WRIGHT'S CONSTANTS IN GRAPH ENUMERATION AND BROWNIAN EXCURSION AREA

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Abstract. This is a collection of various results and formulae. The main purpose is to give explicit relations between the many different similar notations and definitions that have been used by various authors.

There are no new results.

This is an informal note, not intended for publication.

1. Graph enumeration

Let $C(n, q)$ be the number of connected graphs with n given (labelled) vertices and q edges. Recall Cayley's formula $C(n, n-1) = n^{n-2}$ for every $n \geq 1$. Wright [19] proved that for any fixed $k \geq -1$, we have the analoguous asymptotic formula

$$C(n, n+k) \sim \rho_k n^{n+(3k-1)/2} \qquad \text{as } n \to \infty, \tag{1}$$

for some constants ρ_k given by

$$\rho_k = \frac{2^{(1-3k)/2}\pi^{1/2}}{\Gamma(3k/2+1)}\sigma_k, \qquad k \geq -1, \tag{2}$$

with other constants σ_k given by $\sigma_{-1} = -1/2$, $\sigma_0 = 1/4$, $\sigma_1 = 5/16$, and the quadratic recursion relation

$$\sigma_{k+1} = \frac{3(k+1)}{2}\sigma_k + \sum_{j=1}^{k-1} \sigma_j \sigma_{k-j}, \qquad k \geq 1. \tag{3}$$

Note the equivalent recursion formula

$$\sigma_{k+1} = \frac{3k+2}{2}\sigma_k + \sum_{j=0}^{k} \sigma_j \sigma_{k-j}, \qquad k \geq -1. \tag{4}$$

Wright gives in the later paper [20] the same result in the form

$$\rho_k = \frac{2^{(1-5k)/2}3^k\pi^{1/2}(k-1)!}{\Gamma(3k/2)}d_k, \qquad k \geq 1, \tag{5}$$

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(although he now uses the notation $f_k = \rho_k$; we have further corrected a typo in [20, Theorem 2]), where $d_1 = 5/36$ and

$$d_{k+1} = d_k + \sum_{j=1}^{k-1} \frac{d_j d_{k-j}}{(k+1)\binom{k}{j}}, \qquad k \geq 1. \tag{6}$$

See also Bender, Canfield and McKay [3, Corollaries 1 and 2], which gives the result using the same d_k and further numbers w_k defined by $w_0 = \pi/\sqrt{6}$ and

$$w_k = \frac{(8/3)^{1/2}\pi(k-1)!}{\Gamma(3k/2)}\Bigl(\frac{27k}{8e}\Bigr)^{k/2} d_k, \qquad k \geq 1, \tag{7}$$

so that

$$\rho_k = \frac{3^{1/2}}{2\pi^{1/2}}\Bigl(\frac{e}{12k}\Bigr)^{k/2} w_k. \qquad k \geq 0. \tag{8}$$

(Wright [20] and Bender et al. [3] further consider extensions to the case $k \to \infty$, which does not interest us here.)

In the form

$$\rho_k = \frac{2^{-(5k+1)/2} 3^{k+1} \pi^{1/2} k!}{\Gamma(3k/2+1)} d_k, \tag{9}$$

(5) holds for all $k \geq 0$, with $d_0 = 1/6$.

Wright's two versions (2), (3) [19] and (5), (6) [20] are equivalent and we have the relation

$$\sigma_k = \Bigl(\frac{3}{2}\Bigr)^{k+1} k!\, d_k, \qquad k \geq 0. \tag{10}$$

Next, define c_k, $k \geq 1$, as in [9, §8]; c_k is the coefficient for the leading term in an expansion of the generating function for connected graphs (or multigraphs) with n vertices and $n+k$ edges. (Note that $c_k = c_{k0} = \hat{c}_{k0}$ in [9, §8]. c_k is denoted $c_{k,-3k}$ in Wright [19] and b_k in Wright [20].) We have by [20, §5] or comparing (6) and (13) below

