INTERPOLATION OF SUBCOUPLES AND QUOTIENT COUPLES

Svante Janson

Abstract. We extend recent results by Pisier on K-subcouples, i.e. subcouples of an interpolation couple that preserve the K-functional (up to constants) and corresponding notions for quotient couples. Examples include interpolation (in the pointwise sense) and a reinterpretation of the Adamyan-Arov-Krein theorem for Hankel operators.

Introduction

It is well-known that even if we know the interpolation spaces of a certain couple of spaces, by the real or complex method, say, there is no general formula that enables us to directly obtain the interpolation spaces of a couple of subspaces or quotient spaces of a given couple. Indeed, there are examples that show that interpolation of subspaces (or quotient spaces) in general may be ill-behaved, see Triebel [26] and Wallstén [27].

Nevertheless, in many natural examples, there are very simple relations (often with less simple proofs) between the interpolation spaces of a couple and a subcouple (or quotient couple). We will here consider only the real method of interpolation, and the crucial property then is that the K-functional for a subcouple equals (within constants) the K-functional for the supercouple. While this property was recognized a long time ago by Peetre [17], and has been proven in many concrete cases, it has not been studied in detail until Pisier [21] exploited several abstract properties, including relations with quotient couples, duality and approximation. (See also [22], [23], [3].) The purpose of the present paper is to emphasize this part of Pisier's work and to develop it in greater detail.

The basic definition and some simple consequences of it are given in Section 2. Quotient couples are studied in Sections 3 and 4. Section 5 treats simultaneous approximation in several norms. Duality is studied in Section 6. Various examples and counter examples are collected in Section 7, and one example, a reinterpretation of the Adamyan-Arov-Krein theorem for Hankel operators, is studied further in Section 8. Finally, a couple of remarks suggesting possible future extensions are given in Section 9.

1. Preliminaries

For the reader's convenience, we give some definitions and results that will be used later. For details see for example Bergh and Löfström [2] or Brudnyĭ and Krugljak [4].

Although we are interested in vector spaces (real or complex), it is sometimes convenient to forget the scalar multiplication and regard the spaces as Abelian groups (under addition). We have to distinguish, however, between *quasi-normed (Abelian) groups*, defined as in [2], and *quasi-normed vector spaces*, which satisfy the further requirement that the quasi-norm be homogeneous (of degree 1), i.e. $\|tx\| = |t|\,\|x\|$.

This work was done at the Mittag-Leffler Institute. I am particularly grateful to Richard Rochberg for helpful discussions.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

For example, the vector space S_0 of all finite rank operators on a given Hilbert space, with $\|T\|_{S_0} = \operatorname{rank}(T)$, is a quasi-normed group but not a quasi-normed vector space.

It turns out that most of our results are naturally stated for quasi-normed groups, so we will work in that setting; the reader may well assume that all spaces are quasi-normed vector spaces, or even Banach spaces. We will also abuse language by saying space, subspace, etc. instead of group, subgroup, etc.

A quasi-normed group is a metrizable topological group and a quasi-normed vector space is a metrizable topological vector space; the space S_0 defined above is not a topological vector space.

We will sometimes assume that the spaces are complete (it may be simpler to assume so always). A *quasi-Banach space* is a complete quasi-normed vector space. Note that the closed graph theorem holds for such spaces.

Two non-negative functions f and g on some set are *equivalent* if

$$cf \le g \le Cf$$

for some constants $c, C > 0$. In particular, this defines equivalence between two quasi-norms on the same space. We will usually not distinguish between equivalent quasi-norms and let $X = Y$, where X and Y are quasi-normed spaces, signify that X equals Y as sets (and algebraically) and that their quasi-norms are equivalent.

If X is a quasi-normed space and $p > 0$, we let $(X)^p$ denote X with the new quasi-norm $\| \ \|_X^p$. Note that $(X)^p$ and X have the same topology, but (in general) non-equivalent quasi-norms. X is *p-normed* if $(X)^p$ is normed. Every quasi-norm is equivalent to a p-norm for some $p > 0$.

We define $\|x\|_X = \infty$ for $x \notin X$.

A *quasi-normed couple* is a couple $\bar{X} = (X_0, X_1)$ of two quasi-normed spaces that are continuously included as subspaces of some Hausdorff topological group. We may then define $\Sigma(\bar{X}) = X_0 + X_1$ and $\Delta(\bar{X}) = X_0 \cap X_1$ as quasi-normed spaces. We will write $\| \ \|_j$ for $\| \ \|_{X_j}$ when no confusion may occur.

The main functionals of real interpolation and approximation are defined by

$$\begin{aligned}
K(t,x;\bar{X}) &= \inf\big\{\|x_0\|_0 + t\|x_1\|_1 : x = x_0 + x_1\big\}, \quad x \in \Sigma(\bar{X}),\ t > 0,\\
J(t,x;\bar{X}) &= \max\big\{\|x\|_0, t\|x\|_1\big\}, \quad x \in \Delta(\bar{X}),\ t > 0,\\
E(t,x;\bar{X}) &= \inf\big\{\|x_1\|_1 : x = x_0 + x_1, \|x_0\|_0 \le t\big\}, \quad x \in \Sigma(\bar{X}),\ t > 0.
\end{aligned}$$

(The E-functional may be infinite.)

The K-method is defined as follows. Let Φ be a quasi-Banach lattice on $(0,\infty)$ that contains the function $1 \wedge t$. (Such lattices are called *parameters* of the K-method.) Then

$$K_\Phi(\bar{X}) = \big\{x \in \Sigma(\bar{X}) : \|x\|_{K_\Phi(\bar{X})} = \|K(\cdot,x;\bar{X})\|_\Phi < \infty\big\}.$$

$K_\Phi(\bar{X})$ is a quasi-normed space. It is complete if X_0, X_1 are, and a Banach space if X_0, X_1, Φ are. In particular, the choice $\Phi = L^q(t^{-\theta}, dt/t) = \{f : t^{-\theta} f \in L^q(dt/t)\}$ yields the usual interpolation spaces $(\bar{X})_{\theta q}$, $0 < \theta < 1$, $0 < q \le \infty$.

For the general definition of the J-method we refer to [4] (which also discusses the E-method). This method is fully developed for Banach spaces only; the interpolation space $J_\Phi(\bar{X})$ is defined for a Banach couple $\bar{X}$ and a Banach lattice $\Phi \subset L^1(1 \wedge t^{-1}, dt/t)$ with $\Phi \ne \{0\}$. For certain parameters Φ, the method can be defined for all quasi-normed couples. In particular, this is the case for $\Phi = L^q(t^{-\theta}, dt/t)$, (we may here take $0 < \theta < 1$ and $0 < q \le \infty$), and then $J_\Phi(\bar{X}) = K_\Phi(\bar{X}) = (\bar{X})_{\theta q}$, see [2].

If X is an intermediate space, i.e. $\Delta(\bar{X}) \subset X \subset \Sigma(\bar{X})$, we define X^0 to be the closure of $\Delta(\bar{X})$ in X. In particular,

$$x \in X_0^0 \Leftrightarrow x \in X_0 \text{ and } \lim_{t\to 0} K(t,x;\bar{X}) = 0, \tag{1.1}$$

and similarly for X_1^0. The couple $\bar{X}$ is *regular* if $X_0^0 = X_0$ and $X_1^0 = X_1$, i.e. if $\Delta(\bar{X})$ is dense in X_j, $j = 0, 1$.

If X is an intermediate space we also define its *Gagliardo completion* X^c to be the set of all limits in $X_0 + X_1$ of sequences that are bounded in X. A space X is *Gagliardo complete* if $X^c = X$; the couple $\bar{X}$ is Gagliardo complete if X_0 and X_1 are. (We do not require the quasi-norms to be equal, but that may be assumed by renorming.) Note that X_0^c and X_1^c are special cases of the K-method, obtained by choosing $\Phi = L^\infty$ and $L^\infty(t^{-1})$, respectively.

If $\bar{X}$ is a Banach couple we define the dual couple $\bar{X}' = (X_0', X_1')$ by

$$X_j' = \bigl\{x^* \in (X_0 \cap X_1)^* : \|x^*\|_{X_j'} = \sup\bigl\{|\langle x^*, x\rangle| : x \in X_0 \cap X_1 \text{ with } \|x\|_j \le 1\bigr\} < \infty\bigr\}.$$

Then $X_0' + X_1' = (X_0 \cap X_1)^*$ (isometrically). If $\bar{X}$ is regular, we furthermore have natural isometries $X_j' \cong X_j^*$ and $X_0' \cap X_1' \cong (X_0 + X_1)^*$.

It is easily seen that a dual couple is Gagliardo complete.

2. Subcouples

We say that $\bar{Y} = (Y_0, Y_1)$ is a subcouple of a quasi-normed couple $\bar{X} = (X_0, X_1)$ if Y_j is a subspace of X_j with the induced quasi-norm, $j = 0, 1$. Obviously, $\bar{Y}$ then is a quasi-normed couple and

$$K(t, y, \bar{Y}) \ge K(t, y, \bar{X}), \quad t > 0,\ y \in Y_0 + Y_1. \tag{2.1}$$

In general, there is no converse inequality; as remarked in the introduction, we will study the case when there is.

Definition. *A K-subcouple of a quasi-normed couple $\bar{X}$ is a subcouple $\bar{Y}$ such that for some $C < \infty$,*

$$K(t, y; \bar{Y}) \le CK(t, y; \bar{X}), \quad t > 0,\ y \in Y_0 + Y_1. \tag{2.2}$$

Because of (2.1), the condition can equivalently be given as

$$K(t, y; \bar{Y}) \asymp K(t, y; \bar{X}), \quad t > 0,\ y \in Y_0 + Y_1. \tag{2.3}$$

Remark 2.1. This notion was introduced by Peetre [17], although he imposed the stronger requirement

$$K(t, y, \bar{Y}) = K(t, y, \bar{X}), \tag{2.4}$$

i.e. $C = 1$ in (2.2). (Our definition is the same as Pisier's [21], although he uses the term *K-closed* subcouple.) There are several reasons for allowing a constant in the definition; for example, it means that the property is preserved if the norms are replaced by any equivalent ones, and it is (to our knowledge) needed in many of the results and examples below.

We call subcouples satifying (2.4) *exact* K-subcouples.

Problem 2.1. *If $\bar{Y}$ is a K-subcouple of $\bar{X}$, is it possible to renorm $\bar{X}$ (with equivalent quasi-norms) such that $\bar{Y}$ becomes an exact K-subcouple?*

Remark 2.2. If $\bar{X}$ and $\bar{Y}$ are quasi-Banach spaces, condition (2.1) for a fixed t is equivalent to $Y_0 + Y_1$ being a closed subspace of $X_0 + X_1$. The main point of the definition is, thus, that the constant C does not depend on t.

Remark 2.3. We may similarly define (exact) K_p-subcouples ($0 < p \leq \infty$) by replacing K by K_p. Obviously, $\bar{Y}$ is a K_p-subcouples of $\bar{X}$ if and only if it is a K-subcouple. For Banach couples and $p \geq 1$, it follows by Holmstedt and Peetre [11] that the best constant in (2.2) is independent of p; in particular $\bar{Y}$ is an exact K_p-subcouple if and only if it is an exact K-subcouple. This may fail for quasi-normed couples, see Example 7.9.

There is a similar definition using the E-functional.

Definition. *An E-subcouple of a quasi-normed couple $\bar{X}$ is a subcouple $\bar{Y}$ such that for some $C_1, C_2 < \infty$,*

$$E(C_1 t, y; \bar{Y}) \leq C_2 E(t, y; \bar{X}), \quad t > 0,\ y \in Y_0 + Y_1. \tag{2.5}$$

In fact, the definitions are equivalent.

Proposition 2.1. *$\bar{Y}$ is a K-subcouple of $\bar{X}$ if and only if it is an E-subcouple.*

Proof. Suppose that $\bar{Y}$ is a K-subcouple of $\bar{X}$ and let $y \in Y_0 + Y_1$ and $t > 0$. Let $s > E(t, y; \bar{X})$. By definition there exists $x_0 \in X_0$ with $\|x_0\|_{X_0} \leq t$ and $\|y - x_0\|_{X_1} < s$. Hence

$$K(t/s, y; \bar{X}) \leq \|x_0\|_{X_0} + \frac{t}{s}\|y - x_0\|_{X_1} < 2t$$

and thus $K(t/s, y; \bar{Y}) < 2Ct$. Consequently $y = y_0 + y_1$ with $\|y_0\|_{Y_0} + \frac{t}{s}\|y_1\|_{Y_1} < 2Ct$, which implies $E(2Ct, y; \bar{Y}) \leq \|y_1\|_{Y_1} < 2Cs$. Thus $E(2Ct, y; \bar{Y}) \leq 2CE(t, y; \bar{X})$.

Conversely, suppose that $\bar{Y}$ is an E-subcouple, $y \in Y_0 + Y_1$ and $t > 0$. If $s > K(t, y; \bar{X})$, then $y = x_0 + x_1$ with $\|x_0\|_{X_0} + t\|x_1\|_{X_1} < s$ and thus $E(s, y; \bar{X}) \leq \|x_1\|_{X_1} < s/t$. Hence $E(C_1 s, y; \bar{Y}) \leq C_2 s/t$, and $y = y_0 + y_1$ with $\|y_0\|_{Y_0} \leq C_1 s$ and $\|y_1\|_{Y_1} < C_2 s/t$, which yields $K(t, y; \bar{Y}) \leq C_1 s + C_2 s$. Consequently $K(t, y; \bar{Y}) \leq (C_1 + C_2) K(t, y; \bar{X})$. $\square$

Remark 2.4. If we use K_∞ in (2.2), the best constant there equals the infinum of $\max(C_1, C_2)$ for (2.5). In particular, $\bar{Y}$ is an exact K_∞-subcouple if and only if C_1 may be chosen arbitrarily close to 1, and C_2 equal to 1. We do not know whether this is equivalent to having $C_1 = C_2 = 1$ in (2.5), i.e. $E(t, y; \bar{Y}) = E(t, y; \bar{X})$ (such couples are called *exact* E-subcouples).

