STABLE DISTRIBUTIONS

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1. Introduction

We give many explicit formulas for stable distributions, mainly based on Feller [3] and Zolotarev [14] and using several parametrizations; we give also some explicit calculations for convergence to stable distributions, mainly based on less explicit results in Feller [3]. The main purpose is to provide ourselves with easy reference to explicit formulas and examples. (There are probably no new results.)

2. Infinitely divisible distributions

We begin with the more general concept of infinitely divisible distributions.

Definition 2.1. The distribution of a random variable X is *infinitely divisible* if for each $n \geqslant 1$ there exists i.i.d. random variable $Y_1^{(n)}, \dots, Y_n^{(n)}$ such that

$$X \overset{\mathrm{d}}{=} Y_1^{(n)} + \dots Y_n^{(n)}. \tag{2.1}$$

The characteristic function of an infinitely divisible distribution may be expressed in a canonical form, sometimes called the *Lévy–Khinchin representation.* We give several equivalent versions in the following theorem.

Theorem 2.2. *Let $h(x)$ be a fixed bounded measurable real-valued function on $\mathbb{R}$ such that $h(x) = x + O(x^2)$ as $x \to 0$. Then the following are equivalent.*

(i) *$\varphi(t)$ is the characteristic function of an infinitely divisible distribution.*

(ii) *There exist a measure M on $\mathbb{R}$ such that*

$$\int_{-\infty}^{\infty} \bigl(1 \wedge |x|^{-2}\bigr)\,\mathrm{d}M(x) < \infty \tag{2.2}$$

and a real constant b such that

$$\varphi(t) = \exp\Bigl(\mathrm{i}bt + \int_{-\infty}^{\infty} \frac{e^{\mathrm{i}tx} - 1 - \mathrm{i}th(x)}{x^2}\,\mathrm{d}M(x)\Bigr), \tag{2.3}$$

where the integrand is interpreted as $-t^2/2$ at $x = 0$.

(iii) *There exist a measure Λ on $\mathbb{R} \setminus \{0\}$ such that*

$$\int_{-\infty}^{\infty} \bigl(|x|^2 \wedge 1\bigr)\,\mathrm{d}\Lambda(x) < \infty \tag{2.4}$$

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and real constants $a \geqslant 0$ and b such that

$$\varphi(t) = \exp\Bigl(\mathrm{i}bt - \tfrac12 at^2 + \int_{-\infty}^{\infty} \bigl(e^{\mathrm{i}tx} - 1 - \mathrm{i}th(x)\bigr)\,\mathrm{d}\Lambda(x)\Bigr). \tag{2.5}$$

(iv) *There exist a bounded measure K on $\mathbb{R}$ and a real constant b such that*

$$\varphi(t) = \exp\Bigl(\mathrm{i}bt + \int_{-\infty}^{\infty} \Bigl(e^{\mathrm{i}tx} - 1 - \frac{\mathrm{i}tx}{1+x^2}\Bigr)\frac{1+x^2}{x^2}\,\mathrm{d}K(x)\Bigr), \tag{2.6}$$

where the integrand is interpreted as $-t^2/2$ at $x = 0$.

The measures and constants are determined uniquely by φ.

Feller [3, Chapter XVII] uses $h(x) = \sin x$. Kallenberg [8, Corollary 15.8] uses $h(x) = x\mathbf{1}\{|x| \leqslant 1\}$.

Feller [3, Chapter XVII.2] calls the measure M in (ii) the *canonical measure*. The measure Λ in (iii) is known as the *Lévy measure*. The parameters a, b and Λ are together called the *characteristics* of the distribution. We denote the distribution with characteristic function (2.5) (for a given h) by $\mathrm{ID}(a, b, \Lambda)$.

Remark 2.3. Different choices of $h(x)$ yield the same measures M and Λ in (ii) and (iii) but different constants b; changing h to $\tilde h$ corresponds to changing b to

$$\tilde b := b + \int_{-\infty}^{\infty} \frac{\tilde h(x) - h(x)}{x^2}\,\mathrm{d}M(x) = b + \int_{-\infty}^{\infty} \bigl(\tilde h(x) - h(x)\bigr)\,\mathrm{d}\Lambda(x). \tag{2.7}$$

We see also that b is the same in (ii) and (iii) (with the same h), and that (see the proof below) b in (iv) equals b in (ii) and (iii) when $x = x/(1 + x^2)$. $\square$

Proof. (i) $\iff$ (ii): This is shown in Feller [3, Theorem XVII.2.1] for the choice $h(x) = \sin x$. As remarked above, (2.3) for some h is equivalent to (2.3) for any other h, changing b by (2.7).

(ii) $\iff$ (iii): Given M in (ii) we let $a := M\{0\}$ and $\mathrm{d}\Lambda(x) := x^{-2}\,\mathrm{d}M(x)$, $x \neq 0$. Conversely, given a and Λ as in (iii) we define

$$\mathrm{d}M(x) = a\delta_0 + x^2\,\mathrm{d}\Lambda(x). \tag{2.8}$$

The equivalence between (2.3) and (2.5) then is obvious. $\square$

(ii) $\iff$ (iv): Choose $h(x) = x/(1 + x^2)$ and define

$$\mathrm{d}K(x) := \frac{1}{1+x^2}\,\mathrm{d}M(x); \tag{2.9}$$

conversely, $\mathrm{d}M(x) = (1 + x^2)\,\mathrm{d}K(x)$. Then (2.3) is equivalent to (2.6).

Remark 2.4. At least (iii) extends directly to infinitely divisible random vectors in $\mathbb{R}^d$. Moreover, there is a one-to-one correspondence with *Lévy processes*, i.e., stochastic processes X_t on $[0, \infty)$ with stationary independent increments and $X_0 = 0$, given by (in the one-dimensional case)

$$\mathbb{E}\, e^{\mathrm{i}uX_t} = \varphi(u)^t = \exp\Bigl(t\Bigl(\mathrm{i}bu - \tfrac12 au^2 + \int_{-\infty}^{\infty} \bigl(e^{\mathrm{i}ux} - 1 - \mathrm{i}uh(x)\bigr)\,\mathrm{d}\Lambda(x)\Bigr)\Bigr) \tag{2.10}$$

for $t \geqslant 0$ and $u \in \mathbb{R}$. See Bertoin [2] and Kallenberg [8, Corollary 15.8]. $\square$

Example 2.5. The normal distribution $N(\mu,\sigma^2)$ has $\Lambda=0$ and $a=\sigma^2$; thus $M=K=\sigma^2\delta_0$; further, $b=\mu$ for any h. Thus, $N(\mu,\sigma^2)=\mathrm{ID}(\sigma^2,\mu,0)$. $\square$

Example 2.6. The Poisson distribution $\mathrm{Po}(\lambda)$ has $M=\Lambda=\lambda\delta_1$ and $K=\frac{\lambda}{2}\delta_1$; further $b=\lambda h(1)$. (Thus $b=\lambda/2$ in (iv).) $\square$

Example 2.7. The Gamma distribution $\mathrm{Gamma}(\alpha)$ with density function $x^{\alpha-1}e^{-x}/\Gamma(\alpha)$, $x>0$, has the characteristic function $\varphi(t)=(1-\mathrm{i}t)^{-\alpha}$. It is infinitely divisible with

$$\mathrm{d}M(x)=\alpha xe^{-x},\qquad x>0, \tag{2.11}$$

$$\mathrm{d}\Lambda(x)=\alpha x^{-1}e^{-x},\qquad x>0, \tag{2.12}$$

see Feller [3, Example XVII.3.d]. $\square$

Remark 2.8. If X_1 and X_2 are independent infinitely divisible random variables with parameters (a_1,b_1,Λ_1) and (a_2,b_2,Λ_2), then X_1+X_2 is infinitely divisible with parameters $(a_1+a_2,b_1+b_2,\Lambda_1+\Lambda_2)$. In particular, if $X\sim\mathrm{ID}(a,b,\Lambda)$, then

$$X\overset{\mathrm{d}}{=}X_1+Y\qquad\text{with}\qquad X_1\sim\mathrm{ID}(0,0,\Lambda),\ Y\sim\mathrm{ID}(a,b,0)=N(b,a), \tag{2.13}$$

and X_1 and Y independent. Moreover, for any finite partition $\mathbb{R}=\bigcup A_i$, we can split X as a sum of independent infinitely divisible random variables X_i with the Lévy measure of X_i having supports in A_i. $\square$

Example 2.9 (integral of Poisson process)**.** Let Ξ be a Poisson process on $\mathbb{R}\setminus\{0\}$ with intensity Λ, where Λ is a measure with

$$\int_{-\infty}^{\infty}\bigl(|x|\wedge 1\bigr)\,\mathrm{d}\Lambda(x)<\infty. \tag{2.14}$$

Let $X:=\int x\,\mathrm{d}\Xi(x)$; if we regard Ξ as a (finite or countable) set (or possibly multiset) of points $\{\xi_i\}$, this means that $X:=\sum_i\xi_i$. (The sum converges absolutely a.s., so X is well-defined a.s.; in fact, the sum $\sum_{|\xi_i|>1}\xi_i$ is a.s. finite, and the sum $\sum_{|\xi_i|\leqslant 1}|\xi_i|$ has finite expectation $\int_{-1}^{1}|x|\,\mathrm{d}\Lambda(x)$.) Then X has characteristic function

$$\varphi(t)=\exp\Bigl(\int_{-\infty}^{\infty}\bigl(e^{\mathrm{i}tx}-1\bigr)\,\mathrm{d}\Lambda(x)\Bigr). \tag{2.15}$$

(See, for example, the corresponding formula for the Laplace transform in Kallenberg [8, Lemma 12.2], from which (2.15) easily follows.) Hence, (2.5) holds with Lévy measure Λ, $a=0$ and $b=\int_{-\infty}^{\infty}h(x)\,\mathrm{d}\Lambda(x)$. (When (2.5) holds, we can take $h(x)=0$, a choice not allowed in general. Note that (2.15) is the same as (2.5) with $h=0$, $a=0$ and $b=0$.)

By adding an independent normal variable $N(b,a)$, we can obtain any infinitely divisible distribution with a Lévy measure satisfying (2.5); see Example 2.5 and Remark 2.8. $\square$

Example 2.10 (compensated integral of Poisson process)**.** Let Ξ be a Poisson process on $\mathbb{R}\setminus\{0\}$ with intensity Λ, where Λ is a measure with

$$\int_{-\infty}^{\infty}\bigl(|x|^2\wedge|x|\bigr)\,\mathrm{d}\Lambda(x)<\infty. \tag{2.16}$$

Suppose first that $\int_{-\infty}^{\infty}|x|\,\mathrm{d}\Lambda(x)<\infty$. Let X be as in Example 2.9. Then X has finite expectation $\mathbb{E}\,X=\int_{-\infty}^{\infty}x\,\mathrm{d}\Lambda$. Define

$$\widetilde{X}:=X-\mathbb{E}\,X=\int_{-\infty}^{\infty}x\bigl(\mathrm{d}\Xi(x)-\mathrm{d}\Lambda(x)\bigr). \tag{2.17}$$

Then, by (2.15), $\widetilde{X}$ has characteristic function

$$\varphi(t)=\exp\Bigl(\int_{-\infty}^{\infty}\bigl(e^{\mathrm{i}tx}-1-itx\bigr)\,\mathrm{d}\Lambda(x)\Bigr). \tag{2.18}$$

Now suppose that Λ is any measure satisfying (2.16). Then the integral in (2.18) converges; moreover, by considering the truncated measures $\Lambda_n:=\mathbf{1}\{|x|>n^{-1}\}\Lambda$ and taking the limit as $n\to\infty$, it follows that there exists a random variable $\widetilde{X}$ with characteristic function (2.18). Hence, (2.5) holds with Lévy measure Λ, $a=0$ and $b=\int_{-\infty}^{\infty}(h(x)-x)\,\mathrm{d}\Lambda(x)$. (When (2.16) holds, we can take $h(x)=x$, a choice not allowed in general. Note that (2.18) is the same as (2.5) with $h(x)=x$, $a=0$ and $b=0$.)

By adding an independent normal variable $N(b,a)$, we can obtain any infinitely divisible distribution with a Lévy measure satisfying (2.16); see Example 2.5 and Remark 2.8. □

Remark 2.11. Any infinitely divisible distribution can be obtained by taking a sum X_1+X_2+Y of independent random variables with X_1 as in Example 2.9, X_2 as in Example 2.10 and Y normal. For example, we can take the Lévy measures of X_1 and X_2 as the restrictions of the Lévy measure to $\{x:|x|>1\}$ and $\{x:|x|\leqslant 1\}$, respectively. □

Theorem 2.12. *If X is an infinitely divisible random variable with characteristic function given by* (2.5) *and $t\in\mathbb{R}$, then*

$$\mathbb{E}\,e^{tX}=\exp\Bigl(bt+\tfrac12 at^2+\int_{-\infty}^{\infty}\bigl(e^{tx}-1-th(x)\bigr)\,\mathrm{d}\Lambda(x)\Bigr)\leqslant\infty. \tag{2.19}$$

In particular,

$$\begin{aligned}\mathbb{E}\,e^{tX}<\infty\iff&\int_{-\infty}^{\infty}\bigl(e^{tx}-1-th(x)\bigr)\,\mathrm{d}\Lambda(x)<\infty\\ \iff&\begin{cases}\int_1^{\infty}e^{tx}\,\mathrm{d}\Lambda(x)<\infty, & t>0,\\ \int_{-\infty}^{-1}e^{tx}\,\mathrm{d}\Lambda(x)<\infty, & t<0.\end{cases}\end{aligned} \tag{2.20}$$

Proof. The choice of h (satisfying the conditions of Theorem 2.2) does not matter, because of (2.7); we may thus assume $h(x)=x\mathbf{1}\{|x|\leqslant 1\}$. We further assume $t>0$. (The case $t<0$ is similar and the case $t=0$ is trivial.)

Denote the right-hand side of (2.19) by $F_\Lambda(t)$. We study several different cases.

