of k, i.e., that $k \in j\mathbb{Z}$.

For a real number x, let $\lfloor x \rfloor$ and $\lceil x \rceil$ denote x rounded downwards and upwards, respectively, to the nearest integer. Similarly, $\langle x \rangle$ denotes x rounded to the nearest

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integer, for definiteness choosing the larger one if there is a tie. Thus $\lfloor x \rfloor \leqslant x < \lfloor x \rfloor + 1$, $\lceil x \rceil - 1 < x \leqslant \lceil x \rceil$ and $\langle x \rangle - \frac{1}{2} \leqslant x < \langle x \rangle + \frac{1}{2}$; furthermore,

$$\lceil x \rceil = -\lfloor -x \rfloor, \tag{2.1}$$

$$\langle x \rangle = \left\lfloor x + \tfrac{1}{2} \right\rfloor. \tag{2.2}$$

Remark 2.1. Rounding downwards to $\lfloor X \rfloor$ is obviously an asymmetric operation; the mirror operation is to rounding upwards to $\lceil X \rceil$. Since $\lceil X \rceil = -\lfloor -X \rfloor$ by (2.1), we will for simplicity consider only $\lfloor X \rfloor$ and $\langle X \rangle$ below; results for $\lceil X \rceil$ follow from the results $\lfloor X \rfloor$ applied to $-X$. In particular, it is easily seen that (3.5) and (4.1) below hold for $\lceil X \rceil$ if we replace $h_q(t)$ by $h_q(-t)$.

Also rounding to $\langle X \rangle$ is slightly asymmetric since we choose the larger value when there is a tie. The alternative to choose the lower value is again the mirror operation given by $-\langle -X \rangle$, and results follow from the results below, now replacing $\widetilde{h}_q(t)$ by $\widetilde{h}_q(-t)$.

Note that when q is odd and qX is integer-valued, there cannot be a tie for the rounding $\langle X \rangle$, so the two versions coincide. This explains the greater symmetry in some of the results for odd q, in particular that $h_q(t)$ in (3.3) then is a symmetric function. △

3. The characteristic function

Let $q \geqslant 1$ be a fixed integer. Suppose that X is a random variable such that qX is integer-valued, i.e., $X \in \frac{1}{q}\mathbb{Z}$ (a.s.). Note that this assumption implies (and in fact is equivalent to) φ_X havíng period $2\pi q$. We consider the rounded variables $\lfloor X \rfloor$ and $\langle X \rangle$.

We first define two auxilliary functions. Let

$$h_q(t) := \frac{1}{q} \sum_{k=0}^{q-1} e^{-\mathrm{i}tk/q} = \frac{1 - e^{-\mathrm{i}t}}{q(1 - e^{-\mathrm{i}t/q})} = \frac{\sin \frac{t}{2}}{q \sin \frac{t}{2q}} e^{-\mathrm{i}\frac{q-1}{2q}t}, \tag{3.1}$$

where the two last formulas are interpreted by continuity as 1 when the denominator is 0 (i.e., when $t \in 2\pi q\mathbb{Z}$). Furthermore, let, if q is even,

$$\widetilde{h}_q(t) := \frac{1}{q} \sum_{k=-q/2}^{q/2-1} e^{-\mathrm{i}tk/q} = e^{\mathrm{i}t/2} h_q(t) = \frac{\sin \frac{t}{2}}{q \sin \frac{t}{2q}} e^{\frac{\mathrm{i}t}{2q}}, \tag{3.2}$$

and if q is odd

$$\widetilde{h}_q(t) := \frac{1}{q} \sum_{k=-(q-1)/2}^{(q-1)/2} e^{-\mathrm{i}tk/q} = e^{\mathrm{i}\frac{q-1}{2q}t} h_q(t) = \frac{\sin \frac{t}{2}}{q \sin \frac{t}{2q}}. \tag{3.3}$$

Note that these functions may be interpreted as characteristic functions. Let U_q be a random variable that is uniformly distributed on $\{\frac{k}{q} : k = 0, \dots, q-1\}$, and let $\widetilde{U}_q$ be a random variable that is uniformly distributed on $\{\frac{k}{q} : k = -\frac{q}{2}, \dots, \frac{q}{2} - 1\}$ (q even) or $\{\frac{k}{q} : k = -\frac{q-1}{2}, \dots, \frac{q-1}{2}\}$ (q odd). Thus, these random variables are uniformly distributed on the set of the q integer multiples of $1/q$ in $[0, 1)$ (U_q) and $[-\frac{1}{2}, \frac{1}{2})$ ($\widetilde{U}_q$). Then $h_q(-t)$ is the characteristic function $\varphi_{U_q}(t)$ of U_q and $\widetilde{h}_q(-t)$ is the characteristic function $\varphi_{\widetilde{U}_q}(t)$ of $\widetilde{U}_q$; equivalently,

$$h_q(t) = \varphi_{-U_q}(t), \qquad \widetilde{h}_q(t) = \varphi_{-\widetilde{U}_q}(t). \tag{3.4}$$

