LINEAR EXTENSIONS OF A RANDOM PARTIAL ORDER

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Abstract. We study asymptotics of the number of linear extensions of the random $G_{n,p}$ partial order, where p is fixed and $n \to \infty$. In particular, it is shown that the distribution is asymptotically log-normal.

1. Introduction and results

One of the standard models for a random partial order is the $G_{n,p}$ order, defined as follows. Let $<$ denote the natural order on the vertex set $[n] = \{1, \dots, n\}$. A graph G on $[n]$ induces a partial order on that vertex set, viz. the transitive closure of the relation $\{i < j$ and there is an edge $ij\}$. I.e., if each edge ij in G, with $i < j$, is directed from i to j, $i \prec j$ iff there exists a directed path from i to j in G. The random $G_{n,p}$ order is obtained by applying this procedure to the random graph $G_{n,p}$. Here, as throughout, we let $G_{n,p}$ denote a random graph in $\mathcal{G}(n,p)$, i.e., a graph on $[n]$ such that each possible edge appears with probability p, independently of all other edges. We assume throughout that $0 < p < 1$ and set $q = 1 - p$.

This $G_{n,p}$ order has previously been studied by Barak and Erdős [3], who investigated the width of the partial order, and by Albert and Frieze [1], who studied its height and set-up number. Our purpose in this paper is to investigate another fundamental parameter of partial orders, namely the number of linear extensions. Recall that a linear extension of a partial order $(X, <)$ is a total order $\prec$ on the same set X such that $x \prec y$ whenever $x < y$.

We define a *linear extension* of a graph G on $[n]$ to be a linear extension of the corresponding partial order. We denote by $N(G)$ the number of linear extensions of G. The purpose of the present paper is to study the number $N(G_{n,p})$ of linear orders that extend the $G_{n,p}$ partial order; we also use $N_{n,p}$ for this random variable. Obviously, $1 \le N_{n,p} \le n!$.

There is a simple formula for the expectation of $N_{n,p}$. We define

$$\kappa(p) = \prod_{k=1}^{\infty} (1 - q^k). \tag{1.1}$$

Theorem 1. *With notation as above,*

$$\mathrm{E}\,N_{n,p} = \prod_{k=1}^{n} \frac{1-q^k}{1-q} \tag{1.2}$$

Hence, if p is fixed, $0 < p < 1$,

$$\mathrm{E}\,N_{n,p} \sim \kappa(p)\Big(\frac{1}{p}\Big)^n, \qquad \textit{as} \quad n \to \infty. \tag{1.3}$$

The asymptotic distribution of $N_{n,p}$ for fixed p turns out to be log-normal.

Theorem 2. *Let p be fixed and $n \to \infty$. For some $\mu = \mu(p) > 0$ and $\sigma^2 = \sigma^2(p) > 0$,*

$$\frac{\log(N_{n,p}) - \mu n}{\sqrt{n}} \xrightarrow{d} N(0, \sigma^2), \tag{1.4}$$

with convergence of all moments. In particular

$$\frac{\log(N_{n,p}) - \mathrm{E}\log(N_{n,p})}{(\operatorname{Var}\log(N_{n,p}))^{1/2}} \xrightarrow{d} N(0,1). \tag{1.5}$$

We do not have an explicit expression for $\mu(p)$ and $\sigma^2(p)$, but we can give simple bounds for $\mu(p)$.

Theorem 3. *Let Y have the shifted geometric distribution $P(Y = k) = pq^{k-1}$, $k \geq 1$. Then*

$$\mathrm{E}\log(Y) = \sum_{1}^{\infty} \log(k) p q^{k-1} \leq \mu(p) \leq (1-\kappa(p)) \log \frac{1/p - \kappa(p)}{1-\kappa(p)} < \log\frac{1}{p} = \log(\mathrm{E}\,Y). \tag{1.6}$$

For $p = 1/2$ we obtain from (1.6) that $0.507 < \mu < 0.625$ and $1.66 < e^{\mu} < 1.87$.

It may seem surprising that $\mu < \log\frac{1}{p}$, since Theorem 2 says that $N_{n,p}$ is concentrated about $(e^{\mu})^n$ (more precisely, $N_{n,p} = \exp(\mu n + O_p(\sqrt{n}))$, which thus is far less than the mean which is $\asymp (\frac{1}{p})^n$. Some thought, however, shows that Theorems 1 and 2 actually force $e^{\mu} < \frac{1}{p}$, because (1.4) implies that if n is sufficiently large, $\mathrm{P}(N_{n,p} > \exp(\mu n + \sigma\sqrt{n})) > 0.1$, and thus $(\frac{1}{p})^n \geq \mathrm{E}\,N_{n,p} > 0.1 e^{\sigma\sqrt{n}}(e^{\mu})^n > (e^{\mu})^n$.

The same phenomenon, that the variable is concentrated far below its mean, also occurs in the case of a product $X_n = \prod_1^n Y_k$ of i.i.d. positive variables, where (assuming suitable moment conditions) $\log X_n$ is asymptotically normal and X_n is concentrated about $(e^{\mathrm{E}\log Y_1})^n$, which (by Jensen's inequality) is less than $\mathrm{E}\,X_n = (\mathrm{E}\,Y_1)^n$. The variable $N_{n,p}$ is more complicated, but, as we will see in the proof, it has a somewhat similar structure.

