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INTERMEDIATE HANKEL OPERATORS ON THE BERGMAN SPACE

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1. INTRODUCTION AND BACKGROUND

Let $\mathbb{D}$ be the unit disk of the complex plane, $L^2 = L^2(D, \mathrm{d}x\mathrm{d}y)$ the Lebesgue space, and $A^2 = L^2 \cap Hol$ the associated Bergman space. Let P be the orthogonal projection of L^2 onto A^2 and let Q be the orthogonal projection onto $\overline{A_0^2} = \{g \in L^2 : \overline{g} \in A^2,\ g(0) = 0\}$.

For a conjugate holomorphic function f defined on $\mathbb{D}$, define the small Hankel operator, H_f^{small}, to be the linear map from A^2 to $\overline{A_0^2}$ which takes g to $H_f^{small}(g) = Q(fg)$. Define the large Hankel operator, H_f^{big}, to be the map from A^2 into $A^{2\perp}$ which takes g to $H_f^{big}(g) = (I-P)(g)$. There are various ways to think of the names. Note that the "small"Hankel operator involves projection onto the space $\overline{A_0^2}$ which is a rather small subspace of the space $A^{2\perp}$ used in the definition the "large"Hankel operator. Alternatively we can introduce the partial order $\prec$ between operators from A^2 into L^2 by

$$R \prec S \text{ iff } \langle R^*Rf, f\rangle \leqslant \langle S^*Sf, f\rangle \ \forall f \in A^2.$$

We then have $H_f^{small} \prec H_f^{big}$.

One of our main interests in this paper will be with some operators which are intermediate between these two; that is, R for which $H_f^{small} \prec R \prec H_f^{big}$, whence the title.

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For $0 < p < \infty$ denote the Schatten - von Neumann ideal by S_p and the Besov space of holomorphic function in the disk by B_p. The starting point for this paper is the contrast between the following two theorems.

THEOREM 1.1. *Let f be holomorphic. For $0 < p < \infty$ the small Hankel operator $H_{\overline{f}}^{small}$ is in S_p if and only if f is in the Besov space B_p.*

The corresponding result for the large Hankel operator is

THEOREM 1.2. *Let f be holomorphic. For $1 < p < \infty$ the large Hankel operator $H_{\overline{f}}^{big}$ is in S_p if and only if f is in the Besov space B_p. For $p \leqslant 1$, $H_{\overline{f}}^{big}$ is in S_p if and only if f is constant.*

This change in behavior at $p = 1$ is sometimes referred to as a "cut off". A cut off for certain commutators acting on $L^2(\mathbb{R}^n)$ was demonstrated in [6]. Since then similar phenomena have been noted for variety of Hankel operators and commutators (the two classes are closely related) in ([5], [4], [13], [1], [3]). It had been our hope that by considering operators intermediate between the small and large Hankel operators, that is, intermediate between an operator with cut off at 0 and one with cut off at 1, we would gain some insight into this intriguing phenomenon and perhaps even find an instance in which the cut off was strictly between 0 and 1. Although we did not find such an example, the operators we considered did show various interesting types of behavior and that is what we present here.

In the next section we set up notation. In Section 3 we consider an intermediate operator for which the cut off is at 0, that is, for which an analog of Theorem 1.1 holds. The results in Section 4 are a variation of those in Section 3. The operators considered in Section 5 and 6 are not actually intermediate in the sense described above; however, they are of the same general sort. The class of operators considered in Section 5 never contains compact operators (in effect, it is cut off at infinity). The class of operators considered in Section 6 contains operators in S_p for all p but the criteria for membership in S_p is different from those in Theorems 1.1 and 1.2. For those operators we will also show that the boundedness criteria are different from those generally encountered when studying Hankel operators - neither membership in BMO nor membership in the Bloch space is the correct necessary and sufficient condition, the correct result is somewhere between.

(Since this work was done, several instances of Toeplitz and Hankel operators have been found which have a cut off between 0 and 1. Some of these operators are intermediate between the large Hankel operator and the operator we consider in Section 3 ([9], [7], [12], [8]).)

2. NOTATION

Let $L^2 = L^2(D, \mu_\alpha) = L^2\left(D, \frac{\alpha+1}{\pi}\left(1-|z|^2\right)^\alpha \mathrm{d}x\mathrm{d}y\right)$ where $-1 < \alpha < \infty$ and α is fixed throughout. ($\alpha = 0$ is typical for our considerations.) We will be working with operators defined on various (closed) subspaces of L^2. A prime example is the Bergman space $A^2 = L^2 \cap Hol$. Another is $A_0^2 = \{g \in A^2 : g(0) = 0\}$.

For any closed subspaces Z of L^2 let P_Z be the orthogonal projection onto Z. For any function f defined on D we denote by M_f the operator of multiplication by f. Given two closed subspaces, X and Y, of L^2, and given a function f we define the operator $H_f = H_f^{XY}$ mapping X to Y by $H_f^{XY}(g) = P_Y(fg)$. (We will also, when convenient, abuse notation and use the same expression for the operator $P_Y M_f P_X$ which maps L^2 to itself.) (Actually, of course, we must restrict f to be, say, in L^2 and start with H_f^{XY} densely defined. However such issues cause no trouble in our contexts and we will say no more about them.)

Examples of such operators include Toeplitz operators, ($X = Y = A^2$), the small Hankel operator, H_f^{small}, $\left(X = A^2,\ Y = \overline{A^2}\left(\text{or } Y = \overline{A_0^2}\right)\right)$ and the large Hankel operator, H_f^{big}, $\left(X = A^2,\ Y = A^{2\perp},\ \left(\text{or } Y = A_0^{2\perp}\right)\right)$. The one dimensional difference in the choice of Y doesn't matter for us. What is crucial is that $\overline{A_0^2} \subset$ $\subset A^{2\perp}$ $\left(\text{and } \overline{A^2} \subset A_0^{2\perp}\right)$ and hence $H_f^{small} \prec H_f^{big}$. If $X = A^2$ and Y satisfies $\overline{A_0^2} \subset$ $\subset Y \subset A^{2\perp}$ then the associated Hankel operator is "intermediate" in the sense we described; that is, $H_f^{small} \prec H_f^{XY} \prec H_f^{big}$.