Examples of K-subcouples (and thus E-subcouples) are given in Section 7, as well as some counter examples.

We begin with some easy consequences of the definition.

Proposition 2.2.

(i) *(Y_0, Y_1) is a K-subcouple of $(X_0, X_1) \Leftrightarrow (Y_1, Y_0)$ is a K-subcouple of (X_1, X_0)*

(ii) *If $\bar{Y}$ is a K-subcouple of $\bar{X}$ and $\bar{Z}$ is a K-subcouple of $\bar{Y}$, then $\bar{Z}$ is a K-subcouple of $\bar{X}$.*

(iii) *If $\bar{Z}$ is a subcouple of $\bar{Y}$, $\bar{Y}$ is a subcouple of $\bar{X}$ and $\bar{Z}$ is a K-subcouple of $\bar{X}$, then $\bar{Z}$ is a K-subcouple of $\bar{Y}$.*

(iv) *If (X_0, X_1) is a quasi-normed couple, then (X_0^0, X_1) (X_0, X_1^0) and (X_0^0, X_1^0) are K-subcouples of it.*

(v) *If $\bar{Y}$ is a K-subcouple of $\bar{X}$, then (Y_0^0, Y_1) is a K-subcouple of $\bar{X}$ and of (X_0^0, X_1), and similarly for (Y_0, Y_1^0) and (Y_0^0, Y_1^0).*

(vi) *If $\bar{Y}$ is a K-subcouple of $\bar{X}$, then $(\overline{Y}_0, \overline{Y}_1)$ is too. Here $\overline{Y}_j$ is the closure of Y_j in X_j, $j = 0, 1$.)*

(vii) *If $\bar{Y}$ is a K-subcouple of $\bar{X}$, then Y_j^c is a subspace of X_j^c with an equivalent norm, $j = 0, 1$, and (Y_0^c, Y_1^c) is a K-subcouple of (X_0^c, X_1^c).*

(viii) *If (Y_0, Y_1) is a K-subcouple of (X_0, X_1) and $p_0, p_1 > 0$, then $\big((Y_0)^{p_0}, (Y_1)^{p_1}\big)$ is a K-subcouple of $\big((X_0)^{p_0}, (X_1)^{p_1}\big)$.*

Proof. (i), (ii) and (iii) follow directly from the definition.

(iv) follows because, for example, if $x \in X_0^0 + X_1$, then any splitting $x = x_0 + x_1$ with $x_j \in X_j$ has $x_0 \in X_0^0$.

(v) follows from (ii), (iii) and (iv).

(vi) is easy.

For (vii), the definition yields $\|y\|_{Y_0^c} = \sup_t K(t, y; \bar{Y}) \asymp \sup_t K(t, y; \bar{X}) = \|y\|_{X_0^c}$ for $y \in Y_0^c$, and similarly for Y_1^c. This proves the first statement. Hence, if $Y_j^{c'}$ denotes Y_j^c with the quasi-norm of X_j^c,

$$\begin{aligned} K(t, y; Y_0^{c'}, Y_1^{c'}) &\asymp K(t, y; Y_0^c, Y_1^c) = K(t, y; Y_0, Y_1) \asymp K(t, y; X_0, X_1) \\ &= K(t, y; X_0^c, X_1^c) \end{aligned}$$

for $y \in Y_0^c + Y_1^c = Y_0 + Y_1$ and $t > 0$ (assuming, as we may, that the quasi-norms are p-norms).

(viii) follows from Proposition 2.1 and the definition of E-subcouple. □

One important, immediate consequence of the definition is an interpolation formula.

Theorem 2.1. *If $\bar{Y}$ is a K-subcouple of a quasi-normed couple $\bar{X}$, then*

$$K_\Phi(\bar{Y}) = K_\Phi(\bar{X}) \cap (Y_0 + Y_1)$$

for every Φ. The quasi-norm in $K_\Phi(\bar{Y})$ is equivalent to the quasi-norm inherited from $K_\Phi(\bar{X})$. In particular,

$$(\bar{Y})_{\theta q} = (\bar{X})_{\theta q} \cap (Y_0 + Y_1), \quad 0 < \theta < 1,\ 0 < q \le \infty. \quad \square$$

We can also use reiteration. Recall that a couple $\bar{Y}$ is said to be complete when both Y_0 and Y_1 are.

Theorem 2.2. *Suppose that $\bar{Y}$ is a complete K-subcouple of a quasi-normed couple $\bar{X}$.*

(i) *$(K_{\Phi_0}(\bar{Y}), K_{\Phi_1}(\bar{Y}))$ is a K-subcouple of $(K_{\Phi_0}(\bar{X}), K_{\Phi_1}(\bar{X}))$ for any two parameters Φ_0 and Φ_1;*

(ii) *$(Y_0, K_\Phi(\bar{Y}))$ is a K-subcouple of $(X_0, K_\Phi(\bar{X}))$ for any parameter Φ that contains an unbounded positive concave function.*

(iii) *In particular, if $0 < \theta_0 < \theta_1 < 1$ and $0 < q_0, q_1 \le \infty$, then $((\bar{Y})_{\theta_0 q_0}, (\bar{Y})_{\theta_1 q_1})$ is a K-subcouple of $((\bar{X})_{\theta_0 q_0}, (\bar{X})_{\theta_1 q_1})$, and similarly for $\big(Y_0, (\bar{Y})_{\theta q}\big), \big((\bar{Y})_{\theta q}, Y_1\big)$, $0 < \theta < 1$, $0 < q \le \infty$.*

Proof. We observe first that, by Theorem 2.1, the couples constructed from $\bar{Y}$ are subcouples of the couples constructed from $\bar{X}$ (up to equivalent norms).

(i). The generalized Holmstedt formula

$$K\big(t, y; K_{\Phi_0}(\bar{Y}), K_{\Phi_1}(\bar{Y})\big) \asymp K\big(t, K(\cdot, y; \bar{Y}); \hat{\Phi}_0, \hat{\Phi}_1\big), \tag{2.6}$$

where $\hat{\Phi}_j = K_{\Phi_j}(L_\infty, L_\infty(1/s))$, holds for every complete quasi-normed couple $\bar{Y}$ by K-divisibility, cf. [4, proof of 3.3.24 and (3.9.9)]. Part (i) now follows by applying this formula to both $\bar{X}$ and $\bar{Y}$ (provided also $\bar{X}$ is complete; in general we obtain at least an inequality which suffices).

(ii). Choosing $\Phi_0 = L^\infty$, it follows in particular that $(Y_0^c, K_\Phi(\bar{Y}))$ is a K-subcouple of $(X_0^c, K_\Phi(\bar{X}))$, which gives the estimate, for $y \in Y_0^c + K_\Phi(Y)$ and $t > 0$,

$$K\big(t, y; Y_0^c, K_\Phi(\bar{Y})\big) \le C_1 K\big(t, y; X_0^c, K_\Phi(\bar{X})\big) \le C_1 K\big(t, y; X_0, K_\Phi(\bar{X})\big). \tag{2.7}$$

We claim that

$$K\big(t, y; Y_0, K_\Phi(\bar{Y})\big) \le C_2 \|y\|_{Y_0^c}, \quad t > 0,\ y \in Y_0^c \tag{2.8}$$

for some $C_2 < \infty$, which implies that if $y = y_0 + y_1$ then

$$\begin{aligned} K\big(t, y; Y_0, K_\Phi(\bar{Y})\big) &\le C_3 K\big(t, y_0; Y_0, K_\Phi(\bar{Y})\big) + C_3 K\big(t, y_1; Y_0, K_\Phi(\bar{Y})\big) \\ &\le C_4(\|y_0\|_{Y_0^c} + t\|y_1\|_{K_\Phi(\bar{Y})}) \end{aligned}$$

and thus

$$K\big(t, y; Y_0, K_\Phi(\bar{Y})\big) \le C_4 K\big(t, y; Y_0^c, K_\Phi(\bar{Y})\big), \quad t > 0,\ y \in Y_0^c + K_\Phi(Y),$$

which together with (2.7) gives the sought estimate.

In order to prove (2.8), we assume for simplicity that $\bar{Y}$ is normed and that $\|y\|_{Y_0^c} = 1$; the general case is similar. Then there exists a sequence $y_n \in Y_0$ with $\|y_n\|_{Y_0} \le 2$ and $\|y - y_n\|_{Y_1} \le 1/n$. Let ω be a fixed unbounded positive concave function in Φ. Then, if $n, m \ge N$,

$$K(t, y_n - y_m; \bar{Y}) \le \|y_n - y_m\|_0 \wedge t\|y_n - y_m\|_1 \le 4(1 \wedge \frac{t}{N}) \le \frac{4}{\omega(N)}\,\omega(t)$$

and thus

$$\|y_n - y_m\|_{K_\Phi(\bar{Y})} \le \frac{4}{\omega(N)}\|\omega\|_\Phi \to 0 \quad \text{as} \quad N \to \infty.$$

Hence (y_n) is a Cauchy sequence in $K_\Phi(\bar{Y})$. Since $K_\Phi(\bar{Y})$ is complete and y_n converges to y in $\Sigma(\bar{Y})$, $y_n \to y$ in $K_\Phi(\bar{Y})$ too, which yields

$$K\big(t, y; Y_0, K_\Phi(\bar{Y})\big) \le \sup_n \|y_n\|_0 \le 2,$$

thus proving (2.8).

Part (iii) follows from (i) and (ii). $\square$

Remark 2.5. The condition in (ii) that Φ contains an unbounded positive concave function fails if and only if $K_\Phi(\bar{X}) \subset X_0^c$ for every couple $\bar{X}$. The result is not true without this condition, see Example 7.8.

Remark 2.6. Theorem 2.2(iii) follows from the traditional Holmstedt formula, [10] or e.g. [2, Section 3.6]. One can easily check that the formula holds for all quasi-normed couples (and $q_0, q_1 > 0$); hence the last part of the theorem holds also without assuming completeness of $\bar{Y}$. (We do not know whether completeness really is required for the first, more general, parts.)

Problem 2.2. *Is there a converse to Theorem 2.2, similar to Wolff's theorem? More precisely, if $\bar{Y}$ is a subcouple of $\bar{X}$ such that $(Y_0, (\bar{Y})_{\theta_1 q_1})$ and $((\bar{Y})_{\theta_0 q_0}, Y_1)$ are K-subcouples of $(X_0, (\bar{X})_{\theta_1 q_1})$ and $((\bar{X})_{\theta_0 q_0}, X_1)$, where $0 < \theta_0 < \theta_1 < 1$ and $0 < q_0, q_1 \le \infty$, is $\bar{Y}$ a K-subcouple of $\bar{X}$?*

We end this section with some more technical results for K-subcouples.

Proposition 2.3. *If $\bar{Y}$ is a K-subcouple of $\bar{X}$, and $X_0 \cap X_1$ is dense in X_0, then $Y_0 \cap Y_1$ is dense in Y_0.*

Proof. By the assumptions and (1.1),

$$\begin{aligned} y \in Y_0 \Rightarrow y \in Y_0 \text{ and } y \in X_0 = X_0^0 &\Rightarrow y \in Y_0 \text{ and } \lim_{t\to 0} K(t, y; \bar{X}) = 0 \\ &\Rightarrow y \in Y_0 \text{ and } \lim_{t\to 0} K(t, y; \bar{Y}) = 0 \Rightarrow y \in Y_0^0. \quad \square \end{aligned}$$

Proposition 2.4. *Suppose that $\bar{Y}$ is a K-subcouple of $\bar{X}$, with X_j complete and Y_j a closed subspace of X_j, $j = 0, 1$. Then $Y_0 + Y_1$ is a closed subspace of $X_0 + X_1$ (with an equivalent quasi-norm).*

Proof. The standard quasi-norms on $X_0 + X_1$ and $Y_0 + Y_1$ are $K(1, \cdot; \bar{X})$ and $K(1, \cdot; \bar{Y})$, respectively, which are assumed to be equivalent on $Y_0 + Y_1$. Since $Y_0 + Y_1$ is complete in its own quasi-norm, and thus also in the one inherited from $X_0 + X_1$, it is closed in $X_0 + X_1$. $\square$

Proposition 2.5. *If $\bar{Y}$ is a Gagliardo complete K-subcouple of $\bar{X}$, then $Y_j = X_j \cap (Y_0 + Y_1)$, $j = 0, 1$.*

Proof. If $y \in X_0 \cap (Y_0 + Y_1)$, then

$$K(t, y; \bar{Y}) \le C K(t, y; \bar{X}) \le C\|y\|_{X_0}$$

and thus $y \in Y_0^c = Y_0$. Hence $Y_0 = X_0 \cap (Y_0 + Y_1)$, and similarly $Y_1 = X_1 \cap (Y_0 + Y_1)$. $\square$

Remark 2.7. Conversely, if $Y_j = X_j \cap (Y_0 + Y_1)$, $j = 0, 1$, and $\bar{X}$ is Gagliardo complete, then $\bar{Y}$ is Gagliardo complete.

Remark 2.8. We will see in Example 7.8 that Proposition 2.5 is not true without the hypothesis that $\bar{Y}$ is Gagliardo complete, even for Banach couples.

