ON THE SKOROHOD TOPOLOGY FOR FUNCTIONS WITH VALUES IN A COMPLETELY REGULAR SPACE

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Abstract. We correct a gap in the proof of a basic theorem by Jakubowski (1986) on the Skorohod topology on the space of functions on $[0, 1]$ with values in a completely regular topological space.

1. Introduction

The Skorohod (J_1) topology on the space $D([0, 1])$ of functions on $[0, 1]$ that are càdlàg (i.e., right-continuous on $[0, 1]$ and with left limits at all $t \in (0, 1]$) is of fundamental importance in the study of stochastic processes. The topology was introduced by Skorohod [8] for real-valued functions; this is perhaps still the most important case, but it has also been very useful to extend the definition to functions on $[0, 1]$ with values in other spaces. In particular, the extension to functions with values in a metric space is straightforward, see e.g. [3].

A further extension to functions with values in an arbitrary completely regular topological space E was made by Jakubowski [4], generalizing a special case by Mitoma [7]. Unfortunately there is a gap in the proof of the basic theorem [4, Theorem 1.3] showing that the constructed topology depends only on the topology of E (and not on the pseudometrics used in the construction, see Section 3 below). It is easy to give a complete proof, but since we have not been able to find a published proof, we give a detailed proof here (Theorem 3.3). In Section 2, we correct also another error in [4].

Remark 1.1. We consider here functions defined on $[0, 1]$. It is well-known that there is a version of the Skorohod topology for càdlàg functions on $[0, \infty)$. This too was extended by [4] to the space $D([0, \infty), E)$ consisting of the càdlàg functions on $[0, \infty)$ with values in an arbitrary completely regular space E. Using the definition and methods of [4, Section 4] together with the proofs below, it is easy to see that Theorems 3.3 and 3.4 hold also for $D([0, \infty), E)$. △

Remark 1.2. Jakubowski [4, Theorem 1.3] is important also in the standard special case when E is a metric space; in this case it shows that the Skorohod topology does not depend on the choice of metric in E. This important and useful fact seems to be largely ignored or at most implicit in the literature rather than stated explicitly. For example, it is a consequence of (and essentially equivalent to) [3, Problem 3.11.13] or [6, Exercise 16.5] (both given as exercises without proof, and the latter stated only for complete separable metric spaces), but also there it is not stated explicitly. △

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2. Preliminaries

Let $E = (E, \tau)$ be a Hausdorff topological space. Let $D([0,1], E) = D([0,1], E, \tau)$ be the space of all functions $x : [0,1] \to E$ that are càdlàg, i.e., right-continuous on $[0,1]$ and with left limits at all $t \in (0,1]$. (We usually omit τ from the notation.) We denote the left limit at t by

$$f(t-) := \lim_{s \nearrow t} f(s), \tag{2.1}$$

for every function f and $t \in (0,1]$ such that this limit exists. For completeness, we also define $f(0-) := f(0)$.

Let $\widehat{I}$ be the *split interval* or *arrow space* defined by taking two points $t+$ and $t-$ for every $t \in [0,1]$ and letting

$$\widehat{I} := \{t- : t \in (0,1]\} \cup \{t+ : t \in [0,1]\}. \tag{2.2}$$

Remark 2.1. Note the asymmetry at the endpoints of our definition: $0- \notin \widehat{I}$. This is an unfortunate consequence of the standard definition of $D([0,1])$ at the endpoints. For other puposes one usually uses a symmetric version of $\widehat{I}$. $\triangle$

We regard $[0,1]$ as a subset of $\widehat{I}$ by identifying $t+$ with t for every $t \in [0,1]$. We give $\widehat{I}$ the natural order, extending the standard order on $[0,1]$. Formally, for $t, u \in [0,1]$,

$$t+ < u+ \iff t < u, \tag{2.3}$$

$$t- < u- \iff t < u, \tag{2.4}$$

$$t+ < u- \iff t < u, \tag{2.5}$$

$$t- < u+ \iff t \leqslant u. \tag{2.6}$$

We then give $\widehat{I}$ the order topology. It is easily seen that a neighbourhood base at $t-$ is given by the intervals (in $\widehat{I}$ with this order) $(s-, t-]$, $s < t$; similarly a neighbourhood base at $t+$ is given by the intervals $[t+, u+)$, $u > t$, interpreted as $\{1+\}$ when $t = 1$. (Thus $\{1+\}$ is isolated.) It is well known, and easy to se, that $\widehat{I}$ is compact.