We will work with the space

$$
\begin{aligned}
T &= \left\{f \in L^2 : f(r\mathrm{e}^{\mathrm{i}t}) \in H^2(\mathbb{T}) \text{ for a.e. } r\right\} = \\
&= \left\{\sum_0^\infty \hat{f}_n(r)\mathrm{e}^{int} : \sum_0^\infty \int_0^1 \left|\hat{f}_n(r)\right|^2 r(1-r^2)^\alpha \mathrm{d}r < \infty\right\} = \\
&= \overline{\operatorname{span}}\{z^n \overline{z}^m : n \geqslant m\}.
\end{aligned}
$$

Note that

$$
T^\perp = \left\{\sum_{-\infty}^{-1} \hat{f}_n(r)\mathrm{e}^{int} : \sum_0^\infty \int_0^1 \left|\hat{f}_n(r)\right|^2 r(1-r^2)^\alpha \mathrm{d}r < \infty\right\}
$$

and that

$$
\begin{aligned}
\overline{T} &= \left\{\sum_{-\infty}^{0} \hat{f}_n(r)\mathrm{e}^{int} : \sum_0^\infty \int_0^1 \left|\hat{f}_n(r)\right|^2 r(1-r^2)^\alpha \mathrm{d}r < \infty\right\} = \\
&= T^\perp \oplus L^2_{\mathrm{rad}}
\end{aligned}
$$

where

$$
L^2_{\mathrm{rad}} = \left\{f \in L^2 : f(z) = f(|z|)\right\}.
$$

(We note in passing that the subspace of T consisting of functions in T which are continuous on the closed disk is a Banach algebra whose maximal ideal space can be identified with the cone $\{(\zeta,r) : 0 \leqslant |\zeta| \leqslant r \leqslant 1\}$. Specifically, for $f = z^n\overline{z}^m$, $\hat{f}(\zeta,r) = \zeta^{n-m}r^{2m}$.)

All the spaces we will consider are invariant under rotation and hence have a Fourier decomposition $X = \bigoplus_{-\infty}^{\infty} X_n$ where $X_n = \{g \in X : g(re^{it}) = f(r)e^{int}\}$.

For $0 < p < \infty$ we take as known the definitions and basic properties of the Schatten–von Neumann ideals, S_p, and the Besov space of holomorphic function in the disk, $B_p = B_p^{1/p}$. We should note that we will write S_∞ for the bounded operators.

3. AN INTERMEDIATE HANKEL OPERATOR

In this section we consider the case $X = A^2$, $Y = \overline{T}$ or $T^\perp$. In this case we have $H_f^{small} \prec H_f^{XY} \prec H_f^{big}$.

THEOREM 3.1. *For $-1 < \alpha < \infty$, $0 < p < \infty$, and holomorphic f, the intermediate Hankel operators $H_{\overline{f}}^{A^2,\overline{T}}$ and $H_{\overline{f}}^{A^2,T^\perp}$ are in S_p if and only if f is in B_p.*

Proof. There are two cases, $H_{\overline{f}}^{A^2,\overline{T}}$ and $H_{\overline{f}}^{A^2,T^\perp}$, but they are almost identical so we only provide the details for the first case. Set $H_{\overline{f}} = H_{\overline{f}}^{A^2,\overline{T}}$. If $H_{\overline{f}}$ is in S_p then the small Hankel operator also belongs to S_p and hence $f \in B_p$ (see [11]). Similarly, if $f \in B_p$ and $1 < p < \infty$ then, by [1], the big Hankel operator is in S_p and hence so is $H_{\overline{f}}$.

We now consider $f \in B_p$, $0 < p \leqslant 1$. For such f we have the atomic decomposition

$$f(z) = \sum \lambda_i \varphi_{\zeta_i}(z)$$

for sequences $\{\zeta_i\} \subset D$, $(\lambda_i) \in l^p$ where

$$\varphi_\zeta(z) = \frac{1-|\zeta|^2}{1-\zeta z}$$

(see [11]). Thus

$$\|H_{\overline{f}}\|_{S_p}^p \leqslant \sum |\lambda_i|^p \|H_{\overline{\varphi_{\zeta_i}}}\|_{S_p}^p$$

and it suffices to show that $\|H_{\overline{\varphi_\zeta}}\|_{S_p}$ is uniformly bounded in ζ. Furthermore it certainly suffices to consider the case of ζ real and positive. Thus we are reduced to the following lemma.

LEMMA 3.1. *Let $0 < p \leqslant \infty$ and $g(z) = \dfrac{1}{1-a\overline{z}}$, $0 < a < 1$. Then $\|H_g\|_{S_p} \leqslant c_p(1-a^2)^{-1}$.*

Proof. We may assume $p < \infty$. If $n \geqslant 0$ then

$$H_g(z^n) = P\left(\sum_{k=0}^{\infty} a^k \overline{z}^k z^n\right) = \sum_{k=n}^{\infty} a^k \overline{z}^k z^n = \sum_{j=0}^{\infty} a^{n+j} \overline{z}^{n+j} z^n$$

Hence, since $\overline{z}^{m+k} z^n \perp \overline{z}^{n+j} z^n$ whenever $k \neq j$,

$$\begin{aligned}\langle H_g(z^m), H_g(z^n)\rangle &= \sum_{j=0}^{\infty} a^{m+j} a^{n+j} \langle \overline{z}^{m+j} z^m, \overline{z}^{n+j} z^n\rangle = \\ &= \sum_{j=0}^{\infty} a^{m+n+2j} \int |z|^{2m+2n+2j} \mathrm{d}\mu_\alpha(z) = \\ &= \sum_{j=0}^{\infty} a^{m+n+2j} \int_0^1 t^{m+n+j} (1-t)^\alpha (\alpha+1) \mathrm{d}t.\end{aligned}$$

Define

$$b_k = \sum_{j=0}^{\infty} a^{k+2j} \int_0^1 t^{k+j} (1-t)^\alpha \mathrm{d}t = \int_0^1 a^k t^k \frac{1}{1-a^2 t} (1-t)^\alpha \mathrm{d}t.$$

Thus, with respect to the orthonormal basis $\{z^n/\|z^n\|\}_{n=0}^{\infty}$, the operator $H_g^* H_g$ has the matrix representation

$$\left(\left\langle H_g^* H_g \frac{z^m}{\|z^m\|}, \frac{z^n}{\|z^n\|}\right\rangle\right)_{m,n=0}^{\infty} = \left((\alpha+1) b_{m+n} \|z^m\|^{-1} \|z^n\|^{-1}\right)_{m,n=0}^{\infty}.$$

$H_g \in S_p$ if and only if $H_g^* H_g \in S_{p/2}$. Thus after estimating the norm of the monomials the issue becomes whether

$$\left(b_{m+n} (m+1)^{\frac{\alpha+1}{2}} (n+1)^{\frac{\alpha+1}{2}}\right)_{m,n=0}^{\infty}$$

defines an $S_{p/2}$ operator on l^2. By the results of Peller [10] and Semmes [14] this holds if and only if

$$B(z) = \sum_0^{\infty} b_k z^k \in B_{p/2}^{2/p+\alpha+1};$$

in fact

$$\|H_g\|_{S_p} = \|H_g^* H_g\|_{S_{p/2}}^{1/2} \leqslant c \|B(z)\|_{B_{p/2}^{2/p+\alpha+1}}^{1/2}.$$

Using the definition of b_k we find

$$\begin{aligned}B(z) &= \sum_0^{\infty} z^k \int_0^1 a^k t^k \frac{1}{1-a^2 t} (1-t)^\alpha \mathrm{d}t = \\ &= \int_0^1 \frac{1}{1-atz} \frac{1}{1-a^2 t} (1-t)^\alpha \mathrm{d}t.\end{aligned}$$

