SOME INTEGRALS RELATED TO THE GAMMA INTEGRAL

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Abstract. We collect, for easy reference, some formulas related to the Gamma integral.

We collect some formulas related to the Gamma integral. (None of the formulas is new.) See also e.g. [1, Chapter 6] and [2, Section 5.9], where further results are given. (Several of the formulas below appear in [2], but we do not give individual references.)

All integrals are absolutely convergent unless we explicitly say otherwise.

We begin with the standard definition (Euler's integral)

$$\Gamma(\alpha) := \int_0^\infty x^{\alpha-1} e^{-x}\,\mathrm{d}x, \qquad \operatorname{Re}\alpha > 0. \tag{1}$$

I. **Extensions to** $\operatorname{Re}\alpha < 0$**.** For $\operatorname{Re}\alpha < 0$, the integral in (1) does not converge, but if $\operatorname{Re}\alpha \notin \mathbb{Z}$ we have the modifications

$$\int_0^\infty \bigl(e^{-x} - 1\bigr) x^{\alpha-1}\,\mathrm{d}x = \Gamma(\alpha), \qquad -1 < \operatorname{Re}\alpha < 0, \tag{2}$$

$$\int_0^\infty \bigl(e^{-x} - 1 + x\bigr) x^{\alpha-1}\,\mathrm{d}x = \Gamma(\alpha), \qquad -2 < \operatorname{Re}\alpha < -1, \tag{3}$$

and, in general, for any integer $m \geqslant 0$,

$$\int_0^\infty \Bigl(e^{-x} - \sum_{k=0}^m \frac{(-x)^k}{k!}\Bigr) x^{\alpha-1}\,\mathrm{d}x = \Gamma(\alpha), \qquad -m-1 < \operatorname{Re}\alpha < -m. \tag{4}$$

Proof. Denote the integral in (4) by $I_{\alpha,m}$. Then an integration by parts gives

$$\alpha I_{\alpha,m} = \Bigl[\Bigl(e^{-x} - \sum_{k=0}^m \frac{(-x)^k}{k!}\Bigr) x^\alpha\Bigr]_0^\infty + I_{\alpha+1,m-1} = 0 + I_{\alpha+1,m-1}. \tag{5}$$

For $m = 0$ we have $-1 < \operatorname{Re}\alpha < 0$ and then

$$I_{\alpha+1,-1} = \int_0^\infty e^{-x} x^\alpha\,\mathrm{d}x = \Gamma(\alpha+1) = \alpha\Gamma(\alpha);$$

thus (4) for $m = 0$ follows from (5). (This is (2).) The general case now follows by (5) and induction. ∎

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Next we note the following extension of (2).

(6) $$\int_0^\infty \bigl(e^{-tx}-1\bigr)x^{\alpha-1}\,\mathrm{d}x = t^{-\alpha}\Gamma(\alpha), \qquad -1<\operatorname{Re}\alpha<0,\ \operatorname{Re}t\geqslant 0.$$

Proof. For $t>0$, this follows from (2) by a change of variables. The integral in (6) converges for $\operatorname{Re}t\geqslant 0$ and is a continuous function of t in this half-plane, analytic in the open half-plane $\operatorname{Re}t>0$. Hence the result follows by analytic continuation. □

II. sin **and** cos**.**

(7) $$\int_0^\infty x^{\alpha-1}\sin x\,\mathrm{d}x = \sin\frac{\pi\alpha}{2}\,\Gamma(\alpha), \qquad -1<\operatorname{Re}\alpha<0,$$

(8) $$\int_0^\infty x^{\alpha-1}(1-\cos x)\,\mathrm{d}x = -\cos\frac{\pi\alpha}{2}\,\Gamma(\alpha), \qquad -2<\operatorname{Re}\alpha<0,$$

Proof. For $-1<\operatorname{Re}\alpha<0$, these follow from (6) by taking $t=\pm\mathrm{i}$ and using Euler's formulas. (Alternatively, for real α, by taking $t=-\mathrm{i}$ and taking real and imaginary parts.) Then (8) extends to $\operatorname{Re}\alpha>-2$ by analytic continuation. □

In particular, taking $\alpha=-1$ in (8) yields the wellknown

(9) $$\int_0^\infty \frac{1-\cos x}{x^2}\,\mathrm{d}x = \frac{\pi}{2}.$$

In fact, (7) extends to $0\leqslant\operatorname{Re}\alpha<1$, although the integral no longer is absolutely convergent:

(10)
$$\int_0^\infty x^{\alpha-1}\sin x\,\mathrm{d}x := \lim_{A\to\infty}\int_0^A x^{\alpha-1}\sin x\,\mathrm{d}x = \sin\frac{\pi\alpha}{2}\,\Gamma(\alpha), \quad -1<\operatorname{Re}\alpha<1.$$