(i). If $\operatorname{supp}\Lambda$ is bounded, then the integral in (2.19) converges for all complex t and defines an entire function. Thus $F_\Lambda(t)$ is entire and (2.5) shows that $\mathbb{E}\,e^{\mathrm{i}tX}=F_\Lambda(\mathrm{i}t)$. It follows that $\mathbb{E}\,|e^{tX|}<\infty$ and $\mathbb{E}\,e^{tX}=F_\Lambda(t)$ for any complex t, see e.g. Marcinkiewicz [9].

(ii). If $\operatorname{supp}\Lambda \subseteq [1,\infty)$, let Λ_n be the restriction $\Lambda\big|_{[1,n]}$ of the measure Λ to $[1,n]$. By the construction in Example 2.9, we can construct random variables $X_n \sim \mathrm{ID}(0,0,\Lambda_n)$ such that $X_n \nearrow X \sim \mathrm{ID}(0,0,\Lambda)$ as $n \to \infty$. Case (i) applies to each Λ_n, and (2.19) follows for X, and $t > 0$, by monotone convergence.

(iii). If $\operatorname{supp}\Lambda \subseteq (-\infty,1]$, let Λ_n be the restriction $\Lambda\big|_{[-n,-1]}$. Similarly to (ii) we can construct random variables $X_n \sim \mathrm{ID}(0,0,\Lambda_n)$ with $X_n \leqslant 0$ such that $X_n \searrow X \sim \mathrm{ID}(0,0,\Lambda)$ as $n \to \infty$. Case (i) applies to each Λ_n, and (2.19) follows for X; this time by monotone convergence.

(iv). The general case follows by (i)–(iii) and a decomposition as in Remark 2.8. $\square$

3. Stable distributions

Definition 3.1. The distribution of a (non-degenerate) random variable X is *stable* if there exist constants $a_n > 0$ and b_n such that, for any $n \geqslant 1$, if $X_1, X_2, \dots$ are i.i.d. copies of X and $S_n := \sum_{i=1}^n X_i$, then

$$S_n \overset{\mathrm{d}}{=} a_n X + b_n. \tag{3.1}$$

The distribution is *strictly stable* if $b_n = 0$.

(Many authors, e.g. Kallenberg [8], say *weakly stable* for our stable.)

We say that the random variable X is (strictly) stable if its distribution is.

The norming constants a_n in (3.1) are necessarily of the form $a_n = n^{1/\alpha}$ for some $\alpha \in (0,2]$, see Feller [3, Theorem VI.1.1]; α is called the *index* [4], [8] or *characteristic exponent* [3] of the distribution. We also say that a distribution (or random variable) is *α-stable* if it is stable with index α.

The case $\alpha = 2$ is simple: X is 2-stable if and only if it is normal. For $\alpha < 2$, there is a simple characterisation in terms of the Lévy–Khinchin representation of infinitely divisible distributions.

Theorem 3.2. (i) *A distribution is 2-stable if and only if it is normal $N(\mu,\sigma^2)$. (This is an infinitely divisible distribution with $M = \sigma^2\delta_0$, see Example 2.5.)*

(ii) *Let $0 < \alpha < 2$. A distribution is α-stable if and only if it is infinitely divisible with canonical measure*

$$\frac{\mathrm{d}M(x)}{\mathrm{d}x} = \begin{cases} c_+ x^{1-\alpha}, & x > 0, \\ c_- |x|^{1-\alpha}, & x < 0; \end{cases} \tag{3.2}$$

equivalently, the Lévy measure is given by

$$\frac{\mathrm{d}\Lambda(x)}{\mathrm{d}x} = \begin{cases} c_+ x^{-\alpha-1}, & x > 0, \\ c_- |x|^{-\alpha-1}, & x < 0, \end{cases} \tag{3.3}$$

and $a = 0$. Here $c_-, c_+ \geqslant 0$ and we assume that not both are 0.

Proof. See Feller [3, Section XVII.5] or Kallenberg [8, Proposition 15.9]. $\square$

Note that (3.2) is equivalent to

$$M[x_1,x_2] = C_+ x_2^{2-\alpha} + C_- |x_1|^{2-\alpha} \tag{3.4}$$

for any interval with $x_1 \leqslant 0 \leqslant x_2$, with

$$C_\pm = \frac{c_\pm}{2-\alpha}. \tag{3.5}$$

Theorem 3.3. *Let $0 < \alpha \leqslant 2$.*

(i) *A distribution is α-stable if and only if it has a characteristic function*

$$\varphi(t) = \begin{cases} \exp\Bigl(-\gamma^\alpha |t|^\alpha \Bigl(1 - \mathrm{i}\beta \tan\frac{\pi\alpha}{2}\operatorname{sgn}(t)\Bigr) + \mathrm{i}\delta t\Bigr), & \alpha \neq 1, \\ \exp\Bigl(-\gamma |t| \Bigl(1 + \mathrm{i}\beta \frac{2}{\pi}\operatorname{sgn}(t)\log|t|\Bigr) + \mathrm{i}\delta t\Bigr), & \alpha = 1, \end{cases} \tag{3.6}$$

where $-1 \leqslant \beta \leqslant 1$, $\gamma > 0$ and $-\infty < \delta < \infty$. Furthermore, an α-stable distribution exists for any such $\alpha, \beta, \gamma, \delta$. (If $\alpha = 2$, then β is irrelevant and usually taken as 0.)

(ii) *If X has the characteristic function (3.6), then, for any $n \geqslant 1$, (3.1) takes the explicit form*

$$S_n \overset{\mathrm{d}}{=} \begin{cases} n^{1/\alpha} X + (n - n^{1/\alpha})\delta, & \alpha \neq 1, \\ nX + \frac{2}{\pi}\beta\gamma n \log n, & \alpha = 1. \end{cases} \tag{3.7}$$

In particular,

$$X \text{ is strictly stable } \iff \begin{cases} \delta = 0, & \alpha \neq 1, \\ \beta = 0, & \alpha = 1. \end{cases} \tag{3.8}$$

(iii) *An α-stable distribution with canonical measure M satisfying (3.4) has*

$$\gamma^\alpha = \begin{cases} (C_+ + C_-)\frac{\Gamma(3-\alpha)}{\alpha(1-\alpha)}\cos\frac{\pi\alpha}{2}, & \alpha \neq 1, \\ (C_+ + C_-)\frac{\pi}{2}, & \alpha = 1, \end{cases} \tag{3.9}$$

$$\beta = \frac{C_+ - C_-}{C_+ + C_-}. \tag{3.10}$$

(iv) *If $0 < \alpha < 2$, then an α-stable distribution with Lévy measure Λ satisfying (3.3) has*

$$\gamma^\alpha = \begin{cases} (c_+ + c_-)\bigl(-\Gamma(-\alpha)\cos\frac{\pi\alpha}{2}\bigr), & \alpha \neq 1, \\ (c_+ + c_-)\frac{\pi}{2}, & \alpha = 1, \end{cases} \tag{3.11}$$

$$\beta = \frac{c_+ - c_-}{c_+ + c_-}. \tag{3.12}$$

We use the notation $\mathrm{S}_\alpha(\gamma,\beta,\delta)$ for the distribution with characteristic function (3.6), and $X_\alpha(\gamma,\beta,\delta)$ for a random variable with this distribution. We also write $\mathrm{S}_\alpha(\beta)$ and $X_\alpha(\beta)$ for the special case $\gamma = 1$, $\delta = 0$.

Proof. Feller [3, XVII.(3.18)–(3.19) and Theorem XVII.5.1(ii)] gives, in our notation, for a stable distribution satisfying (3.4), the characteristic function

$$\exp\Bigl(-(C_+ + C_-)\frac{\Gamma(3-\alpha)}{\alpha(1-\alpha)}\Bigl(\cos\frac{\pi\alpha}{2} - \mathrm{i}\,\mathrm{sgn}(t)\frac{C_+ - C_-}{C_+ + C_-}\sin\frac{\pi\alpha}{2}\Bigr)|t|^\alpha + \mathrm{i}bt\Bigr) \tag{3.13}$$

if $\alpha \neq 1$ and

$$\exp\Bigl(-(C_+ + C_-)\Bigl(\frac{\pi}{2} + \mathrm{i}\,\mathrm{sgn}(t)\frac{C_+ - C_-}{C_+ + C_-}\log|t|\Bigr)|t| + \mathrm{i}bt\Bigr) \tag{3.14}$$

if $\alpha = 1$. This is (3.6) with (3.9)–(3.10) and $\delta = b$. This proves (i) and (iii), and (iv) follows from (iii) by (3.5).

Finally, (ii) follows directly from (3.6). □

Remark 3.4. If $1 < \alpha \leqslant 2$, then δ in (3.6) equals the mean $\mathbb{E}\, X_\alpha(\gamma,\beta,\delta)$. In particular, (3.8) shows that for $\alpha > 1$, a stable distribution is strictly stable if and only if its expectation vanishes. □

Remark 3.5. If $X_\alpha(\beta) \sim \mathrm{S}_\alpha(\beta) = \mathrm{S}_\alpha(1,\beta,0)$, then, for $\gamma > 0$ and $\delta \in \mathbb{R}$,

$$\gamma X_\alpha(\beta) + \delta \sim \begin{cases} \mathrm{S}_\alpha(\gamma,\beta,\delta), & \alpha \neq 1, \\ \mathrm{S}_\alpha(\gamma,\beta,\delta - \frac{2}{\pi}\beta\gamma\log\gamma), & \alpha = 1. \end{cases} \tag{3.15}$$

Thus, γ is a scale parameter and δ a location parameter; β is a skewness parameter, and α and β together determine the shape of the distribution. □

Remark 3.6. More generally, if $X \sim \mathrm{S}_\alpha(\gamma,\beta,\delta)$, then, for $a > 0$ and $d \in \mathbb{R}$,

$$aX + d \sim \begin{cases} \mathrm{S}_\alpha(a\gamma,\beta,a\delta + d), & \alpha \neq 1, \\ \mathrm{S}_\alpha(a\gamma,\beta,a\delta + d - \frac{2}{\pi}\beta\gamma a\log a), & \alpha = 1. \end{cases} \tag{3.16}$$

□

Remark 3.7. If $X \sim \mathrm{S}_\alpha(\gamma,\beta,\delta)$, then $-X \sim \mathrm{S}_\alpha(\gamma,-\beta,-\delta)$. In other words,

$$-X_\alpha(\gamma,\beta,\delta) \overset{\mathrm{d}}{=} X_\alpha(\gamma,-\beta,-\delta). \tag{3.17}$$

In particular, X has a symmetric stable distribution if and only if $X \sim \mathrm{S}_\alpha(\gamma,0,0)$ for some $\alpha \in (0,2]$ and $\gamma > 0$. □

We may simplify expressions like (3.6) by considering only $t \geqslant 0$ (or $t > 0$); this is sufficient because of the general formula

$$\varphi(-t) = \overline{\varphi(t)} \tag{3.18}$$

for any characteristic function. We use this in our next statement, which is an immediate consequence of Theorem 3.3.

Corollary 3.8. *Let $0 < \alpha \leqslant 2$. A distribution is strictly stable if and only if it has a characteristic function*

$$\varphi(t) = \exp\bigl(-(\kappa - \mathrm{i}\tau)t^\alpha\bigr), \qquad t \geqslant 0, \tag{3.19}$$

where $\kappa > 0$ *and* $|\tau| \leqslant \kappa|\tan\frac{\pi\alpha}{2}|$*; furthermore, a strictly stable distribution exists for any such* κ *and* τ*. (For* $\alpha = 1$*,* $\tan\frac{\pi\alpha}{2} = \infty$*, so any real* τ *is possible. For* $\alpha = 2$*,* $\tan\frac{\pi\alpha}{2} = 0$*, so necessarily* $\tau = 0$*.)*

The distribution $\mathrm{S}_\alpha(\gamma,\beta,0)$ *(*$\alpha \neq 1$*) or* $\mathrm{S}_1(\gamma,0,\delta)$ *(*$\alpha = 1$*) satisfies* (3.19) *with*

$$\kappa = \gamma^\alpha \qquad \textit{and} \qquad \tau = \begin{cases} \beta\kappa\tan\frac{\pi\alpha}{2}, & \alpha \neq 1, \\ \delta, & \alpha = 1. \end{cases} \tag{3.20}$$

Conversely, if (3.19) *holds, then the distribution is*

$$\begin{cases} \mathrm{S}_\alpha(\gamma,\beta,0) \textit{ with } \gamma = \kappa^{1/\alpha},\ \beta = \frac{\tau}{\kappa}\cot\frac{\pi\alpha}{2}, & \alpha \neq 1 \\ \mathrm{S}_1(\kappa,0,\tau), & \alpha = 1. \end{cases} \tag{3.21}$$

□

Remark 3.9. For a strictly stable random variable, another way to write the characteristic function (3.6) or (3.19) is

$$\varphi(t) = \exp\Bigl(-\lambda e^{\mathrm{i}\,\mathrm{sgn}(t)\pi\widetilde{\gamma}/2}|t|^\alpha\Bigr), \tag{3.22}$$

with $\lambda > 0$ and $\widetilde{\gamma}$ real (with $|\widetilde{\gamma}| \leqslant 1$; see further below). A comparison with (3.6) and (3.20) shows that

$$\lambda\cos\frac{\pi\widetilde{\gamma}}{2} = \kappa = \gamma^\alpha, \tag{3.23}$$

$$\tan\frac{\pi\widetilde{\gamma}}{2} = -\frac{\tau}{\kappa} = \begin{cases} -\beta\tan\frac{\pi\alpha}{2}, & \alpha \neq 1, \\ -\frac{\delta}{\gamma^\alpha}, & \alpha = 1. \end{cases} \tag{3.24}$$