We now estimate the Besov norm of $B(z)$. Let $r = p/2$. Fix an integer $m > > 1/r + \alpha + 1$. Let $s = rm - r\alpha - r - 2$.

$$\begin{aligned}
\|B(z)\|^r_{B_r^{1/r+\alpha+1}} &= \int\limits_{|z|<1} \left(|B(z)|^r + |D^m B(z)|^r\right)\left(1-|z|^2\right)^s \mathrm{d}x\mathrm{d}y \leqslant \\
&\leqslant c \int\limits_{|z|<1} \left| \int_0^1 \frac{(at)^m}{(1-atz)^{m+1}} \frac{(1-t)^\alpha}{1-a^2 t} \mathrm{d}t \right|^r \left(1-|z|^2\right)^s \mathrm{d}x\mathrm{d}y \leqslant \\
&\leqslant c \int\limits_{|z|<1} \left(\int_0^1 \frac{1}{|1-atz|^{m+1}} \frac{(1-t)^\alpha}{1-a} \mathrm{d}t \right|^r \left(1-|z|^2\right)^s \mathrm{d}x\mathrm{d}y.
\end{aligned}$$

To estimate this we set $a = 1-\delta,\ t = 1-u$, and $z = (1-x)\mathrm{e}^{\mathrm{i}\theta}$. Using the fact that $|1-atz| \approx \delta + u + x + |\theta|$ we obtain

$$\begin{aligned}
\|B(z)\|^r_{B_r^{1/r+\alpha+1}} &\leqslant C \int_0^1 \int_{-\pi}^{\pi} \left(\int_0^1 (\delta+u+x+|\theta|)^{-m-1} \delta^{-1} u^\alpha \mathrm{d}u \right)^r x^s \mathrm{d}\theta\mathrm{d}x \leqslant \\
&\leqslant C \int_0^\infty \int_0^\infty \left(\int_0^\infty (\delta+u+x+\theta)^{-m-1} \delta^{-1} u^\alpha \mathrm{d}u \right)^r x^s \mathrm{d}\theta\mathrm{d}x \leqslant \\
&\leqslant C\delta^{-2r}.
\end{aligned}$$

(The last inequality follows by homogeniety as soon as we note that the integral is finite). As we noted, this estimate of the function $B(z)$ implies the required Schatten ideal estimate on the operator. Thus the proof of the lemma, and also of the theorem, is finished.

4. ANOTHER INTERMIEDIATE HANKEL OPERATOR

For β with $0 < \beta < 1$ define $T_\beta = \overline{\operatorname{span}}\{z^n \overline{z}^m : \beta n \geqslant m \geqslant 0\}$. Any f in this space can be written

$$f(r\mathrm{e}^{\mathrm{i}t}) = \sum_0^\infty \hat{f}_k(r)\mathrm{e}^{\mathrm{i}kt}$$

where $\hat{f}_k(r) = r^k P_k(r^2),\ P_k$ a polynomial of degree at most $\dfrac{\beta k}{1-\beta}$, and

$$\sum_0^\infty \int_0^1 \left|\hat{f}_k(r)\right|^2 r(1-r^2)^\alpha \mathrm{d}r < \infty.$$

Thus

$$A^2 = T_0 \subset T_\beta \subset T_1 = T.$$

THEOREM 4.1. *For $0 < \beta < 1$, $0 < p < \infty$ and holomorphic f, the intermediate Hankel operator $H_{\overline{f}}^{A^2,\overline{T_\beta}}$ is in S_p if and only if f is in B_p.*

Proof. This is an immediate consequence of the previous result and the inclusions $\overline{A^2} \subset \overline{T_\beta} \subset \overline{T}$.

One reason for mentioning this result is to point to a related question for which we don't know the answer - what is true for $H_{\overline{f}}^{A^2,T_\beta^\perp}$? Using the inclusion $\overline{A_0^2} \subset T_\beta^\perp \subset A^{2\perp}$ and the results in the introduction we find the (expected) answer for $1 < p < \infty$ but the case of $p \leqslant 1$ remains open.

5. A VERY BIG HANKEL OPERATOR

In this section we consider the case $X = T$, $Y = \overline{T}$ (or $Y = T^\perp$). Because $X \supset A^2$ the Hankel operator associated with this pair of subspaces is not intermediate in the sense we described earlier. We denote the classical Hankel operator mapping $H^2(\mathbb{T})$ to $\overline{H^2(\mathbb{T})}$ by $H_{\overline{f}}^{\text{class}}$. We denote the intersection of H^2 with the space of functions of bounded mean oscillation by BMOA and recall that $\text{BMOA} = L^\infty(\mathbb{T})/\overline{H_0^\infty}$.

THEOREM 5.1. *For holomorphic f the Hankel operator $H_{\overline{f}}^{T,\overline{T}}$ is bounded if and only if f is in* BMOA. *In fact $\left\| H_{\overline{f}}^{T,\overline{T}} \right\| = \left\| H_{\overline{f}}^{\text{class}} \right\| = \|f\|_{L^\infty(\mathbb{T})/\overline{H_0^\infty}}$. $H_{\overline{f}}^{T,\overline{T}}$ is not compact unless $f = 0$. Similar statements hold for $H_{\overline{f}}^{T,\overline{T}}$.*

Proof. Any f in T can be written $f(r\mathrm{e}^{\mathrm{i}t}) = \sum_0^\infty f_n(r^2)\mathrm{e}^{int}$. Let $b(z) = \sum_0^\infty b_k z^k$ and $H = H_{\overline{b}}^{T,\overline{T}}$. We now compute $H(f)$.

$$H\left(\sum_0^\infty f_n(r^2)\mathrm{e}^{int}\right) = P_{\overline{T}}\left(\sum_0^\infty \sum_0^\infty \overline{b_k} r^k \mathrm{e}^{-\mathrm{i}kt} f_n(r^2)\mathrm{e}^{int}\right) =$$
$$= \sum_0^\infty \sum_0^\infty \overline{b}_{m+n} r^{m+n} \mathrm{e}^{-\mathrm{i}mt} f_n(r^2).$$

The map from f to (f_n) gives an isometry of T onto

$$\bigoplus_0^\infty L^2\left((\alpha+1)(1-x)^\alpha \mathrm{d}x\right).$$

Using this we find that H is unitarly equivalent to the matrix $\left(M_x^{\frac{m+n}{2}}\overline{b}_{m+n}\right)$ acting on $\bigoplus L^2$. Here we are using M_x to denote multiplication by x. This operator equals $\mathbf{M}\mathbf{H}_{\overline{b}}\mathbf{M}$ where $\mathbf{M}(f_n(x)) = \left(x^{\frac{n}{2}} f_n(x)\right)$ and $\mathbf{H}_{\overline{b}}$ is the Hankel matrix $(\overline{b}_{m+n})$

acting pointwise on $\bigoplus L^2 = L^2(l^2)$. Thus $\|\mathbf{M}\| = 1$ and $\|\mathbf{H}_{\overline{b}}\| = \|H^{\text{class}}_{\overline{b}}\|$ and hence $\|H\| \leqslant \|H^{\text{class}}_{\overline{b}}\|$. In fact equality holds. To see this consider, for large N, the subspace with $f_n \equiv 0$ for $n > N$ and for $n \leqslant N$

$$f_n(x) = \begin{cases} 0 & x < 1-\varepsilon \\ \text{constant} & 1-\varepsilon < x < 1. \end{cases}$$

Letting ε go to zero and N get large gives the required result.