We regard $[0,1]$ as a subset of $\widehat{I}$ by identifying $t \in [0,1]$ with $t+ \in \widehat{I}$. (Note that the subspace topology that this induces on $[0,1]$ regarded as a subset of $\widehat{I}$ is *not* the standard topology.)

Remark 2.2. $\widehat{I}$ is totally disconnected, separable, and first countable, but not second countable and not metrizable, see e.g. [5, Section 9.2]. $\triangle$

If $f \in D([0,1], E)$, then f has a natural extension to $\widehat{I}$ given by (2.1).

Lemma 2.3. *If E is a regular (T_3) topological space, and $f : [0,1] \to E$ is a function, then the following are equivalent:*

(i) *f is càdlàg. In other words, $f \in D([0,1], E)$.*

(ii) *f has a continuous extension to $\widehat{I}$.*

If this holds, then the continuous extension to $\widehat{I}$ is unique, and is the natural extension given by (2.1).

Proof. (ii) $\Longrightarrow$ (i): Suppose that f has a continuous extension (also denoted by f) to $\widehat{I}$.

If $t \in [0,1)$ and $s_n \searrow t$ in $[0,1]$ (with the usual topology), then $s_n \to t$ also in $\widehat{I}$, and thus $f(s_n) \to f(t)$.

Similarly, if $t \in (0,1]$ and $s_n \nearrow t$ in $[0,1]$, then $s_n \to t-$ in $\widehat{I}$, and thus $f(s_n) \to f(t-)$.

Hence, regarded as a function on $[0,1]$, f is càdlàg.

(i) $\Longrightarrow$ (ii): We define an extension to $\widehat{I}$ by (2.1). We claim that this extension is continuous on $\widehat{I}$.

First, let $t \in [0,1)$, and let U be a neighbourhood of $f(t)$. Since E is regular, there exists a closed neighbourhood V of $f(t)$ with $V \subseteq U$. Since f is right-continuous by assumption, there exists $\varepsilon > 0$ such that if $u \in [t, t+\varepsilon]$, then $f(u+) = f(u) \in V$. Furthermore, f has a left limit $f(u-)$ at every such u, and since V is closed, it follows that if $u \in (t, t+\varepsilon]$, then $f(u-) \in V$. Hence, if $v \in \widehat{I}$ with $t \leqslant v < t+\varepsilon$, then $f(v) \in V \subseteq U$. The neighbourhood U was arbitrary, and thus f is continuous at every $t+ \in \widehat{I}$. (Recall that $[t, t+\varepsilon) = (t-, t+\varepsilon)$ is open in $\widehat{I}$.)

Similarly, f is continuous at every $t- \in \widehat{I}$. Thus the extension f is continuous on $\widehat{I}$.

The continuous extension is unique, since $[0,1]$ is dense in $\widehat{I}$. □

Remark 2.4. When $E = \mathbb{R}$ or $\mathbb{C}$, this extension gives an isomorphism $D[0,1] \cong C(\widehat{I})$ (with these denoting spaces of real-valued or complex-valued functions, respectively). $D[0,1]$, equipped with the supremum norm, is a Banach algebra, and its maximal ideal space can be identified with $\widehat{I}$; then this isomorphism $D[0,1] \to C(\widehat{I})$ is the Gelfand transform, see e.g. [5, Section 9.2]. △

Corollary 2.5. *If E is a regular topological space and $f \in D([0,1], E)$, then the set $\{f(t) : t \in [0,1]\} \cup \{f(t-) : t \in (0,1]\}$ is a compact subset of E. In particular, the range $f([0,1])$ is relatively compact.*

Proof. $\widehat{I}$ is a compact space and the extension $f : \widehat{I} \to E$ is continuous by Lemma 2.3. Hence $f(\widehat{I})$ is compact. □

Corollary 2.5 is [4, Proposition 1], but assumes that E is regular. It is erroneously claimed in [4, Proposition 1] that it holds for Hausdorff spaces. The following example shows that this is incorrect; Lemma 2.3 and Corollary 2.5 do not hold for arbitrary Hausdorff spaces E. (Only the completely regular case, which is correct, is used later in [4].)