If $0 < \alpha < 1$, we have $0 < \tan\frac{\pi\alpha}{2} < \infty$ and $|\widetilde{\gamma}| \leqslant \alpha$, while if $1 < \alpha < 2$, then $\tan\frac{\pi\alpha}{2} < 0$ and $\tan\frac{\pi\widetilde{\gamma}}{2} = \beta\tan\frac{\pi(2-\alpha)}{2}$ with $0 < \pi(2-\alpha)/2 < \pi/2$; hence $|\widetilde{\gamma}| \leqslant 2-\alpha$. Finally, for $\alpha = 1$, we have $|\widetilde{\gamma}| < 1$, and for $\alpha = 2$ we have $\widetilde{\gamma} = 0$. These ranges for $\widetilde{\gamma}$ are both necessary and sufficient, except that for $\alpha = 1$, $\widetilde{\gamma} = \pm 1$ is possible in (3.22), but yields a degenerate distribution $X = -\widetilde{\gamma}\lambda$. Summarising, we have the ranges, excluding the degenerate case just mentioned,

$$\begin{cases} |\widetilde{\gamma}| \leqslant \alpha, & 0 < \alpha < 1, \\ |\widetilde{\gamma}| < 1, & \alpha = 1, \\ |\widetilde{\gamma}| \leqslant 2-\alpha, & 1 < \alpha \leqslant 2. \end{cases} \tag{3.25}$$

For $\alpha \neq 1, 2$, note the special cases

$$\beta = 0 \iff \widetilde{\gamma} = 0, \tag{3.26}$$

$$\beta = 1 \iff \widetilde{\gamma} = \begin{cases} -\alpha, & 0 < \alpha < 1, \\ 2-\alpha, & 1 < \alpha < 2. \end{cases} \tag{3.27}$$

$$\beta = -1 \iff \widetilde{\gamma} = \begin{cases} \alpha, & 0 < \alpha < 1, \\ \alpha-2, & 1 < \alpha < 2. \end{cases} \tag{3.28}$$

□

Remark 3.10. For $\alpha = 1$, the general 1-stable characteristic function (3.6) may be written, similarly to (3.19),

$$\varphi(t) = \exp\bigl(-(\kappa - \mathrm{i}\tau)t - \mathrm{i}bt\log t\bigr), \qquad t > 0, \tag{3.29}$$

where $\kappa = \gamma$, $\tau = \delta$ and $b = \frac{2}{\pi}\beta\gamma$. (Thus, $|b| \leqslant 2\kappa/\pi$.) □

3.1. **Positive and spectrally positive stable distributions.**

Definition 3.11. A stable distribution (or random variable) is *spectrally positive* if its Lévy measure is concentrated on $(0,\infty)$, i.e.,

$$\mathrm{d}\Lambda(x) = cx^{-\alpha-1}\,\mathrm{d}x, \qquad x > 0, \tag{3.30}$$

for some $c > 0$ and $\alpha \in (0,2)$. By (3.3) and (3.12), this is equivalent to $c_- = 0$ and to $\beta = 1$, see also (3.27).

Similarly, a stable distribution (or random variable) is *spectrally negative* if its Lévy measure is concentrated on $(-\infty, 0)$.

Thus, X is spectrally negative if and only if $-X$ is spectrally positive. (For this reason, we mainly consider the spectrally positive case.)

Theorem 3.12. *A strictly stable distribution is spectrally positive if and only if it is of the form* $\mathrm{S}_\alpha(\gamma, 1, 0)$ *with* $\alpha \neq 1$.

Equivalently, a strictly stable distribution with characteristic function (3.19) *is spectrally positive if and only if* $\alpha \neq 1$ *and* $\tau = \kappa \tan\frac{\pi\alpha}{2}$.

Proof. This follows from Corollary 3.8, taking $\beta = 1$ in (3.20); note that by (3.21), there is no spectrally positive strictly 1-stable distribution. □

Theorem 3.13. *Let* $0 < \alpha < 2$. *An* α*-stable random variable* $X \sim \mathrm{S}_\alpha(\gamma,\beta,\delta)$ *has finite Laplace transform* $\mathbb{E}\,e^{-tX}$ *for* $t \geqslant 0$ *if and only if it is spectrally positive, i.e., if* $\beta = 1$, *and then*

$$\mathbb{E}\,e^{-tX} = \begin{cases} \exp\Bigl(-\frac{\gamma^\alpha}{\cos\frac{\pi\alpha}{2}}t^\alpha - \delta t\Bigr), & \alpha \neq 1, \\ \exp\Bigl(\frac{2}{\pi}\gamma t\log t - \delta t\Bigr), & \alpha = 1, \end{cases} \tag{3.31}$$

Moreover, then (3.31) *holds for every complex* t *with* $\operatorname{Re} t \geqslant 0$.

Proof. The condition for finiteness follows by Theorem 2.12 and (3.3), together with Definition 3.11. When this holds, the right-hand side of (3.31) is a continuous function of t in the closed right half-plane $\operatorname{Re} t \geqslant 0$, which is analytic in the open half-plane $\operatorname{Re} t > 0$. The same is true for the left-hand side by Theorem 2.12, and the two functions are equal on the imaginary axis $t = \mathrm{i}s$, $s \in \mathbb{R}$ by (3.6) and a simple calculation. By uniqueness of analytic continuation, (3.31) holds for every complex t with $\operatorname{Re} t \geqslant 0$. □

Theorem 3.14. *An stable random variable* $X \sim \mathrm{S}_\alpha(\gamma,\beta,\delta)$ *is positive, i.e.* $X > 0$ *a.s., if and only if* $0 < \alpha < 1$, $\beta = 1$ *and* $\delta \geqslant 0$. *Consequently, the positive strictly stable random variables are* $X_\alpha(\gamma, 1, 0)$ *with* $0 < \alpha < 1$.

Proof. $X > 0$ a.s. if and only if the Laplace transform $\mathbb{E}\, e^{-tX}$ is finite for all $t \geqslant 0$ and $\mathbb{E}\, e^{-tX} \to 0$ as $t \to \infty$. Suppose that this holds. We cannot have $\alpha = 2$, since then X would be normal and therefore not positive; thus Theorem 3.13 applies and shows that $\beta = 1$. Moreover, (3.31) holds. If $1 < \alpha < 2$ or $\alpha = 1$, then the right-hand side of (3.31) tends to infinity as $t \to \infty$, which is a contradiction; hence $0 < \alpha < 1$, and then (3.31) again shows that $\delta \geqslant 0$.

The converse is immediate from (3.31). ☐

Corollary 3.15. *Let X be a stable random variable. Then, $X > 0$ a.s. if and only if $X = Y + \delta$ where $\delta \geqslant 0$ and Y is spectrally positive strictly α-stable with $0 < \alpha < 1$.* ☐

The following examples are the two most important cases of Theorem 3.13.

Example 3.16. If $0 < \alpha < 1$ and $\lambda > 0$, then $X \sim \mathrm{S}_\alpha(\gamma, 1, 0)$ with $\gamma := \bigl(\lambda \cos\frac{\pi\alpha}{2}\bigr)^{1/\alpha}$ is a positive strictly stable random variable with the Laplace transform (extended by analyticity)

$$\mathbb{E}\, e^{-tX} = \exp\bigl(-\lambda t^\alpha\bigr), \qquad \operatorname{Re} t \geqslant 0. \tag{3.32}$$

Note that we have $\widetilde{\gamma} = -\alpha$ by (3.27). ☐

Example 3.17. If $1 < \alpha < 2$ and $\lambda > 0$, then $X \sim \mathrm{S}_\alpha(\gamma, 1, 0)$ with $\gamma := \bigl(\lambda |\cos\frac{\pi\alpha}{2}|\bigr)^{1/\alpha}$ is a spectrally positive strictly stable random variable with the Laplace transform (extended by analyticity)

$$\mathbb{E}\, e^{-tX} = \exp\bigl(\lambda t^\alpha\bigr), \qquad \operatorname{Re} t \geqslant 0. \tag{3.33}$$

Note that in this case $\cos\frac{\pi\alpha}{2} < 0$. Note also that $\mathbb{E}\, e^{-tX} \to \infty$ as $t \to \infty$, which shows that $\mathbb{P}(X < 0) > 0$. ☐

3.2. **Other parametrisations.** Our notation $\mathrm{S}_\alpha(\gamma, \beta, \delta)$ is in accordance with e.g. Samorodnitsky and Taqqu [13, Definition 1.1.6 and page 9]. (Although they use the letters $\mathrm{S}_\alpha(\sigma, \beta, \mu)$.) Nolan [11] uses the notation $S(\alpha, \beta, \gamma, \delta; 1)$; he also defines $S(\alpha, \beta, \gamma, \delta_0; 0) := S(\alpha, \beta, \gamma, \delta_1; 1)$ where

$$\delta_1 := \begin{cases} \delta_0 - \beta\gamma \tan\frac{\pi\alpha}{2}, & \alpha \neq 1, \\ \delta_0 - \frac{2}{\pi}\beta\gamma\log\gamma, & \alpha = 1. \end{cases} \tag{3.34}$$

(Note that our $\delta = \delta_1$.) This parametrisation has the advantage that the distribution $S(\alpha, \beta, \gamma, \delta_0; 0)$ is a continuous function of all four parameters. Note also that $S(\alpha, 0, \gamma, \delta; 0) = S(\alpha, 0, \gamma, \delta; 1)$, and that when $\alpha = 1$, (3.15) becomes $\gamma X_1(\beta) + \delta \sim S(1, \gamma, \beta, \delta; 0)$. Cf. the related parametrisation in [13, Remark 1.1.4], which uses

$$\mu_1 = \begin{cases} \delta_1 + \beta\gamma^\alpha \tan\frac{\pi\alpha}{2} = \delta_0 + \beta(\gamma^\alpha - \gamma)\tan\frac{\pi\alpha}{2}, & \alpha \neq 1, \\ \delta_1 \qquad\qquad\qquad\quad = \delta_0 - \frac{2}{\pi}\beta\gamma\log\gamma, & \alpha = 1; \end{cases} \tag{3.35}$$

again the distribution is a continuous function of $(\alpha, \beta, \gamma, \mu_1)$.

Zolotarev [14] uses three different parametrisations, with parameters denoted $(\alpha, \beta_\bullet, \gamma_\bullet, \lambda_\bullet)$, where $\bullet \in \{A, B, M\}$; these are defined by writing the characteristic function (3.6) as

$$\begin{aligned}\varphi(t) &= \exp\bigl(\lambda_A\bigl(\mathrm{i}t\gamma_A - |t|^\alpha + \mathrm{i}t\omega_A(t,\alpha,\beta_A)\bigr)\bigr) &(3.36)\\ &= \exp\bigl(\lambda_M\bigl(\mathrm{i}t\gamma_M - |t|^\alpha + \mathrm{i}t\omega_M(t,\alpha,\beta_M)\bigr)\bigr) &(3.37)\\ &= \exp\bigl(\lambda_B\bigl(\mathrm{i}t\gamma_B - |t|^\alpha\omega_B(t,\alpha,\beta_B)\bigr)\bigr), &(3.38)\end{aligned}$$

where

$$\omega_A(t,\alpha,\beta) := \begin{cases} |t|^{\alpha-1}\beta\tan\frac{\pi\alpha}{2}, & \alpha \neq 1,\\ -\beta\frac{2}{\pi}\log|t|, & \alpha = 1;\end{cases} \tag{3.39}$$

$$\omega_M(t,\alpha,\beta) := \begin{cases} (|t|^{\alpha-1}-1)\beta\tan\frac{\pi\alpha}{2}, & \alpha \neq 1,\\ -\beta\frac{2}{\pi}\log|t|, & \alpha = 1;\end{cases} \tag{3.40}$$

$$\omega_B(t,\alpha,\beta) := \begin{cases} \exp\bigl(-\mathrm{i}\frac{\pi}{2}\beta K(\alpha)\operatorname{sgn} t\bigr), & \alpha \neq 1,\\ \frac{\pi}{2} + \mathrm{i}\beta\log|t|\operatorname{sgn} t, & \alpha = 1,\end{cases} \tag{3.41}$$

with $K(\alpha) := \alpha - 1 + \operatorname{sgn}(1-\alpha)$, i.e.,

$$K(\alpha) := \begin{cases} \alpha, & 0 < \alpha < 1,\\ \alpha - 2, & 1 < \alpha \leqslant 2.\end{cases} \tag{3.42}$$

The ranges of the parameters are, in all three cases $\bullet \in \{A, B, M\}$,

$$0 < \alpha \leqslant 2, \qquad -1 \leqslant \beta_\bullet \leqslant 1, \qquad -\infty < \gamma_\bullet < \infty, \qquad 0 < \lambda_\bullet < \infty. \tag{3.43}$$

If $\alpha = 2$, we take $\beta_\bullet = 0$.

Here $\alpha \in (0, 2]$ is the same in all parametrisations and, with β, γ, δ is as in (3.6),

$$\beta_A = \beta_M = \beta, \tag{3.44}$$

$$\gamma_A = \delta/\gamma^\alpha, \tag{3.45}$$

$$\gamma_M = \mu_1/\gamma^\alpha = \begin{cases} \gamma_A + \beta\tan\frac{\pi\alpha}{2}, & \alpha \neq 1,\\ \gamma_A, & \alpha = 1,\end{cases} \tag{3.46}$$

$$\lambda_A = \lambda_M = \gamma^\alpha, \tag{3.47}$$

and, for $\alpha \neq 1$,

$$\tan\Bigl(\beta_B\frac{\pi K(\alpha)}{2}\Bigr) = \beta_A\tan\frac{\pi\alpha}{2} = \beta\tan\frac{\pi\alpha}{2}, \tag{3.48}$$

$$\gamma_B = \gamma_A\cos\Bigl(\beta_B\frac{\pi K(\alpha)}{2}\Bigr), \tag{3.49}$$

$$\lambda_B = \lambda_A \,/\, \cos\Bigl(\beta_B\frac{\pi K(\alpha)}{2}\Bigr), \tag{3.50}$$

while for $\alpha = 1$,

$$\beta_B = \beta_A = \beta, \tag{3.51}$$

$$\gamma_B = \frac{\pi}{2}\gamma_A = \frac{\pi\delta}{2\gamma}, \tag{3.52}$$

$$\lambda_B = \frac{2}{\pi}\lambda_A = \frac{2\gamma}{\pi}. \tag{3.53}$$

Note that, for any α, and every $\bullet \in \{A, B, M\}$,

$$\beta_\bullet = 0 \iff \beta = 0 \qquad \text{and} \qquad \beta_\bullet = \pm 1 \iff \beta = \pm 1, \tag{3.54}$$

and that for each fixed α, the mapping $\beta = \beta_A \mapsto \beta_B$ is an increasing homeomorphism of $[-1, 1]$ onto itself.