We shall concentrate on the case where p is a constant, independent of n. In this case, we may assume that a random graph and the corresponding random partial order are defined as above on all of $\mathbf{N}$, or $\mathbf{Z}$, and define $G_{n,p}$ as the restriction $G_{\mathbf{N},p}|_{[n]}$. Let $N(i,j)$ be the number of linear extensions of the restriction of the random partial order to $(i,j]$, so that $N_{n,p} = N(0,n)$. Then $\{N(i,j)\}$ is supermultiplicative,

$$N(i,k) \geq N(i,j)N(j,k), \quad i > j > k, \tag{1.7}$$

and Kingman's subadditive ergodic theorem [10] applies to $-\log(N(i,k))$. Thus we have that $\frac{1}{n}\log(N_{n,p})$ converges a.s. to a random variable ξ a.s., which is easily shown to be constant; in fact, Theorem 2 implies $\xi = \mu$ a.s. This proves the following strong limit theorem.

Theorem 4. *If p is fixed, $N_{n,p}^{1/n} \to e^{\mu(p)}$ a.s. as $n \to \infty$.* □

We can sharpen Theorem 4 as follows.

Theorem 5. *Let p be a constant with $0 < p < 1$. For almost every $G_{\mathbf{N},p}$, there is an $n_0(G,p)$ such that, for every $n \geq n_0$,*

$$|N_{n,p}^{1/n} - e^{\mu(p)}| \leq \frac{15e^{\mu}(\log n)^{3/2}}{n^{1/2}p^2}.$$

Our final result shows that $\log N(G)$ is sharply concentrated around its mean.

Theorem 6. *Suppose that $0 < \lambda \leq n^{1/4}$ and that $q \geq 1/e$. Then*

$$\mathrm{P}\left(|\log N_{n,p} - n\mu(p)| > \frac{\lambda\sqrt{n}\log n}{p^2}\right) \leq 3\exp\left(-\frac{\lambda^2}{6400}\right).$$

The assumption that $q \geq 1/e$ here is purely for convenience: if the assumption is dropped, a similar result with different constants can be obtained. Results concerning deviations larger than those given by setting $\lambda = n^{1/4}$ in Theorem 6 can also be obtained, but the lemmas in the proof below are a little simpler to state if this restriction is made. In Theorem 6, one should think of p as being constant, but no restriction is actually needed on how small $p = p(n)$ can be. However, since $\log N_{n,p} < n \log n$ for all G, the result is only meaningful if p has order larger than $n^{-1/4}$.

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2. Proof of Theorem 1.

There is a $1-1$ correspondence between linear orders $\prec$ on $\{1,\dots,n\}$ and permutations π of $\{1,\dots,n\}$ given by

$$i \prec j \iff \pi(i) < \pi(j) \tag{2.1}$$

(where $<$, as above, denotes the natural order). It is easily seen that $\prec$ extends the $G_{n,p}$ partial order if and only if $\pi(i) < \pi(j)$ for all i, j, $i < j$, connected by an edge in $G_{n,p}$, or, equivalently,

$$\text{there is no edge } ij \text{ in } G_{n,p} \text{ with } i < j \text{ and } \pi(i) > \pi(j). \tag{2.2}$$

Consequently,

$$\mathrm{E}\, N_{n,p} = \sum_{\pi} \mathrm{P}\left((2.2) \text{ holds}\right) = \sum_{\pi} q^{\#\{\text{inversions in } \pi\}}, \tag{2.3}$$

and (1.2) reduces to a well-known formula for the generating function of the number of inversions in permutations.

For later use, we also give a direct proof as follows.

We consider again the random partial order as defined on $\mathbf{N}$, and let $N_n = N(0,n)$, for simplicity omitting the subscript p. Let $\mathcal{F}_n$ be the σ -field generated by the edge indicators $I(ij \text{ is an edge})$, $1 \leq i < j \leq n$; thus N_n is $\mathcal{F}_n$-measurable. Consider a particular linear order $\prec$ on $[n-1] = \{1,\dots,n-1\}$ that extends the partial order there, and order $[n-1]$ according to this order as $i_1 \succ i_2 \succ \cdots \succ i_{n-1}$. This linear order has n extensions to $[n]$, obtained by choosing $k \in \{1,\dots,n\}$ and then inserting n between i_{k-1} and i_k ($n \succ i_1$ if

$k = 1$, and $i_{n-1} \succ n$ if $k = n$). The linear order on $[n]$ constructed in this way extends the partial order on $[n]$ if and only if there are no edges $i_j n$ with $j < k$. Consequently, the number of extensions of $\prec$ to $[n]$ that are extensions of the partial order on $[n]$ equals $n \wedge \min\{j : i_j n \text{ is an edge}\}$. Given $\mathcal{F}_{n-1}$, this is a random variable with the distribution of $Y \wedge n$, where Y is the geometric variable in Theorem 3. Hence, N_n can be written as a sum

$$N_n = \sum_{1}^{N_{n-1}} Y_i^{(n)}, \tag{2.4}$$

where $Y_i^{(n)}$, given $\mathcal{F}_{n-1}$, are (dependent) random variables with the same distribution as $Y \wedge n$. Consequently

$$\mathrm{E}(N_n \mid \mathcal{F}_{n-1}) = \mathrm{E}(Y \wedge n) N_{n-1}$$

and

$$\mathrm{E}\, N_n = \mathrm{E}(Y \wedge n)\, \mathrm{E}\, N_{n-1} = \frac{1-q^n}{1-q}\, \mathrm{E}\, N_{n-1},$$

which gives (1.2) by induction.

3. Proof of Theorem 2.

Suppose that, for a given realization of the random graph $G_{n,p}$, (or $G_{\mathbf{N},p}$), there is a vertex i, $1 \le i \le n$, such that every other vertex is connected to i by a directed path (directed either towards or from i). Then, in the $G_{n,p}$ partial order, every other element is comparable with vertex i. Thus i is a fixed point of every permutation that corresponds to a linear extension of the $G_{n,p}$ order, and the number $N_{n,p} = N(0,n)$ of such orders factorizes as the product $N(0,i)N(i,n)$, i.e., equality holds in (1.7) in this case.