Theorem 3.1. *Let $q \geqslant 1$ be an integer, and suppose that X is a random variable such that qX is integer-valued. Then*

$$\varphi_{\lfloor X \rfloor}(t) = \sum_{j=0}^{q-1} h_q(t+2\pi j)\varphi_X(t+2\pi j) \tag{3.5}$$

and

$$\varphi_{\langle X \rangle}(t) = \sum_{j=0}^{q-1} \widetilde{h}_q(t+2\pi j)\varphi_X(t+2\pi j). \tag{3.6}$$

Proof. It is convenient to use the random variables U_q and $\widetilde{U}_q$ above, assuming as we may that they are independent of X. Then, for any integer k,

$$\begin{aligned}\mathbb{P}(\lfloor X \rfloor = k) &= \sum_{j=0}^{q-1} \mathbb{P}\Bigl(X = k + \frac{j}{q}\Bigr) = q\sum_{j=0}^{q-1} \mathbb{P}\Bigl(X = k + \frac{j}{q}\Bigr)\,\mathbb{P}\Bigl(U_q = \frac{j}{q}\Bigr) \\ &= q\,\mathbb{P}\bigl(X - U_q = k\bigr).\end{aligned} \tag{3.7}$$

Furthermore, if Y is any random variable with values in $\frac{1}{q}\mathbb{Z}$, then

$$\begin{aligned}\sum_{j=0}^{q-1} \varphi_Y(t+2\pi j) &= \sum_{j=0}^{q-1}\sum_{k=-\infty}^{\infty}\sum_{\ell=0}^{q-1} \mathbb{P}\bigl(Y = k + \ell/q\bigr) e^{\mathrm{i}(t+2\pi j)(k+\ell/q)} \\ &= \sum_{k=-\infty}^{\infty}\sum_{\ell=0}^{q-1} e^{\mathrm{i}t(k+\ell/q)}\,\mathbb{P}\bigl(Y = k+\ell/q\bigr)\sum_{j=0}^{q-1} e^{2\pi\mathrm{i}j\ell/q} \\ &= q\sum_{k=-\infty}^{\infty} e^{\mathrm{i}tk}\,\mathbb{P}\bigl(Y = k\bigr)\end{aligned} \tag{3.8}$$

since the inner sum over j vanishes when $\ell \neq 0$ and is q for $\ell = 0$. (This argument can be regarded as Fourier inversion on the finite group $\mathbb{Z}_q$). Taking $Y := X - U_q$, we obtain by combining (3.7) and (3.8) and recalling (3.4),

$$\begin{aligned}\varphi_{\lfloor X \rfloor}(t) &= q\sum_{k=-\infty}^{\infty} e^{\mathrm{i}tk}\,\mathbb{P}\bigl(X - U_q = k\bigr) = \sum_{j=0}^{q-1} \varphi_{X-U_q}(t+2\pi j) \\ &= \sum_{j=0}^{q-1} \varphi_{-U_q}(t+2\pi j)\varphi_X(t+2\pi j) = \sum_{j=0}^{q-1} h_q(t+2\pi j)\varphi_X(t+2\pi j),\end{aligned} \tag{3.9}$$

which shows (3.5).

We obtain (3.6) by the same argument, now using $\widetilde{U}_q$ instead of U_q. □

Remark 3.2. The sums in (3.5) and (3.6) may be taken over any set of j that has exactly one element in each residue class mod j, since the summands have period q in j (because h_q, $\widetilde{h}_q$, and φ_X all are $2\pi q$-periodic). The same applies to (4.1) and (4.2) below. △

4. Moments

Suppose that for some integer $r \geqslant 1$, X has a finite rth moment, i.e., $\mathbb{E}\,[|X|^r] < \infty$. Then obviously $\lfloor X \rfloor$ and $\langle X \rangle$ also have finite rth moments. Theorem 3.1 implies the following formulas.