In the strictly stable case, Zolotarev [14] also uses

$$\varphi(t) = \exp\Bigl(-\lambda_C e^{-\mathrm{i}\,\mathrm{sgn}(t)\pi\alpha\theta/2}|t|^\alpha\Bigr), \tag{3.55}$$

which is the same as (3.22) with

$$\lambda_C = \lambda \tag{3.56}$$

$$\theta = -\widetilde{\gamma}/\alpha; \tag{3.57}$$

thus the ranges of the parameters are (excluding the case $\alpha = 1$ and $\theta = \pm 1$, which is possible in (3.55), but degenerate)

$$\begin{cases} |\theta| \leqslant 1, & \alpha < 1, \\ |\theta| < 1, & \alpha = 1, \\ |\theta| \leqslant 2/\alpha - 1, & \alpha > 1, \end{cases} \tag{3.58}$$

$$0 < \lambda_C < \infty. \tag{3.59}$$

We have

$$\theta = \begin{cases} \beta_B \frac{K(\alpha)}{\alpha}, & \alpha \neq 1, \\ \frac{2}{\pi}\arctan\bigl(\frac{2\gamma_B}{\pi}\bigr), & \alpha = 1. \end{cases} \tag{3.60}$$

$$\lambda_C = \begin{cases} \lambda_B, & \alpha \neq 1, \\ \lambda_B\bigl(\pi^2/4 + \gamma_B^2\bigr)^{1/2}, & \alpha = 1. \end{cases} \tag{3.61}$$

Zolotarev [14] uses in the strictly stable case also the parameters α, ρ, λ_C where

$$\rho := \frac{1+\theta}{2}. \tag{3.62}$$

Thus the range of ρ is

$$\begin{cases} 0 \leqslant \rho \leqslant 1, & \alpha < 1, \\ 0 < \rho < 1, & \alpha = 1, \\ 1 - 1/\alpha \leqslant \rho \leqslant 1/\alpha, & \alpha > 1. \end{cases} \tag{3.63}$$

Zolotarev [14] uses $Y(\alpha, \beta_\bullet, \gamma_\bullet, \lambda_\bullet) = Y_\bullet(\alpha, \beta_\bullet, \gamma_\bullet, \lambda_\bullet)$, where again $\bullet \in \{A, B, M\}$, as a notation for a random variable with the characteristic function (3.36)–(3.38); the parameters $\gamma_\bullet$ and $\lambda_\bullet$ may be omitted when $\gamma_\bullet = 0$ and $\lambda_\bullet = 1$. The distribution is a continuous function of the parameters $(\alpha, \beta_M, \gamma_M, \lambda_M)$. (The representations A and B are discontinuous at $\alpha = 1$.) Similarly, a random variable with the characteristic function (3.55) is denoted $Y(\alpha,\theta,\lambda_C) = Y_C(\alpha,\theta,\lambda_C)$, where λ_C may be omitted when $\lambda_C = 1$. We use $\mathrm{Y}_\bullet(\dots)$ for the distribution of $Y_\bullet(\dots)$.

The parameter ρ has a natural interpretation. (See Theorem 5.1 for a generalization.)

Theorem 3.18. *For a strictly stable random variable $Y_C(\alpha,\theta,\lambda)$,*

$$\mathbb{P}\bigl[Y_C(\alpha,\theta,\lambda) > 0\bigr] = \rho = \frac{1+\theta}{2}. \tag{3.64}$$

Proof. See, e.g., [14, Theorem 2.6.3] (in the special case $s = 0$). ☐

Corollary 3.19. *The strictly stable random variable $Y_C(\alpha,\theta,\lambda)$ is positive $\iff$ $\alpha < 1$ and $\rho = 1$ $\iff$ $\alpha < 1$ and $\theta = 1$.*

Similarly, $Y_C(\alpha,\theta,\lambda)$ is negative $\iff$ $\alpha < 1$ and $\rho = 0$ $\iff$ $\alpha < 1$ and $\theta = -1$.

Proof. By (3.64) and (3.63). ☐

Using Theorem 3.18, (3.60) and (3.48), the probability that a strictly stable random variable is positive can be expressed in α and β_B or β when $\alpha \neq 1$, and in γ_B or (using also (3.49)) γ and δ when $\alpha = 1$. In particular, this yields

$$\mathbb{P}\bigl[X_\alpha(\gamma,\beta,0) > 0\bigr] = \frac12 + \frac{1}{\alpha\pi}\arctan\Bigl(\beta\tan\frac{\pi\alpha}{2}\Bigr), \qquad \alpha \neq 1, \tag{3.65}$$

$$\mathbb{P}\bigl[X_1(\gamma,0,\delta) > 0\bigr] = \frac12 + \frac{1}{\pi}\arctan\Bigl(\frac{\delta}{\gamma}\Bigr). \tag{3.66}$$

Example 3.20. By Corollary 3.19, the positive strictly stable random variable $X_\alpha(\gamma,1,0)$ in Theorem 3.14 can also be described as $Y_C(\alpha,1,\lambda)$; here necessarily $0 < \alpha < 1$. This random variable has, using (3.56)–(3.57), (3.62), (3.27) and (3.23), the parameters

$$\beta = 1, \quad \theta = 1, \quad \rho = 1, \quad \widetilde{\gamma} = -\alpha, \quad \gamma^\alpha = \lambda\cos\frac{\pi\alpha}{2}, \tag{3.67}$$

and, by Theorem 3.13, the Laplace transform

$$\mathbb{E}\, e^{-tY_C(\alpha,1,\lambda)} = e^{-\lambda t^\alpha}, \qquad t \geqslant 0. \tag{3.68}$$

For $0 < \alpha < 1$, $Y_C(\alpha,1,\lambda)$ is thus the random variable in Example 3.16. ☐

We have a similar result for the extreme values in (3.58) and (3.63) also for the case $\alpha > 1$. (The Gaussian case $\alpha = 2$ is trivial; then necessarily $\theta = 0$ and $\rho = 1/2$ by (3.58) and (3.63).)

Theorem 3.21. *Let $1 < \alpha \leqslant 2$. The strictly stable random variable $Y_C(\alpha,\theta,\lambda)$ is spectrally positive $\iff$ $\rho = 1 - 1/\alpha$ $\iff$ $\theta = 1 - 2/\alpha$.*

Similarly, $Y_C(\alpha,\theta,\lambda)$ is spectrally negative $\iff$ $\rho = 1/\alpha$ $\iff$ $\theta = 2/\alpha - 1$.

Note that when $1 < \alpha < 2$, thus $\theta < 0$ in the spectrally positive case, and $\theta > 0$ in the spectrally negative case.

Proof. By Theorem 3.12, (3.54), (3.60), (3.42) and (3.62). ☐

Example 3.22. Let $1 < \alpha < 2$. By Theorem 3.21, the spectrally positive strictly stable random variable $X_\alpha(\gamma, 1, 0)$ in Theorem 3.12 can also be described as $Y_C(\alpha, \theta, \lambda)$ with $\theta = 1 - 2/\alpha$. This random variable has, using (3.27) and (3.23),

$$\beta = 1, \qquad \theta = 1 - \frac{2}{\alpha}, \qquad \rho = 1 - \frac{1}{\alpha}, \qquad \widetilde{\gamma} = 2 - \alpha, \qquad \gamma^\alpha = \lambda\bigl|\cos\frac{\pi\alpha}{2}\bigr|, \tag{3.69}$$

and, by Theorem 3.13, the Laplace transform

$$\mathbb{E}\, e^{-tY_C(\alpha,1,\lambda)} = e^{\lambda t^\alpha}, \qquad t \geqslant 0. \tag{3.70}$$

For $1 < \alpha < 2$, $Y_C(\alpha, 1 - 2/\alpha, \lambda)$ is thus the random variable in Example 3.17.

By (3.64), we have

$$\mathbb{P}\bigl[Y_C(\alpha, 1 - 2/\alpha, \lambda) > 0\bigr] = \rho = 1 - \frac{1}{\alpha}. \tag{3.71}$$

□

4. Stable densities

A stable distribution has by (3.6) a characteristic function that decreases rapidly as $t \to \pm\infty$, and thus the distribution has a density that is infinitely differentiable.

In the case $\alpha < 1$ and $\beta = 1$, $\mathrm{S}_\alpha(\gamma, \beta, \delta)$ has support $[\delta, \infty)$ and in the case $\alpha < 1$ and $\beta = -1$, $\mathrm{S}_\alpha(\gamma, \beta, \delta)$ has support $(-\infty, \delta]$; in all other cases the support is the entire real line. Moreover, the density function is strictly positive in the interior of the support, se Zolotarev [14, Remark 2.2.4].

Feller [3, Section XVII.6] lets, for $\alpha \neq 1$, $p(x; \alpha, \widetilde{\gamma})$ denote the density of the stable distribution with characteristic function (3.22) with $\lambda = 1$. A stable random variable with the characteristic function (3.22) thus has the density function $\lambda^{-1/\alpha} p(\lambda^{-1/\alpha} x; \alpha, \widetilde{\gamma})$. The density of a random variable $X_\alpha(\gamma, \beta, \delta)$ with $\alpha \neq 1$ is thus given by

$$\lambda^{-1/\alpha} p\bigl(\lambda^{-1/\alpha}(x - \delta); \alpha, \widetilde{\gamma}\bigr), \tag{4.1}$$

with λ and $\widetilde{\gamma}$ given by (3.23)–(3.24). (Cf. Remark 3.6.) By Remark 3.7, we have also

$$p(-x; \alpha, \widetilde{\gamma}) = p(x; \alpha, -\widetilde{\gamma}). \tag{4.2}$$

Zolotarev [14] uses $g_\bullet(x; \alpha, \beta_\bullet, \gamma_\bullet, \lambda_\bullet)$ for the density of the random variable $Y_\bullet(\alpha, \beta_\bullet, \gamma_\bullet, \lambda_\bullet)$ with characteristic function (3.36)–(3.38), and $g_\bullet(x; \alpha, \beta_\bullet)$ for the special case $\gamma_\bullet = 0$, $\lambda_\bullet = 1$; the index $\bullet \in \{A, M, B\}$ is often omitted (and often, but not always, taken as B); furthermore, $g(x; \alpha, \theta) = g_C(x; \alpha, \theta)$ is used for the density of the random variable $Y_C(\alpha, \theta)$ with characteristic function (3.55) with $\lambda_C = 1$. Thus, for $\alpha \neq 1$, see (3.56)–(3.57),

$$g_C(x; \alpha, \theta) = p(x; \alpha, -\alpha\theta). \tag{4.3}$$

By (3.55), we have also, in analogy with (4.2) (but now for all $0 < \alpha \leqslant 2$),

$$g_C(-x; \alpha, \theta) = g_C(x; \alpha, -\theta). \tag{4.4}$$

Feller [3, Lemma XVII.6.1] and Zolotarev [14, (2.4.8) and (2.4.6)] give the following series expansions for $p(x; \alpha, \widetilde{\gamma})$ and $g_C(x; , \alpha, \theta)$, repsectively; the latter using $\rho := (1 + \theta)/2$ as in (3.62). These expansions are equivalent by (4.3).

Theorem 4.1. (i) *If* $0 < \alpha < 1$ *and* $x > 0$, *then*

$$p(x;\alpha,\widetilde{\gamma}) = \frac{1}{\pi x}\sum_{k=1}^{\infty}\frac{\Gamma(k\alpha+1)}{k!}(-x^{-\alpha})^k \sin\frac{k\pi}{2}(\widetilde{\gamma}-\alpha), \tag{4.5}$$

$$g_C(x;\alpha,\theta) = \frac{1}{\pi}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\Gamma(k\alpha+1)}{k!}\sin(\pi k\rho\alpha)x^{-k\alpha-1}. \tag{4.6}$$

For $x < 0$, *use* (4.5)–(4.6) *together with* (4.2) *and* (4.4).

(ii) *If* $1 < \alpha \leqslant 2$ *and* $x \in (-\infty,\infty)$, *then*

$$p(x;\alpha,\widetilde{\gamma}) = \frac{1}{\pi x}\sum_{k=1}^{\infty}\frac{\Gamma(k/\alpha+1)}{k!}(-x)^k \sin\frac{k\pi}{2\alpha}(\widetilde{\gamma}-\alpha), \tag{4.7}$$

$$g_C(x;\alpha,\theta) = \frac{1}{\pi}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\Gamma(k/\alpha+1)}{k!}\sin(\pi k\rho)x^{k-1}. \tag{4.8}$$

□

Remark 4.2. The symmetry relations (4.2) and (4.4) are valid for all α, but not needed in Theorem 4.1 for $\alpha > 1$, since then (4.7)–(4.8) hold for all real x (with the obvious interpretation of (4.7) for $x = 0$). It can easily by verified directly that (4.7)–(4.8) satisfy (4.2) and (4.4). □

Example 4.3. The case $\alpha = 2$ is simple; then $\widetilde{\gamma} = 0$, $\theta = 0$ and $\rho = 1/2$ by (3.25), (3.58) and (3.63), and the characteristic function (3.55) shows that $Y_C(2,0) \sim N(0,2)$. Hence,

$$p(x;2,0) = g_C(x;2,0) = \frac{1}{2\sqrt{\pi}}e^{-x^2/4}, \tag{4.9}$$

which indeed has the series expansions (4.7)–(4.8). □

In particular, if $1 < \alpha \leqslant 2$, then (4.7) yields

$$p(0;\alpha,\widetilde{\gamma}) = \frac{1}{\pi}\Gamma(1+1/\alpha)\sin\frac{\pi(\alpha-\widetilde{\gamma})}{2\alpha}. \tag{4.10}$$

In the special case $1 < \alpha < 2$ and $\beta = 1$ we have $\widetilde{\gamma} = 2-\alpha$ by (3.27) and

$$\begin{aligned} p(0;\alpha,2-\alpha) &= \frac{1}{\pi}\Gamma(1+1/\alpha)\sin\frac{\pi(\alpha-1)}{\alpha} = \frac{1}{\pi}\Gamma(1+1/\alpha)\sin\frac{\pi}{\alpha} \\ &= \frac{\Gamma(1+1/\alpha)}{\Gamma(1/\alpha)\Gamma(1-1/\alpha)} = \frac{1}{\alpha\Gamma(1-1/\alpha)} = \frac{1}{|\Gamma(-1/\alpha)|}. \end{aligned} \tag{4.11}$$

For $1 < \alpha < 2$, the distribution $\mathrm{S}_\alpha(\gamma,1,0)$ thus has, by (4.1) and (3.23), the density at $x = 0$

$$\lambda^{-1/\alpha}p(0;\alpha,2-\alpha) = \frac{\lambda^{-1/\alpha}}{|\Gamma(-1/\alpha)|} = \gamma^{-1}\Bigl|\cos\frac{\pi\alpha}{2}\Bigr|^{1/\alpha}|\Gamma(-1/\alpha)|^{-1}. \tag{4.12}$$

4.1. **The case $\alpha = 1$.** The case $\alpha = 1$ was omitted in Theorem 4.1, since there is no similar simple formula, except when $\beta = 0$. However, we have the following power series expansion for $\alpha = 1$ and $\beta \neq 0$, given by Zolotarev [14].