$$c_k = \Bigl(\frac{3}{2}\Bigr)^{k} (k-1)!\, d_k, \qquad k \geq 1. \tag{11}$$

From [9, (8.12)] (which is equivalent to Wright [19, (7)]) follows the recursion

$$3rc_r = \tfrac{1}{2}(3r-1)(3r-3)c_{r-1} + \tfrac{9}{2}\sum_{j=0}^{r-1} j(r-1-j)c_j c_{r-1-j}, \qquad r \geq 1, \tag{12}$$

where jc_j is interpreted as $1/6$ when $j = 0$. In other words, $c_1 = 5/24$ and

$$rc_r = \tfrac{1}{2}r(3r-3)c_{r-1} + \tfrac{3}{2}\sum_{j=1}^{r-2} j(r-1-j)c_j c_{r-1-j}, \qquad r \geq 2. \tag{13}$$

By (10), (11) is equivalent to

$$\sigma_k = \frac{3}{2} k c_k, \qquad k \geq 1 \tag{14}$$

(and also for $k = 0$ with the interpretation $0c_0 = 1/6$ again).

By [9, §§3 and 8], (1) holds for $k \ge 1$ with

$$\rho_k = \frac{2^{(1-3k)/2}\sqrt{\pi}}{\Gamma(3k/2)} c_k, \qquad k \ge 1, \tag{15}$$

which clearly is equivalent to (2) and (5) by (14) and (11).

Finally, we note that (1) can be written

$$C(n, n+k-1) \sim \rho_{k-1} n^{n+3k/2-2} \qquad k \ge 0. \tag{16}$$

2. Brownian excursion area

Let B^{ex} denote a (normalized) Brownian excursion and

$$\mathcal{B} := \int_0^1 B^{\mathrm{ex}}(t)\,dt, \tag{17}$$

the Brownian excursion area. Two variants of this are

$$\mathcal{A} := 2^{3/2}\mathcal{B} \tag{18}$$

defined by Flajolet and Louchard [5], and by them called the *Airy distribution*, and

$$\xi := 2\mathcal{B} \tag{19}$$

used in [7, 4, 8]. (Louchard [11] uses ξ for our $\mathcal{B}$. Takács [13, 14, 15, 16, 17] uses ω^+ for our $\mathcal{B}$.)

The connection between Brownian excursion area and graph enumeration was found by Spencer [12], who gave a new proof of (16), and thus (1), that further shows

$$\rho_{k-1} = \frac{\mathbb{E}\,\mathcal{B}^k}{k!}, \qquad k \ge 0. \tag{20}$$

See also Aldous [1, §6].

Flajolet and Louchard [5] give the formula (further defining $\mu_k := \mathbb{E}\,\mathcal{A}^k$)

$$\mathbb{E}\,\mathcal{A}^k = \frac{2\sqrt{\pi}}{\Gamma((3k-1)/2)}\Omega_k, \qquad k \ge 0, \tag{21}$$

where they define Ω_k by $\Omega_0 := -1$ and the recursion

$$2\Omega_k = (3k-4)k\Omega_{k-1} + \sum_{j=1}^{k-1}\binom{k}{j}\Omega_j\Omega_{k-j}, \qquad k \ge 1. \tag{22}$$

In particular, $\Omega_1 = 1/2$. It is easily seen that (22) is equivalent to (4) and

$$\Omega_k = 2k!\,\sigma_{k-1}, \qquad k \ge 0. \tag{23}$$

Similarly, (21) is equivalent to (2) by (18), (20) and (23).