Theorem 4.1. *Let $q \geqslant 1$ be an integer, and suppose that X is a random variable such that qX is integer-valued. Then, for every integer $r \geqslant 1$ such that $\mathbb{E}\,|X|^r < \infty$,*

$$\mathbb{E}\left(\lfloor X \rfloor^r\right) = \mathrm{i}^{-r} \sum_{j=0}^{q-1} \frac{\mathrm{d}^r}{\mathrm{d}t^r}\bigl(h_q(t)\varphi_X(t)\bigr)\big|_{t=2\pi j} \tag{4.1}$$

and

$$\mathbb{E}\left(\langle X \rangle^r\right) = \mathrm{i}^{-r} \sum_{j=0}^{q-1} \frac{\mathrm{d}^r}{\mathrm{d}t^r}\bigl(\widetilde{h}_q(t)\varphi_X(t)\bigr)\big|_{t=2\pi j}. \tag{4.2}$$

Proof. These follow immediately by differentiating (3.5) and (3.6) r times at $t = 0$. ☐

In the following two sections, we derive more explicit formulas for the first two moments.

5. Mean

Theorem 5.1. *Let $q \geqslant 1$ be an integer, and suppose that X is a random variable such that qX is integer-valued. Suppose also $\mathbb{E}\,|X| < \infty$. Then*

$$\mathbb{E}\,\lfloor X \rfloor = \mathbb{E}\,X - \frac{1}{2} + \frac{1}{2q} + \sum_{j=1}^{q-1} \frac{1}{q(1 - e^{-2\pi \mathrm{i} j/q})}\varphi_X(2\pi j). \tag{5.1}$$

If q is even, then

$$\mathbb{E}\,\langle X \rangle = \mathbb{E}\,X + \frac{1}{2q} + \sum_{j=1}^{q-1} \frac{(-1)^j}{q(1 - e^{-2\pi \mathrm{i} j/q})}\varphi_X(2\pi j). \tag{5.2}$$

If q is odd, then

$$\mathbb{E}\,\langle X \rangle = \mathbb{E}\,X + \sum_{j=1}^{q-1} \frac{(-1)^j}{q(e^{\pi \mathrm{i} j/q} - e^{-\pi \mathrm{i} j/q})}\varphi_X(2\pi j). \tag{5.3}$$

Proof. By taking $r = 1$ in (4.1), or directly by taking the derivative of (3.5) at $t = 0$, we obtain

$$\mathrm{i}\,\mathbb{E}\,\lfloor X \rfloor = \varphi'_{\lfloor X \rfloor}(0) = \sum_{j=0}^{q-1} h_q(2\pi j)\varphi'_X(2\pi j) + \sum_{j=0}^{q-1} h'_q(2\pi j)\varphi_X(2\pi j). \tag{5.4}$$

We have $\varphi'_X(0) = \mathrm{i}\,\mathbb{E}\,X$, and (3.1) yields

$$h_q(0) = 1 \quad \text{and} \quad h_q(2\pi j) = 0 \text{ for } 1 \leqslant j \leqslant q-1. \tag{5.5}$$

Hence, (5.4) simplifies to

$$\mathrm{i}\,\mathbb{E}\,\lfloor X \rfloor = \mathrm{i}\,\mathbb{E}\,X + \sum_{j=0}^{q-1} h'_q(2\pi j)\varphi_X(2\pi j). \tag{5.6}$$

The first expression in (3.1) yields

$$h_q'(0)=\frac{1}{q}\sum_{k=0}^{q-1}\frac{-\mathrm{i}k}{q}=-\mathrm{i}\frac{q(q-1)/2}{q^2}=-\mathrm{i}\frac{q-1}{2q}. \tag{5.7}$$

(Alternatively, by (3.4) this follows from, and is equivalent to, $\mathbb{E}\,U_q=\frac{q-1}{2q}$, which easily is seen directly.) Furthermore, the second expression in (3.1) yields, noting that the numerator there vanishes at $t=2\pi j$,

$$h_q'(2\pi j)=\frac{\mathrm{i}}{q(1-e^{-2\pi \mathrm{i}j/q})},\qquad 1\leqslant j\leqslant q-1. \tag{5.8}$$

Consequently, (5.6) yields (5.1).

For $\mathbb{E}\,\langle X\rangle$, we argue in the same way using (3.6) or (4.2). We note that (3.2)–(3.3) and (5.5) yield

$$\widetilde{h}_q(0)=1\quad\text{and}\quad \widetilde{h}_q(2\pi j)=0 \text{ for } 1\leqslant j\leqslant q-1. \tag{5.9}$$

Hence, taking derivatives in (3.6) yields

$$\mathrm{i}\,\mathbb{E}\,\langle X\rangle=\mathrm{i}\,\mathbb{E}\,X+\sum_{j=0}^{q-1}\widetilde{h}_q'(2\pi j)\varphi_X(2\pi j). \tag{5.10}$$