Theorem 4.4. *Let $\alpha = 1$.*

(i) *If $\beta = 0$, then $\mathrm{S}_1(\gamma, 0, \delta)$ has the density function*

$$\frac{\gamma/\pi}{(x-\delta)^2+\gamma^2}, \qquad -\infty < x < \infty. \tag{4.13}$$

(ii) *If $\beta > 0$, then $Y_B(1, \beta, 0, 1) = X_1\big(\frac{\pi}{2}, \beta, 0\big)$ has the density function*

$$g_B(x; 1, \beta) = \frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^n c_n x^n, \tag{4.14}$$

with

$$c_n := \frac{1}{n!}\int_0^\infty e^{-\beta u \log u} \sin\big[(1+\beta)\frac{\pi}{2}u\big] u^n \,\mathrm{d}u \tag{4.15}$$

(iii) *If $\beta < 0$, then $Y_B(1, \beta, 0, 1) = X_1\big(\frac{\pi}{2}, \beta, 0\big)$ has the density function*

$$g_B(x; 1, \beta) = g_B(-x; 1, -\beta), \tag{4.16}$$

which is given by (4.14).

Proof. (i): This well-known formula follows directly by Fourier inversion of the characteristic function $\varphi(t) = e^{-\gamma|t|+\mathrm{i}\delta t}$.

(ii): Note first that if $\alpha = 1$, then (3.51)–(3.53) show that $\beta_B = \beta$, $\gamma_B = 0 \iff \delta = 0$, and $\lambda_B = 1 \iff \gamma = \pi/2$. Hence, $Y_B(1, \beta, 0, 1) = X_1\big(\frac{\pi}{2}, \beta, 0\big)$ as asserted.

The expansion (4.14)–(4.15) is [14, (2.4.7)] (with our c_n equal to $(n+1)b_{n+1}$ there).

(iii): This follows by (3.17). ◻

4.2. **Analyticity.** The density of any stable distribution $\mathrm{S}_\alpha(\gamma, \beta, \delta)$ is, as said above, infinitely differentiable. Moreover, it is easy to see from Theorems 4.1 and 4.4 that this density is real analytic for $x \neq \delta$. At $x = \delta$, the situation differs for $\alpha < 1$ and $\alpha \geqslant 1$, as shown by the following result.

Theorem 4.5. *Consider the density $p(x)$ of $X \sim \mathrm{S}_\alpha(\gamma, \beta, \delta)$.*

(i) *If $\alpha \geqslant 1$, then $p(x)$ is real analytic on $(-\infty, \infty)$.*

(ii) *If $\alpha < 1$, then $p(x)$ is real analytic on $\mathbb{R} \setminus \{\delta\}$, but not at δ (although it is infinitely differentiable there too).*

Proof. (i): For $\alpha \neq 1$, by (4.1), it suffices to consider $p(x; \alpha, \widetilde{\gamma})$, and the analyticity follows from (4.7).

For $\alpha = 1$, analyticity follows from (4.13), (4.14) or (4.16) (depending on β), together with a liear change of variable.

(ii): Again, by (4.1) it suffices to consider $p(x; \alpha, \widetilde{\gamma})$, and thus $\delta = 0$. The analyticity for $x > 0$ follows from (4.5), and then for $x < 0$ from (4.2). These also show that $p(x) = p(x; \alpha, \widetilde{\gamma})$ extends to an analytic function $p(z)$ in each of the half planes $\operatorname{Re} z < 0$ and $\operatorname{Re} z > 0$, with

$$|p(z)| = O\big(|z|^{-1-\alpha}\big), \qquad |z| \geqslant 1. \tag{4.17}$$

Suppose that p is real analytic also at $x=0$. Then p would extend to an analytic function in a neighbourhood of 0, and thus the extensions would combine to an analytic extension in a strip $|\operatorname{Im} z|<2\varepsilon$ for some $\varepsilon>0$. The characteristic function $\varphi(t)$ then would be given by, by a shift of the line of integration using Cauchy's integral formula and the bound (4.17),

$$\varphi(t)=\int_{-\infty}^{\infty} e^{\mathrm{i}tx}p(x)\,\mathrm{d}x=\int_{-\infty}^{\infty} e^{\mathrm{i}t(x+\mathrm{i}\varepsilon)}p(x+\mathrm{i}\varepsilon)\,\mathrm{d}x, \qquad t\in\mathbb{R}, \tag{4.18}$$

and thus, by (4.17) again,

$$\bigl|\varphi(t)\bigr|\leqslant e^{-\varepsilon t}\int_{-\infty}^{\infty}\bigl|p(x+\mathrm{i}\varepsilon)\bigr|\,\mathrm{d}x=Ce^{-\varepsilon t}, \qquad t\in\mathbb{R}, \tag{4.19}$$

which for $\alpha<1$ contradicts the explicit expression (3.6). This contradiction shows that $p(x)$ is not analytic at $x=0=\delta$. □

Remark 4.6. The proof yields also the following. For $\alpha>1$, and for $\alpha=1$ and $\beta\neq 0$, the density $p(x)$ of $\mathrm{S}_\alpha(\gamma,\beta,\delta)$ extends to an entire analytic function on $\mathbb{C}$. In the (strictly stable) case $\alpha=1$ and $\beta=0$, the explicit formula (4.13) shows that that $p(x)$ extends to a meromorphic, but not entire, function on $\mathbb{C}$. For $\alpha<1$, the restrictions of $p(x)$ to $(-\infty,\delta)$ and (δ,∞) extend to analytic functions $p_+(z)$ and $p_-(z)$ in the slit planes $\mathbb{C}\setminus[\delta,\infty)$ and $\mathbb{C}\setminus[-\infty,\delta)$, respectively, but these two extensions are not equal.

To verify the claim that $p_+\neq p_-$ when $\alpha<1$, it again suffices to consider the case $\lambda=1$ and $\delta=0$, when the density is $p(x;\alpha,\widetilde{\gamma})$. Note that $p_+(x)$ is obtained by extending (4.5) to complex $x\notin(-\infty,0]$. In particular, it has a jump across the cut that satisfies

$$\begin{aligned}
&\lim_{x\to-\infty}|x|^{1+\alpha}\bigl[p_+(x+0\mathrm{i};\alpha,\widetilde{\gamma})-p_+(x-0\mathrm{i};\alpha\widetilde{\gamma})\bigr]\\
&\qquad=\frac{\Gamma(\alpha+1)}{\pi}\bigl(e^{-\mathrm{i}\alpha\pi}-e^{\mathrm{i}\alpha\pi}\bigr)\sin\bigl(\frac{\pi}{2}(\widetilde{\gamma}-\alpha)\bigr)\\
&\qquad=2\mathrm{i}\frac{\Gamma(\alpha+1)}{\pi}\sin(\alpha\pi)\sin\bigl(\frac{\pi}{2}(\alpha-\widetilde{\gamma})\bigr).
\end{aligned} \tag{4.20}$$

If $\widetilde{\gamma}\in[-\alpha,\alpha)$, then this limit is non-zero, and thus p_+ has a jump across the cut at least for large $|x|$. On the other hand, p_- is analytic across the negative real axis. If $\widetilde{\gamma}=\alpha$, we have $p_+(x)=0$, and again we see that p_+ and p_- are different. □

Example 4.7. Consider a positive strictly stable variable; thus $\alpha<1$, $\delta=0$ and $\widetilde{\gamma}=-\alpha$ by Theorem 3.14 and Example 3.16. We then have $p(x;\alpha,-\alpha)=0$ for $x\leqslant 0$ but $p(x;\alpha,-\alpha)>0$ for $x>0$; hence, it is in this case obvious that the density p is not analytic at 0, as claimed in Theorem 4.5. (See Example 6.3 for a concrete example.) □

4.3. **Duality.** There is a duality due to Zolotarev between the densities of the distributions of strictly stable random variables with parameters α and $1/\alpha$, valid at least for part of the ranges.

Theorem 4.8 (Zolotarev [14], Feller [3])**.** *Let* $1 \leqslant \alpha \leqslant 2$ *and* $|\theta| \leqslant 2/\alpha - 1$*, cf.* (3.58)*. Define* θ' *by*

$$\theta' = \alpha(1+\theta) - 1 \in [2\alpha - 3, 1]. \tag{4.21}$$

Then,

$$g_C(x;\alpha,\theta) = x^{-\alpha-1} g_C\bigl(x^{-\alpha};\alpha^{-1},\theta'\bigr), \qquad x > 0. \tag{4.22}$$

Equivalently, if $0 \leqslant A \leqslant B \leqslant \infty$*, then*

$$\mathbb{P}\bigl[A < Y_C(\alpha,\theta) < B\bigr] = \frac{1}{\alpha}\,\mathbb{P}\bigl[B^{-\alpha} < Y_C(\alpha^{-1},\theta') < A^{-\alpha}\bigr]. \tag{4.23}$$

Hence,

$$\bigl(Y_C(\alpha,\theta)^{-\alpha} \mid Y_C(\alpha,\theta) > 0\bigr) \overset{\mathrm{d}}{=} \bigl(Y_C\bigl(\alpha^{-1},\theta'\bigr) \mid Y_C\bigl(\alpha^{-1},\theta'\bigr) > 0\bigr). \tag{4.24}$$

If $1 < \alpha < 2$*, we have, equivalently,*

$$p(x;\alpha,\widetilde{\gamma}) = x^{-\alpha-1} p\bigl(x^{-\alpha};\alpha^{-1},\gamma^*\bigr) \tag{4.25}$$

with

$$\gamma^* := \alpha^{-1}(\widetilde{\gamma}+1) - 1. \tag{4.26}$$

Note that the spectrally negative case $\theta = 2/\alpha - 1$ corresponds to the positive case $\theta' = 1$. (See Theorem 3.21 and Corollary 3.19.)

Proof. The relation (4.22) is [14, (2.3.3)], and it is equivalent to (4.23) by integration (or, conversely, by differentiating (4.23) with respect to B). The conditional version (4.24) follows from (4.23) (and is equivalent to it if we also use Theorem 3.18).

Furthermore, for $1 < \alpha < 2$, (4.25) is [3, Lemma XVII.6.2] (with a change of variable), and it is equivalent to (4.22) by (4.3).

Note also that the cases $\alpha = 1$ and $\alpha = 2$ in (4.23) follow by continuity from the case $1 < \alpha < 2$, since the distribution of $Y_C(\alpha,\theta)$ is a continuous function of (α,θ) by (3.55) (with $\lambda = 1$). ☐

The relation (4.21) can also be written, using (3.62) and (3.63),

$$\rho' = \alpha\rho \in [\alpha - 1, 1]. \tag{4.27}$$

(The case $A = 0$, $B = \infty$ in (4.23) thus is in accordance with Theorem 3.18.)

Note that for $1 < \alpha \leqslant 2$, (4.21) does not cover the whole range of θ' allowed for $Y_C(\alpha^{-1},\theta',1)$, and similarly for (4.26).

For $x < 0$, we may as usual change signs by (4.2) and (4.4), but note that this will change the relations (4.21) and (4.26). Theorem 4.8 implies, still for $1 \leqslant \alpha \leqslant 2$ and $|\theta| \leqslant 2/\alpha - 1$,

$$\begin{aligned} g_C(x,\alpha,\theta) &= g_C(|x|;\alpha,-\theta) = |x|^{-1-\alpha} g_C\bigl(|x|^{-\alpha};\alpha^{-1},-\theta''\bigr) \\ &= |x|^{-1-\alpha} g_C\bigl(-|x|^{-\alpha};\alpha^{-1},\theta''\bigr), \qquad x < 0, \end{aligned} \tag{4.28}$$

with

$$\theta'' = 1 - \alpha(1-\theta) = 1 - \alpha + \alpha\theta \in [-1, 3 - 2\alpha]. \tag{4.29}$$

4.4. **Density at 0 and ∞.** As said above, the density $g_C(x;\alpha,\theta)$ of a strictly stable distribution $\mathrm{Y}_C(\alpha,\theta)=\mathrm{Y}_C(\alpha,\theta,1)$ is always continuous at $x=0$ (although not always analytic there). Its value is given by a simple formula.

Theorem 4.9. *For every $\alpha\in(0,2]$ and θ satisfying* (3.58),

$$g_C(0;\alpha,\theta)=\frac{1}{\pi}\Gamma\Bigl(1+\frac{1}{\alpha}\Bigr)\cos\Bigl(\frac{\pi}{2}\theta\Bigr)=\frac{1}{\pi}\Gamma\Bigl(1+\frac{1}{\alpha}\Bigr)\sin\bigl(\pi\rho\bigr). \tag{4.30}$$

Proof. The case $\alpha\neq 1$ is [14, (2.2.11)], together with (3.60) and (3.62).

If $\alpha=1$, then $\mathrm{Y}_C(1,\theta)=\mathrm{S}_1(\cos\frac{\pi\theta}{2},0,\sin\frac{\pi\theta}{2})$ by (3.55) and (3.6) (or by (3.56)–(3.57) and (3.23)–(3.24)), and (4.30) follows by (4.13). ☐

As $x\to\infty$, we have a corresponding simple asymptotic formula.