We call such a vertex a *post*, and obtain, letting $i_1, i_2, \ldots, i_m$ be the successive posts in $\{1, \ldots, n\}$, a decomposition of $N_{n,p}$ as a product of $N(i_j, i_{j+1})$ (with $i_0 = 1$ and $i_{m+1} = n$). The number of factors is random, and we shall show later that the different factors are independent. However, before we can make such a statement, it is necessary to show that, almost surely, there do exist posts in the $G_{\mathbf{N},p}$ order, indeed infinitely many of them.

Lemma 3.1. *For every $0 < p < 1$ there is a constant $C = C(p) > 1$ such that, for every sufficiently large k, the probability that none of the k elements $2k, 4k, 6k, \ldots 2k^2$ are posts in $G_{\mathbf{Z},p}$ is at most C^{-k}.*

Proof. Observe first that, for $j \ge 1$,

$$\mathrm{P}(j \text{ comparable with } 0 \mid 1, \ldots, j-1 \text{ comparable with } 0) = 1 - q^j,$$

so the probability that 0 is comparable with all elements $1, 2, \ldots$ is

$$\prod_{j=1}^{\infty} (1 - q^j) = \kappa(p).$$

The same argument holds independently for the negative elements, so the probability that 0 (or, by symmetry, any other element) is a post is $\kappa(p)^2$.

The event that none of the k elements $2k, 4k, 6k, \ldots 2k^2$ are posts in $G_{\mathbf{Z},p}$ is equivalent to the event that, for each of the elements $2jk$ $(j = 1, \ldots, k)$, there is an element n_j of $\mathbf{Z}$ incomparable with $2jk$ in the $G_{\mathbf{Z},p}$ partial order. We consider two events whose union contains this event, and show that each of the two events has small probability.

(i) For each j, there is an element n_j in the interval $[(2j-1)k, (2j+1)k]$ incomparable with $2jk$ in the partial order.

This event is the conjunction of k independent events, since the existence of a suitable element n_j depends only on those edges with both endpoints in the interval $[(2j-1)k, (2j+1)k]$. Each of these events has probability at most the probability that $2jk$ is not a post, which is $1-\kappa(p)^2$. Hence the probability of this case is at most $(1-\kappa(p)^2)^k$.

(ii) For some j, every element in the interval $[(2j-1)k, (2j+1)k]$ is comparable with $2jk$ in the partial order, but there is an element n_j outside this interval incomparable with $2jk$.

The probability of this event is at most $2k$ times the probability that 0 is comparable with every element $1, 2, \ldots, k$, but is incomparable with some element beyond k. This probability is at most

$$\sum_{j=k}^{\infty} \mathrm{P}(j+1 \text{ incomparable with } 0 \mid 1, \ldots, j \text{ comparable with } 0) = \sum_{j=k}^{\infty} q^{j+1} = \frac{q^{k+1}}{p}.$$

Combining the probabilities of events (i) and (ii), we see that

$$\mathrm{P}(\text{none of } 2k, 4k, \ldots 2k^2 \text{ is a post}) \leq (1-\kappa(p)^2)^k + 2kq^{k+1}/p.$$

Taking $C(p)$ to be any constant such that $C^{-1} > \max\{1-\kappa(p)^2, q\}$, the result follows. □

Equipped with this lemma, we are now in a position to prove a fair amount about the sequence of posts and the structure of the random partial order.

Lemma 3.2. *The posts form a.s. a two-way infinite sequence, and the indicator variables $I(i$ is a post$)$, $-\infty < i < \infty$, form a stationary sequence.*

Proof. It is obvious that the sequence $I(i$ is a post$)$ is stationary. Consequently, the events $\{i$ is the first post$\}$ have the same probability; since these events are disjoint, the probability has to be 0. Hence the sequence of posts has a.s. no first element. Similarly, there is a.s. no last element, so the sequence of posts is either empty or two-way infinite.

But Lemma 3.1 tells us that the probability that there is no post is less than C^{-k} for every k, and hence is 0. Therefore the sequence of posts is a.s. two-way infinite. □

Let $\ldots, U_{-1}, U_0, U_1, \ldots$ denote the random variables giving the positions of the infinite sequence of posts, with (to be definite) U_0 being the first post at or to the right of 0. We call the posets induced on the intervals $[U_j, U_{j+1}]$ the *segments* of the partial order. Note that segments overlap at the posts, but that each edge is in at most one segment.

Lemma 3.3. *The various distances $U_{j+1} - U_j$, $j \geq 0$, are mutually independent, identically distributed, random variables. In particular the events $\{m$ is a post$\}$ are recurrent.*

Given the set of posts, the distribution of the edges is as follows: The set of edges inside each segment has a distribution depending only on the length of the segment (ignoring an obvious translation); the sets of edges inside different segments are independent; the edges that do not lie in a segment occur with probability p, independently of each other and of the edges inside segments.

Proof. We prove that the distances between successive posts are mutually independent. By translation invariance, it suffices to prove that, if $\mathcal{E}$ is any event depending only on the set of edges one of whose end-points is to the left of 0, then

$$\begin{aligned}\mathrm{P}(&m \text{ is the first post to the right of } 0 \mid \mathcal{E} \text{ and } 0 \text{ is a post}) \\ &= \mathrm{P}(m \text{ is the first post to the right of } 0 \mid 0 \text{ is a post}), \quad (3.1)\end{aligned}$$

since this implies that the distribution of U_1 ($= U_1 - U_0$) is independent of the distribution of the posts to the left of $U_0 = 0$. But (3.1) holds, since an element m to the right of the post at 0 is itself a post iff there is a directed path from m to every non-negative element of $\mathbf{Z}$ (since other elements then have a directed path to m via 0), and this last event is independent of $\mathcal{E}$.