Proof. Integration by parts yields, using (8) and letting $A\to\infty$,

$$\begin{aligned}\int_0^A x^{\alpha-1}\sin x\,\mathrm{d}x &= \bigl[x^{\alpha-1}(1-\cos x)\bigr]_0^A-(\alpha-1)\int_0^A x^{\alpha-2}(1-\cos x)\,\mathrm{d}x\\ &\to 0+(\alpha-1)\cos\frac{\pi(\alpha-1)}{2}\,\Gamma(\alpha-1)=\sin\frac{\pi\alpha}{2}\,\Gamma(\alpha).\end{aligned}$$

□

In particular, taking $\alpha=0$ in (10) yields the conditionally convergent

(11) $$\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x := \lim_{A\to\infty}\int_0^A \frac{\sin x}{x}\,\mathrm{d}x = \frac{\pi}{2}.$$

There is also a corresponding conditionally convergent cosine integral, related to (8):

(12)
$$\int_0^\infty x^{\alpha-1}\cos x\,\mathrm{d}x := \lim_{A\to\infty}\int_0^A x^{\alpha-1}\cos x\,\mathrm{d}x = \cos\frac{\pi\alpha}{2}\,\Gamma(\alpha), \quad 0<\operatorname{Re}\alpha<1.$$

Proof. Integration by parts yields, using (7) and letting $A \to \infty$,

$$\int_0^A x^{\alpha-1} \cos x \,\mathrm{d}x = \bigl[x^{\alpha-1} \sin x\bigr]_0^A - (\alpha-1) \int_0^A x^{\alpha-2} \sin x \,\mathrm{d}x$$
$$\to 0 - (\alpha-1) \sin \frac{\pi(\alpha-1)}{2}\, \Gamma(\alpha-1) = \cos \frac{\pi\alpha}{2}\, \Gamma(\alpha).$$

□

Another formula is:

$$\int_0^\infty \bigl(e^{-ax} - 1 + a \sin x\bigr) x^{\alpha-1} \,\mathrm{d}x = \Bigl(a^{-\alpha} + a \sin \frac{\pi\alpha}{2}\Bigr)\Gamma(\alpha), \qquad -2 < \operatorname{Re}\alpha < 0,\ \operatorname{Re} a \geqslant 0. \tag{13}$$

Proof. If $-1 < \operatorname{Re}\alpha < 0$, this follows by (6) and (7). The general case follows by analytic continuation. □

In particular, taking $\alpha = -1$ in (13) yields

$$\int_0^\infty \bigl(e^{-ax} - 1 + a \sin x\bigr) x^{-2} \,\mathrm{d}x = a \log a, \qquad \operatorname{Re} a \geqslant 0. \tag{14}$$

Proof. If $f(a) := a^{-\alpha} + a \sin \frac{\pi\alpha}{2}$, then $f(-1) = 0$ and $f'(-1) = -a \log a$, and if $g(a) := 1/\Gamma(a) = a(a+1)/\Gamma(a+2)$, then $g(-1) = 0$ and $g'(-1) = -1$. The result follows by l'Hôpital's rule. □

III. **Subtracting on $[0,1]$ only.**

$$\int_0^1 \bigl(e^{-x} - 1\bigr) x^{\alpha-1} \,\mathrm{d}x + \int_1^\infty e^{-x} x^{\alpha-1} \,\mathrm{d}x = \int_0^\infty \bigl(e^{-x} - \mathbf{1}\{x<1\}\bigr) x^{\alpha-1} \,\mathrm{d}x = \Gamma(\alpha) - \alpha^{-1}, \qquad -1 < \operatorname{Re}\alpha. \tag{15}$$

Proof. For $\operatorname{Re}\alpha > 0$, this follows from (1). The general case $\operatorname{Re}\alpha > -1$ follows by analytic continuation. □

In particular, taking $\alpha = 0$,

$$\int_0^1 \frac{e^{-x} - 1}{x} \,\mathrm{d}x + \int_1^\infty \frac{e^{-x}}{x} \,\mathrm{d}x = \int_0^\infty \frac{e^{-x} - \mathbf{1}\{x<1\}}{x} \,\mathrm{d}x = -\gamma. \tag{16}$$

Proof. As $\alpha \to 0$,

$$\Gamma(\alpha) - \alpha^{-1} = \frac{\Gamma(\alpha+1) - 1}{\alpha} \to \Gamma'(1) = -\gamma.$$

□

We have also similar results with $\sin x$ and $\cos x$ in the integral.

$$\int_0^\infty x^{\alpha-1} \bigl(\sin x - x \mathbf{1}\{x<1\}\bigr) \,\mathrm{d}x = \sin \frac{\pi\alpha}{2}\, \Gamma(\alpha) - \frac{1}{\alpha+1}, \qquad -3 < \operatorname{Re}\alpha < 1. \tag{17}$$

Here the integral is absolutely convergent if $-3 < \operatorname{Re}\alpha < 0$, and otherwise conditionally convergent.