The first formula for the moments of $\mathcal{B}$ was given by Louchard [11] (using formulas in [10]), who showed (using β_n for $\mathbb{E}\,\mathcal{B}^n$)

$$\mathbb{E}\,\mathcal{B}^k = (36\sqrt{2})^{-k}\frac{2\sqrt{\pi}}{\Gamma((3k-1)/2)}\gamma_k, \qquad k \ge 0, \tag{24}$$

where γ_k satisfies (60) below. Clearly, (24) is equivalent to (21) and

$$\gamma_k = 18^k \Omega_k, \qquad k \geq 0. \tag{25}$$

It is easily seen from (22) that $2^k \Omega_k$ is an integer for all $k \geq 0$, and thus by (25), γ_k is an integer for $k \geq 0$.

Takács [13, 14, 15, 16, 17] give the formula (using M_r for $\mathbb{E}\,\mathcal{B}^r$)

$$\mathbb{E}\,\mathcal{B}^k = \frac{4\sqrt{\pi}\,2^{-k/2}k!}{\Gamma((3k-1)/2)} K_k, \qquad k \geq 0, \tag{26}$$

where $K_0 = -1/2$ and

$$K_k = \frac{3k-4}{4} K_{k-1} + \sum_{j=1}^{k-1} K_j K_{k-j}, \qquad k \geq 1. \tag{27}$$

It is easily seen that (26) and (27) are equivalent to (21) and (22) and the relation

$$K_k = \frac{\Omega_k}{2^{k+1}k!}, \qquad k \geq 0. \tag{28}$$

By (28) and (23) we further have

$$K_k = 2^{-k}\sigma_{k-1}, \qquad k \geq 0. \tag{29}$$

Flajolet, Poblete and Viola [6] define

$$\omega_k := \frac{\Omega_k}{k!}, \qquad k \geq 0, \tag{30}$$

$$\omega_k^* := 2^{2k-1}\omega_k = \frac{2^{2k-1}}{k!}\Omega_k, \qquad k \geq 0. \tag{31}$$

Note that ω_k^* is an integer for $k \geq 1$. The numbers ω_k^* are the same as ω_{k0}^* in Janson [7]. (This is a special case of ω_{kl}^* in [7], but we will only need the case $l = 0$.) The sequence (ω_k^*) is called the Wright–Louchard–Takács sequence in [6].

By (31), the recursion (22) translates to

$$\omega_k^* = 2(3k-4)\omega_{k-1}^* + \sum_{j=1}^{k-1} \omega_j^* \omega_{k-j}^*, \qquad k \geq 1; \tag{32}$$

with $\omega_0^* := -1/2$ and $\omega_1^* = 1$. By (23), or by comparing (32) and (4), it follows that

$$\omega_k^* = 2^{2k}\sigma_{k-1}, \qquad k \geq 0, \tag{33}$$

and thus also, by (31),

$$\omega_k = 2\sigma_{k-1}, \qquad k \geq 0. \tag{34}$$

By [7, Theorem 3.3],

$$\mathbb{E}\,\xi^k = \frac{2^{2-5k/2}\sqrt{\pi}\,k!}{\Gamma((3k-1)/2)}\omega_k^*, \qquad k \geq 0, \tag{35}$$

which is equivalent to (21) by (18), (19) and (31). By (38) below, this is further equivalent to the formula (which also follows by (20) and (15))

$$\mathbb{E}\,\xi^k = \frac{2^{1-k/2}3\sqrt{\pi}\,k!}{\Gamma((3k-1)/2)}(k-1)c_{k-1} = \frac{2^{2-k/2}\sqrt{\pi}\,k!}{\Gamma(3(k-1)/2)}c_{k-1}, \qquad k \geq 2, \quad (36)$$

as claimed in [8, Remark 2.5].

3. Further relations

Further relations are immediately obtained by combining the ones above; we give some examples.