The remaining assertions are immediate. □

Remark 3.1. Note that, as for all stationary recurrent sequences, the distribution of $U_0 - U_{-1}$, the length of the segment containing 0 and -1, differs from the distribution of $U_1 - U_0$ ("the waiting time paradox"). See (3.3) for the precise relation.

Lemma 3.3 implies that, if $(X, <)$ is a finite poset with a minimal and a maximal element, but no other elements comparable to every element of X, then the probability that the segment between U_j and U_{j+1} is isomorphic to $(X, <)$ is independent of the nature of any other segment. Hence we can consider the random $G_{n,p}$ partial order as built up by successively choosing segments, independently, from some fixed distribution, and amalgamating them at the successive posts. (This picture is in fact slightly distorted by edge effects at both ends.)

Before completing the proof of Theorem 2, we need a moment condition on the lengths of the segments.

Lemma 3.4. *Let L be the random variable describing the length of the segments:*

$$\mathrm{P}(L = l) = \mathrm{P}(U_1 = l \mid U_0 = 0), \quad l \geq 1.$$

Then $\mathrm{E}\, L^r < \infty$ *for every* $r < \infty$.

Similarly, if $\tilde{L} = U_0 - U_{-1}$ *is the length of the segment containing* 0 *and* -1, $\mathrm{E}\, \tilde{L}^r < \infty$ *for every* $r < \infty$.

Proof. Lemma 3.1 gives us that, for some $C = C(p) > 1$ and sufficiently large k,

$$\begin{aligned} &\mathrm{P}(\text{none of } 2k, 4k, \dots, 2k^2 \text{ is a post} \mid 0 \text{ is a post}) \\ &\qquad \leq \mathrm{P}(\text{none of } 2k, 4k, \dots, 2k^2 \text{ is a post}) / \mathrm{P}(0 \text{ is a post}) \leq C^{-k}/\kappa(p)^2. \end{aligned}$$

Hence the probability that L is greater than $2k^2$ is at most C_1^{-k} for some $C_1(p) > 1$ and sufficiently large k, which yields, for some $C_2(p) > 1$,

$$P(L > l) \leq C_2^{-\sqrt{l}}, \tag{3.2}$$

and the claim follows.

For $\tilde{L}$ we observe that

$$\begin{aligned} \mathrm{P}(\tilde{L} = l) &= \sum_1^l \mathrm{P}(-j \text{ is a post and } l - j \text{ is the next}) \\ &= l\, \mathrm{P}(0 \text{ is a post and } l \text{ is the next}) \\ &= l \kappa(p)^2\, \mathrm{P}(L = l). \end{aligned} \tag{3.3}$$

Hence also $\mathrm{E}\, \tilde{L}^r < \infty$. □

Remark 3.2. Summing (3.3) over l, we obtain $\mathrm{E}\, L = \kappa(p)^{-2}$.

Remark 3.3. Using a different argument, we can improve (3.2) to show that

$$\mathrm{P}(L > l) \le c_1 e^{-c_2 l/\log l},$$

but we do not know whether this can be improved to an exponential bound.

We may now complete the proof of Theorem 2. Let $V_i = \log(N(0, U_i))$, $i \ge 0$. Further, let $W_0 = U_0$, $Z_0 = V_0$, and, for $i \ge 1$, $W_i = U_i - U_{i-1}$, $Z_i = V_i - V_{i-1} = \log(N(U_{i-1}, U_i))$, where the last equality follows because U_{i-1} is a post. It follows from Lemma 3.3 that the two-dimensional random variables $(W_i, Z_i), i \ge 0$, are independent, and that they have the same distribution for $i \ge 1$. By Lemma 3.4, for any $r < \infty$, $EW_i^r = EL^r < \infty$ for $i \ge 1$, and $EW_0^r \le E\tilde{L}^r < \infty$. Furthermore, $0 \le Z_i \le \log(W_i!) \le W_i \log(W_i)$, so $EZ_i^r < \infty$, $r < \infty, i \ge 0$.

We also observe that the events $\{W_i = 3, Z_i = \log 1\}$ and $\{W_i = 3, Z_i = \log 2\}$ both have positive probability, so W_i and Z_i are not proportional and $\gamma^2 = \mathrm{Var}(\mathrm{E}(W_1)Z_1 - \mathrm{E}(Z_1)W_1) > 0$.

Define $\tau(n) = \min\{i : U_i > n\}$, the index of the first post after n. We have placed ourselves in the setting of Gut [8, Section IV.2], except for the minor complication that here we have variables (W_0, Z_0) with a distribution different from $(W_i, Z_i), i \ge 1$. Theorem IV.2.3 in [8] is easily generalized to this case by standard arguments which we omit; the theorem then yields

$$\frac{V_{\tau(n)} - \mu n}{\sqrt{n}} \xrightarrow{d} N(0, \sigma^2), \quad \text{as} \quad n \to \infty, \tag{3.4}$$

with convergence of all moments, where $\mu = \mathrm{E}(Z_1)/\,\mathrm{E}(W_1)$ and $\sigma^2 = \gamma^2(\mathrm{E}\,W_1)^{-3} > 0$.