Theorem 4.10. *For every $\alpha\in(0,2]$ and θ satisfying* (3.58),

$$g_C(x;\alpha,\theta)=\frac{1}{\pi}\Gamma(1+\alpha)\sin\bigl(\pi\alpha\rho\bigr)x^{-1-\alpha}+O\bigl(x^{-1-2\alpha}\bigr),\qquad x\to+\infty. \tag{4.31}$$

Proof. If $\alpha<1$, then (4.31) is immediate from (4.6).

If $\alpha=1$, then (4.31) follows from (4.13), noting again that $\mathrm{Y}(1,\theta)=\mathrm{S}_1(\cos\frac{\pi\theta}{2},0,\sin\frac{\pi\theta}{2})$ and that $\cos\bigl(\pi\theta/2\bigr)=\sin(\pi\rho)$.

If $\alpha>1$, then (4.31) follows from (4.22), (4.27), and (4.30) (applied to α^{-1} and $\rho':=\alpha\rho$). ☐

5. One-sided moments

It is well-known, that for an α-stable random variable X with $\alpha\neq 2$, and $s>0$, we have

$$\mathbb{E}\,|X|^s<\infty \iff 0<s<\alpha. \tag{5.1}$$

For strictly stable random variables, these absolute moments can be calculated explicitly. Moreover, in this case, we can find the moments of the positive and negative parts of X. We use the general notation $\mathbb{E}\bigl[X;\mathcal{E}\bigr]:=\mathbb{E}\bigl[X\cdot\mathbf{1}\{\mathcal{E}\}\bigr]=\int_{\mathcal{E}}X\,\mathrm{d}\,\mathbb{P}$ for a random variable X and an event $\mathcal{E}$. We then have the following formulas. Recall that $\lambda_C=\lambda$ by (3.56).

Theorem 5.1. *If $Y=Y_C(\alpha,\theta,\lambda)$ and $\rho=(1+\theta)/2$, then, for complex s with $-1<\operatorname{Re}s<\alpha$,*

$$\mathbb{E}\bigl[Y^s;Y>0\bigr]=\lambda^{s/\alpha}\frac{\sin\pi\rho s}{\sin\pi s}\frac{\Gamma(1-s/\alpha)}{\Gamma(1-s)} \tag{5.2}$$

$$=\frac{1}{\pi}\lambda^{s/\alpha}\sin(\pi\rho s)\Gamma(s)\Gamma(1-s/\alpha), \tag{5.3}$$

$$=\lambda^{s/\alpha}\frac{\Gamma(s)\Gamma(1-s/\alpha)}{\Gamma(\rho s)\Gamma(1-\rho s)}, \tag{5.4}$$

and

$$\mathbb{E}\bigl[|Y|^s;Y<0\bigr]=\lambda^{s/\alpha}\frac{\sin\pi(1-\rho)s}{\sin\pi s}\frac{\Gamma(1-s/\alpha)}{\Gamma(1-s)} \tag{5.5}$$

$$= \frac{1}{\pi}\lambda^{s/\alpha}\sin(\pi(1-\rho)s)\Gamma(s)\Gamma(1-s/\alpha) \tag{5.6}$$

$$= \lambda^{s/\alpha}\frac{\Gamma(s)\Gamma(1-s/\alpha)}{\Gamma((1-\rho)s)\Gamma(1-(1-\rho)s)}, \tag{5.7}$$

Proof. Zolotarev [14, Theorem 2.6.3] and homogeneity give (5.2), and then (5.3)–(5.4) follow from the reflection formula $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$.

Since $-Y \overset{\mathrm{d}}{=} Y_C(\alpha, -\theta, \lambda)$, and $(1-\theta)/2 = 1-\rho$, then (5.5)–(5.7) follow. ☐

The absolute moment $\mathbb{E}\,|Y|^s$ is obtained by summing (5.2) and (5.5).

Note that the special case $s = 0$ (when the formulas are interpreted in the obvious ways, taking limits) yields $\mathbb{P}[Y > 0] = \rho$ and $\mathbb{P}[Y < 0] = 1-\rho$, as stated in Theorem 3.18. Consequently, we obtain the conditional moments $\mathbb{E}\bigl[Y^s \mid Y > 0\bigr]$ and $\mathbb{E}\bigl[|Y|^s \mid Y < 0\bigr]$ by dividing (5.2)–(5.4) and (5.5)–(5.7) by ρ and $1-\rho$, respectively.

When $\operatorname{Re} s > 0$, we can also interpret (5.2)–(5.7) as the moments of $Y_+ := \max\{Y, 0\}$ and $Y_- := \max\{-Y, 0\}$.

Remark 5.2. If Y has density $p(x)$, then $\mathbb{E}\bigl[Y^s; Y > 0\bigr] = \int_0^\infty x^s p(x)\,\mathrm{d}x$ and $\mathbb{E}\bigl[|Y|^s; Y < 0\bigr] = \int_0^\infty x^s p(-x)\,\mathrm{d}x$. Hence, (5.2)–(5.7) can be regarded as formulas for the Mellin transforms of p restricted to the positive and negative half-axes. ☐

Remark 5.3. The range $-1 < \operatorname{Re} s < \alpha$ in Theorem 5.1 is in most cases optimal. In fact, it follows from (5.3) that $\mathbb{E}\bigl[Y^s; Y > 0\bigr]$ has a pole as $s = -1$ unless $\sin(-\pi\rho) = 0$, i.e., $\rho = 0$ or $\rho = 1$; in both cases $\alpha < 1$ by (3.63). Similarly, (5.3) shows that $s = \alpha$ is a pole unless $\sin(\pi\rho\alpha) = 0$, i.e., $\rho = 0$ (and then $\alpha < 1$), or $\rho = 1/\alpha$ (and then $\alpha > 1$). These exceptional cases are treated in the examples below. In all other cases, we thus have poles at -1 and α, and, consequently, $\mathbb{E}\bigl[Y^s; Y > 0\bigr] = \infty$ for $s \leqslant -1$ or $s \geqslant \alpha$. ☐

Example 5.4. If $\alpha < 1$ and $\rho = 0$, then $Y < 0$ a.s. by Theorem 3.18, and thus, trivially, $\mathbb{E}\bigl[Y^s; Y > 0\bigr] = 0$ for all s, which agrees with (5.2). ☐

Example 5.5. If $\alpha < 1$ and $\rho = 1$, then $Y > 0$ a.s. by Theorem 3.18, i.e., Y is a positive strictly stable random variable as in Example 3.16. Hence its infinitely differentiable density $p(x)$ vanishes on $(-\infty, 0)$, and thus has all derivates $= 0$ at 0, whence $p(x) = O(x^N)$ as $x \to 0$ for any $N > 0$. It follows that $\mathbb{E}\bigl[Y^s\bigr]$ is finite for all $s < 0$, and thus analytic in $\operatorname{Re} s < \alpha$. By Remark 5.3, there is a pole at α. By (5.2) and analytic continuation,

$$\mathbb{E}\bigl[Y^s\bigr] = \lambda^{s/\alpha}\frac{\Gamma(1-s/\alpha)}{\Gamma(1-s)}, \qquad \operatorname{Re} s < \alpha. \tag{5.8}$$

☐

Example 5.6. If $1 < \alpha < 2$ and $\rho = 1/\alpha$, then Y is spectrally negative by Theorem 3.21. Hence, by Theorem 3.13 and a change of signs, the moment generating function $\mathbb{E}\,e^{tY} < \infty$ for every $t \geqslant 0$, and it follows that $\mathbb{E}\bigl[Y^s; Y > 0\bigr] < \infty$ for all

$s > 0$. Hence, (5.4) and analytic continuation yield

$$\mathbb{E}\big[Y^s; Y > 0\big] = \lambda^{s/\alpha} \frac{\Gamma(s)}{\Gamma(s/\alpha)}, \qquad \operatorname{Re} s > -1. \tag{5.9}$$

This holds for $\alpha = 2$ too, when Theorem 3.13 as stated does not apply, because then Y is normal and the moment generating function is finite everywhere. $\square$

Example 5.7. If $1 < \alpha < 2$ and $\rho = 1 - 1/\alpha$, then Y is spectrally positive by Theorem 3.21, and $-Y$ is as in Example 5.6. Hence, $\mathbb{E}\big[|Y|^s; Y < 0\big]$ is finite for $\operatorname{Re} s > -1$, while $\mathbb{E}\big[Y^s; Y > 0\big]$ has a pole at α. $\square$

6. Some examples

Example 6.1 ($\alpha = 2$)**.** The case $\alpha = 2$ is simple, and also exceptional in several ways. By (3.6), the distribution $\mathrm{S}_2(\gamma, \beta, \delta)$ has characteristic function

$$\varphi(t) = e^{\mathrm{i}\delta t - \gamma^2 t^2}, \tag{6.1}$$

and thus a 2-stable distribution is nornal: $\mathrm{S}_2(\gamma, \beta, \delta) = N(\delta, 2\gamma^2)$. As said in Theorem 3.3, this distribution does not depend on β, and we take $\beta = 0$.

Conversely, we see that a normal distribution $N(\mu, \sigma^2)$ is 2-stable, with, by (6.1) and (3.36)–(3.38),

$$\gamma = \frac{1}{\sqrt{2}}\sigma, \quad \delta = \mu, \quad \lambda_A = \lambda_B = \lambda_M = \frac{1}{2}\sigma^2, \quad \gamma_A = \gamma_B = \gamma_M = \frac{2\mu}{\sigma^2}. \tag{6.2}$$

The distribution is strictly stable if and only if its mean $\mu = 0$ (see Remark 3.4), and then we further have, by (3.19), (3.22), (3.55), and (3.62),

$$\kappa = \lambda = \lambda_C = \frac{1}{2}\sigma^2, \quad \tau = \widetilde{\gamma} = \theta = 0, \quad \rho = \frac{1}{2}. \tag{6.3}$$

(Cf. (3.25), (3.58), (3.63).)

In particular, $\mathrm{S}_2(1, 0, 0) = \mathrm{Y}_C(\frac{1}{2}, 0)$ has the density

$$p(x; 2, 0) = g_C(x; 2, 0) = \frac{1}{2\sqrt{\pi}} e^{-x^2/4}. \tag{6.4}$$

The normal distribution has Lévy measure $\Lambda = 0$, and the canonical measure M is a point mass at $\{0\}$, with $M\{0\} = \sigma^2$; see (2.5) and (2.3). $\square$

Example 6.2 ($\alpha = 1$)**.** The Cauchy distribution has density

$$f(x) = \frac{1}{\pi(1 + x^2)}, \qquad -\infty < x < \infty, \tag{6.5}$$

and characteristic function

$$\varphi(t) = e^{-|t|}, \qquad -\infty < t < \infty. \tag{6.6}$$

The Cauchy distribution is thus strictly 1-stable. More precisely, by (3.6), it is $\mathrm{S}_1(1, 0, 0) = \mathrm{S}_1(0)$; see also Theorem 4.4(i). We thus have, using also (3.19) or (3.20), (3.22)–(3.24), (3.44)–(3.47), (3.51)–(3.53), (3.56)–(3.57), and (3.62),

$$\gamma = 1, \quad \beta = \delta = 0, \quad \kappa = 1, \quad \tau = 0, \quad \lambda = 1, \quad \widetilde{\gamma} = 0,$$

$$\beta_A = \beta_B = \beta_M = 0, \quad \gamma_A = \gamma_B = \gamma_M = 0, \quad \lambda_A = \lambda_M = \lambda_C = 1, \quad \lambda_B = 2/\pi,$$
$$\theta = 0, \quad \rho = \frac{1}{2}. \tag{6.7}$$

By Theorem 3.3, the strictly 1-stable distributions are $\mathrm{S}_1(\gamma, 0, \delta)$, and by (3.15),

$$X_1(\gamma, 0, \delta) \overset{\mathrm{d}}{=} \gamma X_1(0) + \delta. \tag{6.8}$$

In other words, the strictly 1-stable distributions are precisely the linear transformations of the Cauchy distribution.

If we normalize to $\gamma = 1$, we have, generalizing (6.7), that the strictly stable distribution $\mathrm{S}_1(1, 0, \delta)$ has, by Remark 3.10, (3.19), (3.22), (3.24), (3.44)–(3.47), (3.51)–(3.54), (3.56)–(3.57), and (3.62),

$$\kappa = \gamma = 1, \quad \tau = \delta, \quad \lambda = \lambda_C = \sqrt{1+\delta^2}, \quad \widetilde{\gamma} = -\frac{2}{\pi}\arctan\delta,$$
$$\beta = \beta_A = \beta_B = \beta_M = 0, \quad \gamma_A = \gamma_M = \delta, \quad \gamma_B = \frac{\pi\delta}{2}, \quad \lambda_A = \lambda_M = 1,$$
$$\lambda_B = \frac{2}{\pi}, \quad \theta = \frac{2}{\pi}\arctan\delta, \quad \rho = \frac{1}{2} + \frac{1}{\pi}\arctan\delta. \tag{6.9}$$

□

Example 6.3 ($\alpha = 1/2$)**.** The positive $\frac{1}{2}$-stable distribution is closely connected to the normal distribution and Brownian motion.

One way to see this is to consider a standard Brownian motion B_t, $0 \leqslant t < \infty$, and for $a \geqslant 0$ let T_a be the hitting time $T_a := \inf\{t \geqslant 0 : B_t \geqslant a\}$. Then, by Brownian scaling, $T_a \overset{\mathrm{d}}{=} a^2 T_1$, and by the strong Markov property, $T_{a+b} - T_a \overset{\mathrm{d}}{=} T_b$, for $a, b \geqslant 0$. Hence, if $X = T_1$, then

$$S_n := \sum_{i=1}^n X_i \overset{\mathrm{d}}{=} T_n \overset{\mathrm{d}}{=} n^2 X, \tag{6.10}$$

which shows that $X = T_1$ is strictly $\frac{1}{2}$-stable. Obviously, $T_1 > 0$. More generally, $(T_a)_{a \geqslant 0}$ is an increasing stable process (i.e., a Lévy process with stable increments, see Remark 2.4 and e.g. [2]).