Furthermore, (3.2) and (3.3) together with (5.5) and (5.7)–(5.8) imply that: If q is even, then

$$\widetilde{h}_q'(0)=h_q'(0)+\frac{\mathrm{i}}{2}h_q(0)=\frac{\mathrm{i}}{2q} \tag{5.11}$$

and

$$\widetilde{h}_q'(2\pi j)=e^{\mathrm{i}\pi j}h_q'(2\pi j)=\mathrm{i}\frac{(-1)^j}{q(1-e^{-2\pi \mathrm{i}j/q})},\qquad 1\leqslant j\leqslant q-1; \tag{5.12}$$

if q is odd, then

$$\widetilde{h}_q'(0)=h_q'(0)+\mathrm{i}\frac{q-1}{2q}h_q(0)=0 \tag{5.13}$$

(which also follows from (3.4) since $\mathbb{E}\,\widetilde{U}_q=0$ when q is odd), and

$$\widetilde{h}_q'(2\pi j)=e^{\mathrm{i}\frac{q-1}{q}\pi j}h_q'(2\pi j)=\mathrm{i}\frac{(-1)^j}{q(e^{\pi \mathrm{i}j/q}-e^{-\pi \mathrm{i}j/q})}=\frac{(-1)^j}{2q\sin(\pi j/q)},\qquad 1\leqslant j\leqslant q-1. \tag{5.14}$$

We obtain (5.2) and (5.3) by substituting (5.11)–(5.14) into (5.10). □

6. Second moment

Theorem 6.1. *Let $q\geqslant 1$ be an integer, and suppose that X is a random variable such that qX is integer-valued. Suppose also $\mathbb{E}\,[X^2]<\infty$. Then*

$$\begin{split}\mathbb{E}\left[\lfloor X\rfloor^2\right]=\mathbb{E}\,[X^2]+\frac{2q^2-3q+1}{6q^2}-\frac{q-1}{q}\,\mathbb{E}\,X-2\sum_{j=1}^{q-1}\frac{\mathrm{i}}{q(1-e^{-2\pi \mathrm{i}j/q})}\varphi_X'(2\pi j)\\ -\sum_{j=1}^{q-1}\Bigl(\frac{1}{q(1-e^{-2\pi \mathrm{i}j/q})}+2\frac{e^{-2\pi \mathrm{i}j/q}}{q^2(1-e^{-2\pi \mathrm{i}j/q})^2}\Bigr)\varphi_X(2\pi j).\end{split} \tag{6.1}$$

If q is even, then

$$\mathbb{E}\left[\langle X\rangle^2\right] = \mathbb{E}\,[X^2] + \frac{1}{12} + \frac{1}{6q^2} + \frac{1}{q}\,\mathbb{E}\,X - 2\sum_{j=1}^{q-1} \mathrm{i}\frac{(-1)^j}{q(1-e^{2\pi\mathrm{i}j/q})}\varphi_X'(2\pi j) + \sum_{j=1}^{q-1}\frac{(-1)^j}{2q^2\sin^2(\pi j/q)}\varphi_X(2\pi j). \tag{6.2}$$

If q is odd, then

$$\mathbb{E}\left[\langle X\rangle^2\right] = \mathbb{E}\,[X^2] + \frac{1}{12} - \frac{1}{12q^2} - \sum_{j=1}^{q-1}\frac{(-1)^j}{q\sin(\pi j/q)}\varphi_X'(2\pi j) + \sum_{j=1}^{q-1}\frac{(-1)^j\cos(\pi j/q)}{2q^2\sin^2(\pi j/q)}\varphi_X(2\pi j). \tag{6.3}$$

Proof. By taking $r=2$ in (4.1), or directly by taking the second derivative of (3.5) at $t=0$, we obtain, using $\varphi_X''(0) = -\,\mathbb{E}\,[X^2]$ and (5.5),

$$\begin{aligned}\mathbb{E}\,(\lfloor X\rfloor^2) &= -\varphi_{\lfloor X\rfloor}''(0)\\ &= -\sum_{j=0}^{q-1} h_q(2\pi j)\varphi_X''(2\pi j) - 2\sum_{j=0}^{q-1} h_q'(2\pi j)\varphi_X'(2\pi j) - \sum_{j=0}^{q-1} h_q''(2\pi j)\varphi_X(2\pi j)\\ &= \mathbb{E}\,[X^2] - 2\sum_{j=0}^{q-1} h_q'(2\pi j)\varphi_X'(2\pi j) - \sum_{j=0}^{q-1} h_q''(2\pi j)\varphi_X(2\pi j).\end{aligned} \tag{6.4}$$

We obtain from (3.1)

$$h_q''(0) = \frac{1}{q}\sum_{k=0}^{q-1}\frac{-k^2}{q^2} = -\frac{q(q-1)(2q-1)}{6q^3} = -\frac{2q^2-3q+1}{6q^2} \tag{6.5}$$

and, since $e^{2\pi j\mathrm{i}} = 1$,

$$h_q''(2\pi j) = \frac{1}{q(1-e^{-2\pi\mathrm{i}j/q})} + 2\frac{e^{-2\pi\mathrm{i}j/q}}{q^2(1-e^{-2\pi\mathrm{i}j/q})^2}, \qquad j=1,\dots,q-1. \tag{6.6}$$

Consequently, using also (5.7)–(5.8), $\varphi_X'(0) = \mathrm{i}\,\mathbb{E}\,X$ and $\varphi_X(0)=1$, (6.1) follows from (6.4).