3. Definition of quotient couples

Given a closed subcouple (Y_0, Y_1) of a quasi-normed couple (X_0, X_1) (i.e. Y_j is a closed subspace of X_j), we can form the two quotient spaces X_0/Y_0 and X_1/Y_1. These are quasi-normed spaces, but in order to regard them as a quasi-normed couple, we also have to regard them as subspaces of a common containing space. (A simple example of the problem that may arise is given by $X_0 = X_1 = R^2$, $Y_0 = \{(x, 0)\}$, $Y_1 = \{(0, x)\}$, where natural definitions would give $X_0/Y_0 + X_1/Y_1 = \{0\}$.) We, of course, require the quotient mappings $X_j \to X_j/Y_j$ to define a mapping $(X_0, X_1) \to (X_0/Y_0, X_1/Y_1)$ of the couples, which means that they are restrictions of a continuous map $\pi : X_0 + X_1 \to (X_0/Y_0) + (X_1/Y_1)$. This map is onto and thus it defines a continuous bijection $(X_0 + X_1)/Z \to (X_0/Y_0) + (X_1/Y_1)$, where $Z = \ker(\pi)$ is a closed subspace of $X_0 + X_1$. Furthermore, the diagram

$$\begin{array}{ccc} X_0 \oplus X_1 & \longrightarrow & (X_0/Y_0) \oplus (X_1/Y_1) \\ \downarrow & & \downarrow \\ X_0 + X_1 & \xrightarrow{\pi} & (X_0/Y_0) + (X_1/Y_1) \end{array}$$

commutes, all mappings in it are surjective, and all except π, by definition, induce the quasi-norms on their respective ranges as the corresponding quotient norm. Hence, if $u \in (X_0/Y_0) + (X_1/Y_1)$,

$$\begin{aligned}\|u\| &= \inf\left\{\|u_0\|_{X_0/Y_0} + \|u_1\|_{X_1/Y_1} : u = u_0 + u_1\right\} \\ &= \inf\left\{\|x_0\|_{X_0} + \|x_1\|_{X_1} : u = \pi(x_0 + x_1)\right\} = \inf\left\{\|x\|_{X_0+X_1} : u = \pi(x)\right\};\end{aligned}$$

that is, the mapping $(X_0 + X_1)/Z \to (X_0/Y_0) + (X_1/Y_1)$ is an isometry.

This shows that we may take the containing space for the quotient couple as $(X_0+X_1)/Z$ for some closed subspace Z of $X_0 + X_1$. Conversely, if $Z \subset X_0 + X_1$ is closed and $\pi : X_0 + X_1 \to (X_0 + X_1)/Z$ is the quotient mapping, π induces continuous embeddings $X_j/(Z \cap X_j) \to (X_0 + X_1)/Z$, $(j = 0, 1)$; if $Y_j = Z \cap X_j$, we thus have defined a compatible couple $(X_0/Y_0, X_1/Y_1)$.

Given X_0, X_1, Y_0, Y_1, the conditions $Z \cap X_j = Y_j$ may be satisfied by several subspaces Z of X_0+X_1. This happens already when $Y_0 = Y_1 = \{0\}$, where we may take any Z which intersects X_0 and X_1 in $\{0\}$ only, and regard X_0 and X_1 as subspaces of $(X_0 + X_1)/Z$; this introduces new identifications of elements in X_0 and X_1. This should be avoided, and returning to the general case, two elements in X_0/Y_0 and X_1/Y_1 should be identified only when we can choose a common representative in $X_0 \cap X_1$ for them; in other words π should map $X_0 \cap X_1$ *onto* $(X_0/Y_0) \cap (X_1/Y_1)$. The following (purely algebraic) lemma shows that this requires Z to equal $Y_0 + Y_1$.

Lemma 3.1. *Suppose that (Y_0, Y_1) is a subcouple of (X_0, X_1) and that Z is a subspace of $X_0 + X_1$. Then X_0/Y_0 and X_1/Y_1 may be regarded as subspaces of $(X_0 + X_1)/Z$ if and only if $Y_0 = Z \cap X_0$ and $Y_1 = Z \cap X_1$. Furthermore, in this case π (defined above) maps $X_0 \cap X_1$ onto $(X_0/Y_0) \cap (X_1/Y_1)$ if and only if $Z = Y_0 + Y_1$.*

Proof. The first part is obvious. For the second part, assume first that π maps $X_0 \cap X_1$ onto $(X_0/Y_0) \cap (X_1/Y_1)$. If $z \in Z$, then $z = x_0 - x_1$ with $x_j \in X_j$. Since $\pi(x_0 - x_1) = \pi(z) = 0$, $\pi(x_0) = \pi(x_1) \in (X_0/Y_0) \cap (X_1/Y_1)$ and thus $\pi(x_0) = \pi(x_1) = \pi(x)$ for some $x \in X_0 \cap X_1$. This means that $x_j - x \in Z \cap X_j = Y_j$ and $z = x_0 - x_1 = (x_0 - x) - (x_1 - x) \in Y_0 + Y_1$. Conversely, if $Z = Y_0 + Y_1$ and $\pi(x_0) = \pi(x_1)$ with $x_j \in X_j$, then $x_0 - x_1 \in Z$ and thus $x_0 - x_1 = y_0 + y_1$ for some $y_j \in Y_j \subset X_j$. Thus $x_0 - y_0 = x_1 + y_1 \in X_0 \cap X_1$ and $\pi(x_0) = \pi(x_0 - y_0) \in \pi(X_0 \cap X_1)$. $\square$

The subcouples that allow us to form nice quotient couples are thus those described in the following definition.

Definition. *A normal subcouple of a quasi-normed couple (X_0, X_1) is a subcouple (Y_0, Y_1) such that*

(i) $Y_0 = X_0 \cap (Y_0 + Y_1)$ *and* $Y_1 = X_1 \cap (Y_0 + Y_1)$*;*
(ii) $Y_0 + Y_1$ *is closed in* $X_0 + X_1$*.*

(It follows that Y_0 is closed in X_0 and Y_1 in X_1.)

Proposition 3.1. *Suppose that (Y_0, Y_1) is a normal subcouple of (X_0, X_1). Then the quotient spaces $Q_0 = X_0/Y_0$ and $Q_1 = X_1/Y_1$ may be regarded as an interpolation couple $\bar{Q} = (Q_0, Q_1)$ (also denoted by $\bar{X}/\bar{Y}$), with $Q_0 + Q_1 = (X_0 + X_1)/(Y_0 + Y_1)$ isometrically and $Q_0 \cap Q_1 = (X_0 \cap X_1)/(Y_0 \cap Y_1)$ algebraically.* $\square$

Note that (ii) in the definition above only says that $Y_0 + Y_1$ is a closed subset of $X_0 + X_1$; we do not require its intrinsic quasi-norm to be equivalent to the one induced by $X_0 + X_1$, although that is implied by the closed graph theorem whenever X_0 and X_1 are quasi-Banach spaces.

Similarly, if X_0 and X_1 are quasi-Banach spaces in Proposition 3.1, it follows by the closed graph theorem that $Q_0 \cap Q_1$ and $(X_0 \cap X_1)/(Y_0 \cap Y_1)$ have equivalent quasi-norms, but the quasi-norms are in general not equal. This is related to the notion of J-quotient couples discussed in the next section.

The first condition in the definition of normal subcouples has several equivalent formulations.

Proposition 3.2. *Let (Y_0, Y_1) be a subcouple of (X_0, X_1). Then the following are equivalent.*

(i) $Y_j = X_j \cap (Y_0 + Y_1)$, $j = 0, 1$,
(ii) $Y_j = X_j \cap Z$, $j = 0, 1$, *for some* $Z \subset X_0 + X_1$
(iii) $Y_0 \cap X_1 = Y_0 \cap Y_1 = X_0 \cap Y_1$
(iv) $Y_0 \cap X_1 \subset Y_1$ *and* $Y_1 \cap X_0 \subset Y_0$.

Proof. (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(iv) are obvious.

Assume (iv) and suppose that $x \in X_0 \cap (Y_0 + Y_1)$. Then $x = y_0 + y_1$ with $y_j \in Y_j$. Since also $y_1 = x - y_0 \in X_0$, $y_1 \in Y_1 \cap X_0 \subset Y_0$ and thus $x = y_0 + y_1 \in Y_0$. Hence $X_0 \cap (Y_0 + Y_1) \subset Y_0$ and since the converse inclusion is obvious, and $X_1 \cap (Y_0 + Y_1) = Y_1$ follows by symmetry, (i) holds. □

Corollary 3.1. *Let (X_0, X_1) be a quasi-normed couple with $X_0 \subset X_1$ (continuous inclusion). Then a subcouple (Y_0, Y_1) is a normal subcouple if and only if*

(i) $Y_0 = X_0 \cap Y_1$
(ii) Y_1 *is a closed subspace of* X_1. □

4. Interpolation of quotient couples

An important part of Pisier's work [21], is the equivalence between the K-subcouple condition for a subcouple and a 'dual' condition for quotient couples. We will in this section describe this equivalence; several of the results below are contained in [21], at least implicity, and are included here for completeness.

Lemma 4.1 (Pisier). *Let $\bar{Y}$ be a subcouple of a $\bar{X}$. Then the following are equivalent.*

(i) *$\bar{Y}$ is a K-subcouple of $\bar{X}$.*
(ii) *There exists $C < \infty$ such that for every $x \in X_0 \cap X_1$, $y_0 \in Y_0$, $y_1 \in Y_1$ and $t > 0$, there exists $y \in Y_0 \cap Y_1$ such that*

$$\|x - y\|_{X_0} \vee t\|x - y\|_{X_1} \le C\big(\|x - y_0\|_{X_0} \vee t\|x - y_1\|_{X_1}\big). \tag{4.1}$$

(iii) *Let $\overline{\overline{X}} = X_0 \times X_1$, $\overline{\overline{Y}} = Y_0 \times Y_1 \subset \overline{\overline{X}}$ and $\overline{\overline{\Delta}} = \{(x, x) : x \in X_0 \cap X_1\} \subset \overline{\overline{X}}$. Then there exist $C < \infty$ such that if $t > 0$ and $\overline{\overline{X}}$ is given the norm $\|(x_0, x_1)\|_t = \|x_0\|_{X_0} \vee t\|x_1\|_{X_1}$, every $\overline{\overline{x}} \in \overline{\overline{Y}} + \overline{\overline{\Delta}}$ has a splitting $\overline{\overline{x}} = \overline{\overline{y}} + \overline{\overline{z}}$ with $\overline{\overline{y}} \in \overline{\overline{Y}}$, $\overline{\overline{z}} \in \overline{\overline{\Delta}}$ and*

$$\|\overline{\overline{y}}\|_t + \|\overline{\overline{z}}\|_t \le C\|\overline{\overline{x}}\|_t. \tag{4.2}$$

Proof. (i)$\Leftrightarrow$(iii). By the quasi-triangular inequality, (4.2) may be replaced by $\|\overline{\overline{y}}\|_t \le C\|\overline{\overline{x}}\|_t$ (for a different C). Since $(x_0, x_1) = (y_0, y_1) + \overline{\overline{z}}$ with $\overline{\overline{z}} \in \overline{\overline{\Delta}}$ if and only if $x_0 - x_1 = y_0 - y_1$, (iii) can be restated as:

If $x_0 - x_1 \in Y_0 + Y_1$, then $x_0 - x_1 = y_0 - y_1$ for some $y_j \in Y_j$ with

$$\|y_0\|_0 \vee t\|y_1\|_1 \le C\big(\|x_0\|_0 \vee t\|x_1\|_1\big). \tag{4.3}$$

This is equivalent to (2.2) (or, more directly, the definition of K_∞-subcouple, cf. Remark 2.3).

(ii)$\Leftrightarrow$(iii). Similarly, (4.2) can be replaced by $\|\overline{\overline{z}}\|_t \le C\|\overline{\overline{x}}\|_t$. Since $(x_0, x_1) = \overline{\overline{y}} + (x, x)$ with $\overline{\overline{y}} \in \overline{\overline{Y}}$ if and only if $x_0 - x \in Y_0$ and $x_1 - x \in Y_1$, (iii) can be restated as:

If $x \in X_0 \cap X_1$, $y_0 \in Y_0$ and $y_1 \in Y_1$, there exists z with

$$(x - y_0, x - y_1) - (z, z) \in \overline{\overline{Y}} \tag{4.4}$$

and $\|(z, z)\|_t \le C\|(x - y_0, x - y_1)\|_t$.

Since (4.4) is equivalent to $x - z \in Y_0$ and $x - z \in Y_1$, this gives (ii) by letting $y = x - z$ (and conversely). $\square$

Theorem 4.1 (Pisier). *Suppose that $\bar{Y}$ is a normal subcouple of $\bar{X}$ and let $\bar{Q} = (X_0/Y_0, X_1/Y_1)$. The following are equivalent.*

(i) *$\bar{Y}$ is a K-subcouple of $\bar{X}$.*

(ii) *There exists $C < \infty$ such that for every $x \in X_0 \cap X_1$ there exists $y \in Y_0 \cap Y_1$ with*

$$\|x - y\|_{X_j} \le C \inf\left\{\|x - u\|_{X_j} : u \in Y_j\right\}, \quad j = 0, 1. \tag{4.5}$$

(iii) *There exists $C < \infty$ such that for every $x \in X_0 + X_1$, there exists $y \in Y_0 + Y_1$ with*

$$\|x - y\|_{X_j} \le C \inf\left\{\|x - u\|_{X_j} : u \in Y_0 + Y_1\right\}, \quad j = 0, 1. \tag{4.6}$$

(iv) *There exists $C < \infty$ such that for every $z \in Q_0 \cap Q_1$ and $t > 0$, there exists $x \in X_0 \cap X_1$ with $\pi(x) = z$ and*

$$J(t, x; \bar{X}) \le C J(t, z; \bar{Q}). \tag{4.7}$$

(v) *There exists $C < \infty$ such that if $z \in Q_0 \cap Q_1$ then there exists $x \in X_0 \cap X_1$ with $\pi(x) = z$ and*

$$\|x\|_{X_j} \le C\|z\|_{Q_j}, \quad j = 0, 1. \tag{4.8}$$

(vi) *There exists $C < \infty$ such that if $z \in Q_0 + Q_1$ then there exists $x \in X_0 + X_1$ with $\pi(x) = z$ and*

$$\|x\|_{X_j} \le C\|z\|_{Q_j}, \quad j = 0, 1. \tag{4.9}$$

(The norms in (iii) and (vi) may be infinite.)