Now suppose H is compact. If b is not identically zero then for some k, $b_k \neq 0$. By the computations in the previous paragraph, we would then have that $\overline{b_k} M_x^{k/2}$ is a compact operator on $L^2\left((1-x)^\alpha \mathrm{d}x\right)$. This contradiction completes the proof.

6. A BAD HANKEL OPERATOR

We now return to the spaces $T_\beta = \overline{\operatorname{span}}\{z^n \overline{z}^m : \beta n \geqslant m \geqslant 0\}$ with $0 < \beta < 1$ and consider the situation $X = T_\beta,\ Y = \overline{T_\beta}$. Our partial results in this case are quite different from other known results about Hankel operators.

The operator $H^{T_\beta, \overline{T_\beta}}_{\overline{z}^n}$ has finite rank and hence there will be operators of the form $H^{T_\beta, \overline{T_\beta}}_{\overline{f}}$ in all S_p. However the rank is $\sim n^2$ in contrast to the estimate $\sim n$ for the small Hankel operator. This suggests that the usual criterion for membership in S_p no longer holds. In fact that is what we now show. We only present the case of the unweighted Bergman space ($\alpha = 0$). We have not investigated the general situation.

THEOREM 6.1. $\left\| H^{T_\beta, \overline{T_\beta}}_{\overline{z}^n} \right\|_{S_2} \sim n^{3/4}$. *Consequently, for holomorphic f, $H^{T_\beta, \overline{T_\beta}}_{\overline{f}} \in$ $\in S_2$ if and only if $f \in B_2^{3/4}$.*

Proof. Let P_k be the space of polynomials of degree at most k. Set $\gamma = \beta/(1-\beta)$. The individual terms in the Fourier decomposition of the space T_β are of the form

$$X_m = \left\{ z^m p\left(|z|^2\right) : p \in P_{\gamma m} \right\}.$$

Let

$$U = U_m : P_{\beta m} \to X_m$$

denote this isomorphism. Then

$$\begin{aligned} \|Up\|_X &= \|Up\|_{L^2(D, \pi^{-1} \mathrm{d}x \mathrm{d}y)} = \\ &= \left(\int_0^1 r^{2m} |p(r^2)|^2 2r \mathrm{d}r \right)^{1/2} = \\ &= \left(\int_0^1 x^m |p(x)|^2 \mathrm{d}x \right)^{1/2} = \\ &= \left\| x^{m/2} p \right\|_{L^2(0,1)}. \end{aligned}$$

Similarly,

$$\|\overline{z}^m Up\|_{L^2} = \left\| x^{\frac{n+m}{2}} p \right\|_{L^2(0,1)}. \tag{6.2}$$

Let M_x denote multiplication by x on $L^2(0,1)$ and regard P_k as a subspace of $L^2(0,1)$. By (6.1) U is a contraction from $P_{\beta m}$ to X_m. Furthermore, if $n > cm$ (for some c which depends on β), then $H_{\overline{z}^n} g = \overline{z}^n g \ \forall\, g \in X_m$. Thus

$$\begin{aligned}\|H_{\overline{z}^n}\|_{S_2(X_m,Y)} &= \|M_{\overline{z}^n}\|_{S_2(X_m,Y)} \geqslant \\ &\geqslant \|M_{\overline{z}^n} U\|_{S_2(P_{\beta m},Y)} = \left\| M_x^{\frac{n+m}{2}} \right\|_{S_2(P_{\beta m},L^2(0,1))}.\end{aligned}$$

The last equality by (6.2). Now using Lemma 6.1 (which is stated and proved at the end of this section) we see that, if n is large enough,

$$\begin{aligned}\|H_{\overline{z}^n}\|^2_{S_2(X,Y)} &= \sum_0^\infty \|H_{\overline{z}^n}\|^2_{S_2(X_m,Y)} \geqslant \sum_0^{cn} \left\| M_x^{\frac{n+m}{2}} \right\|^2_{S_2(P_{\beta m},L^2(0,1))} \geqslant \\ &\geqslant \sum_{c\sqrt{n}}^{cn} c\frac{\beta m}{\sqrt{n+m}} \geqslant \sum_{c\sqrt{n}}^{cn} c\frac{m}{\sqrt{n}} \geqslant cn^{3/2}.\end{aligned}$$

(Here we are, of course, following the custom of using "c"to denote various inessential positive constants, not necessarily all the same.)

In the converse direction, let $m' = \left[\dfrac{m+1}{2}\right]$ and let $Q = x^{m'} P_{\beta m} \subset P_{(r+1)m}$. By (6.1) $\|Up\|_{X_m} \geqslant \left\|x^{m'}p\right\|_Q \ \forall p \in P_{(r+1)m}$. Thus $\left\|UM_x^{-m'}q\right\|_{X_m} \geqslant \|q\|_Q \ \forall q \in Q$. Hence $\left(UM_x^{-m'}\right)^{-1} : X_m \to Q$ is a contraction. Thus

$$\begin{aligned}\|H_{\overline{z}^n}\|_{S_2(X_m,L^2)} &\leqslant \|M_{\overline{z}^n}\|_{S_2(X_m,L^2)} \leqslant \\ &\leqslant \left\|M_{\overline{z}^n}UM_x^{-m'}\right\|_{S_2(Q,L^2)} \left\|\left(UM_x^{-m'}\right)^{-1}\right\|_{S_\infty(X_m,Q)} \leqslant \\ &\leqslant \left\|M_{\overline{z}^n}UM_x^{-m'}\right\|_{S_2(Q,L^2)} = \left\|M_x^{\frac{n+m}{2}}M_x^{-m'}\right\|_{S_2(Q,L^2)} \leqslant \\ &\leqslant \left\|M_x^{\frac{n+m}{2}-m'}\right\|_{S_2(P_{(\gamma+1)m},L^2)} \leqslant cm^{1/2}\left(\frac{n-1}{2}\right)^{-1/4}.\end{aligned} \tag{6.3}$$

Here the final equality used (6.2) and the final inequality used Lemma 6.1 again. Thus, for $n > 0$,

$$\begin{aligned}\|H_{\overline{z}^n}\|^2_{S_2(X,L^2)} &\leqslant \sum \|H_{\overline{z}^n}\|^2_{S_2(X_m,L^2)} \leqslant \\ &\leqslant C + C\sum_1^n \frac{m}{\sqrt{n}} \leqslant Cn^{3/2}.\end{aligned}$$

This proves the first assertion, the second follows because the Hankel operators generated by the monomials are orthogonal.