By (23) and (14), or comparing (12) and (22), we find

$$\Omega_k = 3(k-1)k!\,c_{k-1}, \qquad k \geq 2. \quad (37)$$

By (33) and (14), or by (31) and (37), or by (12) and (32),

$$\omega_k^* = 2^{2k-1}\,3(k-1)c_{k-1}, \qquad k \geq 2. \quad (38)$$

By (28), (30) and (31) we further have

$$K_k = 2^{-k-1}\omega_k = 2^{-3k}\omega_k^*, \qquad k \geq 0. \quad (39)$$

By (2) and (33),

$$\rho_{k-1} = \frac{2^{2-3k/2}\sqrt{\pi}}{\Gamma((3k-1)/2)}\sigma_{k-1} = \frac{2^{2-7k/2}\sqrt{\pi}}{\Gamma((3k-1)/2)}\omega_k^*, \qquad k \geq 0. \quad (40)$$

Note further that (15) can be written

$$\rho_{k-1} = \frac{2^{2-3k/2}\sqrt{\pi}}{\Gamma(3(k-1)/2)}c_{k-1}, \qquad k \geq 2. \quad (41)$$

4. Asymptotics

Wright [20] proved that the limit $\lim_{k\to\infty} d_k$ exists, and gave the approximation 0.159155. The limit was later identified by Bagaev and Dmitriev, [2] as $1/(2\pi)$, i.e.

$$d_k \to \frac{1}{2\pi} \qquad \text{as } k \to \infty. \quad (42)$$

See [9, p. 262] for further history and references.

It follows by Stirling's formula that for w_k in (7),

$$w_k \to 1 \qquad \text{as } k \to \infty. \quad (43)$$

Hence, by (8)

$$\rho_k \sim \frac{3^{1/2}}{2\pi^{1/2}}\Bigl(\frac{e}{12k}\Bigr)^{k/2} \qquad \text{as } k \to \infty, \quad (44)$$

and, equivalently,

$$\rho_{k-1} \sim 3\pi^{-1/2}k^{1/2}\Bigl(\frac{e}{12k}\Bigr)^{k/2} \qquad \text{as } k \to \infty. \quad (45)$$

By (20) follows further, as stated in Takács [13, 14, 15, 16, 17]

$$\mathbb{E}\,\mathcal{B}^k \sim 3\sqrt{2}\,k\Big(\frac{k}{12e}\Big)^{k/2} \qquad \text{as } k\to\infty, \tag{46}$$

and equivalently, as stated by Flajolet and Louchard [5] and Chassaing and Janson [4], respectively,

$$\mathbb{E}\,\mathcal{A}^k \sim 2^{1/2}3k\Big(\frac{2k}{3e}\Big)^{k/2} \qquad \text{as } k\to\infty, \tag{47}$$

$$\mathbb{E}\,\xi^k \sim 2^{1/2}3k\Big(\frac{k}{3e}\Big)^{k/2} \qquad \text{as } k\to\infty. \tag{48}$$

By (11) and (42), or by [9, Theorem 8.2],

$$c_k \sim \frac{1}{2\pi}\Big(\frac{3}{2}\Big)^k(k-1)! \qquad \text{as } k\to\infty. \tag{49}$$

Further, by (10) and (42),

$$\sigma_k \sim \frac{1}{2\pi}\Big(\frac{3}{2}\Big)^{k+1}k! \qquad \text{as } k\to\infty, \tag{50}$$

and thus by (33), (34) and (30),

$$\omega_k^* \sim \frac{1}{2\pi}6^k(k-1)! \qquad \text{as } k\to\infty, \tag{51}$$

$$\omega_k \sim \frac{1}{\pi}\Big(\frac{3}{2}\Big)^k(k-1)! \qquad \text{as } k\to\infty, \tag{52}$$

$$\Omega_k \sim \frac{1}{\pi k}\Big(\frac{3}{2}\Big)^k(k!)^2 \sim 2\Big(\frac{3k^2}{2e^2}\Big)^k \qquad \text{as } k\to\infty. \tag{53}$$

Similarly by (50) and (29), as stated in Takács [13, 14, 15, 16, 17]

$$K_k \sim \frac{1}{2\pi}\Big(\frac{3}{4}\Big)^k(k-1)! \qquad \text{as } k\to\infty. \tag{54}$$