Proof. (i)$\Rightarrow$(ii). If $x \in Y_0$ or $x \in Y_1$, then $x \in Y_0 \cap Y_1$ because $\bar{Y}$ is a normal subcouple, and we may take $y = x$. Otherwise, the infima in the right hand side of (4.5) are positive, because Y_j is closed in X_j, and we may choose $y_j \in Y_j$ with $\|x - y_j\|_{X_j} \le 2\inf\{\|x - u\|_{X_j} : u \in Y_j\}$. We apply Lemma 4.1 (ii) with $t = \|x - y_0\|_{Y_0} / \|x - y_1\|_{Y_1}$ and obtain $y \in Y_0 \cap Y_1$ satisfying (4.1), which easily yields (4.5).

(ii)$\Rightarrow$(i). By Lemma 4.1, since (4.5)$\Rightarrow$(4.1).

(ii)$\Leftrightarrow$(v). Writing $x' = x - y$, (ii) may be restated as:

For every $x \in X_0 \cap X_1$, there exists $x' \in X_0 \cap X_1$ with $\pi(x') = \pi(x)$ and $\|x'\|_{X_j} \le C\|\pi(x)\|_{Q_j}$, $j = 0, 1$. Replacing $\pi(x)$ by $z \in Q_0 \cap Q_1$ (recalling that $\pi(X_0 \cap X_1) = Q_0 \cap Q_1$ by Proposition 3.1), this is (v) (with x' instead of x).

(iii)$\Leftrightarrow$(vi). Similar.

(v)$\Rightarrow$(vi). Clear if $z \in Q_0 \cap Q_1$, and otherwise trivial.

(vi)$\Rightarrow$(v). Trivial.

(v)$\Rightarrow$(iv). Trivial.

(iv)⇒(v). If $z \neq 0$, let $t = \|z\|_{Q_0}/\|z\|_{Q_1}$; (4.7) then implies (4.8). $\square$

We say that $\bar{Q} = \bar{X}/\bar{Y}$ is a *J-quotient couple* of $\bar{X}$ if (iv) holds. (Pisier [21] uses the term *J-closed*.) Hence, if $\bar{Y}$ is a normal subcouple of $\bar{X}$, $\bar{Y}$ is a K-subcouple if and only if $\bar{X}/\bar{Y}$ is a J-quotient couple of $\bar{X}$.

Remark 4.1. Although x in Theorem 4.1 (iv) a priori may depend on t, it can, in fact, be chosen independently of t by (v).

Remark 4.2. The condition (iv) for a fixed t says that $(X_0/Y_0) \cap (X_1/Y_1) = (X_0 \cap X_1)/(Y_0 \cap Y_1)$ with equivalent quasi-norms. Again, the main point of the condition is that C does not depend on t, cf. Remark 2.2.

We obtain easily an interpolation theorem for quotient couples using the J-method.

Theorem 4.2. *Suppose that $\bar{Y}$ is a normal K-subcouple of a complete quasi-normed couple $\bar{X}$. Then*

$$(\bar{X}/\bar{Y})_{\theta q} = (\bar{X})_{\theta q}/(\bar{Y})_{\theta q}, \quad 0 < \theta < 1,\ 0 < q \leq \infty. \tag{4.10}$$

More precisely, the quotient mapping $\pi : \bar{X} \to \bar{X}/\bar{Y}$ maps $(\bar{X})_{\theta q}$ onto $(\bar{X}/\bar{Y})_{\theta q}$, and induces an isomorphism $(\bar{X})_{\theta q}/(\bar{Y})_{\theta q} \cong (\bar{X}/\bar{Y})_{\theta q}$ with equivalence of quasi-norms.

More generally, if $\bar{X}$ is a Banach couple,

$$J_\Phi(\bar{X}/\bar{Y}) = J_\Phi(\bar{X})/\big(J_\Phi(\bar{X}) \cap (Y_0 + Y_1)\big) \tag{4.11}$$

for every parameter Φ.

Proof. Use the discrete definition of the J-method [2], [4] and Theorem 4.1 (iv); for (4.10) we also use Theorem 2.1. We leave the details to the reader. $\square$

We can also consider several interpolation methods simultaneously.

Theorem 4.3. *Let $\bar{Y}$ be a normal K-subcouple of a Banach couple $\bar{X}$. Then there exists $C < \infty$ such that if $\{\Phi_\alpha\}_{\alpha \in A}$ is a finite set of parameters for the J-method and $z \in \Sigma(\bar{X}/\bar{Y})$, there exists $x \in \Sigma(\bar{X})$ with $\pi(x) = z$ and*

$$\|x\|_{J_{\Phi_\alpha}(\bar{X})} \leq C\|z\|_{J_{\Phi_\alpha}(\bar{X}/\bar{Y})}, \quad \alpha \in A. \tag{4.12}$$

Proof. We only have to consider α with $0 < \|z\|_{J_{\Phi_\alpha}(\bar{X}/\bar{Y})} < \infty$, and changing the norm in Φ_α by a constant factor we may assume that $\|z\|_{J_{\Phi_\alpha}(\bar{X}/\bar{Y})} = 1$. We may also assume that Φ_α is an interpolation space for $(L^1(dt/t), L^1(dt/t^2))$ [4, Corollary 3.4.6]. We now apply Theorem 4.2 to the parameter $\Phi = \Delta(\Phi_\alpha) = \bigcap_\alpha \Phi_\alpha$, using $J_{\Delta(\Phi_\alpha)} = \Delta(J_{\Phi_\alpha})$ [4, Theorem 3.4.9], observing that the equivalence constants in these theorems do not depend on the set $\{\Phi_\alpha\}$, which gives $\|z\|_{J_\Phi(\bar{X}/\bar{Y})} \leq C_1$ and the existence of an $x \in \pi^{-1}(z)$ with $\|x\|_{J_\Phi(\bar{X})} \leq C_2$. $\square$

We next consider the K-method.

Definition. *Let $\bar{Y}$ be a normal subcouple of $\bar{X}$ Then $\bar{X}/\bar{Y}$ is a K-quotient couple of $\bar{X}$ if there exists $C < \infty$ such that for every $z \in \Sigma(\bar{X}/\bar{Y})$ there exists $x \in \Sigma(\bar{X})$ with $\pi(x) = z$ and*

$$K(t, x; \bar{X}) \leq CK(t, z; \bar{X}/\bar{Y}), \quad t > 0. \tag{4.13}$$

Remark 4.3. If $\pi(x) = z$ then $K(t, x; \bar{X}) \geq K(t, z; \bar{X}/\bar{Y})$. Hence (4.13) says that the K-functionals are equivalent (uniformly in z).

Remark 4.4. We may similarly define *E-quotient couples*, cf. (2.5), and prove as in Proposition 2.1 that E-quotient couples are the same as K-quotient couples.

Theorem 4.4. *Let $\bar{Y}$ be a normal subcouple of a complete and Gagliardo complete couple $\bar{X}$. Then the following are equivalent,*

(i) *$\bar{X}/\bar{Y}$ is a K-quotient couple of $\bar{X}$.*
(ii) *$\bar{X}/\bar{Y}$ is a Gagliardo complete J-quotient couple of $\bar{X}$.*
(iii) *$\bar{Y}$ is a K-subcouple of $\bar{X}$ and $\bar{X}/\bar{Y}$ is Gagliardo complete.*

The implications (ii) $\Longleftrightarrow$ (iii) $\Longrightarrow$ (i) hold also if $\bar{X}$ is not Gagliardo complete.

Proof. (ii)$\Leftrightarrow$(iii) holds by Theorem 4.1.

(i)$\Rightarrow$(ii). Let $Q_j = X_j/Y_j$ and assume that $z \in Q_0 \cap Q_1$. By assumption there exists $x \in \pi^{-1}(z)$ such that

$$K(t, x; \bar{X}) \le CK(t, z; \bar{Q}) \le C\big(\|z\|_{Q_0} \wedge t\|z\|_{Q_1}\big).$$

Hence $x \in X_0^c \cap X_1^c$ and $\|x\|_{X_j^c} \le C\|z\|_{Q_j}$, $j = 0, 1$. Since $X_j^c = X_j$, this shows that (4.8) holds, and thus $\bar{Q}$ is a J-quotient couple. Similarly, if $z \in Q_j^c$, $z = \pi(x)$ with $\|x\|_{X_j^c} \le C\|z\|_{Q_j^c}$, and since X_j is Gagliardo complete, Q_j is too.

(ii)$\Rightarrow$(i). By first replacing the quasi-norms on X_0, X_1 by equivalent p-norms, for some small $p > 0$, and then raising all quasi-norms to power p, we may assume that the spaces are normed groups. We use K-divisibility in the form of Cwikel's lemma [5, Theorem V.3.4] (which holds for Gagliardo complete normed couples by the same proof) and obtain that $z = \sum_{-\infty}^{\infty} z_k$ (in $Q_0 + Q_1$) for some sequence $\{z_k\}$ such that

$$\sum_k \|z_k\|_{Q_0} \wedge t\|z_k\|_{Q_1} \le C_1 K(t, z; \bar{Q}). \tag{4.14}$$

By Theorem 4.1 (vi) there exist $x_k \in X_0 + X_1$ with $\pi(x_k) = z_k$ and $\|x_k\|_{X_j} \le C_2\|z_k\|_{Q_j}$, and thus

$$\begin{aligned}\sum_k K(t, x_k; \bar{X}) &\le \sum_k \|x_k\|_{X_0} \wedge t\|x_k\|_{X_1} \\ &\le C_2 \sum_k \|z_k\|_{Q_0} \wedge t\|z_k\|_{Q_1} \le C_3 K(t, z; \bar{Q}).\end{aligned} \tag{4.15}$$

In particular, $\sum_k \|x_k\|_{X_0+X_1} = \sum_k K(1, x_k; \bar{X}) < \infty$, and thus (since $X_0 + X_1$ is complete) $x = \sum x_k$ exists in $X_0 + X_1$. Clearly, $\pi(x) = \sum \pi(x_k) = z$ and, by (4.15),

$$K(t, x; \bar{X}) \le \sum_k K(t, x_k; \bar{X}) \le C_3 K(t, z; \bar{Q}). \quad \square$$

Interpolation results for K-quotient couples follow directly from the definition. In particular, Theorem 4.4 yields, with Theorem 2.1, the following theorems.

Theorem 4.5. *Let $\bar{Y}$ be a normal K-subcouple of a complete quasi-normed couple $\bar{X}$ such that the couple $\bar{X}/\bar{Y}$ is Gagliardo complete. Then, for any parameter Φ,*

$$K_\Phi(\bar{X}/\bar{Y}) = K_\Phi(\bar{X})/\big(K_\Phi(\bar{X}) \cap (Y_0 + Y_1)\big) = K_\Phi(\bar{X})/K_\Phi(\bar{Y}). \quad \square \tag{4.16}$$

Theorem 4.6. *Let $\bar{Y}$ and $\bar{X}$ be as in Theorem 4.5. Then there exists $C < \infty$ such that if $z \in \sum(\bar{X}/\bar{Y})$, there exists $x \in X_0 + X_1$ such that $\pi(x) = z$ and*

$$\|x\|_{K_\Phi(\bar{X})} \le C\|z\|_{K_\Phi(\bar{X}/\bar{Y})} \tag{4.17}$$

for any parameter Φ of the K-method. □

The condition that $\bar{X}/\bar{Y}$ be Gagliardo complete in the last theorems is annoying for several reasons: it is often difficult to check in applications; we do not know whether it really is required for the results; we do not know whether it can fail to hold in Theorem 4.4.

Problem 4.1. *If $\bar{Y}$ is a normal K-subcouple of a complete, Gagliardo complete couple $\bar{X}$, is $\bar{X}/\bar{Y}$ necessarily Gagliardo complete?*

We give some results that may help to show Gagliardo completeness in applications.