We do not know if there is a similar characterization of the Hankel operators in S_p for $p \neq 2$, but we doubt it. The next result shows that in contrast to the results for more familiar types of Hankel operators, neither BMOA nor the Bloch space, B, characterizes the class of symbols for which these operators will be bounded.

THEOREM 6.2.

$$\mathrm{BMOA} \subsetneq \left\{ f : f \in Hol,\ H_{\overline{f}}^{T_\beta, \overline{T_\beta}} \text{ is bounded} \right\} \subsetneq B.$$

Proof. The inclusion follow from the previous result and the inclusions $A \subset T_\beta \subset \subset \overline{T}$. In order to see that the inclusions are strict we use lacunary series. First, assume that f is holomorphic and that $H_{\overline{f}}$ is bounded. Let $N > 0$, and let $m = [cN]$, where c is sufficiently small so that $H_{\overline{z}^n} g = \overline{z}^n g$ for all $g \in X_m,\ n \geqslant N$. Let $p \in P_{\gamma m}$ be such that $\|p\|_{L^2(0,1)} = 1$ and

$$p(x) \geqslant c\gamma m \geqslant cN \text{ on } \left[1 - (\gamma m)^{-2},\ 1\right] \supset \left[1 - N^{-2}, 1\right].$$

(That we can find such polynomials is shown in Lemma 6.2 below.) Let $g = Up \in X_m$. Then $\|g\|_X \leqslant 1$ and for all n between N and N^2 we have

$$\begin{aligned} \|H_{\overline{z}^n} g\|_{L^2}^2 = \|\overline{z}^n g\|^2 = \left\| x^{\frac{n+m}{2}} p \right\|_{L^2(0,1)}^2 \geqslant \\ \geqslant \int_{1-N^{-2}}^{1} x^{n+m} p(x)^2 \mathrm{d}x \geqslant cN^2 N^{-2} = c. \end{aligned}$$

Here we used the estimate $p(x) > cN$ and $x^{n+m} > c$ when $x > 1 - N^{-2}$. Furthermore $H_{\overline{z}^n} g \in L^2_{m-n}$ and thus $\{H_{\overline{z}^n} g\}$ are orthogonal. Consequently

$$\begin{aligned} \left\| H_{\overline{f}} \right\|_{S_\infty} \geqslant \left\| H_{\overline{f}} g \right\| = \left(\sum_0^\infty \left\| \overline{\hat{f}(n)} H_{\overline{z}^n} g \right\|^2 \right)^{1/2} \geqslant \\ \geqslant \left(\sum_N^{N^2} \left| \hat{f}(n) \right|^2 \right) \end{aligned}$$

for some constant c that is independent of N. That is

$$\sup \left\{ \sum_N^{N^2} \left| \hat{f}(n) \right|^2 : N = 0,\ 1,\ \ldots \right\} \leqslant c \left\| H_{\overline{f}} \right\|_{S_\infty}^2 .$$

It follows that, for instance, if $f(z) = \sum_0^\infty z^{2^k}$ then $H_{\overline{f}}$ cannot be bounded although, as is well known, f is in the Bloch space.

On the other hand, if $f(z) = \sum_0^\infty z^{2^{2^k}}$ then f is not in BMOA (it is not even in H^2) but, as we now show, $H_{\overline{f}}$ is bounded.

Set $X^{(0)} = X_0 \oplus X_1$; and for $k = 1,\ 2,\ \ldots$ set

$$X^{(k)} = \bigoplus_{m=2^{2^{k-1}}}^{2^{2^k}-1} X_m.$$

Thus $X = \bigoplus_0^\infty X^{(k)}$. Let $Y^{(k)} = \overline{X^{(k)}}$ and $H_{\overline{f}}^{k,l} = P_{Y^{(l)}} H_{\overline{f}} P_{X^{(k)}}$. Note that for positive k and l, this operator only depends on the terms z^n with

$$2^{2^{k\vee l-1}} < 2^{2^{k-1}} + 2^{2^{l-1}} \leqslant n < 2^{2^k} + 2^{2^l} < 2^{2^{k\vee l+1}} < 2^{2^{k\vee l+1}},$$

and thus, for positive k and l and for $f_j = z^{2^{2^j}}$, $H_{\overline{f}}^{k,l} = H_{\overline{f_s}}^{k,l}$, $s = k \vee l$. Trivially $\left\| H_{\overline{f_s}}^{k,l} \right\| \leqslant \|f_s\|_{L^\infty} = 1$. Furthermore, if $l > k > 0$ and $2^{2^{k-1}} \leqslant m < 2^{2^k}$ then

$$\begin{aligned} \left\| H_{\overline{f}}^{k,l} \right\|_{S_\infty(X_m, L^2)} &\leqslant \left\| H_{\overline{f_l}}^{k,l} \right\|_{S_2(X_m, L^2)} \leqslant \left\| H_{\overline{f_l}} \right\|_{S_2(X_m, L^2)} \leqslant \\ &\leqslant c m^{1/2} \left(2^{2^l}\right)^{-1/4} \leqslant c 2^{2^{k-1} - 2^{l-2}} \leqslant c 2^{-(l-k)}. \end{aligned} \tag{6.4}$$

where we used (6.3). (Actually, the remark after Lemma 6.2 below implies a better bound but we have room to spare.)

Because $H_{\overline{f}}^{k,l} = H_{\overline{f_l}}^{k,l}$ decomposes into orthogonal parts $H_{\overline{f_l}}^{k,l} P_{X_m}$, $2^{2^{k-1}} \leqslant m < 2^{2^k}$, (6.4) implies

$$\left\| H_{\overline{f}}^{k,l} \right\|_{X^{(k)}, Y^{(l)}} \leqslant c 2^{-(l-k)},\ l > k \geqslant 1.$$

If $k > 1$ then, by duality, $\left\| H_{\overline{f}}^{k,l} \right\| = H_{\overline{f}}^{l,k} \leqslant c 2^{-(k-l)}$. Consequently, for all positive $k,\ l$ $\left\| H_{\overline{f}}^{l,k} \right\| \leqslant c 2^{-|k-j|}$. Hence, by orthogonality, for every integer j,

$$\left\| \sum_{k-l=j} H_{\overline{f}}^{l,k} \right\| \leqslant c 2^{-|j|}.$$

Thus, summing over all positive k and l produces a bounded operator. It only remains to check that the two "edges",

$$\sum_{l=0}^\infty H_{\overline{f}}^{0,l} \text{ and } \sum_{k=1}^\infty H_{\overline{f}}^{k,0},$$

are bounded operators. This follows, for instance, because f is in L^2 and $X^{(0)} \subset H^\infty$.