Turning to $\langle X\rangle$, we obtain as above, from (3.6) or (4.2),

$$\mathbb{E}\,(\langle X\rangle^2) = \mathbb{E}\,[X^2] - 2\sum_{j=0}^{q-1}\widetilde{h}_q'(2\pi j)\varphi_X'(2\pi j) - \sum_{j=0}^{q-1}\widetilde{h}_q''(2\pi j)\varphi_X(2\pi j). \tag{6.7}$$

For even q, we obtain from (3.2), (6.5)–(6.6), (5.7)–(5.8), and (5.5),

$$\widetilde{h}_q''(0) = h_q''(0) + \mathrm{i}h_q'(0) - \tfrac{1}{4}h_q(0) = -\frac{q^2+2}{12q^2} \tag{6.8}$$

and

$$\widetilde{h}_q''(2\pi j) = (-1)^j h_q''(2\pi j) + 2\frac{\mathrm{i}}{2}(-1)^j h_q'(2\pi j)$$

$$= 2\frac{(-1)^j e^{-2\pi \mathrm{i} j/q}}{q^2(1-e^{-2\pi \mathrm{i} j/q})^2} = \frac{(-1)^{j+1}}{2q^2\sin^2(\pi j/q)}, \qquad j = 1,\dots,q-1. \tag{6.9}$$

For odd j, we obtain similarly (or directly from the last formula in (3.3))

$$\widetilde{h}_q''(0) = -\frac{q^2-1}{12q^2} \tag{6.10}$$

and

$$\widetilde{h}_q''(2\pi j) = \frac{(-1)^{j+1}\cos(\pi j/q)}{2q^2\sin^2(\pi j/q)}, \qquad j = 1,\dots,q-1. \tag{6.11}$$

The results (6.2) and (6.3) now follow from (6.7), using (5.11)–(5.14) and (6.8)–(6.11). □

7. Examples

Example 7.1. Consider the simplest case $q = 2$, i.e., assume that X a.s. is an integer or half-integer. Then $\langle X\rangle = \lceil X\rceil$. For the mean, we obtain from Theorem 5.1

$$\mathbb{E}\lfloor X\rfloor = \mathbb{E}X - \frac{1}{4} + \frac{\varphi_X(2\pi)}{4}, \tag{7.1}$$

$$\mathbb{E}\lceil X\rceil = \mathbb{E}\langle X\rangle = \mathbb{E}X + \frac{1}{4} - \frac{\varphi_X(2\pi)}{4}. \tag{7.2}$$

(This can also easily be seen directly.) Similarly, for the second moment, from Theorem 6.1,

$$\mathbb{E}\left[\lfloor X\rfloor^2\right] = \mathbb{E}[X^2] + \frac{1}{8} - \frac{1}{2}\mathbb{E}X - \frac{\mathrm{i}}{2}\varphi_X'(2\pi) - \frac{1}{8}\varphi_X(2\pi), \tag{7.3}$$

$$\mathbb{E}\left[\lceil X\rceil^2\right] = \mathbb{E}\left[\langle X\rangle^2\right] = \mathbb{E}[X^2] + \frac{1}{8} + \frac{1}{2}\mathbb{E}X + \frac{\mathrm{i}}{2}\varphi_X'(2\pi) - \frac{1}{8}\varphi_X(2\pi). \tag{7.4}$$

Note that (7.1)–(7.2) and (7.3)–(7.4) agree with the relation $\lceil X\rceil = -\lfloor -X\rfloor$. △

Example 7.2. Let $q \geqslant 1$, and let $X := U_q$. This eaxmple is trivial, since obviously $\lfloor U_q\rfloor = 0$; nevertheless it is interesting to see how this is reflected in the formulas above. Recall that by (3.4), we now have

$$\varphi_X(t) = \varphi_{U_q}(t) = h_q(-t) = \overline{h_q(t)}. \tag{7.5}$$

In particular, (5.5) shows that

$$\varphi_{U_q}(2\pi j) = 0 \quad \text{for } 1 \leqslant j \leqslant q-1. \tag{7.6}$$

Hence the sum in (5.1) vanishes, and (5.1) reduces to

$$0 = \mathbb{E}\lfloor U_q\rfloor = \mathbb{E}U_q - \frac{1}{2} + \frac{1}{2q} = \mathbb{E}U_q - \frac{q-1}{2q}, \tag{7.7}$$

or

$$\mathbb{E}U_q = \frac{q-1}{2q}, \tag{7.8}$$

as is easily seen (and was observed after (5.7)).