Lemma 4.2. *Let $\bar{X}$ be a quasi-normed couple and let $0 < \theta < 1,\ 0 < q \le \infty$. Then X_0 is Gagliardo complete in (X_0, X_1) iff X_0 is Gagliardo complete in $(X_0, (\bar{X})_{\theta q})$.*

Proof. By Holmstedt's formula [10], for $x \in X_0 + (\bar{X})_{\theta q}$,

$$K\big(t, x; X_0, (\bar{X})_{\theta q}\big) \asymp t\Big(\int_{t^{1/\theta}}^{\infty} \big(s^{-\theta} K(s, x; \bar{X})\big)^q \frac{ds}{s}\Big)^{1/q}$$

(with the obvious modification if $q = \infty$), and it follows directly that $K(\cdot, x; X_0, (\bar{X})_{\theta q})$ is bounded if and only if $K(\cdot, x; X_0, X_1)$ is. Since $X_0^c = X_0 + (X_0 \cap X_1)^c \subset X_0 + (\bar{X})_{\theta q}$, this completes the proof. □

Proposition 4.1. *Suppose that $\bar{Y}$ is a normal K-subcouple of a complete quasi-normed couple $\bar{X}$.*

(i) *If $0 < \theta_0, \theta_1 < 1$ and $0 < q_0, q_1 < \infty$, then $\big((\bar{X})_{\theta_0 q_0}/(\bar{Y})_{\theta_0 q_0}, (\bar{X})_{\theta_1 q_1}/(\bar{Y})_{\theta_1 q_1}\big)$ is a Gagliardo complete K-quotient couple of $\big((\bar{X})_{\theta_0 q_0}, (\bar{X})_{\theta_1 q_1}\big)$.*

(ii) *If $(X_0/Y_0, (\bar{X})_{\theta q}/(\bar{Y})_{\theta q})$ is Gagliardo complete for some θ, q with $0 < \theta < 1$, $0 < q \le \infty$, then $(X_0/Y_0, (\bar{X})_{\theta q}/(\bar{Y})_{\theta q})$ is Gagliardo complete for all such θ, q, and it is a K-quotient couple of $(X_0, (\bar{X})_{\theta q})$.*

Proof. (i). For any couple $\bar{Z}$, $(\bar{Z})_{\theta_j q_j}$ is Gagliardo complete in (Z_0, Z_1), and thus also in $((\bar{Z})_{\theta_0 q_0}, (\bar{Z})_{\theta_1 q_1})$. Taking $\bar{Z} = \bar{X}/\bar{Y}$, the result follows by Theorems 4.2 and 4.4.

(ii). By Lemma 4.2 and Theorem 4.2, X_0/Y_0 is Gagliardo complete in $(X_0/Y_0, X_1/Y_1)$ and thus in $(X_0/Y_0, (\bar{X})_{\theta q}/(\bar{Y})_{\theta q})$ for any θ, q. The result now follows as for (i). □

We can obtain partial results even without Gagliardo completeness of $\bar{X}/\bar{Y}$.

Theorem 4.7. *Let $\bar{Y}$ be a normal K-subcouple of a complete quasi-normed couple $\bar{X}$.*

(i) *There exists $C < \infty$ such that if $z \in \sum(\bar{X}/\bar{Y})$ and $\omega(t)$ is any positive function on $(0, \infty)$ with*

$$\omega(t) \to \infty \text{ as } t \to \infty \qquad \text{and} \qquad \frac{\omega(t)}{t} \to \infty \text{ as } t \to 0, \tag{4.18}$$

then there exist $x \in X_0 + X_1$ with $\pi(x) = z$ and

$$K(t, x; \bar{X}) \le CK(t, z; \bar{X}/\bar{Y}) + \omega(t), \quad t > 0. \tag{4.19}$$

(ii) *For any parameter* Φ *that contains a function* ω *as in (4.18),*

$$K_\Phi(\bar{X}/\bar{Y}) = K_\Phi(\bar{X})/K_\Phi(\bar{Y}). \tag{4.20}$$

(iii) *If* $(\Phi_\alpha)_{\alpha\in A}$ *is a set of parameters of the* K*-method such that* $\cap_\alpha \Phi_\alpha$ *contains a function* ω *as in (4.18), there exist* $C_\alpha < \infty$ *such that if* $z \in \sum(\bar{X}/\bar{Y})$*, there exists* $x \in X_0 + X_1$ *such that* $\pi(x) = z$ *and*

$$\|x\|_{K_{\Phi_\alpha}(\bar{X})} \le C_\alpha \|z\|_{K_{\Phi_\alpha}(\bar{X}/\bar{Y})}, \quad \alpha \in A. \tag{4.21}$$

Proof. (i). The Gagliardo completeness of $\bar{X}/\bar{Y}$ is used in the proof of Theorem 4.4 only through Cwikel's lemma, where it is needed when $K(t,z;\bar{Q})/(1\wedge t) \to 0$ as $t\to\infty$ or as $t \to 0$; however, a weaker version with the right hand side of (4.14) replaced by $C_1K(t,z;\bar{Q}) + \omega(t)$ holds in any case. (Check the proof in [5], first replacing $\omega(t)$ by a smaller positive increasing concave function.) The same argument as for Theorem 4.4 now yields (4.19) (with ω replaced by $C\omega$).

(ii) and (iii) follow from (i) by replacing ω by e.g. $K(1,z;\bar{X}/\bar{Y})\omega(t)$ in (4.19). □

Remark 4.5. The parameters Φ that are excluded in Theorem 4.7 (ii) are the degenerate ones for which $K_\Phi(\bar{X}) \subset X_0^c$ or X_1^c for every couple $\bar{X}$. In particular, part (ii) applies to the usual methods $(\ )_{\theta q}$, which also was proved in Theorem 4.2 using the J-method. Part (iii) applies, for example, to any finite or countable set of parameters that satisfy the condition in (ii), and to the whole scale $(\ )_{\theta q}$, $0<\theta<1$, $0<q\le\infty$.

Remark 4.6. If, say, X_0/Y_0 is Gagliardo complete in $\bar{X}/\bar{Y}$, (4.18) may be weakened to $\omega(t)/t \to \infty$ as $t \to 0$. In particular, we may then take $\Phi = L^\infty$ in Theorem 4.7 (ii) and (iii).

5. Simultaneous approximation

In this section we consider the problem of approximating an element x in a quasi-normed space by an element y in a subspace Y such that the error $\|x-y\|$ is small. We do not look for best approximations, where the error attains its infinum, but we require the error to be at most a constant times this.

Definition. *Let* Y *be a subspace of some vector space (or abelian group)* V *and let* $\|\ \|$ *be a quasi-norm defined on another subspace of* V*. An element* $y \in Y$ *is a good approximation in* Y *with respect to* $\|\ \|$*, of an element* $x \in V$*, if*

$$\|x-y\| \le C \inf_{u\in Y} \|x-u\| \tag{5.1}$$

for some constant $C<\infty$*. (We include the case when the right hand side of (5.1) is infinite; then every* $y\in Y$ *is a good approximation of* x*.) When* $\|\ \| = \|\ \|_X$ *for a quasi-normed space* X*, we also say good approximation with respect to* X*.*

The definition is stated somewhat informally. It is meaningless as it stands for a single pair y and x (unless $x \in \bar{Y}$), but we will use it for a set of approximations, meaning that the same error constant C can be chosen for the whole set.

As long as we consider only a single quasi-normed space X and a closed subspace Y, every element in X has trivially a good approximation in Y (for any $C>1$ in (5.1)). The situation becomes more complicated if we consider two or more different quasi-norms; in general, the sets of good approximation in the different quasi-norms may be very different, and there is no guarantee that there exists a simultaneous good approximation with respect to the different quasi-norms. The following result for two quasi-norms is a reformulation of Theorem 4.1 (i)–(iii).

Theorem 5.1 (Pisier). *Suppose that $\bar{Y}$ is a normal subcouple of a quasi-normed couple $\bar{X}$. The following are equivalent.*

(i) *$\bar{Y}$ is a K-subcouple of $\bar{X}$.*

(ii) *Every $x \in X_0 \cap X_1$ has a simultaneous good approximation $y \in Y_0 \cap Y_1$, which is a good approximation in Y_j with respect to $\| \ \|_{X_j}$, $j = 0, 1$.*

(iii) *Every $x \in X_0 + X_1$ has a simultaneous good approximation $y \in Y_0 + Y_1$ with respect to both $\| \ \|_{X_0}$ and $\| \ \|_{X_1}$.* □

We next consider simultaneous approximations with respect to more than two quasi-norms, assuming that these quasi-norms come from an interpolation family. Note that if y is a good approximation of x with respect to both X_0 and X_1, we have, for example, the estimate

$$\|x - y\|_{(\bar{X})_{\theta q}} \le C_1 \|x - y\|_{X_0}^{1-\theta} \|x - y\|_{X_1}^{\theta} \le C_2 d_{X_0}(x, Y_0 + Y_1)^{1-\theta} d_{X_1}(x, Y_0 + Y_1)^{\theta} \tag{5.2}$$

for the error in $(\bar{X})_{\theta q}$. Nevertheless, y does not have to be a good approximation with respect to this quasi-norm, because much better approximations may exist, see Example 7.10.

Theorem 5.2. *Let $\bar{Y}$ be a normal K-subcouple of a complete quasi-normed couple $\bar{X}$ such that the couple $\bar{X}/\bar{Y}$ is Gagliardo complete. Then every $x \in X_0 + X_1$ has a simultaneous good approximation $y \in Y_0 + Y_1$ with respect to the quasi-norms $\| \ \|_{K_\Phi(\bar{X})}$, where Φ ranges over all parameters for the K-method. (The error constant in (5.1) may be chosen the same for all Φ.) Furthermore, $y \in K_\Phi(\bar{Y})$ for all parameters Φ such that $x \in K_\Phi(\bar{X})$.*

Remark 5.1. If $\bar{X}$ is Gagliardo complete, the original quasi-norms $\| \ \|_0$ and $\| \ \|_1$ are included among $\| \ \|_{K_\Phi(\bar{X})}$.

Proof. Let $z = \pi(x)$, where $\pi : \bar{X} \to \bar{X}/\bar{Y}$ is the quotient mapping. By Theorem 4.6 there exists $x' \in X_0 + X_1$ with $\pi(x') = z = \pi(x)$ and

$$\|x'\|_{K_\Phi(\bar{X})} \le C \|z\|_{K_\Phi(\bar{X}/\bar{Y})} \le C \|x - u\|_{K_\Phi(\bar{X})}, \tag{5.3}$$

for every $u \in Y_0 + Y_1$. Then $y = x - x' \in \pi^{-1}\{0\} = Y_0 + Y_1$, and (5.3) shows that y is a good approximation with respect to every $K_\Phi(\bar{X})$.

If $x \in K_\Phi(\bar{X})$, we may choose $u = 0$ in (5.3) and obtain a finite right hand side. Hence $x' \in K_\Phi(\bar{X})$ and $y \in K_\Phi(\bar{X}) \cap (Y_0 \cap Y_1) = K_\Phi(\bar{Y})$ by Theorem 2.1. □

As remarked in the preceding section, we do not know whether the condition that $\bar{X}/\bar{Y}$ be Gagliardo complete really is needed here. We can give some partial results without this condition, proved in the same way using Theorems 4.7 and 4.3.

Theorem 5.3. *Let $\bar{Y}$ be a normal K-subcouple of a complete quasi-normed couple $\bar{X}$.*

(i) *If $(\Phi_\alpha)_{\alpha \in A}$ is a set of parameters of the K-method such that there exists a positive function ω as in (4.18) with $\omega \in \cap_\alpha \Phi_\alpha$, then every $x \in X_0 + X_1$ has a simultaneous good approximation $y \in Y_0 + Y_1$ with respect to the quasi-norms $\| \ \|_{K_{\Phi_\alpha}(\bar{X})}$, $\alpha \in A$. (The error constant in (5.1) may depend on α.) In particular, this applies to the quasi-norms $\| \ \|_{(\bar{X})_{\theta q}}$, $0 < \theta < 1$, $0 < 1 \le \infty$.*

(ii) *If $\bar{X}$ is a Banach couple, the same conclusion holds for the J-method and any finite set of parameters.* □

Remark 5.2. If X_0/Y_0 is Gagliardo complete in $\bar{X}/\bar{Y}$, we may weaken (4.18) as in Remark 4.6. In particular, if furthermore X_0 is Gagliardo complete in $\bar{X}$, we may also require that y be a good approximation with respect to X_0. (By symmetry, the same thing is true for X_1 if X_1 and X_1/Y_1 are Gagliardo complete.)

6. Duality

In this section we exclusively consider Banach couples. The dual of a Banach couple $\bar{X}$ was defined in Section 1. If furthermore $\bar{X}$ is regular and $\bar{Y}$ is a regular closed subcouple of $\bar{X}$, we define

$$(Y_0 \cap Y_1)^{\perp} = \{x^* \in (X_0 \cap X_1)^* : \langle x^*, y\rangle = 0 \text{ for } y \in Y_0 \cap Y_1\} \tag{6.1}$$

and

$$Y_j^{\perp} = X_j' \cap (Y_0 \cap Y_1)^{\perp} = \{x^* \in X_j' : \langle x^*, y\rangle = 0 \text{ for } y \in Y_0 \cap Y_1\}, \quad j = 0, 1. \tag{6.2}$$

Then $(Y_0 \cap Y_1)^{\perp}$ is a closed subspace of $(X_0 \cap X_1)^* = X_0' + X_1'$ and $Y_j^{\perp}$ is a closed subspace of X_j'; thus $\bar{Y}^{\perp} = (Y_0^{\perp}, Y_1^{\perp})$ is a closed subcouple of $\bar{X}'$. Moreover, there are natural isometries

$$(Y_0 \cap Y_1)^{\perp} \cong (X_0 \cap X_1/Y_0 \cap Y_1)^*$$

and

$$Y_j^{\perp} \cong \{x^* \in X_j^* : \langle x^*, y\rangle = 0 \text{ for } y \in Y_j\} \cong (X_j/Y_j)^*, \quad j = 0, 1.$$

Note that, although not shown in our notation, the spaces $Y_0^{\perp}, Y_1^{\perp}$ and $(Y_0 \cap Y_1)^{\perp}$ depend on $\bar{X}$ as well as on $\bar{Y}$.

Proposition 6.1. *Suppose that $\bar{Y}$ is a normal regular subcouple of a regular Banach couple $\bar{X}$. Then the isomorphism*

$$\big(\Delta(\bar{X}/\bar{Y})\big)^* = (X_0 \cap X_1/Y_0 \cap Y_1)^* \cong (Y_0 \cap Y_1)^{\perp}$$

yields an isometry between the couples $(\bar{X}/\bar{Y})'$ and $\bar{Y}^{\perp}$.