Proof. This follows from (7) when $-1 < \operatorname{Re}\alpha < 0$, and extends to $-3 < \operatorname{Re}\alpha < 0$ by analytic continuation (with absolutely convergent integrals). The case $-1 < \operatorname{Re}\alpha < 1$ follows similarly from (10). ☐

In particular, taking $\alpha = 0$, -1 and -2, cf. (11),

$$\int_0^\infty \frac{\sin x - x\mathbf{1}\{x<1\}}{x}\,\mathrm{d}x = \frac{\pi}{2} - 1, \tag{18}$$

$$\int_0^\infty \frac{\sin x - x\mathbf{1}\{x<1\}}{x^2}\,\mathrm{d}x = 1 - \gamma, \tag{19}$$

$$\int_0^\infty \frac{\sin x - x\mathbf{1}\{x<1\}}{x^3}\,\mathrm{d}x = 1 - \frac{\pi}{4}. \tag{20}$$

Proof. As $\varepsilon \to 0$,

$$\sin\frac{\pi\varepsilon}{2}\,\Gamma(\varepsilon) - \frac{1}{\varepsilon+1} \to \frac{\pi}{2} - 1,$$

$$\begin{aligned}
\sin\frac{\pi(\varepsilon-1)}{2}\,\Gamma(\varepsilon-1) - \frac{1}{\varepsilon} &= \frac{\sin\frac{\pi(\varepsilon-1)}{2}(\varepsilon-1)^{-1}\Gamma(\varepsilon+1) - 1}{\varepsilon} \\
&= \frac{\sin\frac{\pi(1-\varepsilon)}{2}(1-\varepsilon)^{-1}\Gamma(\varepsilon+1) - 1}{\varepsilon} \\
&\to \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Bigl(\sin\frac{\pi(1-\varepsilon)}{2}(1-\varepsilon)^{-1}\Gamma(\varepsilon+1)\Bigr)\Big|_{\varepsilon=0} \\
&= -\frac{\pi}{2}\cos\frac{\pi}{2} + 1 + \Gamma'(1) = 1 - \gamma,
\end{aligned}$$

and

$$\sin\frac{\pi(\varepsilon-2)}{2}\,\Gamma(\varepsilon-2) - \frac{1}{\varepsilon-1} = \frac{-\sin\frac{\pi\varepsilon}{2}\Gamma(\varepsilon+1)}{(\varepsilon-2)(\varepsilon-1)\varepsilon} + \frac{1}{1-\varepsilon} \to -\frac{\pi}{4} + 1.$$

☐

Similarly for $\cos x$, with the integral absolutely convergent for $-2 < \operatorname{Re}\alpha < 0$ and conditionally convergent for $0 \leqslant \operatorname{Re}\alpha < 1$:

$$\int_0^\infty x^{\alpha-1}\bigl(\cos x - \mathbf{1}\{x<1\}\bigr)\,\mathrm{d}x = \cos\frac{\pi\alpha}{2}\,\Gamma(\alpha) - \frac{1}{\alpha}, \qquad -2 < \operatorname{Re}\alpha < 1. \tag{21}$$

Proof. This follows from (8) when $-2 < \operatorname{Re}\alpha < 0$. The case $0 < \operatorname{Re}\alpha < 1$ follows directly from (12). The general case follows by integration by parts and (17), which yield

$$\begin{aligned}
&\int_0^\infty x^{\alpha-1}\big(\cos x - \mathbf{1}\{x<1\}\big)\,\mathrm{d}x \\
&\qquad = -\int_0^\infty (\alpha-1)x^{\alpha-2}\big(\sin x - x\mathbf{1}\{x<1\} - \mathbf{1}\{x\geqslant 1\}\big)\,\mathrm{d}x \\
&\qquad = -(\alpha-1)\Big(\sin\frac{\pi(\alpha-1)}{2}\,\Gamma(\alpha-1) - \frac{1}{\alpha}\Big) + \int_1^\infty (\alpha-1)x^{\alpha-2}\,\mathrm{d}x \\
&\qquad = \sin\frac{\pi(1-\alpha)}{2}\,\Gamma(\alpha) + \frac{\alpha-1}{\alpha} - 1 \\
&\qquad = \cos\frac{\pi\alpha}{2}\,\Gamma(\alpha) - \frac{1}{\alpha}. \qquad \square
\end{aligned}$$