Let τ_0 be the standard topology on $\mathbb{R}$.

Example 2.6. Let τ_K (often called the K-topology or the Smirnov topology) be the topology on $\mathbb{R}$ defined by letting $K := \{\frac{1}{n} : n \in \mathbb{N}\}$ and declaring a set $O \subseteq \mathbb{R}$ to be open if $O = U \setminus H$ where U is open in the usual topology τ_0 and $\emptyset \subseteq H \subseteq K$. Then the subspace topology on $\mathbb{R} \setminus \{0\}$ equals the standard topology, but the topology at 0 is different: a neigbourhood base at 0 is given by the sets $(-\varepsilon, \varepsilon) \setminus K$. The topology τ_K is Hausdorff (since it is finer than the standard topology), but it is not regular since K is a closed set and $0 \notin K$, but 0 and K cannot be separated by two open sets. (See e.g. [9, Counterexample 64] or [2, Example 1.5.6, with a trivial modification].)

Note, for later use, that K is closed but discrete and infinite, and therefore not compact.

We let $\mathbb{R}_K = (R, \tau_K)$ denote $\mathbb{R}$ with this topology.

Define a function $f : [0,1] \to \mathbb{R}_K$ by

$$\begin{cases} f(0) := f(1) := 0, \\ f(t) := \frac{1}{2}(t + \frac{1}{n}), \qquad t \in [\frac{1}{n+1}, \frac{1}{n}),\ n \in \mathbb{N}. \end{cases} \tag{2.7}$$

Note that $f(t) \notin K$ for all $t \in [0,1]$.

Evidently $f \in D([0,1], \mathbb{R}, \tau_0)$ (with the standard topology on $\mathbb{R}$), with

$$\begin{cases} f(x-) = f(x), & x \notin K, \\ f(\frac{1}{n}-) = \frac{1}{n}, & n \in \mathbb{N}. \end{cases} \tag{2.8}$$

Thus $f(t) \in \mathbb{R} \setminus \{0\}$ for every $t \in (0,1)$ and $f(t-) \in \mathbb{R} \setminus \{0\}$ for every $t \in (0,1]$; since τ_K equals the standard topology on $\mathbb{R} \setminus \{0\}$, it is easily seen that f is right-continuous and has left limits $f(t-)$ also in τ_K everywhere on $(0,1]$. Furthermore, since $f(t) \notin K$ for all t, it follows that f is (right-)continuous at 0 too in τ_K. Hence f is càdlàg also for τ_K, and thus $f \in D([0,1], \mathbb{R}_K)$.

This means that f can be extended to $\widehat{I}$ using (2.8), also for τ_K. (This extension is necessarily the same as for the standard topology on $\mathbb{R}$). However, this extension of f to $\widehat{I}$ is *not* continuous, since $\frac{1}{n}- \to 0$ in $\widehat{I}$ as $n \to \infty$, but $f(\frac{1}{n}-) = \frac{1}{n} \not\to 0$ in $\mathbb{R}_K$: by construction, $U := \mathbb{R} \setminus K$ is a neighbourhood of 0 such that $f(\frac{1}{n}-) \notin U$ for every n. Hence, Lemma 2.3 does not hold for $\mathbb{R}_K$.

Similarly, Corollary 2.5 does not hold for $\mathbb{R}_K$ and the function $f \in D([0,1], \mathbb{R}_K)$ above. In fact, the set $\{f(t) : t \in [0,1]\} \cup \{f(t-) : t \in (0,1]\} = f(\widehat{I})$ is not compact (and not even relatively compact), since it contains $\{f(\frac{1}{n}-) : n \in \mathbb{N}\} = K$ as a closed but non-compact subset. Moreover, $f(\widehat{I}) \subseteq \overline{f([0,1])}$ (for any càdlàg f), and thus $\overline{f([0,1])}$ is not compact, i.e., $f([0,1])$ is not relatively compact. △

3. The Skorohod topology on $D([0,1], E)$

Assume from now on that $E = (E, \tau)$ is a completely regular space.