Proof. By definition, identifying $(\Delta(\bar{X}/\bar{Y}))^*$ with $(Y_0 \cap Y_1)^{\perp} \subset (X_0 \cap X_1)^*$ by the isomorphism above, for $x^* \in (Y_0 \cap Y_1)^{\perp}$

$$\|x^*\|_{(X_j/Y_j)'} = \sup\big\{|\langle x^*, x\rangle| : x \in X_0 \cap X_1,\ \|\pi(x)\|_{X_j/Y_j} \le 1\big\}$$

and

$$\|x^*\|_{Y_j^{\perp}} = \|x^*\|_{X_j'} = \sup\big\{|\langle x^*, x\rangle| : x \in X_0 \cap X_1,\ \|x\|_{X_j} \le 1\big\}.$$

Hence $\|x^*\|_{Y_j^{\perp}} \le \|x^*\|_{(X_j/Y_j)'}$. On the other hand, let $x^* \in Y_j^{\perp}$ and $x \in X_0 \cap X_1$ with $\|\pi(x)\|_{X_j/Y_j} < 1$; then $x = x_1 + y_1$ with $\|x_1\|_{X_j} < 1$ and $y_1 \in Y_j$. Since $\bar{Y}$ is regular, there exists $y_2 \in Y_0 \cap Y_1$ with $\|y_1 - y_2\|_{Y_j} < 1 - \|x_1\|_{X_j}$, and thus $\|x - y_2\|_{X_j} \le \|x_1\|_{X_j} + \|y_1 - y_2\|_{X_j} < 1$. Since $x - y_2 \in X_0 \cap X_1$, this yields

$$|\langle x^*, x\rangle| = |\langle x^*, x - y_2\rangle| \le \|x^*\|_{Y_j^{\perp}},$$

which implies $\|x^*\|_{(X_j/Y_j)'} \le \|x^*\|_{Y_j^{\perp}}$. $\square$

Suppose that $\bar{Y}$ and $\bar{X}$ are as in Proposition 6.1. It is then easy to see that $\bar{Y}^{\perp} \cong (\bar{X}/\bar{Y})'$ is a K-subcouple of $\bar{X}'$ if and only if $\bar{X}/\bar{Y}$ is a J-quotient couple of $\bar{X}$, which, by Theorem 4.1, holds if and only if $\bar{Y}$ is a K-subcouple of $\bar{X}$.

We give a different proof that works also when $\bar{Y}$ is not a normal subcouple.

We begin with a preliminary result.

Proposition 6.2. *Suppose that $\bar{Y}$ is a regular closed subcouple of a regular Banach couple $\bar{X}$. Then the following are equivalent.*

(i) $Y_0 + Y_1$ *is a closed subspace of* $X_0 + X_1$.
(ii) $Y_0^\perp + Y_1^\perp$ *is a closed subspace of* $(X_0 \cap X_1)^* = X_0' + X_1'$.
(iii) $Y_0^\perp + Y_1^\perp$ *is a closed subspace of* $(Y_0 \cap Y_1)^\perp$.
(iv) $Y_0^\perp + Y_1^\perp = (Y_0 \cap Y_1)^\perp$.
(v) $\bar{Y}^\perp$ *is a normal subcouple of* $\bar{X}'$.

Proof. (i)$\Leftrightarrow$(iv). We apply Lemma 3.1 to the subcouple $\bar{Y}^\perp$ of $\bar{X}'$ with $Z = (Y_0 \cap Y_1)^\perp$. We identify $(X_0' + X_1')/Z = (X_0 \cap X_1)^*/(Y_0 \cap Y_1)^\perp$ with $(Y_0 \cap Y_1)^*$. Then the quotient map $\pi : X_0' + X_1' \to (X_0' + X_1')/Z$ is the natural (restriction) map $X_0' + X_1' = (X_0 \cap X_1)^* \to Y_0' + Y_1' = (Y_0 \cap Y_1)^*$, and it induces the natural identifications $X_j'/Y_j^\perp \cong Y_j' \cong Y_j^*$. By Lemma 3.1, (iv) holds if and only if π maps $X_0' \cap X_1' \cong (X_0 + X_1)^*$ onto $Y_0' \cap Y_1' \cong (Y_0 + Y_1)^*$. Since $\pi : (X_0 + X_1)^* \to (Y_0 + Y_1)^*$ is the adjoint of the embedding $Y_0 + Y_1 \to X_0 + X_1$, this happens if and only if $Y_0 + Y_1$ is a closed subspace of $X_0 + X_1$.

(iv)$\Rightarrow$(iii) is trivial.

(iii)$\Rightarrow$(iv). Let $T : (X_0 \cap X_1)/(Y_0 \cap Y_1) \to (X_0/Y_0) \oplus (X_1/Y_1)$ be the natural mapping. Then T is injective; furthermore

$$T^* : (X_0/Y_0)^* \oplus (X_1/Y_1)^* \cong Y_0^\perp \oplus Y_1^\perp \to \big((X_0 \cap X_1)/(Y_0 \cap Y_1)\big)^* \cong (Y_0 \cap Y_1)^\perp$$

may be identified with the mapping $(x_0', x_1') \to x_0' + x_1'$ of $Y_0^\perp \oplus Y_1^\perp$ into $(Y_0 \cap Y_1)^\perp$. Hence (iii) $\Rightarrow$ $\mathrm{im}(T^*)$ is closed $\Rightarrow$ $\mathrm{im}(T)$ is closed. $\Rightarrow$ T is an isomorphism into $\Rightarrow$ T^* is onto $\Rightarrow$ (iv).

(iii)$\Leftrightarrow$(ii) is trivial.

(ii)$\Leftrightarrow$(v) follows by the definitions. $\square$

Theorem 6.1 (Pisier). *Suppose that $\bar{Y}$ is a closed subcouple of a regular Banach couple $\bar{X}$. Then $\bar{Y}$ is a K-subcouple of $\bar{X}$ if and only if $\bar{Y}$ is regular and $(Y_0^\perp, Y_1^\perp)$ is a K-subcouple of (X_0', X_1').*

Proof. We may assume that $\bar{Y}$ is regular, because Proposition 2.3 implies that $\bar{Y}$ is regular if it is a K-subcouple of $\bar{X}$. Similarly, by Proposition 6.2 (i)$\Rightarrow$(v) ($\bar{Y}$ is a K-subcouple) or (ii)$\Rightarrow$(v) ($\bar{Y}^\perp$ is a K-subcouple), we may assume that $\bar{Y}^\perp$ is a normal subcouple of $\bar{X}'$. It is easily seen (cf. the proof of Proposition 6.2) that the quotient couple $\bar{X}'/\bar{Y}^\perp$ is isometric to $\bar{Y}'$, and that the quotient map

$$\pi : X_0' \cap X_1' \cong (X_0 + X_1)^* \to (X_0'/Y_0^\perp) \cap (X_1'/Y_1^\perp) \cong Y_0' \cap Y_1' \cong (Y_0 + Y_1)^*$$

is the adjoint of the embedding $Y_0 + Y_1 \to X_0 + X_1$. Since the dual of the norm $K(t, \cdot; \bar{X})$ on $X_0 + X_1$ is $J(t^{-1}, \cdot; \bar{X}')$, and similarly for $\bar{Y}$, it follows that, for each t and C, $K(t, y; \bar{Y}) \leq CK(t, y; \bar{X})$ for all $y \in Y_0 + Y_1$, if and only if each $y' \in Y_0' \cap Y_1'$ can be lifted to an $x' \in X_0' \cap X_1'$ with $J(t^{-1}, x'; \bar{X}') \leq CJ(t^{-1}, y'; \bar{Y}')$. Consequently, $\bar{Y}$ is a K-subcouple of $\bar{X}$ if and only if $\bar{Y}' \cong \bar{X}'/\bar{Y}^\perp$ is a J-quotient couple of $\bar{X}'$, and the result follows by Theorem 4.1. $\square$

We collect some further consequences of the argument above.

Theorem 6.2. *Suppose that $\bar{Y}$ is a closed K-subcouple of a regular Banach couple $\bar{X}$. Then $\bar{Y}^\perp$ is a normal K-subcouple of $\bar{X}'$ and there is a natural isometry between the couples $\bar{Y}'$ and $\bar{X}'/\bar{Y}^\perp$ which allows us to regard $\bar{Y}'$ as a J-quotient couple and K-quotient couple of $\bar{X}'$.*

Proof. All assertions follow by Theorem 6.1 and its proof except that $\bar{Y}'$ is a K-quotient couple. For this we use Theorem 4.4 and the fact that dual couples are Gagliardo complete. $\square$

Remark 6.1. It does not follow from the assumptions in Theorem 6.2 that $\bar{Y}$ is a normal subcouple of $\bar{X}$, see Example 7.8.

7. Examples

Example 7.1. A complemented subcouple, i.e. a subcouple $\bar{Y}$ of $\bar{X}$ such that there is a bounded projection $\bar{X} \to \bar{Y}$, is a K-subcouple. $\bar{Y}$ is normal and $I - P$ induces a linear lifting $\bar{X}/\bar{Y} \to \bar{X}$ that yields liftings as in the definitions of J-quotient couple and K-quotient couple. P gives a linear good approximation. Conversely, if, for example, $\bar{X}$ and $\bar{Y}$ are regular and there is a linear lifting satisfying the J-quotient condition, or a linear good approximation, then $\bar{Y}$ is a complemented subcouple of $\bar{X}$.

Example 7.2. Regard the Hardy space H^p on the unit disc as a closed subspace of L^p on the circle. Then (H^p, H^q) is a K-subcouple of (L^p, L^q), $0 < p, q \leq \infty$, see Pisier [21] and (for $p = 1$) Bennett and Sharpley [5]. (See also [22], [23], [3] for generalizations and applications.) The subcouple is normal, e.g. by Corollary 3.1.

By Theorem 4.1, in particular, $(L^p/H^p, L^\infty/H^\infty)$ is a J-quotient couple of (L^p, L^∞). If $p > 1$, this is a dual couple and hence Gagliardo complete. By Proposition 4.1 applied to $(L^\infty/H^\infty, L^{p_0}/H^{p_0})$ for some $p_0 < p$, it follows that $(L^p/H^p, L^\infty/H^\infty)$ is Gagliardo complete for each $p > 0$, and thus a K-quotient couple of (L^p, L^∞).

Theorem 5.2 yields, for example, that for each $p_0 < 0$, every $f \in L^{p_0}$ has a simultaneous good analytic approximation with respect to all L^p, $p_0 \leq p \leq \infty$. The error constants may depend on p here, since the equalities $(L^{p_0}, L^\infty)_{\theta p} = L^p$ are not isometric; it seems likely that they can be taken to depend on p_0 only, but we have not made any detailed analysis. We do not know whether the result corresponding to the limit $p_0 \to 0$ also holds, i.e., whether every $f \in \bigcup_{p>0} L^p$ has a simultaneous good analytic approximation with respect to all L^p, $0 < p \leq \infty$.

Example 7.3. Let Θ be an inner function in the unit disc. Since multiplication by Θ is an isometry in each L^p, it follows from the preceding example that, if $0 < p < q \leq \infty$, $(\Theta H^p, \Theta H^q)$ is a K-subcouple of $(\Theta L^p, \Theta L^q) = (L^p, L^q)$; hence $(\Theta H^p, \Theta H^q)$ is a K-subcouple of (H^p, H^q) by Proposition 2.2 (iii), and $(H^p/\Theta H^p, H^q/\Theta H^q)$ is a J-quotient couple of (H^p, H^q) by Theorem 4.1.

Suppose now that Θ is a Blaschke product with simple zeros $(z_n)_1^\infty$. Then $H^p/\Theta H^p$ may be identified with the sequence space $Z_p = \{(f(z_n))_1^\infty : f \in H^p\}$. It follows, for example, that $(Z_{p_0}, Z_{p_1})_{\theta p} = Z_p$ when $1/p = (1-\theta)/p_0 + \theta/p_1$, $0 < \theta < 1$, $0 < p_0, p_1 \leq \infty$ (Theorem 4.2), and that if $(a_n)_1^\infty \in Z_{p_0} \cap Z_{p_1}$, then $a_n = f(z_n)$ for some analytic f with $\|f\|_{H^{p_j}} \leq C \|(a_n)\|_{Z_{p_j}}$ (Theorem 4.1(v)).

If furthermore Θ is an interpolating Blaschke product, i.e. (z_n) is an interpolating sequence, then $Z_p = \ell^p(d\mu)$, $0 < p \leq \infty$, where $\mu(n) = 1 - |z_n|^2$, see e.g. [8], [24]. Thus (Z_p, Z_q) is Gagliardo complete, and is thus a K-quotient couple of (H^p, H^q), $0 < p \leq q \leq \infty$. In fact, in this case there is a linear lifting that maps $Z_p = \ell^p(d\mu)$ boundedly into H^p for all p, $0 < p \leq \infty$, for example

$$(a_n)_1^\infty \to \sum_1^\infty a_n f_n(z) \exp\big(1 - \big(\frac{1 - \bar{z}_k z}{1 - |z_k|^2}\big)^{1/2}\big), \tag{7.1}$$

where $f_n \in H^\infty$ are as in the P. Beurling interpolation theorem [8, Theorem VII.2.1] (this is easily verified for $p = \infty$ or $p \leq 1$; for $1 < p < \infty$ we use interpolation). Hence $(\Theta H^p, \Theta H^q)$ is a complemented subcouple of (H^p, H^q). We do not know whether this extends to Θ that are not interpolating Blaschke products. (Note that this example combines two different meanings of 'interpolation'.)