In the previous proof we needed certain facts about polynomials. We now state and prove those lemmas. It seems quite plausible that some of these facts, in particular Lemma 6.3, are in the literature, but we were not able to find them. If these lemmas were extended to cover weighted Bergman spaces (using Jacobi polynomials in place of Legendre polynomials) then the previous proof would extended to weighted Bregman spaces. However we have not pursued the issue.

LEMMA 6.1. *For all $\lambda,\ n \geqslant 1$,*

$$\|M_x^\lambda\|_{S_2(P_n,L^2)} \asymp \min\left(\frac{n}{\lambda^{1/2}}, \frac{n^{1/2}}{\lambda^{1/4}}\right) = \begin{cases} \left(\dfrac{n^2}{\lambda}\right)^{1/4} & \lambda \leqslant n^2 \\ \left(\dfrac{n^2}{\lambda}\right)^{1/2} & \lambda \geqslant n^2 \end{cases}$$

and the implicit constants do not depend on n or λ.

LEMMA 6.2. *For each positive integer n there is a p in P_n such that $\|p\|_{L^2} \leqslant 1$ and $p(x) \geqslant cn$ when $1 - n^{-2} \leqslant x \leqslant 1$.*

Remark: With p as in Lemma 6.2,

$$\left\|x^\lambda p\right\|_{L^2} \geqslant c\left(\int_{1-n^{-2}}^1 x^{2\lambda} n^2 \mathrm{d}x\right)^{1/2} \geqslant \left(cn^2\left(\frac{1}{n^2} \wedge \frac{1}{\lambda}\right)\right)^{1/2}.$$

Consequently (using Lemma 6.1 for an upper bound when $\lambda \geqslant n^2$) we have

$$\|M_x^\lambda\|_{S_\infty(P_n,L^2)} \asymp \min\left(1, \frac{n}{\lambda^{1/2}}\right) = \begin{cases} 1 & \lambda \leqslant n^2 \\ \dfrac{n}{\lambda^{1/2}} & \lambda \geqslant n^2 \end{cases}.$$

Is it true that

$$\|M_x^\lambda\|_{S_p(P_n,L^2)} \asymp \min\left(\frac{n}{\lambda^{1/2}}, \left(\frac{n}{\lambda^{1/2}}\right)^{1/p}\right)$$

for all p greater than 2? What happens if p is less than 2? Less than 1?

The proof of these two lemmas uses orthogonal polynomials. In order to conform with the usual notations in the subject we will work with polynomials in $L^2\left((-1,1), \frac{1}{2}\mathrm{d}x\right)$. Those polynomials are related to the P_n we have been considering by the elementary changes of variables $x \to \dfrac{x+1}{2}$, or $x \to 2x - 1$.

The Legendre polynomials $\{P_n(x)\}_{n=0}^\infty$ are orthogonal on $[-1,1]$ with respect to Lebesgue measure. Using the standardization in Erdelyi et al [2] Chapter 10.10 (but using the normalized measure $\frac{1}{2}\mathrm{d}x$) we have

$$P_n(1) = 1,\ \int P_n^2(x)\frac{\mathrm{d}x}{2} = \frac{1}{2n+1};$$

hence an orthonormal basis is given by $\tilde{P}_n = \sqrt{2n+1}P_n$ and the reproducing kernel for polynomials of degree n is given by

$$K_n(x,y) = \sum_0^n \tilde{P}_j(x)\tilde{P}_j(y) = \sum_0^n (2j+1)P_j(x)P_j(y).$$

Proof of Lemma 6.2. Let

$$p(x) = \frac{1}{n+1}\sum_0^n (2k+1)P_k(x) \quad -1 \leqslant x \leqslant 1.$$

Then

$$\|p\|_{L^2} = \frac{1}{(n+1)^2}\sum_0^n (2k+1) = 1$$

and $p(1) = n+1$. (In fact, p is the polynomial of degree n which maximizes $p(1)$ subject to $\|p\|_{L^2\left((-1,1),\ \frac{dx}{2}\right)} \leqslant 1$.) Furthermore, by [2] (10.18.6),

$$\begin{aligned}|p'(x)| &\leqslant \frac{1}{n+1}\sum_0^n (2k+1)\,|P_k'(x)| \leqslant \\ &\leqslant \frac{1}{n+1}\sum_0^n \frac{1}{2}(2k+1)k(k+1) = \frac{n(n+1)(n+2)}{4}.\end{aligned}$$

Thus, if $1-2n^{-2} \leqslant x \leqslant 1$ then, for $n > 2$,

$$p(x) \geqslant p(1) - 2n^{-2}\,\|p'\|_{L^\infty} = (n+1)\left(1 - \frac{n+2}{2n}\right) \geqslant cn.$$

For $n \leqslant 2$ we can take p identically one. To get the required polynomial on $(0,1)$ we take $p(2x-1)$. The proof is done.

Before proceeding to the proof of Lemma 6.1 we need a technical estimate on the reproducing kernel associated to the Legendre polynomials.

Lemma 6.3.

$$K_n(x,x) \asymp n\left(\left(1-x^2\right)^{-1/2} \wedge n\right). \tag{6.5}$$

The implicit constants are uniform in x and n.

Proof. We start with the generating function ((10.10.39) of [2])

$$\sum_0^\infty P_n(x)z^n = (1-2xz+z^2)^{-1/2} \quad x \in [-1,1],\ |z| < 1.$$

Differentiation gives

$$\sum_1^\infty nP_n(x)z^{n-1} = \frac{x-z}{(1-2xz+z^2)^{3/2}}.$$

Hence, by Plancherel,

$$\sum_1^\infty n^2 P_n(x)^2 r^{2(n-1)} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{|x-r\mathrm{e}^{\mathrm{i}t}|^2}{|1-2xr\mathrm{e}^{\mathrm{i}t}+r^2\mathrm{e}^{2\mathrm{i}t}|^3}\mathrm{d}t \tag{6.6}$$

for $x \in [-1,1],\ 0 \leqslant r \leqslant 1$. For simplicity we now assume that $0 \leqslant x < 1,\ \frac{1}{2} \leqslant r < 1$ and write $y = 1-x,\ \delta = 1-r$. By symmetry we need only consider the integral in (6.6) for positive t. Furthermore, if $\frac{3\pi}{4} \leqslant t \leqslant \pi$ then

$$\operatorname{Re}\left(1-2xr\mathrm{e}^{\mathrm{i}t}+r^2\mathrm{e}^{2\mathrm{i}t}\right) \geqslant 1$$

and thus

$$\int_{\frac{3\pi}{4}}^{\pi} \frac{|x-r\mathrm{e}^{\mathrm{i}t}|^2}{|1-2xr\mathrm{e}^{\mathrm{i}t}+r^2\mathrm{e}^{2\mathrm{i}t}|^3}\mathrm{d}t < 4\frac{\pi}{4}.$$

Hence we will assume that $0 < t < \frac{3\pi}{4}$.