The variance formula is more interesting. The last sum in (6.1) vanishes, again because of (7.6). For the first sum we have, by (5.8) and (7.5),

$$\sum_{j=1}^{q-1}\frac{\mathrm{i}}{q(1-e^{-2\pi \mathrm{i} j/q})}\varphi_{U_q}'(2\pi j) = \sum_{j=1}^{q-1} h_q'(2\pi j)\overline{h_q'(2\pi j)} = \sum_{j=1}^{q-1}\left|h_q'(2\pi j)\right|^2$$

$$= \sum_{j=1}^{q-1} \frac{1}{|2q\sin(\pi j/q)|^2} = \frac{1}{4q^2} \sum_{j=1}^{q-1} \frac{1}{\sin^2(\pi j/q)}. \tag{7.9}$$

Hence (6.1) yields, using also (7.8),

$$0 = \mathbb{E}\left[\lfloor U_q \rceil^2\right] = \mathbb{E}\,[U_q^2] + \frac{2q^2-3q+1}{6q^2} - \frac{(q-1)^2}{2q^2} - \frac{1}{2q^2}\sum_{j=1}^{q-1} \frac{1}{\sin^2(\pi j/q)}. \tag{7.10}$$

Furthermore, by (7.5) and (6.5),

$$\mathbb{E}\,[U_q^2] = -\varphi''_{U_q}(0) = -h''_q(0) = \frac{2q^2-3q+1}{6q^2}, \tag{7.11}$$

which equals the second term on the right-hand side of (7.10). (See the proof of Theorem 6.1.) Hence (7.10) is equivalent to

$$\sum_{j=1}^{q-1} \frac{1}{\sin^2(\pi j/q)} = \frac{2}{3}\bigl(2q^2-3q+1\bigr) - (q-1)^2 = \frac{q^2-1}{3}. \tag{7.12}$$

This non-obvious formula can also be shown more directly by applying Parseval's formula to the restriction of h'_q to $\{2\pi j : 0 \leqslant j < q\}$, regarded as a function on the cyclic group $\mathbb{Z}_q$; this function is given by (5.7)–(5.8), and it follows from (3.1) that its inverse Fourier transform (also a function on $\mathbb{Z}_q$) is given by $-\mathrm{i}\frac{k}{q}$, $0 \leqslant k < q$; calculations similar to (and related to) the ones above then yield (7.12).

We obtain the same result (7.12) by similar calculations if we instead take $X := \widetilde{U}_q$ and consider $\langle \widetilde{U}_q \rangle = 0$ in Theorem 6.1. △

Example 7.3. As a more complicated example, we consider the following problem; we thank Seungki Kim for asking it, which was the original motivation for the present paper.

Let q be odd, let $n \geqslant 1$, and let $\xi_1, \dots, \xi_n$ be i.i.d. with each ξ_k having the distribution of $\widetilde{U}_q$ in Section 3, i.e., ξ_k is uniformly distributed on $\{-\frac{q-1}{2q}, -\frac{q-3}{2q}, \dots, \frac{q-3}{2q}, \frac{q-1}{2q}\}$; in other words each $q\xi_k$ is uniformly distributed on the set $\{-\frac{q-1}{2}, -\frac{q-3}{2}, \dots, \frac{q-3}{2}, \frac{q-1}{2}\}$ of integers in $(-\frac{q}{2}, \frac{q}{2})$. Let $s_1, \dots, s_n$ be positive integers, and consider

$$X := s_1\xi_1 + \dots + s_n\xi_n. \tag{7.13}$$

Since q is odd we have $\mathbb{E}\,\xi_i = \mathbb{E}\,\widetilde{U}_q = 0$ and, using (3.4) and (6.10),

$$\operatorname{Var}\xi_i = \mathbb{E}\,[\xi_i^2] = \mathbb{E}\left[\widetilde{U}_q^2\right] = -\widetilde{h}''_q(0) = \frac{q^2-1}{12q^2}. \tag{7.14}$$

Consequently,

$$\operatorname{Var} X = \sum_{i=1}^{n} s_i^2 \operatorname{Var}\xi_i = \frac{q^2-1}{12q^2} \sum_{i=1}^{n} s_i^2. \tag{7.15}$$

Consider now the rounded variable $\langle X \rangle$, and in particular its variance. A natural approximation to $\operatorname{Var}[\langle X \rangle]$ is $\operatorname{Var} X + \frac{1}{12}$. (For any X. In statistics this is known as Sheppard's correction (for the variance) when dealing with grouped data, see e.g. [1, Section 27.9]. See also [2] for continuous X.) How good is this approximation for the variable X in (7.13)?