Example 7.4. Consider functions in R^n and let $k \geq 1$. Define $Tf = (D^\alpha f)_{|\alpha| \leq k}$, where α ranges over the $N = \binom{n+k}{k}$ multi-indices with $|\alpha| \leq k$. Then T yields an isomorphism of the Sobolev space W_k^p into $(L^p)^N$; in particular, we may regard (W_k^1, W_k^∞) as a closed subcouple of $((L^1)^N, (L^\infty)^N)$. By a result by DeVore and Scherer [7], [5], it is a K-subcouple.

Example 7.5. Consider again functions on R^n and let $T(g_0, \dots, g_n) = g_0 + \sum_{i=1}^n R_i g_i$, where R_i are the Riesz transforms. Then $T : (L^p)^{n+1} \to L^p$, $1 < p < \infty$, and $T : (L^\infty)^{n+1} \to BMO$. Miyachi [15] proved (somewhat more generally) that if $1 < p < \infty$ and $f \in L^p \cap BMO$, then $f = Tg$ for some $g \in (L^p \cap L^\infty)^{n+1}$ with $\|g\|_{(L^p)^{n+1}} \leq C\|f\|_{L^p}$ and $\|g\|_{(L^\infty)^{n+1}} \leq C\|f\|_{BMO}$ (C depends on p but not on f). In other words, (L^p, BMO) may be regarded as a J-quotient couple of $((L^p)^{n+1}, (L^\infty)^{n+1})$, $1 < p < \infty$. (L^p, BMO) is Gagliardo complete, so by Theorem 4.4 it is a K-quotient couple as well. By Theorem 4.1, $\bar{Y} = (\ker T \cap (L^p)^{n+1}, \ker T \cap (L^\infty)^{n+1})$ is a K-subcouple of $((L^p)^{n+1}, (L^\infty)^{n+1})$, $1 < p < \infty$.

Note also that $T^* : f \to (f, -R_1 f, \dots, -R_n f)$ yields isomorphisms $H^1 \to (L^1)^{n+1}$ and $L^p \to (L^p)^{n+1}$, $1 < p < \infty$. (Here H^1 is the real variable Hardy space.) Hence we can regard (H^1, L^p) as a closed subcouple of $((L^1)^{n+1}, (L^p)^{n+1})$. It is easily seen that $(H^{1\perp}, L^{p\perp}) = (\ker T \cap (L^\infty)^{n+1}, \ker T \cap (L^{p'})^{n+1})$, which by the above is a K-subcouple of $((L^\infty)^{n+1}, (L^{p'})^{n+1})$; Theorem 6.1 now yields that (H^1, L^p) is a K-subcouple of $((L^1)^{n+1}, (L^p)^{n+1})$.

Example 7.6. Consider the space of all bounded linear operators on ℓ^2, regarded as matrices $(a_{ij})_{i,j=1}^\infty$. We let S_∞ be the space of all bounded linear operators, S_0 the space of finite rank operators with $\|T\|_{S_0} = \operatorname{rank}(T)$, and S_p, for $0 < p < \infty$, the Schatten p-class. S_p can be defined by interpolation (the E-method):

$$S_p = \{T : \big(\int_0^\infty E(t, T; S_0, S_\infty)^p \, dt\big)^{1/p} < \infty\}$$
$$= \big((S_0, S_\infty)_{p/(p+1), p+1}\big)^{1+1/p}, \quad 0 < p < \infty, \tag{7.2}$$

see [2, Section 7.3]. ($E(t, T; S_0, S_\infty)$ is known as the singular number $s_{[t]}(T)$.) S_p is a quasi-Banach space for $0 < p \leq \infty$ and a Banach space for $p \geq 1$.

Let T_p denote the subspace of S_p consisting of (upper) triangular matrices, i.e. matrices (a_{ij}) with $a_{ij} = 0$ when $i > j$. It is well known that the natural projection τ maps S_p onto T_p if and only if $1 < p < \infty$. In particular, (T_p, T_q) is a complemented subcouple of (S_p, S_q) for $1 < p_0, p_1 < \infty$.

Pisier [21] proved that (T_p, T_q) is a K-subcouple of (S_p, S_q) whenever $0 < p \leq q \leq \infty$. Let $Q_p = S_p / T_p$. If $p' < p < q$, then $Q_p = (Q_{p'}, Q_q)_{\theta q}$ for some θ by Theorem 4.2. Hence $Q_p^c = Q_p$ in $Q_{p'} + Q_q$ and thus in $Q_p + Q_q$. Furthermore, $Q_p + Q_q = Q_q$, so $Q_q^c = Q_q$ (in $Q_p + Q_q$) is trivial. Consequently, (Q_p, Q_q) is Gagliardo complete for $0 < p < q \leq \infty$. By Theorem 4.4 it is a K-quotient couple of (S_p, S_q) and by Theorem 5.2, for every $p_0 > 0$, every bounded operator has a simultaneous good triangular approximation with respect to (for example) all S_p, $p_0 < p \leq \infty$. (This extends results by Pisier [21] and Kaftal, Larson and Weiss [13].)

We do not know whether there exists a simultaneous good triangular approximation with respect to S_p for all $p > 0$.

Example 7.7. In the preceding example we only considered $p > 0$. We will here show that (T_0, T_∞) is *not* a K-subcouple of (S_0, S_∞). (The same argument shows that (T_0, T_p) is not a K-subcouple of (S_0, S_p) for any $p > 0$.)

Let $n \geq 1$ and define $A = (a_{ij})_{i,j=1}^{\infty}$ with

$$a_{ij} = \begin{cases} n^{3(j-i)} & 1 \leq i,j \leq n+1 \\ 0 & \text{otherwise.} \end{cases} \tag{7.3}$$

Let B be the triangular projection of A. Since $A - B$ has $n(n-1)/2$ non-zero entries, each of them at most n^{-3}, and $\operatorname{rank}(A) = 1$,

$$E(1, B; S_0, S_\infty) \leq \|B - A\|_{S_\infty} \leq n^2 n^{-3} = n^{-1}.$$

On the other hand, any triangular matrix of rank $\leq n$ has at most n non-zero entries on the diagonal. Hence, if $C \in T_0$ with $\operatorname{rank}(C) \leq n$, $B - C$ has at least one diagonal entry equal to 1, so $\|B - C\|_{S_\infty} \geq 1$. Consequently,

$$E(n, B; T_0, T_\infty) \geq 1 \geq nE(1, B; S_0, S_\infty). \tag{7.4}$$

Since n is arbitrary, (T_0, T_∞) is not an E-subcouple of (S_0, S_∞); by Proposition 2.1 it is not a K-subcouple.

Example 7.8. This example shows that the conditions in Theorem 2.2 (ii) and Proposition 2.5 can not be omitted. Let $X_0 = \ell^\infty$, $Y_0 = c_0$ and $X_1 = Y_1 = c_0(w) = \{(a_n)_1^\infty : w(n)a_n \to 0 \text{ as } n \to \infty$, where $w(n)$ is a sequence of positive numbers such that $w(n) \to 0$ as $n \to \infty$. Then $X_0 = Y_0^c$, and $\bar{Y}$ is an exact K-subcouple of $\bar{X}$. Let $\Phi = L^\infty$. Then $K_\Phi(\bar{Z}) = Z_0^c$ for every couple $\bar{Z}$, and it is easy to check that $(Y_0, K_\Phi(\bar{Y})) = (Y_0, Y_0^c) = (c_0, \ell_\infty)$ is not a K-subcouple of $(X_0, K_\Phi(\bar{X})) = (\ell_\infty, \ell_\infty)$.

Note also that $Y_0 \neq X_0 \cap (Y_0 + Y_1)$, cf. Proposition 2.5, so $\bar{Y}$ is not a normal subcouple of $\bar{X}$. Note that $X_0/Y_0 = \ell^\infty/c_0$ and $X_1/Y_1 = \{0\}$ do not form a couple in a natural way, while their duals $Y_0^\perp \cong c_0^\perp$ and $Y_1^\perp = \{0\}$ form a normal K-subcouple of $\bar{X}' = ((\ell^\infty)^*, \ell^1(w^{-1}))$ by Theorem 6.2.

Example 7.9. Let $X_0 = X_1 = R^2$ with the norms $\|(x_1, x_2)\|_0 = \|(x_1, -x_2)\|_1 = I(x_1 \neq 0) + I(x_1 \neq x_2)$, where I is the indicator function. Let $Y_0 = Y_1 = \{(x_1, 0)\}$. Thus $\|(x_1, 0)\|_{Y_j} = 2I(x_1 \neq 0)$. It is easily seen that for any $y = (x_1, 0) \neq 0$, $K(t, y; \bar{Y}) = K_\infty(t, y; \bar{Y}) = K(t, y; \bar{X}) = 2 \wedge 2t$, while $K_\infty(t, y; \bar{X}) = \min(2, 2t, 1 \vee t)$. Hence $\bar{Y}$ is an exact K-subcouple of $\bar{X}$ but not an exact K_∞-subcouple. Similarly, $\bar{Y}$ is not an exact E-subcouple.

Example 7.10. This example shows that a simultaneous good approximation with respect to two norms, may be a bad approximation with respect to an intermediate norm.

Let $\bar{Y}$ be a normal K-subcouple of a Banach couple $\bar{X}$ and assume that $Y_0^0 \not\subset Y_1$ and $Q_0^0 \not\subset Q_1$, where $\bar{Q} = \bar{X}/\bar{Y}$; for example $\bar{Y} = (H^{p_0}, H^{p_1})$ and $\bar{X} = (L^{p_0}, L^{p_1})$, $1 \leq p_0 < p_1 \leq \infty$, as in Example 7.2. Fix $\theta \in (0,1)$ and $q \leq \infty$. Then $\|y\|_{(\bar{Y})_{\theta q}}/\|y\|_{Y_0}$ can be arbitrarily large for $y \in Y_0 \cap Y_1$, because otherwise $K(t, y; \bar{Y}) \leq C_1 t^\theta \|y\|_{(\bar{Y})_{\theta q}} \leq C_2 t^\theta \|y\|_{Y_0} \leq \frac{1}{2}\|y\|_{Y_0}$ if t is small enough, and this would, as is easily seen, imply $Y_0^0 \subset Y_1$. Consequently there exist $y_n \in Y_0 \cap Y_1$ with $\|y_n\|_{Y_0} = 1$ and $\|y_n\|_{(\bar{Y})_{\theta q}} \geq n$. Similarly there exist $z_n \in Q_0 \cap Q_1$ with $\|z_n\|_{Q_0} = 1$ and $\|z_n\|_{Q_1} = \lambda_n \to \infty$; we may further assume that $\lambda_n \geq \|y_n\|_{Y_1}^{1/(1-\theta)}$.

By Theorem 4.1 there exist $u_n \in X_0 \cap X_1$ with $z_n = \pi(u_n)$ and $\|u_n\|_{X_j} \asymp \|z_n\|_{Q_j} = \lambda_n^j$, $j = 0, 1$. Define $x_n = \lambda_n^{-\theta} u_n + u_1$. Then (if n is large enough)

$$\|x_n\|_{X_0} \asymp 1 \asymp \|\pi(x_n)\|_{Q_0} = d_{X_0}(x_n, Y_0)$$

and
$$\|x_n\|_{X_1} \asymp \lambda_n^{1-\theta} \asymp \|\pi(x_n)\|_{Q_1} = d_{X_1}(x_n, Y_1).$$
Since also $\|x_n - y_n\|_{X_0} \leq C$ and $\|x_n - y_n\|_{X_1} \leq C\lambda_n^{1-\theta}$, y_n is a good approximation of x_n with respect to both X_0 and X_1. On the other hand,
$$\|x_n\|_{(\bar{X})_{\theta q}} \leq C\lambda_n^{-\theta}\|u_n\|_{X_0}^{1-\theta}\|u_n\|_{X_1}^{\theta} + \|u_1\|_{(\bar{X})_{\theta q}} \leq C$$
and thus
$$\|x_n - y_n\|_{(\bar{X})_{\theta q}} \geq \|y_n\|_{(\bar{X})_{\theta q}} - C \geq c\|y_n\|_{(\bar{Y})_{\theta q}} - C \geq cn - C \to \infty,$$
while
$$d_{(\bar{X})_{\theta q}}(x_n, (\bar{Y})_{\theta q}) \leq \|x_n\|_{(\bar{X})_{\theta q}} \leq C.$$
Hence y_n is not a good approximation of x_n with respect to $(\bar{X})_{\theta q}$.

8. The Adamyan-Arov-Krein theorem

Consider again the spaces S_p of matrices acting on ℓ^2 defined in Example 7.6, $0 \leq p \leq \infty$, and let Γ_p be the subspaces of Hankel matrices, i.e. matrices of the form $(a_{i+j})_{i,j=1}^{\infty}$. By a well-known unitary equivalence, we may equivalently let S_p be the Schatten classes of operators from H^2 to $H^2_- = L^2 \ominus H^2$, where $H^2 \subset L^2 = L^2(T)$ is the Hardy space as in Example 7.2, and $\Gamma_p = \Gamma_\infty \cap S_p$ consists of the Hankel operators
$$\Gamma_\infty = \{T : Tf = P_-(\varphi f) \text{ for some } \varphi \in L^\infty\}. \tag{8.1}$$

The Adamyan-Arov-Krein theorem (or AAK for short), see e.g. [16], says that if T is a Hankel operator, then
$$\inf\{\|T - K\|_{S_\infty} : \operatorname{rank}(K) \leq n\} = \inf\{\|T - K\|_{S_\infty} : \operatorname{rank}(K) \leq n,\ K \in \Gamma_0\}, \tag{8.2}$$
i.e., in our notation,
$$E(t, T; S_0, S_\infty) = E(t, T; \Gamma_0, \Gamma_\infty), \quad t > 0 \tag{8.3}$$
(and furthermore, the infima are attained). In other words, $(\Gamma_0, \Gamma_\infty)$ is an exact E-subcouple of (S_0, S_∞). Proposition 2.1 yields the following.