A simple calculation using the martingale $e^{\sqrt{2t}B_x - tx}$, $x \geqslant 0$, see e.g. [12, Proposition II.3.7], gives the Laplace transform

$$\mathbb{E}\, e^{-tT_1} = e^{-\sqrt{2t}}, \qquad t \geqslant 0. \tag{6.11}$$

Hence, by Example 3.16 (with $\lambda = \sqrt{2}$), $T_1 \sim \mathrm{S}_{1/2}(1, 1, 0)$. Using also Theorem 3.14, (3.20), (3.27), (3.44)–(3.50), (3.56)–(3.57), and (3.62),

$$\gamma = 1, \quad \beta = 1, \quad \delta = 0, \quad \kappa = 1, \quad \tau = 1, \quad \lambda = \lambda_C = \sqrt{2}, \quad \widetilde{\gamma} = -\frac{1}{2},$$
$$\beta_A = \beta_B = \beta_M = 1, \quad \gamma_A = \gamma_B = 0, \quad \gamma_M = 1, \quad \lambda_A = \lambda_M = 1, \quad \lambda_B = \sqrt{2},$$
$$\theta = \rho = 1. \tag{6.12}$$

More generally, for any $a > 0$, $T_a \sim \mathrm{S}_{1/2}(a^2, 1, 0) = \mathrm{Y}_C(1/2, 1, a\sqrt{2})$.

Moreover, using the reflection principle [12, Proposition III.3.7], for any $x > 0$,

$$\mathbb{P}(T_1 \leqslant x) = \mathbb{P}\bigl(\sup_{0\leqslant t\leqslant x} B_t \geqslant 1\bigr) = 2\,\mathbb{P}\bigl(B_x \geqslant 1\bigr) = \mathbb{P}\bigl(|B_x| \geqslant 1\bigr)$$
$$= \mathbb{P}\bigl(x^{1/2}|B_1| \geqslant 1\bigr) = \mathbb{P}\bigl(|B_1|^2 \geqslant 1/x\bigr) = \mathbb{P}\bigl(|B_1|^{-2} \leqslant x\bigr). \qquad (6.13)$$

Hence,

$$T_1 \overset{\mathrm{d}}{=} B_1^{-2}, \qquad \text{where } B_1 \sim N(0,1). \qquad (6.14)$$

In other words, if $Z \sim N(0,1)$, then $Z^{-2} \sim \mathrm{S}_{1/2}(1,1,0) = \Upsilon_C(1/2,1,\sqrt{2})$.

From (6.14), T_1 has the density

$$f_{T_1}(x) = \frac{1}{\sqrt{2\pi x^3}} e^{-1/(2x)}, \qquad x > 0. \qquad (6.15)$$

This follows also from (4.22). Hence, if $X \sim \mathrm{S}_{1/2}(\gamma,1,0) = \Upsilon_C(1/2,1,\sqrt{2\gamma})$, then $X \overset{\mathrm{d}}{=} \gamma T_1$ has density

$$f_X(x) = \frac{\gamma^{1/2}}{\sqrt{2\pi x^3}} e^{-\gamma/(2x)}, \qquad x > 0. \qquad (6.16)$$

Taking $\gamma = 1/2$, we find

$$g_C(x;1/2,1) = \frac{1}{2\sqrt{\pi x^3}} e^{-1/(4x)}, \qquad x > 0, \qquad (6.17)$$

which agrees with (4.22) and (6.4). □

Example 6.4 ($\alpha = 3/2$)**.** Banderier, Flajolet, Schaeffer and Soria [1] define a $\frac{3}{2}$-stable distribution, by them called *the Airy distribution of map type*; it has a density $\mathcal{A}(x)$ given by [1, (B.2)]

$$\mathcal{A}(x) = \frac{1}{2\pi\mathrm{i}} \int_{-\infty\mathrm{i}}^{\infty\mathrm{i}} e^{-xt+t^{3/2}/3}\,\mathrm{d}t = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\mathrm{i}xt+(\mathrm{i}t)^{3/2}/3}\,\mathrm{d}t, \qquad (6.18)$$

which can be recognized as the inversion formula for a distribution with characteristic function

$$\varphi(t) = e^{(\mathrm{i}t)^{3/2}/3} = \exp\Bigl(-\frac{1}{3} e^{-\mathrm{i}\pi\,\mathrm{sgn}(t)/4}|t|^{3/2}\Bigr) = \exp\Bigl(-\frac{1}{3\sqrt{2}}(1-\mathrm{i}\,\mathrm{sgn}(t))|t|^{3/2}\Bigr). \qquad (6.19)$$

This is thus (as noted in [1]) a $\frac{3}{2}$-stable distribution; more precisely, by comparing with (3.19) and (3.55), we see that this is the strictly stable distribution with, using also (3.62),

$$\alpha = 3/2, \quad \kappa = \tau = \frac{1}{3\sqrt{2}} = \frac{1}{\sqrt{18}}, \quad \lambda_C = \frac{1}{3}, \quad \theta = \frac{1}{3}, \quad \rho = \frac{2}{3}. \qquad (6.20)$$

We find also, using (3.21), (3.22) or (3.28), (3.44)–(3.50), (3.56), (3.61),

$$\gamma = 2^{-1/3}3^{-2/3} = 18^{-1/3}, \quad \beta = -1, \quad \delta = 0, \quad \lambda = \frac{1}{3}, \quad \widetilde{\gamma} = -\frac{1}{2},$$
$$\beta_A = \beta_B = \beta_M = -1, \quad \gamma_A = \gamma_B = 0, \quad \gamma_M = 1, \quad \lambda_A = \lambda_M = \frac{1}{3\sqrt{2}},$$

$$\lambda_B = \lambda_C = \frac{1}{3}. \tag{6.21}$$

The distribution is thus spectrally negative. If X has this distribution, then by (3.33) applied to $-X$,

$$\mathbb{E}\, e^{tX} = \exp\bigl(\tfrac{1}{3}t^{3/2}\bigr), \qquad \operatorname{Re} t \geqslant 0. \tag{6.22}$$

It is shown in [1] that the density (6.18) also can be expressed as

$$\mathcal{A}(x) := 2e^{-2x^3/3}\bigl(x\operatorname{Ai}(x^2) - \operatorname{Ai}'(x^2)\bigr), \qquad -\infty < x < \infty, \tag{6.23}$$

where $\operatorname{Ai}(x)$ is the Airy function [10, Chapter 9].

This distribution is of the type in Example 5.6, and (5.9) yields

$$\int_0^\infty x^s \mathcal{A}(x)\,\mathrm{d}x = 3^{-2s/3}\frac{\Gamma(s)}{\Gamma(2s/3)}, \qquad \operatorname{Re} s > -1. \tag{6.24}$$

For the negative side, we have by (5.6) and the reflection formula for the Gamma function,

$$\begin{aligned}\int_{-\infty}^0 |x|^s \mathcal{A}(x)\,\mathrm{d}x &= \frac{1}{\pi}3^{-2s/3}\sin\frac{\pi s}{3}\,\Gamma(s)\Gamma(1-2s/3)\\ &= 3^{-2s/3}\frac{\sin\frac{\pi s}{3}}{\sin\frac{2\pi s}{3}}\frac{\Gamma(s)}{\Gamma(2s/3)}\\ &= 2^{-1}3^{-2s/3}\frac{1}{\cos\frac{\pi s}{3}}\frac{\Gamma(s)}{\Gamma(2s/3)}, \qquad -1 < \operatorname{Re} s < 3/2. \end{aligned} \tag{6.25}$$

The formulas (6.24) and (6.25) are equivalent to [1, (B.5)–(B.6)].

By (6.21) and (4.1), the density

$$\mathcal{A}(x) = g_C\bigl(x; 3/2, 1/3, 1/3\bigr) = 3^{2/3}p\bigl(3^{2/3}x; 3/2, -1/2\bigr) \tag{6.26}$$

and thus, by (4.3) and (6.23),

$$\begin{aligned} g_C\bigl(x; 3/2, 1/3\bigr) &= p\bigl(x; 3/2, -1/2\bigr) = 3^{-2/3}\mathcal{A}\bigl(3^{-2/3}x\bigr)\\ &= 2\cdot 3^{-2/3}e^{-2x^3/27}\Bigl(3^{-2/3}x\operatorname{Ai}\bigl(3^{-4/3}x^2\bigr) - \operatorname{Ai}'\bigl(3^{-4/3}x^2\bigr)\Bigr). \end{aligned} \tag{6.27}$$

An alternative formula using the Whittaker function $W_{\kappa,\mu}$ [10, §13.14] is [14, (2.8.34) with a typo]:

$$g_C\bigl(x; 3/2, 1/3\bigr) = \frac{\sqrt{3}}{\sqrt{\pi}}x^{-1}e^{-2x^3/27}W_{1/2,1/6}\Bigl(\frac{4x^3}{27}\Bigr), \qquad x > 0. \tag{6.28}$$

For the negative side we have, by (4.4) and [14, (2.8.35)],

$$\begin{aligned} g_C\bigl(x; 3/2, 1/3\bigr) &= g_C\bigl(|x|; 3/2, -1/3\bigr)\\ &= \frac{1}{2\sqrt{3\pi}}|x|^{-1}e^{2|x|^3/27}W_{-1/2,1/6}\Bigl(\frac{4|x|^3}{27}\Bigr), \qquad x < 0. \end{aligned} \tag{6.29}$$

Of course, the corresponding spectrally positive distribution $\mathrm{Y}_C(3/2, -1/3)$ has density $g_C(-x; 3/2, 1/3)$ obtained by switching (6.28) and (6.29). □

Example 6.5 ($\alpha = 2/3$)**.** The positive strictly $\frac{2}{3}$-stable distribution with Laplace transform

$$\mathbb{E}\, e^{-tX} = \exp\bigl(-t^{2/3}\bigr), \qquad \operatorname{Re} t \geqslant 0, \tag{6.30}$$

is $\mathrm{S}_{2/3}(2^{-3/2}, 1, 0) = \mathrm{Y}_C(2/3, 1) = \mathrm{Y}_C(2/3, 1, 1)$ by Examples 3.16 and 3.20. By (4.3) and (4.22) (with $\alpha = 3/2$ and $\theta = 1/3$), its density function is

$$g_C(x; 2/3, 1) = p\bigl(x; 2/3, -2/3\bigr) = x^{-5/3} g_C\bigl(x^{-2/3}; 3/2, 1/3\bigr), \qquad x > 0. \tag{6.31}$$

By (6.27), this yields the density, for $x > 0$,

$$g_C(x; 2/3, 1) = 6 e^{-\frac{2}{27x^2}} \Bigl((3x)^{-7/3} \mathrm{Ai}\bigl((3x)^{-4/3}\bigr) - (3x)^{-5/3} \mathrm{Ai}'\bigl((3x)^{-4/3}\bigr) \Bigr). \tag{6.32}$$

Similarly, (6.31) and (6.28) yield [14, (2.8.33) with typo]

$$g_C\bigl(x; 2/3, 1\bigr) = \frac{\sqrt{3}}{\sqrt{\pi}} x^{-1} e^{-\frac{2}{27x^2}} W_{1/2,1/6}\Bigl(\frac{4}{27x^2}\Bigr), \qquad x > 0. \tag{6.33}$$

□

Example 6.6 ($\alpha = 2/3$)**.** The symmetric $\frac{2}{3}$-stable distribution with characteristic function

$$\mathbb{E}\, e^{\mathrm{i}tX} = \exp\bigl(-|t|^{2/3}\bigr), \qquad -\infty < t < \infty, \tag{6.34}$$

is $\mathrm{S}_{2/3}(1, 0, 0) = \mathrm{Y}_C(2/3, 0) = \mathrm{Y}_C(2/3, 0, 1)$ by (3.6) and (3.55).

By symmetry, (4.3) and (4.22) (with $\alpha = 3/2$ and $\theta = -1/3$), the density function is

$$g_C(x; 2/3, 0) = p\bigl(x; 2/3, 0\bigr) = |x|^{-5/3} g_C\bigl(|x|^{-2/3}; 3/2, -1/3\bigr), \tag{6.35}$$

which by (6.29) yields [14, (2.8.32)]

$$g_C(x; 2/3, 0) = \frac{1}{2\sqrt{3\pi}} |x|^{-1} e^{\frac{2}{27x^2}} W_{-1/2,1/6}\Bigl(\frac{4}{27x^2}\Bigr), \qquad x \neq 0. \tag{6.36}$$

□

Example 6.7 ($\alpha = 1/3$)**.** The positive strictly $\frac{1}{3}$-stable distribution with Laplace transform

$$\mathbb{E}\, e^{-tX} = \exp\bigl(-t^{1/3}\bigr), \qquad \operatorname{Re} t \geqslant 0, \tag{6.37}$$

is $\mathrm{S}_{1/3}((3/4)^{3/2}, 1, 0) = \mathrm{Y}_C(1/3, 1) = \mathrm{Y}_C(1/3, 1, 1)$ by Examples 3.16 and 3.20.

The density function is, by [14, (2.8.31)] and [10, (9.6.1)],

$$g_C\bigl(x; 1/3, 1\bigr) = p\bigl(x; 1/3, -1/3\bigr) = 3^{-1/3} x^{-4/3} \mathrm{Ai}\bigl((3x)^{-1/3}\bigr), \qquad x > 0, \tag{6.38}$$

where $\mathrm{Ai}(x)$ again is the Airy function. Equivalently, $3\mathrm{Ai}(x)$, $x > 0$, is the density of the random variable $(3Y_C(1/3, 1))^{-1/3}$. (The distribution of this variable, apart from the factor $3^{-1/3}$, is known as a Mittag–Leffler distribution).

The moment formula (5.8) with $\alpha = 1/3$ is by (6.38) and a change of variables equivalent to the integral formula [10, (9.10.17)]

$$\int_0^\infty x^{a-1} \mathrm{Ai}(x)\, \mathrm{d}x = 3^{-(\alpha+2)/3} \frac{\Gamma(a)}{\Gamma((\alpha+2)/3)}, \qquad \operatorname{Re} a > 0. \tag{6.39}$$

□

7. Domains of attraction

Definition 7.1. A random variable X belongs to the *domain of attraction* of a stable distribution $\mathcal{L}$ if there exist constants $a_n > 0$ and b_n such that

$$\frac{S_n - b_n}{a_n} \xrightarrow{\mathrm{d}} \mathcal{L} \tag{7.1}$$

as $n \to \infty$, where $S_n := \sum_{i=1}^n X_i$ is a sum of n i.i.d. copies of X.