We will in (7.37) below give an upper bound of the error that is valid for all $n \geqslant 2$ and $s_1, \dots, s_n$; it is presumably not sharp but it seems to be rather good when the

s_i are much smaller than q. We leave it to the readers to find better bounds, in particular for cases with larger s_i.

Each ξ_k has the characteristic function

$$\varphi_\xi(t) = \frac{1}{q}\sum_{j=-(q-1)/2}^{(q-1)/2} e^{\mathrm{i}jt/q} = \widetilde{h}_q(t) = \frac{\sin\frac{t}{2}}{q\sin\frac{t}{2q}}, \tag{7.16}$$

see (3.3) and (3.4). Consequently, X has the characteristic function

$$\varphi_X(t) = \prod_{k=1}^n \varphi_\xi(s_k t) = \prod_{k=1}^n \widetilde{h}_q(s_k t). \tag{7.17}$$

It follows from (5.9) and the fact that $\widetilde{h}_q$ has period $2\pi q$ that for every $j \in \mathbb{Z}$, we have $\varphi_X(2\pi j) \in \{0,1\}$, and, letting $\gcd(\dots)$ denote the greatest common divisor,

$$\begin{aligned}\varphi_X(2\pi j) \neq 0 \iff \varphi_X(2\pi j) = 1 \iff \varphi_\xi(2\pi j s_k) = 1\ \forall k \leqslant n \iff q \mid j s_k\ \forall k \leqslant n \\ \iff q \mid j\gcd(s_1,\dots,s_n,q). \end{aligned}\tag{7.18}$$

Let

$$d := \gcd(s_1,\dots,s_n,q), \tag{7.19}$$
$$J := q/d. \tag{7.20}$$

Note that d and J are divisors of q, and thus odd positive integers with $d, J \leqslant q$. Furthermore, (7.18) can be written

$$\varphi_X(2\pi j) = \mathbf{1}\{q \mid jd\} = \mathbf{1}\{J \mid j\}. \tag{7.21}$$

Each ξ_k has a distribution that is symmetric: $-\xi_k \overset{\mathrm{d}}{=} \xi_k$. Thus the same holds for X, and thus also for $\langle X\rangle$ (since q is odd, cf. Remark 2.1). In particular,

$$\mathbb{E}\,\langle X\rangle = 0. \tag{7.22}$$

For the second moment, we use (6.3). Consider first the last sum in (6.3). By (7.21), we may sum over $j = \ell J$, $\ell = 1,\dots,d-1$, only, and thus the absolute value of the sum is

$$\left|\sum_{j=1}^{q-1}\frac{(-1)^j\cos(\pi j/q)}{2q^2\sin^2(\pi j/q)}\varphi_X(2\pi j)\right| \leqslant \sum_{\ell=1}^{d-1}\frac{1}{2q^2\sin^2(\pi\ell J/q)} = \frac{1}{2q^2}\sum_{\ell=1}^{d-1}\frac{1}{\sin^2(\pi\ell/d)}. \tag{7.23}$$

The final sum is $(d^2-1)/3$ by (7.12). Hence, (7.23) implies

$$\left|\sum_{j=1}^{q-1}\frac{(-1)^j\cos(\pi j/q)}{2q^2\sin^2(\pi j/q)}\varphi_X(2\pi j)\right| < \frac{d^2}{6q^2} \leqslant \frac{(\min_k s_k)^2}{6q^2}. \tag{7.24}$$

For the first sum in (6.3), we first define

$$X_k := \sum_{i\neq k} s_i\xi_i, \tag{7.25}$$

the sum (7.13) with the term $s_k\xi_k$ omitted. Then we differentiate (7.17) and obtain

$$\varphi_X'(t) = \sum_{k=1}^n s_k\varphi_\xi'(s_k t)\prod_{i\neq k}\varphi_\xi(s_i t) = \sum_{k=1}^n s_k\varphi_\xi'(s_k t)\varphi_{X_k}(t). \tag{7.26}$$

Let

$$d_k := \gcd(\{s_i : i \neq k\} \cup \{q\}), \tag{7.27}$$

$$J_k := q/d_k. \tag{7.28}$$

Then (7.21) applied to X_k yields, for all $j \in \mathbb{Z}$,

$$\varphi_{X_k}(2\pi j) = \mathbf{1}\{q \mid jd_k\} = \mathbf{1}\{J_k \mid j\}. \tag{7.29}$$