Since $V_{\tau(n)-1} \le \log(N_{n,p}) = \log(N(0,n)) \le V_{\tau(n)}$, $|\log(N_{n,p}) - V_{\tau(n)}| \le Z_{\tau(n)} \le \tilde{L}_n \log \tilde{L}_n$, where $\tilde{L}_n$ is the length of the segment containing n and $n+1$. Since $\tilde{L}_n \overset{\mathrm{d}}{=} \tilde{L}$ for all n, Lemma 3.4 and Cramér's theorem implies that we can replace $V_{\tau(n)}$ by $\log(N_{n,p})$ in (3.4), and the theorem follows. □

Remark 3.4. Say that a parameter f of partial orders is *additive* if whenever the partially ordered set P is the linear sum of P_1 and P_2, i.e. P is obtained by putting all of P_2 above P_1, then $f(P) = f(P_1) + f(P_2)$. Note that the logarithm of the number of linear extensions is an example of an additive parameter. Our proof of Theorem 2 shows also that other additive parameters of $G_{n,p}$, for example the height and the number of incomparable pairs, have asymptotic normal distributions, see [7, Theorem 8].

4. Proof of Theorem 3.

We continue the argument in Section 2, introducing $Q_n = \frac{N_n}{N_{n-1}}$, $n \ge 1$, (with $N_0 = 1$). If m is the last post before n, $Q_n = N(0,n)/N(0,n-1)$ depends only on the set of edges $\{ij : m \le i < j \le n\}$, and it follows easily that Q_n converges in distribution to some variable Q_∞. Using (2.4) and Jensen's (or Cauchy's) inequality, we obtain

$$Q_n^2 \le \frac{1}{N_{n-1}} \sum_1^{N_{n-1}} (Y_i^{(n)})^2, \tag{4.1}$$

and thus

$$\mathrm{E}\,Q_n^2 \le \mathrm{E}(Y \wedge n)^2 \le \mathrm{E}\,Y^2 < \infty,$$

which implies that $\{Q_n\}$ and $\{\log Q_n\}$ are uniformly integrable and thus

$$\mathrm{E}\,Q_\infty = \lim_{n\to\infty} \mathrm{E}\,Q_n = \lim_{n\to\infty} \mathrm{E}(Y \wedge n) = \mathrm{E}\,Y = \frac{1}{p}, \tag{4.2}$$

$$\mathrm{E}\log(N_n) - \mathrm{E}\log(N_{n-1}) = \mathrm{E}\log(Q_n) \to \mathrm{E}\log(Q_\infty), \tag{4.3}$$

which yields

$$\mu = \lim_{n\to\infty} \frac{1}{n} \operatorname{E}\log(N_n) = \operatorname{E}\log(Q_\infty). \tag{4.4}$$

By (2.4) and Jensen's inequality again,

$$\log(Q_n) \geq \frac{1}{N_{n-1}} \sum_{1}^{N_{n-1}} \log(Y_i^{(n)}), \tag{4.5}$$

and thus $\operatorname{E}\log(Q_n|\mathcal{F}_{n-1}) \geq \operatorname{E}\log(Y \wedge n)$, which yields

$$\mu = \operatorname{E}\log(Q_\infty) \geq \operatorname{E}\log(Y). \tag{4.6}$$

Finally we observe that $Q_n = 1$ if and only if every i with $1 \leq i < n$ may be connected to n by a directed path (cf. the definition of posts in Section 3). Hence

$$\operatorname{P}(Q_\infty = 1) = \lim_{n\to\infty} \operatorname{P}(Q_n = 1) = \lim_{n\to\infty} \prod_{1}^{n-1} (1 - q^{n-i}) = \kappa(p), \tag{4.7}$$

and, by Jensen's inequality,

$$\begin{aligned} \operatorname{E}\log Q_\infty &= \operatorname{P}(Q_\infty \neq 1)\operatorname{E}(\log Q_\infty \mid Q_\infty \neq 1) \\ &\leq \operatorname{P}(Q_\infty \neq 1) \log \operatorname{E}(Q_\infty \mid Q_\infty \neq 1) \\ &= (1-\kappa(p)) \log \frac{\operatorname{E} Q_\infty - \kappa(p)}{1-\kappa(p)}. \end{aligned} \tag{4.8}$$

The theorem follows by (4.4), (4.6), (4.8), and the concavity of the logarithm.

5. Proof of Theorems 5 and 6.

Our proofs of these results are based on the following theorem, which in turn is based upon martingale inequalities due to Azuma [2] and Hoeffding [9]. For further details, the reader is referred to the articles by Bollobás [4], [5] or McDiarmid [11].

Theorem A. *Suppose that $H_1 \cup H_2 \cup \cdots \cup H_m$ is a partition of $[n]^{(2)}$ into m parts. Let $Z(G)$ be a random variable depending on the random graph $G = G_{n,p}$ with vertex set $[n]$, such that $|Z(G) - Z(G')| \leq h$ whenever G and G' differ only on one of the H_i. Then, for any real a, we have*

$$\operatorname{P}\left(|Z(G_{n,p}) - \operatorname{E}Z(G_{n,p})| > a\right) \leq 2\exp(-a^2/2mh^2). \qquad \square$$

Ideally, we would like to apply Theorem A to the random variable $\log N(G)$, with $m = n$ and H_j being the set of pairs (i,j) with $i < j$. The condition $|Z(G) - Z(G')| \leq h$ then amounts to saying that $\log N(G)$ does not change much when the edges "downward" from one vertex are changed. Unfortunately this is not quite true: in the worst case the addition of the single edge between the two middle vertices can change $N(G)$ from $\binom{n}{n/2}$ to 1. Our solution, as is often the case in applications of Theorem A, is to consider a slightly modified random variable in place of $\log N(G)$.

For G a graph with vertex set $[n]$, and w a positive real number, let $N^*(G,w)$ be the number of linear extensions $\prec$ of G in which $j \prec k$ whenever $j < k - 2w$. In other words, $N^*(G,w)$ is the number of linear extensions of the the graph G^* obtained from G by adding an edge between vertices j and k whenever $j < k - 2w$.