5. Numerical values

Numerical values for small k are given in Table 1. See further Louchard [11] ($\mathbb{E}\,\mathcal{B}^k$), Takács [13, 16, 17] ($\mathbb{E}\,\mathcal{B}^k$, K_k), Janson, Knuth, Łuczak and Pittel [9, p. 259 or 262] (c_k), Flajolet, Poblete and Viola [6, Table 1 and p. 503] ($\mathbb{E}\,\mathcal{A}^k$, Ω_k, ω_k, ω_k^*), Flajolet and Louchard [5, Table 1] ($\mathbb{E}\,\mathcal{A}^k$, Ω_k), and Janson [7, p. 343] (ω_k^*).

6. Power series

Define the formal power series

$$C(z) := \sum_{r=1}^{\infty} c_r z^r. \tag{55}$$

By [9, §8 and (7.2)], we then have

$$e^{C(z)} = \sum_{r=0}^{\infty} e_r z^r, \tag{56}$$

$\rho_{-1} = 1$	$\rho_0 = \frac{\sqrt{2\pi}}{4}$	$\rho_1 = \frac{5}{24}$	$\rho_2 = \frac{5\sqrt{2\pi}}{256}$	$\rho_3 = \frac{221}{24192}$
	$\mathbb{E}\,\mathcal{B} = \frac{\sqrt{2\pi}}{4}$	$\mathbb{E}\,\mathcal{B}^2 = \frac{5}{12}$	$\mathbb{E}\,\mathcal{B}^3 = \frac{15\sqrt{2\pi}}{128}$	$\mathbb{E}\,\mathcal{B}^4 = \frac{221}{1008}$
	$\mathbb{E}\,\mathcal{A} = \sqrt{\pi}$	$\mathbb{E}\,\mathcal{A}^2 = \frac{10}{3}$	$\mathbb{E}\,\mathcal{A}^3 = \frac{15\sqrt{\pi}}{4}$	$\mathbb{E}\,\mathcal{A}^4 = \frac{884}{63}$
$\sigma_{-1} = -\frac{1}{2}$	$\sigma_0 = \frac{1}{4}$	$\sigma_1 = \frac{5}{16}$	$\sigma_2 = \frac{15}{16}$	$\sigma_3 = \frac{1105}{256}$
	$d_0 = \frac{1}{6}$	$d_1 = \frac{5}{36}$	$d_2 = \frac{5}{36}$	$d_3 = \frac{1105}{7776}$
		$c_1 = \frac{5}{24}$	$c_2 = \frac{5}{16}$	$c_3 = \frac{1105}{1152}$
$\Omega_0 = -1$	$\Omega_1 = \frac{1}{2}$	$\Omega_2 = \frac{5}{4}$	$\Omega_3 = \frac{45}{4}$	$\Omega_4 = \frac{3315}{16}$
$\gamma_0 = -1$	$\gamma_1 = 18$	$\gamma_2 = 405$	$\gamma_3 = 65610$	$\gamma_4 = 21749715$
$K_0 = -\frac{1}{2}$	$K_1 = \frac{1}{8}$	$K_2 = \frac{5}{64}$	$K_3 = \frac{15}{128}$	$K_4 = \frac{1105}{4096}$
$\omega_0 = -1$	$\omega_1 = \frac{1}{2}$	$\omega_2 = \frac{5}{8}$	$\omega_3 = \frac{15}{8}$	$\omega_4 = \frac{1105}{128}$
$\omega_0^* = -\frac{1}{2}$	$\omega_1^* = 1$	$\omega_2^* = 5$	$\omega_3^* = 60$	$\omega_4^* = 1105$