Theorem 8.1. *$(\Gamma_0, \Gamma_\infty)$ is a K-subcouple of (S_0, S_∞).* □

Conversely, Theorem 8.1 yields the AAK theorem up to constants and may thus be regarded as a weak AAK theorem.

By reiteration (Theorems 2.1 and 2.2 and Proposition 2.2 (viii)) we easily obtain a generalization. This result is due to Peller [18], [19] (at least for $q = \infty$), where also applications to interpolation of Besov spaces are given.

Theorem 8.2. *(Γ_p, Γ_q) is a K-subcouple of (S_p, S_q), $0 \leq p \leq q \leq \infty$.* □

Theorem 8.2 implies in particular, using the equivalence of K-subcouples and E-subcouples again, that (8.3) and (8.2) hold within constants (in the norm and the rank) also if S_∞ is replaced by S_p for any $p > 0$, which can be regarded as a weak AAK theorem for S_p. In this case the constants are really needed; the (strong) AAK theorem, with equalities in (8.2) and (8.3), does not hold for $S_p,\ p < \infty$, as the following example shows.

Example 8.1. Let A be the Hankel matrix (a_{ij}) with $a_{11}=1,\ a_{12}=a_{21}=\varepsilon>0$ (small) and $a_{ij}=0,\ i+j>3$, and let $B=(b_{ij})$ with $b_{11}=1,\ b_{12}=b_{21}=\varepsilon,\ b_{22}=\varepsilon^2$ and $b_{ij}=0,\ i\vee j>2$. Then B has rank 1 so $E(1,A;S_0,S_p)\le\|A-B\|_{S_p}=\varepsilon^2$.

On the other hand, suppose $C=(c_{ij})$ is any rank 1 Hankel matrix with $\|A-C\|_{S_p}\le 2\varepsilon^2$. Then $c_{11}=a_{11}+O(\varepsilon^2)=1+O(\varepsilon^2)$, $c_{12}=\varepsilon+O(\varepsilon^2)$, and $c_{12}=\varepsilon+O(\varepsilon^2)$. Hence, since C has rank 1, $c_{22}=c_{12}c_{21}/c_{11}=\varepsilon^2+O(\varepsilon^3)$. Since A and C are Hankel, this gives $a_{13}-c_{13}=a_{22}-c_{22}=a_{31}-c_{31}=-\varepsilon^2+O(\varepsilon^3)$, and, for example, $\|A-C\|_{S_2}\ge\sqrt{3}|a_{22}-c_{22}|=\sqrt{3}\varepsilon^2+O(\varepsilon^3)$. Consequently $E(1,A;\Gamma_0,\Gamma_2)\ge\sqrt{3}\varepsilon^2+O(\varepsilon^3)$, which shows that (Γ_0,Γ_2) is *not* an exact E-subcouple of (S_0,S_2). In fact, it is not difficult to show that $\|A-C\|_{S_p}\ge 3^{1/(p\vee 1)}\varepsilon^2+O(\varepsilon^3)$, and thus (Γ_0,Γ_p) is not an exact E-subcouple of (S_0,S_p) for any $p<\infty$.

We do not know whether the AAK theorem for S_p holds in the form

$$E(n,T;\Gamma_0,\Gamma_p)\overset{?}{\le} CE(n,T;S_0,S_p), \tag{8.4}$$

i.e., if we may take $C_1=1$ in (2.5) in this case.

It follows as in Example 7.6 that $(S_p/\Gamma_p,S_\infty/\Gamma_\infty)$ is Gagliardo complete for every $p>0$.

Remark 8.1. Christian Le Merdy (personal communication) has shown that S_0/Γ_0 is Gagliardo complete in S_∞/Γ_∞, which answers a question that was asked in a draft of this paper. More precisely he showed that if $R_n=\{T\in S_0:\operatorname{rank}(T)\le n\}$, then $\overline{\Gamma_\infty+R_n}\subset\Gamma_\infty+R_{2n}$, which implies that for any $\bar T\in(S_0/\Gamma_0)^c$, $\|\bar T\|_{S_0/\Gamma_0}\le 2\|\bar T\|_{(S_0/\Gamma_0)^c}$. He also showed that the constant 2 is necessary; in fact, the matrix $A=(a_{ij})$ with $a_{23}=a_{32}=1$ and all other $a_{ij}=0$, belongs to $\overline{\Gamma_\infty+R_1}\setminus(\Gamma_\infty+R_1)$, and the corresponding element $\bar A\in S_\infty/\Gamma_\infty$ satisfies $\|\bar A\|_{S_0/\Gamma_0}=2$ and $\|\bar A\|_{(S_0/\Gamma_0)^c}=1$.

Theorem 5.3 (with Remark 5.2 for $p=\infty$) yields the following approximation result.

Theorem 8.3. *Every bounded operator on ℓ^2 has a simultaneous good Hankel approximation with respect to all S_p, $0<p\le\infty$. (The error constants may depend on p.)* □

Remark 8.2. There are several similarities between the scales (Γ_p) and (T_p); for example, the natural projection is bounded $S_p\to\Gamma_p$ when $1<p<\infty$ [18]. Note however one difference: The AAK theorem does *not* hold for (T_p) (not even weakly), as was shown in Example 7.7.

We have so far dealt with the classical Hankel operators or, equivalently, Hankel matrices. There are several generalizations of Hankel operators where the AAK theorem remains valid, and the considerations above hold for them as well.

One such generalization, due to Ball and Helton [1] and Treil [25], is to vector-valued spaces, where we consider operators from $H^2(E_1)$ to $H^2_-(E_2)$ for two separable Hilbert spaces E_1 and E_2, taking $\varphi\in L^\infty(B(E_1,E_2))$ in (8.1); cf. Peller [20].

Other generalizations are given by Cotlar and Sadosky [6], in particular they prove the AAK theorem for weighted spaces $H^2(\mu)\subset L^2(\mu)$, where μ is a (finite) measure on the circle.

There are other generalizations of Hankel operators for which it is not known whether the AAK theorem holds (even in its weak version). We do not know of any case where the weak AAK theorem (Theorem 8.1) is proved but not the strong form.

9. Final remarks

Remark 9.1. This paper is in the usual framework of (compatible) couples of spaces. It seems plausible, however, that the notions in this paper could be defined and studied more generally for Doolittle diagrams [14], [4, section 2.7.2], which might simplify the treatment of quotients and duals.

Remark 9.2. We do not know if it is possible to develop analoguous results for the complex method or other interpolation methods. From an abstract point of view, the following attempt seems reasonable.

Definition. *Let $\bar{A}$ be a Banach couple and let $\bar{Y}$ be a closed subcouple of a Banach couple $\bar{X}$. Then $\bar{Y}$ is an $\bar{A}$-subcouple of $\bar{X}$ if every linear mapping $\bar{Y} \to \bar{A}$ may be extended to a mapping $\bar{X} \to \bar{A}$, and (if $\bar{Y}$ is a normal subcouple) $\bar{X}/\bar{Y}$ is an $\bar{A}$-quotient couple of $\bar{X}$ if every linear mapping $\bar{A} \to \bar{X}/\bar{Y}$ may be lifted to a mapping $\bar{A} \to \bar{X}$. (By the closed graph theorem, we automatically have norm estimates.)*

The definition is motivated by the following simple result, whose proof we leave to the reader. Here $\bar{\ell^p}$ denotes the couple $(\ell^p, \ell^p(2^n))$ of weighted spaces of two-sided sequences.

Proposition 9.1. *Let $\bar{Y}$ be a closed subcouple of a Banach couple $\bar{X}$. Then $\bar{Y}$ is a K-subcouple of $\bar{X}$ if and only if it is an $\bar{\ell}^\infty$-subcouple, and (if $\bar{Y}$ is normal) $\bar{X}/\bar{Y}$ is a J-quotient couple of $\bar{X}$ if and only if it is an $\bar{\ell}^1$-quotient couple.* □

Recalling that the K-method can be defined as a coorbit functor using the couple $\bar{\ell}^\infty$, and that the J-method can be defined as an orbit functor using $\bar{\ell}^1$ [4], it looks promising to consider $\bar{A}$-subcouples or $\bar{A}$-quotient couples for other coorbit or orbit functors, for example $\overline{FL^\infty}$-subcouples and $\overline{FL^1}$-quotient couples for the complex method, where $\overline{FL^p}$ is a weighted couple of Fourier sequence spaces, cf. [12].

Remark 9.3. Interpolation of subspaces and quotient spaces by the complex method for infinite families of spaces have been considered by Hernandez, Rochberg and Weiss [9], who in particular give a duality theorem. We do not know whether the methods of this paper can be extended to families of spaces.

References

1. J.A. Ball and J.W. Helton, A Beurling-Lax theorem for the Lie group $U(m,n)$ which contains most classical interpolation theory, *J. Operator Th.* **9** (1983), 107–142.
2. J. Bergh and J. Löfström, *Interpolation Spaces*, Springer-Verlag, Berlin, 1976.
3. J. Bourgain, Some consequences of Pisier's approach to interpolation, Preprint 1991.
4. Yu.A. Brudnyĭ and N.Y. Krugljak, *Interpolation functors and interpolation spaces*, North-Holland, Amsterdam, 1991.
5. C. Bennett and R. Sharpley, *Interpolation of operators*, Academic Press, Orlando, 1988.
6. M. Cotlar and C. Sadosky, Weighted and two-dimensional Adamjan-Arov-Krein theorems and analogues for Sarason commutants, *Mittag-Leffler Report 24* (1990/91).
7. R.A. DeVore and K. Scherer, Interpolation of operators on Sobolev spaces., *Ann. Math.* **109** (1979), 583–599.
8. J. Garnett, *Bounded analytic functions*, Academic Press, New York, 1981.
9. E. Hernandez, R. Rochberg and G. Weiss, Interpolation of subspaces and quotient spaces by the complex method, *Function spaces and applications, Proceedings, Lund 1986* (M. Cwikel, J. Peetre, Y. Sagher, H. Wallin, eds.), Lecture Notes in Math. 1302, Springer-Verlag, Berlin, 1988, pp. 253–289.
10. T. Holmstedt, Interpolation of quasi-normed spaces, *Math. Scand.* **26** (1970), 177–199.
11. T. Holmstedt and J. Peetre, On certain functionals arising in the theory of interpolation spaces, *J. Funct. Anal.* **4** (1969), 88–94.
12. S. Janson, Minimal and maximal methods of interpolation, *J. Funct. Anal.* **14** (1981), 50–72.
13. V. Kaftal, D. Larson and G. Weiss, Quasitriangular subalgebras of semifinite von Neumann algebras are closed (to appear).
14. S. Kaijser and J.W. Pelletier, *Interpolation functors and duality*, Lecture Notes in Math. 1208, Springer-Verlag, Berlin, 1986.
15. A. Miyashi, Some Littlewood-Paley type inequalities and their application to the Fefferman-Stein decomposition of BMO, *Indiana Univ. Math. J.* **39** (1990), 563–583.

16. N.K. Nikolskiĭ, *Treatise on the shift operator*, Springer-Verlag, Berlin, 1986.
17. J. Peetre, Interpolation functors and Banach couples, *Actes Congrès Intern. Math. 1970*, vol. 2, 1971, pp. 373–378.
18. V.V. Peller, Hankel operators of class $\mathfrak{S}_p$ and their applications (rational approximation, Gaussian processes, the majorization problems for operators), *Mat. Sb.* **113** (1980), 538–581 (Russian); English transl., *Math. USSR Sbornik* **41** (1982), 443–479.
19. V.V. Peller, A description of Hankel operators of class $\mathfrak{S}_p$ for $p > 0$, an investigation of the rate of rational approximation, and other applications, *Mat. Sb.* **122** (1983), 481–510 (Russian); English transl., *Math. USSR Sbornik* **50** (1985), 465–494.
20. V.V. Peller, A remark on interpolation in spaces of vector-valued functions, *Zapiski Nauchn. Semin. LOMI* **141** (1985), 162–164. (Russian)
21. G. Pisier, Interpolation between H^p spaces and non-commutative generalizations I, Preprint 1991.
22. G. Pisier, Interpolation between H^p spaces and non-commutative generalizations II, Preprint 1991.
23. G. Pisier, A simple proof of a theorem of Jean Bourgain, Preprint 1991.
24. H.S. Shapiro and A.L. Shields, On some interpolation problems for analytic functions, *Amer. J. Math.* **83** (1961), 513–532.
25. S.R. Treil, The theorem of Adamyan-Arov-Krein: vector variant, *Zapiski Nauchn. Semin. LOMI* **141** (1985), 56–71. (Russian)
26. H. Triebel, Allgemeine Legendresche Differentialoperatoren II, *Ann. Scuola Norm. Pisa (3)* **24** (1970), 1–35.
27. R. Wallstén, Remarks on interpolation of subspaces, *Function spaces and applications, Proceedings, Lund 1986* (M. Cwikel, J. Peetre, Y. Sagher, H. Wallin, eds.), Lecture Notes in Math. 1302, Springer-Verlag, Berlin, 1988, pp. 410–419.

Department of Mathematics, Uppsala University, P.O. Box 480, 751 06 Uppsala, Sweden.
E-mail address: `svante.janson@math.uu.se`