We now apply Theorem A to $Z(G) = \log N^*(G,w)$.

Lemma 5.1. *Let w and a be positive real numbers. Then*

$$\mathrm{P}\left(|\log N^*(G_{n,p},w) - \mathrm{E}\log N^*(G_{n,p},w)| > a\right) \le 2\exp\left(-\frac{a^2}{16nw^2}\right).$$

Proof. As indicated, we apply Theorem A with $m = n$, $Z(G) = \log N^*(G,w)$, and $H_j = \{(i,j) : i < j\}$, for $j = 1,\dots,n$.

We wish to bound $|Z(G) - Z(G')|$, under the assumption that G and G' differ only on the pairs in H_j. Clearly we may assume that G contains none of the edges from j to the vertices in $\{j - \lfloor 2w \rfloor, \dots j-1\}$, whilst G' contains all of them. Then, given a linear extension $\prec$ of G^*, we can convert it to a linear extension $\prec'$ of $(G')^*$ by putting all the vertices $1,\dots,j-1$ below all the vertices $j,\dots n$, with the ordering inside each of the two subsets $\{1,\dots,j-1\}$ and $\{j,\dots,n\}$ the same in $\prec'$ as in $\prec$. Under this map, at most $\binom{2\lfloor 2w \rfloor}{\lfloor 2w \rfloor}$ linear extensions $\prec$ of G^* are mapped onto any linear extension $\prec'$ of $(G')^*$, since $\prec$ can be determined from $\prec'$ by specifying the relative positions of the vertices $j - \lfloor 2w \rfloor, \dots, j + \lfloor 2w \rfloor - 1$. Hence $N^*(G,w) \le N^*(G',w)\binom{2\lfloor 2w \rfloor}{\lfloor 2w \rfloor}$, and so $Z(G) - Z(G') \le \log \binom{2\lfloor 2w \rfloor}{\lfloor 2w \rfloor} \le 4w\log 2 < 2\sqrt{2}w$.

The result now follows from Theorem A. □

In order to make use of Lemma 5.1, we need to show that, for an appropriate choice of $w = w(n,p)$, the random variable $\log N^*(G,w)$ is not too different from $\log N(G)$. Let us first make the following easy observation.

Lemma 5.2. *Let p be any constant. Then there is a constant $C(p)$ such that, for all n,*

$$\mathrm{P}\left(N(G_{n,p}) \ne N^*(G_{n,p},w)\right) \le n^{-2},$$

where $w = \lceil C(p) + 3\log n/2p^2 \rceil$.

Proof. If j and k are any vertices of G, the probability that j and k are not related in the induced partial order is at most $(1-p^2)^{|j-k|-1}$, since that is the probability that no vertex between j and k is adjacent to both. Hence the probability that some pair (j,k) with $k - j > 2w$ is not related is at most

$$\begin{aligned} n\sum_{i=2w}^{\infty}(1-p^2)^i &= \frac{n}{p^2}(1-p^2)^{2w} \\ &\le \frac{n}{p^2}(1-p^2)^{2C(p)}\left[(1-p^2)^{1/p^2}\right]^{3\log n} \\ &\le \frac{n}{p^2}(1-p^2)^{2C(p)}n^{-3}. \end{aligned}$$

For a suitable value of $C(p)$, this is the required result. □

In order to prove Theorem 6, we shall need some results which are rather stronger than Lemma 5.2, to get the probability of failure down to the required level. However, Lemma 5.2 is already sufficient for us to prove Theorem 5.

Proof of Theorem 5. For p fixed, take w as in Lemma 5.2, and n large enough so that $w \le 2\log n/p^2$. Now set $a = 12n^{1/2}(\log n)^{3/2}p^{-2}$. Lemma 5.1 tells us that

$$\mathrm{P}\left(|\log N^*(G_{n,p},w) - \mathrm{E}\log N^*(G_{n,p},w)| > a\right) \le 2n^{-9/4}$$

for every such n, and Lemma 5.2 gives

$$\mathrm{P}\left(N(G_{n,p}) \neq N^*(G_{n,p}, w)\right) \leq n^{-2}.$$

Since $\log N(G) \leq n \log n$ for every G, this implies that

$$\mathrm{E}(\log N(G_{n,p}) - \log N^*(G_{n,p}, w)) \leq \log n/n.$$

Combining these, we see that

$$\mathrm{P}\left(|\log N(G_{n,p}) - \mathrm{E}\log N(G_{n,p})| > 13n^{1/2}(\log n)^{3/2}p^{-2}\right) \leq 2n^{-2},$$

for n sufficiently large.

If we now consider the infinite random graph $G_{\mathbf{N},p}$, the Borel–Cantelli lemma implies that, almost surely, there is an $n_1 = n_1(G,p)$ such that

$$|\log N(G_{n,p}) - \mathrm{E}\log N(G_{n,p})| \leq 13n^{1/2}(\log n)^{3/2}p^{-2}$$

for all $n \geq n_1$.

To complete the proof, we require some bounds on the convergence of $\frac{1}{n}\mathrm{E}\log N(G_{n,p})$ to its limit $\mu(p)$. We obtain such bounds by relating $N_{2n,p}$ to $N_{n,p}$.