In particular, taking $\alpha = 0$ and -1, with the first integral conditionally convergent and the second absolutely convergent,

$$\int_0^\infty \frac{\cos x - \mathbf{1}\{x<1\}}{x}\,\mathrm{d}x = -\gamma, \tag{22}$$

$$\int_0^\infty \frac{\cos x - \mathbf{1}\{x<1\}}{x^2}\,\mathrm{d}x = 1 - \frac{\pi}{2}. \tag{23}$$

Proof. As $\varepsilon \to 0$,

$$\begin{aligned}
\cos\frac{\pi\varepsilon}{2}\,\Gamma(\varepsilon) - \frac{1}{\varepsilon} &= \frac{\cos\frac{\pi\varepsilon}{2}\Gamma(\varepsilon+1) - 1}{\varepsilon} \\
&\to \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big(\cos\frac{\pi\varepsilon}{2}\Gamma(\varepsilon+1)\Big)\Big|_{\varepsilon=0} \\
&= \Gamma'(1) = -\gamma
\end{aligned}$$

and

$$\cos\frac{\pi(\varepsilon-1)}{2}\,\Gamma(\varepsilon-1) - \frac{1}{\varepsilon-1} = \frac{\sin\frac{\pi\varepsilon}{2}\Gamma(\varepsilon+1)}{(\varepsilon-1)\varepsilon} + \frac{1}{1-\varepsilon} \to -\frac{\pi}{2} + 1.$$

$\square$

IV. **Differences for different exponents.**

$$\int_0^\infty \big(e^{-ax} - e^{-bx}\big)x^{\alpha-1}\,\mathrm{d}x = \big(a^{-\alpha} - b^{-\alpha}\big)\Gamma(\alpha), \qquad \operatorname{Re}\alpha > -1,\ \operatorname{Re}a > 0,\ \operatorname{Re}b > 0. \tag{24}$$

Proof. If $\operatorname{Re}\alpha > 0$ and $a > 0$, $b > 0$, this follows immediately from (1) by separating the integral into two and changing variables. The case $\operatorname{Re}\alpha > 0$ now follows by analytic continuation in a and b, and this extends to $\operatorname{Re}\alpha > -1$ by analytic continuation in α. (Cf. also (6).) $\square$

In particular, taking $\alpha = 0$ we find:

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\,\mathrm{d}x = \log b - \log a = \log\frac{b}{a}, \qquad \operatorname{Re}a, \operatorname{Re}b > 0. \tag{25}$$

V. **Another formula for γ.**

$$\int_0^\infty \Bigl(\frac{1}{1-e^{-x}}-\frac{1}{x}\Bigr)e^{-x}\,\mathrm{d}x=\int_0^\infty \Bigl(\frac{e^{-x}}{1-e^{-x}}-\frac{e^{-x}}{x}\Bigr)\,\mathrm{d}x=\gamma. \tag{26}$$

Proof. We have, using (25),

$$\begin{aligned}\int_0^\infty \Bigl(\frac{1}{1-e^{-x}}-\frac{1}{x}\Bigr)e^{-x}\,\mathrm{d}x&=\int_0^\infty \frac{e^{-x}-1+x}{x(1-e^{-x})}e^{-x}\,\mathrm{d}x\\ &=\int_0^\infty \frac{e^{-x}-1+x}{x}\sum_{n=1}^\infty e^{-nx}\,\mathrm{d}x\\ &=\sum_{n=1}^\infty\int_0^\infty \Bigl(\frac{e^{-(n+1)x}-e^{-nx}}{x}+e^{-nx}\Bigr)\,\mathrm{d}x\\ &=\sum_{n=1}^\infty\Bigl(\log n-\log(n+1)+\frac{1}{n}\Bigr)=\lim_{N\to\infty}\Bigl(\sum_{n=1}^N\frac{1}{n}-\log(N+1)\Bigr)=\gamma.\end{aligned}$$

□

VI. **Other powers in the exponent.** The change of variables $x=y^{1/\beta}$ yields immediately

$$\int_0^\infty x^{\alpha-1}e^{-x^\beta}\,\mathrm{d}x=\frac{1}{\beta}\Gamma\Bigl(\frac{\alpha}{\beta}\Bigr),\qquad \operatorname{Re}\alpha>0. \tag{27}$$

and, in particular,

$$\int_0^\infty e^{-x^\beta}\,\mathrm{d}x=\Gamma(1+1/\beta),\qquad \beta>0. \tag{28}$$

References

[1] M. Abramowitz & I. A. Stegun, eds., *Handbook of Mathematical Functions.* 9th printing. Dover, New York, 1972. Also available at `http://people.maths.ox.ac.uk/~macdonald/aands/abramowitz_and_stegun.pdf`

[2] Frank W. J. Olver, Daniel W. Lozier, Ronald F. Boisvert and Charles W. Clark, *NIST Handbook of Mathematical Functions.* Cambridge Univ. Press, 2010.
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