$$1-2xr\mathrm{e}^{\mathrm{i}t}+r^2\mathrm{e}^{2\mathrm{i}t} = (1-r^2)(1-\mathrm{e}^{\mathrm{i}t}) + \mathrm{e}^{\mathrm{i}t}(1-r^2+2r^2\cos t - 2xr).$$

For $0 < t < \frac{3\pi}{4}$ the angle between $1-\mathrm{e}^{\mathrm{i}t}$ and $\mathrm{e}^{\mathrm{i}t}$ is $\frac{\pi - t}{2}$ which is bounded away from 0 and π and hence

$$\left|1-2xr\mathrm{e}^{\mathrm{i}t}+r^2\mathrm{e}^{2\mathrm{i}t}\right| \asymp (1-r^2)\left|1-\mathrm{e}^{\mathrm{i}t}\right| + \left|1-r^2+2r^2\cos t-2xr\right|.$$

We set $u = |1-\mathrm{e}^{\mathrm{i}t}| = 2\sin\frac{t}{2}$ and hence $\cos t = 1-\frac{u^2}{2}$.

$$\begin{aligned}\left|1-2xr\mathrm{e}^{\mathrm{i}t}+r^2\mathrm{e}^{2\mathrm{i}t}\right| &\asymp (1+r)\delta u + \left|\delta^2+2ry-r^2u^2\right| \asymp \\ &\asymp \delta u + \left|\delta^2+2ry-ru^2+r\delta u^2\right| \asymp \\ &\asymp \delta u + \left|\delta^2+r(2y-u^2)\right|.\end{aligned}$$

Furthermore

$$\begin{aligned}\left|x-r\mathrm{e}^{\mathrm{i}t}\right|^2 &= (x-r\cos t)^2 + r^2\sin^2 t \asymp \\ &\asymp \left(x-r+r\frac{u^2}{2}\right)^2 + r^2u^2 \asymp \\ &\asymp (x-r)^2+u^2 = (\delta-y)^2+u^2.\end{aligned} \tag{6.7}$$

Thus

$$\sum_1^\infty n^2 P_n(x)^2 r^{2(n-1)} \asymp O(1) + \int_0^{2\sin\frac{3\pi}{8}} \frac{(\delta-y)^2+u^2}{(\delta u + |\delta^2+2ry-ru^2|)^3}\mathrm{d}u \asymp$$

$$\asymp \int_0^2 \frac{(\delta-y)^2+u^2}{(\delta u + |\delta^2+2ry-ru^2|)^3}\mathrm{d}u =$$

$$= \int_0^2 \varphi(u)\mathrm{d}u,$$

say. If $u<\delta$ then $ru^2 \leqslant u^2 \leqslant \delta u$ and thus

$$\varphi(u) \asymp \frac{(\delta-y)^2+u^2}{(\delta u + |\delta^2+2ry|)^3} \asymp \frac{(\delta-y)^2+u^2}{(\delta^2+y)^3}$$

and

$$\int_0^\delta \varphi(u)\mathrm{d}u \asymp \frac{\delta(\delta-y)^2+\dfrac{\delta^3}{3}}{(\delta^2+y)^3} \asymp \frac{\delta^3+\delta y^2}{\delta^6+y^3}. \tag{6.8}$$

For the remaining part of the integral we consider two cases. Case (i) is $y \leqslant 2\delta^2$. In this case, for $u>\delta$, $\delta^2+2ry \leqslant 5\delta u$ and thus

$$\varphi(u) \asymp \frac{u^2}{(\delta u + ru^2)^3} \asymp \frac{u^2}{u^6} = u^{-4}.$$

Hence

$$\int_\delta^2 \varphi(u) \asymp \delta^{-3}.$$

By (6.8) we also have

$$\int_0^\delta \varphi(u)\mathrm{d}u \asymp \delta^{-3}$$

and thus

$$\int_0^2 \varphi(u)\mathrm{d}u \asymp \delta^{-3}.$$

Case (ii) is $y>2\delta^2$ and, still, $u>\delta$. In this case

$$\varphi(u) \asymp \frac{y^2+u^2}{(\delta u + |2ry-ru^2|)^3} \asymp \frac{y^2+u^2}{\left(\delta u + \left(\sqrt{2y}+u\right)\left|\sqrt{2y}-u\right|\right)^3}.$$

Hence

$$\begin{aligned}
\int_{\delta}^{\sqrt{2y}-\delta} \varphi(u)\mathrm{d}u &\asymp \int_{\delta}^{\sqrt{2y}-\delta} \frac{y^2+u^2}{\left(\sqrt{y}\,|\sqrt{2y}-u|\right)^3}\mathrm{d}u \leqslant \\
&\leqslant \int_{\delta}^{\sqrt{2y}-\delta} \frac{3y}{\left(\sqrt{y}\,|\sqrt{2y}-u|\right)^3}\mathrm{d}u = O\left(y^{-1/2}\delta^{-2}\right), \\
\int_{\sqrt{2y}-\delta}^{\sqrt{2y}+\delta} \varphi(u)\mathrm{d}u &\asymp \frac{y}{\left(\delta\sqrt{y}\right)^3}2\delta = 2\delta^{-2}y^{-1/2}, \text{ and} \\
\int_{\sqrt{2y}+\delta}^{2} \varphi(u)\mathrm{d}u &\asymp \int_{\sqrt{2y}+\delta}^{2} \frac{u^2}{\left(u|\sqrt{2y}-u|\right)^3}\mathrm{d}u \leqslant \\
&\leqslant \int_{\sqrt{2y}+\delta}^{2} \frac{\mathrm{d}u}{\sqrt{2y}(u-\sqrt{2y})^3} = O\left(y^{-1/2}\delta^{-2}\right).
\end{aligned}$$

Also by (6.8)

$$\int_0^{\delta} \varphi(u)\mathrm{d}u \asymp \frac{\delta^3+\delta y^2}{y^3} \leqslant \frac{\delta^3}{y^{1/2}\delta^5} + \frac{\delta}{y} = O\left(y^{-1/2}\delta^{-2}\right),$$

and thus

$$\int_0^2 \varphi(u)\mathrm{d}u \asymp y^{-1/2}\delta^{-2}.$$

We have proved that for $0 < y \leqslant 1,\ 0 < \delta \leqslant \frac{1}{2}$, we have

$$1 + \sum_1^{\infty} n^2 P_n(x)^2 r^{2(n-1)} \asymp \min\left(\delta^{-3},\ y^{-1/2}\delta^{-2}\right). \tag{6.9}$$

It is clear from the integral representation that this estimate extends to $\frac{1}{2} < \delta \leqslant 1$. We now multiply (6.9) by r and integrate:

$$\begin{aligned}
\sum_0^{\infty}(2n+1)P_n(x)^2t^{2n} &\asymp 1 + \sum_1^{\infty} nP_n(x)^2t^{2n} = \\
&= 1 + 2\int_0^t \sum_1^{\infty} n^2P_n(x)^2r^{2n-1}\mathrm{d}r \asymp \\
&\asymp 1 + \int_{1-t}^1 \min\left(\delta^{-3}, y^{-1/2}\delta^{-2}\right)\mathrm{d}\delta \asymp \\
&\asymp \min\left((1-t)^{-2}, (1-x)^{-1/2}(1-t)^{-1}\right)
\end{aligned} \tag{6.10}$$

for $0 \leqslant x, t < 1$.