For G a graph with vertex set $[2n]$, let G_1 be the graph $G|_{[n]}$ and G_2 be the graph $G|_{\{n+1,\dots,2n\}}$. Also let $A(G)$ denote the average, over all linear extensions $\prec_1$ and $\prec_2$ of G_1 and G_2 respectively, of the number of linear extensions of G extending both $\prec_1$ and $\prec_2$. Then we have

$$\begin{aligned} N(G) &= \sum_{\prec_1}\sum_{\prec_2}(\text{number of linear extensions of } G \text{ extending both } \prec_1 \text{ and } \prec_2) \\ &= N(G_1)N(G_2)A(G) \end{aligned}$$

and

$$\begin{aligned} \mathrm{E}\log N_{2n,p} &= 2\mathrm{E}\log N_{n,p} + \mathrm{E}\log A(G_{n,p}) \\ &\leq 2\mathrm{E}\log N_{n,p} + \log(\mathrm{E}A(G_{n,p})). \end{aligned}$$

The expectation of $A(G_{n,p})$ is just the expected number of linear extensions of G extending any pair $\prec_1$ and $\prec_2$ of linear extensions of the two halves. It was observed in [6] that this expectation increases to $\prod_{i=1}^{\infty}(1-q^i)^{-1} = 1/\kappa(p)$, as $n \to \infty$; moreover $A(G) \geq 1$.

This means that $\mu_n = \frac{1}{n}\mathrm{E}\log N_{n,p}$ satisfies:

$$\begin{aligned} 2n\mu_n \leq 2n\mu_{2n} &\leq 2n\mu_n + \log(1/\kappa(p)) \\ \mu_n \leq \mu_{2n} &\leq \mu_n + \frac{\log(1/\kappa(p))}{2n}. \end{aligned}$$

Hence, summing terms $\mu_{2^k n} - \mu_{2^{k-1}n}$, we have

$$\mu(p) \geq \mu_n \geq \mu(p) - \frac{1}{n}\log(1/\kappa(p)). \tag{5.1}$$

We now have that, for almost every $G_{\mathbf{N},p}$, there is an $n_2(G,p)$ such that

$$|\log N(G_{n,p}) - n\mu(p)| \leq 14n^{1/2}(\log n)^{3/2}p^{-2}$$

for all $n \geq n_2$. This implies the result. □

We now turn our attention to completing the proof of Theorem 6. Recall that we wish to relate $N(G)$ to $N^*(G,w)$, for suitable w, and that we require our estimates to be correct with extremely high probability.

From now on, we set $v = \lceil 4\log n/\log(1/q)\rceil$ and $w = 4v/p$.

Let **P** and **Q** be the following properties of graphs with vertex set $[n]$ $(n \geq 2)$.

P: There is a set S of order at most $n^{1/2}/\log n$ such that any two disjoint sets of vertices of size v, not intersecting S, have an edge between them.

Q: There is a set T of order at most $2n^{1/2}/\log n$ such that (i) for every vertex j of G, not in T, with $j \leq n-w$, there are at least $2v$ neighbours of j among the vertices $j+1,\dots,j+\lceil w\rceil$, and (ii) for every vertex j of G, not in T, with $j > w$, there are at least $2v$ neighbours of j among the vertices $j-\lceil w\rceil,\dots,j-1$.

Lemma 5.3. (i) *The probability that $G_{n,p}$ fails to have property* **P** *is at most* $\exp(-n^{1/2})$.
(ii) *The probability that $G_{n,p}$ fails to have property* **Q** *is at most* $2\exp(-n^{1/2})$.

Proof. (i). We construct S by the following procedure. If the graph contains disjoint sets A_1, A_2, both of size v, without an edge between them, remove them from the graph, add the vertices in $A_1 \cup A_2$ to S, and repeat. If the procedure halts after at most $n^{1/2}/2v\log n$ steps, then the original graph clearly has property **P**. If not, then the graph contains at least $n^{1/2}/2v\log n$ vertex disjoint pairs of sets of size v, with no edges between any of the pairs of sets. The probability of this is at most

$$\binom{n}{v}^{n^{1/2}/v\log n}\left(q^{v^2}\right)^{n^{1/2}/2v\log n} \leq \exp\left[\frac{n^{1/2}}{2\log n}\left(2\log n - v\log 1/q\right)\right] \leq \exp\left(-n^{1/2}\right),$$

as desired.

(ii). Let us take any vertex $j \leq n-w$, and estimate the probability that it fails to have at least $2v$ neighbours in $\{j+1,\dots,j+\lceil w\rceil\}$. Applying the Chernoff bound $\mathrm{P}\left(\mathrm{Bi}(N,p) \leq Np(1-\epsilon)\right) \leq \exp(-\epsilon^2 pN/2)$, we see that this is at most $e^{-pw/8}$, since $2v \leq p\lceil w\rceil/2$. Since $\log(1/q) \leq 1$, we have $w \geq 16\log n/p$, and so the probability that j is "bad" is at most n^{-2}. Hence the probability that more than $n^{1/2}/\log n$ vertices fail to have $2v$ neighbours as required is at most

$$\binom{n}{\lceil n^{1/2}/\log n\rceil}(n^{-2})^{\lceil n^{1/2}/\log n\rceil} \leq (n^{-1})^{n^{1/2}/\log n} = e^{-n^{1/2}}.$$

Similarly the probability that more than $n^{1/2}/\log n$ vertices $j \geq w$ fail to have at least $2v$ neighbours in $\{j-\lceil w\rceil,\dots,j-1\}$ is also at most $e^{-n^{1/2}}$, and the result follows. □

Lemma 5.4. *Let G be a graph on $[n]$ with properties* **P** *and* **Q**. *Then there is a set U of at most $11n^{1/2}/p\log n$ vertices such that any two vertices j,k not in U with $|j-k| > 2\lceil w\rceil$ are comparable in the induced partial order.*