The Skorohod topology on $D([0,1], E)$ is defined by [4] (generalizing [7]) as follows. We use the fact that any completely regular topology is generated by a family of pseudometrics $\{d_i\}_{i \in I}$ satisfying

$$\forall_{a,b \in E} \exists_{i \in I} \quad d_i(a,b) > 0, \tag{3.1}$$

$$\forall_{i,j \in I} \exists_{k \in I} \quad \max(d_i, d_j) \leqslant d_k. \tag{3.2}$$

More precisely, the topology on E is generated by the functions $d_i(a, \cdot) : E \to \mathbb{R}$ with $i \in I$ and $a \in E$; equivalently, the set of open balls for the pseudometrics d_i forms a base of the topology. Furthermore, every pseudometric d_i is continuous $E \times E \to \mathbb{R}$.

Remark 3.1. Conversely, any such family of pseudometrics on a set defines a completely regular topology. Note also that (3.2) is mainly for convenience, and can be assumed without loss of generality, since for any family of pseudometrics $\{d_i\}$ satisfying (3.1), we may add all finite maxima $\max(d_{i_1}, \dots, d_{i_m})$ to the family; then (3.2) holds, and the enlarged family defines the same topology as the original family. (For the original family, the set of balls is a subbase for the topology.) △

To define the topology on $D([0,1], E)$, we choose such a family $\{d_i\}_{i \in I}$. For any pseudometric d on E, we define a corresponding pseudometric $\tilde{d}$ on $D([0,1], E)$ by

$$\tilde{d}(x,y) := \inf_{\lambda \in \Lambda} \max\Bigl\{ \sup_{t \in [0,1]} |\lambda(t) - t|, \sup_{t \in [0,1]} d\bigl(x(\lambda(t)), y(t)\bigr) \Bigr\}, \tag{3.3}$$

where Λ is the set of strictly increasing continuous functions λ mapping $[0,1]$ onto itself. Finally, $D([0,1],E)$ is given the topology generated by the family of pseudometrics $\{\tilde{d}_i\}_{i\in I}$.

Lemma 3.2. *Let the topology on E be defined by a family of pseudometrics $\{d_i\}_{i\in I}$ satisfying* (3.1)–(3.2)*. Let ρ be any continuous pseudometric on E, and suppose that K is a compact subset of E. Then, for every $\varepsilon > 0$, there exists a pseudometric d_i in the given family and $\delta > 0$, such that if $x \in K$ and $y \in E$ with $d_i(x,y) < \delta$, then $\rho(x,y) < \varepsilon$.*

Proof. For every $z \in K$, the set $\{y \in E : \rho(y,z) < \varepsilon/2\}$ is an open neighbourhood of z, and thus there exists $j_z \in I$ and $\delta_z > 0$ such that

$$U_z := \{y : d_{j_z}(y,z) < 2\delta_z\} \subseteq \{y \in E : \rho(y,z) < \varepsilon/2\}. \tag{3.4}$$

The open sets $U'_z := \{y : d_{j_z}(y,z) < \delta_z\}$ cover K, so we may select an finite subcover $U'_{z_1},\dots,U'_{z_n}$. It follows from (3.2) that there exists $i \in I$ such that $d_{j_{z_k}} \leqslant d_i$ for every $k = 1,\dots,n$.

Let $\delta := \min_{1\leqslant k\leqslant n} \delta_{z_{j_k}}$. If $x \in K$ and $y \in E$ with $d_i(x,y) < \delta$, then choose z_k such that $x \in U'_{z_k}$. We have

$$d_{j_{z_k}}(y,z) \leqslant d_{j_{z_k}}(x,y) + d_{j_{z_k}}(x,z) \leqslant d_i(x,y) + \delta_{z_k} < \delta + \delta_{z_k} \leqslant 2\delta_{z_k}; \tag{3.5}$$

thus $y \in U_{z_k}$. Furthermore, $x \in U'_{z_k} \subseteq U_{z_k}$. Consequently, (3.4) shows that $\rho(x,z_k) < \varepsilon/2$ and $\rho(y,z_k) < \varepsilon/2$, and thus $\rho(x,y) < \varepsilon$. □