Consequently, for the first sum in (6.3), using (7.26), (7.16), and (5.13)–(5.14), and taking $j = \ell J_k$ similarly to the argument in (7.23),

$$\begin{aligned}\sum_{j=1}^{q-1} \frac{(-1)^j}{q\sin(\pi j/q)}\varphi_X'(2\pi j) &= \sum_{j=1}^{q-1} \frac{(-1)^j}{q\sin(\pi j/q)} \sum_{k=1}^{n} s_k \varphi_\xi'(2\pi s_k j)\varphi_{X_k}(2\pi j) \\ &= \sum_{k=1}^{n} s_k \sum_{j=1}^{q-1} \frac{(-1)^j}{q\sin(\pi j/q)} \cdot \frac{(-1)^{s_k j}\mathbf{1}\{q \nmid s_k j\}}{2q\sin(\pi s_k j/q)}\mathbf{1}\{J_k \mid j\} \\ &= \sum_{k=1}^{n} \frac{s_k}{2q^2} \sum_{\ell=1}^{d_k-1} \frac{(-1)^{(s_k+1)\ell}\mathbf{1}\{d_k \nmid s_k\ell\}}{\sin(\pi\ell/d_k)\sin(\pi s_k\ell/d_k)}.\end{aligned} \tag{7.30}$$

Denote the inner sum on the last line by S_k. Then the Cauchy–Schwarz inequality yields

$$S_k^2 \leqslant \sum_{\ell=1}^{d_k-1} \frac{1}{\sin^2(\pi\ell/d_k)} \cdot \sum_{\ell=1}^{d_k-1} \frac{\mathbf{1}\{d_k \nmid s_k\ell\}}{\sin^2(\pi s_k\ell/d_k)}. \tag{7.31}$$

The first sum is $< d_k^2/3$ by (7.12). For the second sum, we note that by (7.19) and (7.27),

$$d = \gcd(d_k, s_k) \tag{7.32}$$

and thus

$$d_k \mid s_k\ell \iff d_k \mid \gcd(s_k\ell, d_k\ell) = d\ell \iff (d_k/d) \mid \ell. \tag{7.33}$$

Let $r_k := d_k/d$ (an odd integer by (7.32) and (7.27)) and $s_k' := s_k/d$ (also an integer by (7.32)). It follows that $\gcd(s_k', r_k) = 1$; thus $s_k'\ell$ runs through the equivalence classes modulo r_k once each as $\ell = 1, \dots, r_k$, and $d_k/r_k = d$ times each as $\ell = 1, \dots, d_k$. Consequently, since $s_k\ell/d_k = s_k'\ell/r_k$, using (7.12) again,

$$\sum_{\ell=1}^{d_k-1} \frac{\mathbf{1}\{d_k \nmid s_k\ell\}}{\sin^2(\pi s_k\ell/d_k)} = d\sum_{j=1}^{r_k-1} \frac{1}{\sin^2(\pi j/r_k)} < \frac{dr_k^2}{3} = \frac{d_k^2}{3d} \leqslant \frac{d_k^2}{3}. \tag{7.34}$$

Hence, (7.31) yields $S_k^2 \leqslant (d_k^2/3)^2$ and thus $|S_k| \leqslant d_k^2/3$, and (7.30) yields

$$\left|\sum_{j=1}^{q-1} \frac{(-1)^j}{q\sin(\pi j/q)}\varphi_X'(2\pi j)\right| \leqslant \frac{1}{6q^2}\sum_{k=1}^{n} s_k d_k^2. \tag{7.35}$$

Assume now $n \geqslant 2$. Then $d_k \leqslant s_{k+1}$ (with the index taken modulo n), and thus Hölder's inequality (with exponents 3 and 3/2) shows

$$\sum_{k=1}^{n} s_k d_k^2 \leqslant \Bigl(\sum_{k=1}^{n} s_k^3\Bigr)^{1/3}\Bigl(\sum_{k=1}^{n} d_k^3\Bigr)^{2/3} \leqslant \sum_{k=1}^{n} s_k^3. \tag{7.36}$$

Consequently, (6.3) yields, using (7.35)–(7.36) and (7.24),

$$\left|\mathbb{E}\left[\langle X\rangle^2\right]-\left(\mathbb{E}\left[X^2\right]+\tfrac{1}{12}\right)\right| \leqslant \frac{1+\sum_{k=1}^n s_k^3+(\min_k s_k)^2}{6q^2} \leqslant \frac{\sum_{k=1}^n s_k^3}{3q^2}. \tag{7.37}$$

△

References

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