Proof. Let S and T be sets as in the definitions of properties **P** and **Q** respectively. Set $V = \{j :$ there are more than v vertices of S in either the set $\{j+1,\dots,j+\lceil w\rceil\}$ or the set $\{j-\lceil w\rceil,\dots,j-1\}\}$, and set $U = T \cup V$. Note that $|V| \leq 2\lceil w\rceil|S|/v \leq 9|S|/p$. So

$$|U| \leq |T| + 9|S|/p \leq \frac{n^{1/2}}{\log n}(2+9/p) \leq \frac{11n^{1/2}}{p\log n}.$$

Suppose that neither j nor k is in U, and that $k-j>2\lceil w\rceil$. Then j is adjacent to at least $2v$ vertices among $\{j+1,\dots,j+\lceil w\rceil\}$, and at most v of these vertices are in S. Let A be a set of v neighbours of j in $\{j+1,\dots,j+\lceil w\rceil\}\setminus S$. Similarly let B be a set of v neighbours of k in $\{k-\lceil w\rceil,\dots,k-1\}\setminus S$. Then there is an edge between some vertex x of A and some vertex y of B. Thus we have $j<x<y<k$ in the partial order induced by G. □

Lemma 5.5. *Suppose that G satisfies properties* **P** *and* **Q**. *Then $N(G)/N^*(G,w)\le \exp(11n^{1/2}/p)$.*

Proof. Let U be a set of vertices as given by Lemma 5.4. Note that G and G^* induce the same partial order on $\overline{U}=[n]\setminus U$. Hence for any linear extension $\prec$ of G, there is a linear extension $\prec^*$ of G^* extending $\prec|_{\overline{U}}$. Any such map, taking $\prec$ to $\prec^*$, is such that at most $n^{|U|}$ linear extensions $\prec$ are mapped onto any one linear extension $\prec^*$, since $\prec$ is determined by $\prec^*$ and the positions of the vertices in U. The result now follows. □

Proof of Theorem 6. First recall Lemma 5.1, which says that

$$\mathrm{P}\left(|\log N^*(G_{n,p},w)-\mathrm{E}\log N^*(G_{n,p},w)|>a\right)\le 2\exp\left(-\frac{a^2}{16nw^2}\right) \tag{5.2}$$

for every positive real a,w.

If G satisfies properties **P** and **Q**, then, by Lemma 5.5, we have $0\le \log N(G)-\log N^*(G,w)\le 11n^{1/2}/p$. By Lemma 5.3, the probability that $G_{n,p}$ fails to satisfy properties **P** and **Q** is at most $3\exp(-n^{1/2})$; hence

$$\mathrm{P}\big(\log N(G_{n,p})-\log N^*(G_{n,p},w)>11n^{1/2}/p\big)\le 3\exp(-n^{1/2}). \tag{5.3}$$

Since $N(G)\le n!$ for every G, this also implies that

$$\mathrm{E}\big(\log N(G_{n,p})-\log N^*(G_{n,p},w)\big)\le 11n^{1/2}/p+3\exp(-n^{1/2})n\log n\le 12n^{1/2}/p. \tag{5.4}$$

Recall that $|\mathrm{E}\log N(G_{n,p})-n\mu(p)|\le \log(1/\kappa(p))$ by (5.1); moreover, rather crudely,

$$\log\big(1/\kappa(p)\big)=\sum_{k=1}^{\infty}-\log(1-q^k)\le\sum_{k=1}^{\infty}\frac{q^k}{p}\le\frac{1}{p^2}. \tag{5.5}$$

Since the conclusion is trivially true for $\lambda\le 80$, we may assume $80\le\lambda\le n^{1/4}$. We set

$$a=\frac{\lambda\sqrt{n}w}{20}; \tag{5.6}$$

recall that $w=4\lceil 4\log n/\log(1/q)\rceil/p\le 16\log n/p^2+4/p$. Now, if $|\log N^*(G_{n,p},w)-\mathrm{E}\log N^*(G_{n,p},w)|\le a$ and $|\log N(G_{n,p})-\log N^*(G_{n,p},w)|\le 11n^{1/2}/p$, then by these estimates

$$\begin{aligned}
&|\log N(G_{n,p})-n\mu(p)|\\
&\quad\le|\log N(G_{n,p})-\log N^*(G_{n,p},w)|+|\log N^*(G_{n,p},w)-\mathrm{E}\log N^*(G_{n,p},w)|\\
&\qquad+|\mathrm{E}\big(\log N^*(G_{n,p},w)-\log N(G_{n,p})\big)|+|\mathrm{E}\log N(G_{n,p})-n\mu(p)|\\
&\quad\le 11n^{1/2}/p+a+12n^{1/2}/p+\log(1/\kappa(p))\le a+23n^{1/2}/p+1/p^2\\
&\quad\le\frac{\lambda\sqrt{n}\log n}{p^2}\Big(\frac{16}{20}+\frac{4p}{20\log n}+\frac{23p}{\lambda\log n}+\frac{1}{\lambda\sqrt{n}\log n}\Big)\le\frac{\lambda\sqrt{n}\log n}{p^2}.
\end{aligned}$$

Therefore, using (5.2), (5.6) and (5.3),

$$\begin{aligned}
\mathrm{P}\Big(|\log N(G_{n,p}) - n\mu(p)| > \frac{\lambda\sqrt{n}\log n}{p^2}\Big) \\
&\le \mathrm{P}\big(|\log N^*(G_{n,p},w) - \mathrm{E}\log N^*(G_{n,p},w)| > a\big) \\
&\qquad + \mathrm{P}\big(\log N(G_{n,p}) - \log N^*(G_{n,p},w) > 11n^{1/2}/p\big) \\
&\le 2\exp(-\lambda^2/6400) + 3\exp(-n^{1/2}) \le 3\exp(-\lambda^2/6400),
\end{aligned}$$

as desired. □

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