TABLE 1. Some numerical values

where

$$\begin{aligned} e_r &:= \frac{(6r)!}{2^{5r}3^{2r}(3r)!\,(2r)!} = \Big(\frac{3}{2}\Big)^r \frac{\Gamma(r+5/6)\Gamma(r+1/6)}{2\pi r!} = \Big(\frac{3}{2}\Big)^r \frac{(5/6)^{\overline{r}}(1/6)^{\overline{r}}}{r!} \\ &= 18^{-r}\frac{\Gamma(3r+1/2)}{\Gamma(r+1/2)\,r!}. \end{aligned} \tag{57}$$

(The last formula follows by the triplication formula for the Gamma function, or by induction.) We have $e_0 = 1$, $e_1 = 5/24$, $e_2 = 385/1152$, $e_3 = 85085/82944$. The constants e_r are the coefficient for the leading term in an expansion of the generating function for all graphs (or multigraphs) with n vertices and $n+k$ edges, see [9, §7] (where e_r is denoted e_{r0}).

7. LINEAR RECURSIONS

From (56) follows the linear recursion, see [9, §8],

$$c_r = e_r - \frac{1}{r}\sum_{j=1}^{r-1} jc_je_{r-j}, \qquad r \geq 1, \tag{58}$$

where e_k are given explicitly by (57).

By (37) together with (57) and simple calculations, (58) is equivalent to

$$18^r\Omega_r = \frac{12r}{6r-1}\frac{\Gamma(3r+1/2)}{\Gamma(r+1/2)} - \sum_{j=1}^{r-1}\binom{r}{j}\frac{\Gamma(3j+1/2)}{\Gamma(j+1/2)}18^{r-j}\Omega_{r-j}, \qquad r \geq 1, \tag{59}$$

which is given by Flajolet and Louchard [5], and, equivalently, see (25),

$$\gamma_r = \frac{12r}{6r-1}\frac{\Gamma(3r+1/2)}{\Gamma(r+1/2)} - \sum_{j=1}^{r-1}\binom{r}{j}\frac{\Gamma(3j+1/2)}{\Gamma(j+1/2)}\gamma_{r-j}, \qquad r \geq 1, \tag{60}$$

which is given by Louchard [11]. Changing the upper summation limit we can also write these as

$$18^r\Omega_r = \frac{6r+1}{6r-1}\frac{\Gamma(3r+1/2)}{\Gamma(r+1/2)} - \sum_{j=1}^{r}\binom{r}{j}\frac{\Gamma(3j+1/2)}{\Gamma(j+1/2)}18^{r-j}\Omega_{r-j}, \quad r \geq 0, \tag{61}$$

$$\gamma_r = \frac{6r+1}{6r-1}\frac{\Gamma(3r+1/2)}{\Gamma(r+1/2)} - \sum_{j=1}^{r}\binom{r}{j}\frac{\Gamma(3j+1/2)}{\Gamma(j+1/2)}\gamma_{r-j}, \qquad r \geq 0. \tag{62}$$

By (28), these are further equivalent to the linear recursion in Takács [13, 14]

$$K_r = \frac{6r+1}{2(6r-1)}\alpha_r - \sum_{j=1}^{r}\alpha_j K_{r-j}, \qquad r \geq 1, \tag{63}$$

where, cf. (57),

$$\alpha_j := 36^{-j}\frac{\Gamma(3j+1/2)}{\Gamma(j+1/2)\,j!} = 2^{-j}e_j, \qquad j \geq 0. \tag{64}$$

8. Airy and Bessel functions

See [9, (8.14) and (8.15)] and Flajolet and Louchard [5].

9. Continued fractions

See [9, (8.15) and (8.16)].

10. Related results

Note the related formulas, involving related quadratic or linear recursions, for the moments of the integrals $\int_0^1 |B^{\mathrm{br}}(t)|\,dt$ and $\int_0^1 |B(t)|\,dt$ of absolute values of a Brownian bridge and Brownian motion, respectively, by Takács [16, 18], and for another functional of a Brownian excursion by Janson [7] (see also [4]).

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