Theorem 3.3 (Jakubowski [4])**.** *Let $\{d_i\}_{i\in I}$ and $\{\zeta_j\}_{j\in J}$ be two families of pseudometrics on E satisfying* (3.1) *and* (3.2)*. Let the topology τ generated by $\{d_i\}_{i\in I}$ be coarser than the topology σ generated by $\{\zeta_j\}_{j\in J}$. Then obviously $D([0,1],E,\tau) \supseteq D([0,1],E,\sigma)$ and the topology on $D([0,1],E,\sigma)$ generated by the pseudometrics $\{\tilde{\zeta}_j\}_{j\in J}$ is finer than the topology induced by $D([0,1],E,\tau)$ generated by the pseudometrics $\{\tilde{d}_i\}_{i\in I}$.*

In particular, if the families $\{d_i\}_{i\in I}$ and $\{\zeta_j\}_{j\in J}$ define the same topology on E, then the families $\{\tilde{d}_i\}_{i\in I}$ and $\{\tilde{\zeta}_j\}_{j\in J}$ define the same topology on $D([0,1],E)$.

Proof. Suppose that $x \in D([0,1],E,\sigma)$. Then obviously $x \in D([0,1],E,\tau)$.

Let $i \in I$. Then the pseudometric d_i is continuous on (E,τ), and thus also on (E,σ). By Corollary 2.5, the range $x([0,1])$ is relatively compact in (E,σ), and is thus a subset of some compact set K in (E,σ).

Let $\varepsilon > 0$. Lemma 3.2 (applied to (E,σ), $\{\zeta_j\}_{j\in J}$, and d_i) shows that there exists $j \in J$ and $\delta > 0$ such that if $y \in D([0,1],E,\sigma)$, $t \in [0,1]$, $\lambda \in \Lambda$, and $\zeta_j(x(\lambda(t)),y(t)) < \delta$, then $d_i(x(\lambda(t)),y(t)) < \varepsilon$. Consequently, recalling (3.3), if $\tilde{\zeta}_j(x,y) < \min(\delta,\varepsilon)$, then $\tilde{d}_i(x,y) \leqslant \varepsilon$. It follows that $\tilde{d}_i$ is continuous on $D([0,1],E,\sigma)$ for the topology defined by $\{\tilde{\zeta}_j\}_{j\in J}$, and the result follows. □

This theorem shows that we can unambiguously talk about $D([0,1],E)$ as a topological space, for any completely regular topological space E.

The theorem has the following corollary.

Theorem 3.4. *Suppose that E and F are two completely regular spaces, and that $\psi : E \to F$ is a continuous function. Define $\Psi : D([0,1],E) \to D([0,1],F)$ by*

$$\Psi(x) := \Psi \circ x, \qquad i.e., \qquad \Psi(x)(t) := \Psi(x(t)), \quad t \in [0,1]. \tag{3.6}$$

Then Ψ is continuous $D([0,1],E) \to D([0,1],F)$.

Proof. It is clear that Ψ maps $D([0,1],E)$ into $D([0,1],F)$.

Let the topologies of E and F be defined by families of pseudometrics $\{d_i\}_{i\in I}$ and $\{\zeta_j\}_{j\in J}$, respectively. For $j\in J$, define $\zeta_j^*(a,b):=\zeta_j(\psi(a),\psi(b))$ for $a,b\in E$; then ζ_j^* is a continuous pseudometric on E. Hence, for every $i\in I$ and $j\in J$, $\max(d_i,\zeta_j^*)$ is a continuous pseudometric on E. It follows that $\{d_i\}_{i\in I}\cup\{\zeta_j^*\}_{j\in J}\cup\{\max(d_i,\zeta_j^*)\}_{(i,j)\in I\times J}$ is a family of pseudometrics on E that satisfies (3.1)–(3.2), and that this family defines the same topology on E as $\{d_i\}_{i\in I}$.

By Theorem 3.3, the corresponding pseudometrics on $D([0,1],E)$ generate the topology on $D([0,1],E)$. In particular, every $\widetilde{\zeta}_j^*$ is a continuous semimetric on $D([0,1],E)$. It follows from the definitions (3.3) and (3.6) that for any $x,y\in D([0,1],E)$,

$$\widetilde{\zeta}_j\bigl(\Psi(x),\Psi(y)\bigr)=\widetilde{\zeta}_j^*(x,y). \tag{3.7}$$

We have shown that this is a continuous function of $(x,y)\in D([0,1],E)^2$. Since the pseudometrics $\widetilde{\zeta}_j$ generate the topology on $D([0,1],F)$, it follows that Ψ is continuous. ☐

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