We conclude the proof with a standard Tauberian argument. Take $t = 1 - \dfrac{1}{n}$ and we obtain (for $n > 1$)

$$K_n(x) = \sum_0^n (2k+1)P_k(x)^2 \leqslant$$
$$\leqslant C \sum_0^n (2k+1)P_k(x)^2 \left(1 - \frac{1}{n}\right)^{2k} \leqslant C\min\left(n^2, n(1-x)^{-1/2}\right).$$

For the converse take $t = 1 - \dfrac{\gamma}{n}$ where γ is a large constant which will be chosen later. Then (with C's independent of γ and with $n > 2\gamma$)

$$\gamma^{-2}\min\left(n^2, n(1-x)^{1/2}\right) \leqslant \min\left(\left(\frac{n}{\gamma}\right)^2, (1-x)^{-1/2}\left(\frac{n}{\gamma}\right)\right) \leqslant$$
$$\leqslant C \sum_0^\infty (2k+1)P_k(x)^2 \left(1 - \frac{\gamma}{n}\right)^{2k} \leqslant$$
$$\leqslant CK_n(x) + C\sum_{j=1}^\infty \sum_{k=jn+1}^{(j+1)n} (2k+1)P_k(x)^2 \left(1 - \frac{\gamma}{n}\right)^{2jn} \leqslant$$
$$\leqslant CK_n(x) + C\sum_{j=1}^\infty K_{(j+1)n}(x)\mathrm{e}^{-2j\gamma} \leqslant$$
$$\leqslant CK_n(x) + C\sum_{j=1}^\infty (j+1)^2\mathrm{e}^{-2j\gamma}\min\left(n^2, n(1-x)^{-1/2}\right).$$

If γ is large enough then the right hand side is

$$\leqslant CK_n(x) + \frac{1}{2}\gamma^{-2}\min\left(n^2, n(1-x)^{-1/2}\right)$$

and hence, for x between 0 and 1

$$K_n(x) \asymp \min\left(n^2, n(1-x)^{-1/2}\right)$$

and we are done.

Proof of Lemma 6.1

$$\left\|M_x^\lambda\right\|_{S_2(P_n,L^2)} = \left\|M^\lambda_{\frac{1+x}{2}}\right\|_{S_2\left(P_n, L^2\left((-1,1),\frac{\mathrm{d}x}{2}\right)\right)} =$$
$$= \sum_0^n \left\|\left(\frac{1+x}{2}\right)^\lambda \tilde{P}_k(x)\right\|^2_{L^2\left((-1,1),\frac{\mathrm{d}x}{2}\right)} =$$
$$= \int_{-1}^1 \left(\frac{1+x}{2}\right)^{2\lambda} K_n(x,x)\frac{\mathrm{d}x}{2}.$$

By Lemma 6.3 we can continue with

$$\int_0^1 \left(\frac{1+x}{2}\right)^{2\lambda} n \min\left(n,(1-x)^{-1/2}\right) \mathrm{d}x \asymp n\int_{\frac12}^1 t^{2\lambda}\min\left(n,(1-t)^{-1/2}\right)\mathrm{d}t.$$

Hence

$$\|M_x^\lambda\|_{S_2(P_n,L^2)} \leqslant Cn^2\int_0^1 t^{2\lambda}\mathrm{d}t \asymp \frac{n^2}{\lambda},$$
$$\|M_x^\lambda\|_{S_2} \leqslant Cn\int_0^1 t^{2\lambda}(1-t)^{-1/2}\mathrm{d}t \asymp n\lambda^{-1/2},$$

and

$$\begin{aligned}\|M_x^\lambda\|_{S_2} &\geqslant Cn\int_{1-\frac{1}{2\lambda}}^1 \min\left(n,(1-t)^{-1/2}\right)\mathrm{d}t = \\ &= Cn\int_0^{\frac{1}{2\lambda}}\min\left(n,s^{-1/2}\right)\mathrm{d}s \geqslant Cn\int_{\frac{1}{4\lambda}}^{\frac{1}{2\lambda}}\min\left(s^{-1/2},n\right)\mathrm{d}s \asymp \\ &\asymp n\frac{1}{\lambda}\min\left(\lambda^{1/2},n\right) = \min\left(\frac{n}{\lambda^{1/2}},\frac{n^2}{\lambda}\right).\end{aligned}$$

and we are done.

REFERENCES

1. Arazy, J.; Fisher, S.; Peetre, J., Hankel operators on weighted Bergman Spaces, *Amer. J. Math.*, **110**(1988), 989–1054.
2. Erdelyi et al, *Higher Transcedental Functions*, McGraw Hill, 1953.
3. Feldman, M.; Rochberg, R., *Singular value estimates for commutators and Hankel operators on the unit ball and the Heisenberg group, Analysis and Partial Differential Equations*, C. Sadosky ed. Dekker, 1990, 121–160.
4. Janson, S., Hankel operators between weighted Bergman spaces, *Ark. Math.*, **26**(1988), 205–219.
5. Janson, S.; Peetre, J., Paracommutators - boundedness and Schatten-von Neuman properties, *Trans. Amer. Math. Soc.*, **305**(1988), 467–504.
6. Janson, S.; Wolff, T., Schatten classes and commutators of singular integral operators, *Ark. Mat.*, **20**(1982), 301–310.
7. Peng, L., *Ha-plitz operators on Bergman space*, preprint 1990.
8. Peng, L.; Rochberg, R.; Wu, Z., *Orthogonal polynomials and middle Hankel operators on Bergman spaces*, *Studia Math.*, **102**(1992), 57–75.
9. Peng, L.; Zhang, G, *Middle Hankel operators on Bergman space*, preprint 1990.
10. Peller, V., Hankel operators of class S_p and their applications (rational approximation, Gaussian processes, and the problem of majorizing operators), *Math. USSR-Sb.*, **451**(1982), 443–479.
11. Rochberg, R., *Decomposition theorems for Bergman spaces and their applications, Operators and Function Theory*, S. C. Power ed, Reidel, 1985, 225–278.

12. Rochberg, R., *Eigenvalue estimates for Calderón-Toeplitz operators*, preprint 1990.
13. Rochberg, R.; Semmes, S., *End point results for estimates of singular values of singular integral operators*, *Contributions to Operator Theory and its Applications*, Gohberg et al eds, Birkhauser, 1988, 217–232.
14. Semmes, S., Trace ideal criteria for Hankel operators and applications to Besov spaces, *Integral Equations and Operator Theory*, **7**(1984), 241–281.

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