We will in the sequel always use the notation S_n in the sense above (as we already have done in Section 3). All unspecified limits are as $n \to \infty$.

Theorem 7.2. *Let $0 < \alpha \leqslant 2$. A (non-degenerate) random variable X belongs to the domain of attraction of an α-stable distribution if and only if the following two conditions hold:*

(i) *the truncated moment function*

$$\mu(x) := \mathbb{E}\big(X^2 \mathbf{1}\{|X| \leqslant x\}\big) \tag{7.2}$$

varies regularly with exponent $2-\alpha$ as $x \to \infty$, i.e.,

$$\mu(x) \sim x^{2-\alpha} L_1(x), \tag{7.3}$$

where $L_1(x)$ varies slowly;

(ii) *either $\alpha = 2$, or the tails of X are balanced:*

$$\frac{\mathbb{P}(X > x)}{\mathbb{P}(|X| > x)} \to p_+, \qquad x \to \infty, \tag{7.4}$$

for some $p_+ \in [0,1]$.

Proof. Feller [3, Theorem XVII.5.2]. □

For the case $\alpha < 2$, the following version is often more convenient.

Theorem 7.3. *Let $0 < \alpha < 2$. A random variable X belongs to the domain of attraction of an α-stable distribution if and only if the following two conditions hold:*

(i) *the tail probability $\mathbb{P}(|X| > x)$ varies regularly with exponent $-\alpha$ as $x \to \infty$, i.e.,*

$$\mathbb{P}(|X| > x) \sim x^{-\alpha} L_2(x), \tag{7.5}$$

where $L_2(x)$ varies slowly;

(ii) *the tails of X are balanced:*

$$\frac{\mathbb{P}(X > x)}{\mathbb{P}(|X| > x)} \to p_+, \qquad x \to \infty, \tag{7.6}$$

for some $p_+ \in [0,1]$.

Proof. Feller [3, Corollary XVII.5.2]. □

We turn to identifying the stable limit distributions in Theorems 7.2–7.3 explicitly.

7.1. **The case $\alpha < 2$.** If the conditions of Theorem 7.2 or 7.3 hold for some $\alpha < 2$, then the conditions of the other hold too, and we have, by [3, (5.16)],

$$L_2(x) \sim \frac{2-\alpha}{\alpha} L_1(x), \qquad x \to \infty. \tag{7.7}$$

Furthermore, by [3, (5.6)], with a_n, b_n as in (7.1) and M and Λ the canonical measure and Lévy measure of the limit distribution $\mathcal{L}$,

$$n\,\mathbb{P}(X > a_n x) \to \Lambda(x,\infty) = \int_x^\infty y^{-2}\,\mathrm{d}M(y), \qquad x > 0, \tag{7.8}$$

and, by symmetry,

$$n\,\mathbb{P}(X < -a_n x) \to \Lambda(-\infty,-x) = \int_{-\infty}^x y^{-2}\,\mathrm{d}M(y), \qquad x > 0. \tag{7.9}$$

In particular,

$$n\,\mathbb{P}(|X| > a_n) \to \Lambda\{y : |y| > 1\} \in (0,\infty); \tag{7.10}$$

conversely, we may in (7.1) choose any sequence (a_n) such that $n\,\mathbb{P}(|X| > a_n)$ converges to a positive, finite limit. (Any two such sequences (a_n) and (a'_n) must satisfy $a_n/a'_n \to c$ for some $c \in (0,\infty)$, as a consequence of (7.5).)

If

$$n\,\mathbb{P}(|X| > a_n) \to C > 0 \tag{7.11}$$

and (7.5)–(7.6) hold, then (7.8)–(7.9) hold with $\Lambda(x,\infty) = p_+ C x^{-\alpha}$ and $\Lambda(-\infty,-x) = p_- C x^{-\alpha}$, where $p_- := 1 - p_+$. Hence, (3.2)–(3.3) hold with

$$c_+ = p_+ C\alpha, \qquad\qquad c_- = p_- C\alpha. \tag{7.12}$$

Consequently, the limit distribution is given by (3.6) where, by (3.11)–(3.12),

$$\gamma = \bigl(C\alpha\bigl(-\Gamma(-\alpha)\cos\tfrac{\pi\alpha}{2}\bigr)\bigr)^{1/\alpha} = \bigl(C\Gamma(1-\alpha)\cos\tfrac{\pi\alpha}{2}\bigr)^{1/\alpha}, \tag{7.13}$$

$$\beta = p_+ - p_-. \tag{7.14}$$

For $\alpha = 1$ we interpret (7.13) by continuity as

$$\gamma = C\tfrac{\pi}{2}. \tag{7.15}$$

Theorem 7.4. *Let $0 < \alpha < 2$. Suppose that (7.5)–(7.6) hold and that a_n are chosen such that (7.11) holds, for some C. Let γ and β be defined by (7.13)–(7.14).*

(i) *If $0 < \alpha < 1$, then*

$$\frac{S_n}{a_n} \xrightarrow{\mathrm{d}} \mathrm{S}_\alpha(\gamma,\beta,0). \tag{7.16}$$

(ii) *If $1 < \alpha < 2$, then*

$$\frac{S_n - n\,\mathbb{E}\,X}{a_n} \xrightarrow{\mathrm{d}} \mathrm{S}_\alpha(\gamma,\beta,0). \tag{7.17}$$

(iii) *If $\alpha = 1$, then*

$$\frac{S_n - nb_n}{a_n} \xrightarrow{\mathrm{d}} \mathrm{S}_1(\gamma,\beta,0), \tag{7.18}$$

where γ is given by (7.15) and

$$b_n := a_n\,\mathbb{E}\sin(X/a_n). \tag{7.19}$$

Proof. Feller [3, Theorem XVII.5.3] together with the calculations above. ☐

Example 7.5. Suppose that $0 < \alpha < 2$ and that X is a random variable such that, as $x \to \infty$,

$$\mathbb{P}(X > x) \sim Cx^{-\alpha}, \tag{7.20}$$

with $C > 0$, and $\mathbb{P}(X < -x) = o(x^{-\alpha})$. Then (7.5)–(7.6) hold with $L_2(x) := C$ and $p_+ = 1$, and thus $p_- := 1 - p_+ = 0$. We take $a_n := n^{1/\alpha}$; then (7.11) holds, and thus (3.2)–(3.3) hold with

$$c_+ = C\alpha, \qquad\qquad c_- = 0; \tag{7.21}$$

hence, (7.13)–(7.14) yield

$$\gamma = \bigl(C\Gamma(1-\alpha)\cos\tfrac{\pi\alpha}{2}\bigr)^{1/\alpha}, \tag{7.22}$$

and $\beta = 1$. Consequently, Theorem 7.4 yields the following.

(i) If $0 < \alpha < 1$, then

$$\frac{S_n}{n^{1/\alpha}} \xrightarrow{\mathrm{d}} \mathrm{S}_\alpha(\gamma, 1, 0). \tag{7.23}$$

The limit variable Y is positive and has by Theorem 3.13 and (7.22) the Laplace transform

$$\mathbb{E}\, e^{-tY} = \exp\bigl(-C\Gamma(1-\alpha)t^\alpha\bigr), \qquad \operatorname{Re} t \geqslant 0. \tag{7.24}$$

(ii) If $1 < \alpha < 2$, then

$$\frac{S_n - n\,\mathbb{E}\,X}{n^{1/\alpha}} \xrightarrow{\mathrm{d}} \mathrm{S}_\alpha(\gamma, 1, 0). \tag{7.25}$$

The limit variable Y has by Theorem 3.13 and (7.22) the finite Laplace transform

$$\mathbb{E}\, e^{-tY} = \exp\bigl(C|\Gamma(1-\alpha)|t^\alpha\bigr), \qquad \operatorname{Re} t \geqslant 0. \tag{7.26}$$

By (4.12) and (7.22), the density function f_Y of the limit variable satisfies

$$f(0) = C^{-1/\alpha}|\Gamma(1-\alpha)|^{-1/\alpha}|\Gamma(-1/\alpha)|^{-1}. \tag{7.27}$$

(iii) If $\alpha = 1$, then

$$\frac{S_n - nb_n}{n} = \frac{S_n}{n} - b_n \xrightarrow{\mathrm{d}} \mathrm{S}_1(\gamma, 1, 0), \tag{7.28}$$

where, by (7.15), $\gamma = C\pi/2$ and

$$b_n := n\,\mathbb{E}\sin(X/n). \tag{7.29}$$

We return to the evaluation of b_n in Section 7.2.

☐

Example 7.6. Suppose that $0 < \alpha < 2$ and that $X \geqslant 0$ is an integer-valued random variable such that, as $n \to \infty$,

$$\mathbb{P}(X = n) \sim cn^{-\alpha-1}. \tag{7.30}$$

Then (7.20) holds with

$$C = c/\alpha \tag{7.31}$$

and the results of Example 7.5 hold, with this C. In particular, (7.22) yields

$$\gamma^\alpha = -c\,\Gamma(-\alpha)\cos\tfrac{\pi\alpha}{2}, \tag{7.32}$$

and both (7.24) and (7.26) can be written

$$\mathbb{E}\,e^{-tY} = \exp\bigl(c\,\Gamma(-\alpha)t^\alpha\bigr), \qquad \operatorname{Re} t \geqslant 0; \tag{7.33}$$

note that $\Gamma(-\alpha) < 0$ for $0 < \alpha < 1$ but $\Gamma(-\alpha) > 0$ for $1 < \alpha < 2$.

Taking t imaginary in (7.33), we find the characteristic function

$$\mathbb{E}\,e^{\mathrm{i}tY} = \exp\bigl(c\,\Gamma(-\alpha)(-\mathrm{i}t)^\alpha\bigr) = \exp\bigl(c\,\Gamma(-\alpha)e^{-\mathrm{i}\operatorname{sgn}(t)\pi\alpha/2}|t|^\alpha\bigr), \qquad t \in \mathbb{R}. \tag{7.34}$$

□

7.2. **The special case $\alpha = 1$.** Suppose that, as $x \to \infty$,

$$\mathbb{P}(X > x) \sim Cx^{-1} \tag{7.35}$$

and $\mathbb{P}(X < -x) = o(x^{-1})$, with $C > 0$. Then Example 7.5 applies, and (7.28)–(7.29) hold. We calculate the normalising quantity b_n in (7.28) for some examples.

Example 7.7. Let $X := 1/U$, where $U \sim \mathrm{U}(0,1)$ has a uniform distribution. Then $\mathbb{P}(X > x) = x^{-1}$ for $x \geqslant 1$ so (7.35) holds with $C = 1$ and (7.15) yields $\gamma = \pi/2$. Furthermore, X has a Pareto distribution with the density

$$f(x) = \begin{cases} x^{-2}, & x > 1, \\ 0, & x \leqslant 1. \end{cases} \tag{7.36}$$

Consequently, by (7.29),

$$\begin{aligned} b_n = n\sin(X/n) &= n\int_1^\infty \sin(x/n)x^{-2}\,\mathrm{d}x = \int_{1/n}^\infty \sin(y)y^{-2}\,\mathrm{d}y \\ &= \log n + \int_{1/n}^1 \frac{\sin y - y}{y^2}\,\mathrm{d}y + \int_1^\infty \frac{\sin y}{y^2}\,\mathrm{d}y \\ &= \log n + \int_0^\infty \frac{\sin y - y\mathbf{1}\{y<1\}}{y^2}\,\mathrm{d}y + o(1) = \log n + 1 - \bar\gamma + o(1), \end{aligned}$$

where $\bar\gamma$ is Euler's gamma. (For the standard evaluation of the last integral, see e.g. [7].) Hence, (7.28) yields

$$\frac{S_n}{n} - \bigl(\log n + 1 - \bar\gamma\bigr) \xrightarrow{\mathrm{d}} \mathrm{S}_1(\pi/2, 1, 0). \tag{7.37}$$

or

$$\frac{S_n}{n} - \log n \xrightarrow{\mathrm{d}} \mathrm{S}_1(\pi/2, 1, 1 - \bar\gamma). \tag{7.38}$$

□

Example 7.8. Let $X := 1/Y$, where $Y \sim \mathrm{Exp}(1)$ has an exponential distribution. Then $\mathbb{P}(X > x) = 1 - \exp(-1/x) \sim x^{-1}$ as $x \to \infty$ so $C = 1$ and (7.15) yields $\gamma = \pi/2$. In this case we do not calculate b_n directly from (7.29). Instead we define

$U := 1 - e^{-Y}$ and $X' := 1/U$ and note that U has a uniform distribution on $[0,1]$ as in Example 7.7; furthermore

$$X' - X = \frac{1}{1-e^{-Y}} - \frac{1}{Y} = \frac{e^{-Y} - 1 + Y}{(1-e^{-Y})Y}. \tag{7.39}$$

This is a positive random variable with finite expectation

$$\mathbb{E}(X' - X) = \int_0^\infty \frac{e^{-y} - 1 + y}{(1-e^{-y})y} e^{-y}\,\mathrm{d}y = \int_0^\infty \Bigl(\frac{e^{-y}}{1-e^{-y}} - \frac{e^{-y}}{y}\Bigr)\,\mathrm{d}y = \bar\gamma, \tag{7.40}$$

see e.g. [10, (5.9.18)] or [7].

Taking i.i.d. pairs $(X_i, X_i') \overset{\mathrm{d}}{=} (X, X')$ we thus have, with $S_n' := \sum_{i=1}^n X_i'$, by the law of large numbers,

$$\frac{S_n' - S_n}{n} \xrightarrow{\mathrm{p}} \mathbb{E}(X' - X) = \bar\gamma. \tag{7.41}$$

Since Example 7.7 shows that $S_n'/n - \log n \xrightarrow{\mathrm{d}} \mathrm{S}_1(\pi/2, 1, 1-\bar\gamma)$, it follows that

$$S_n/n - \log n \xrightarrow{\mathrm{d}} \mathrm{S}_1(\pi/2, 1, 1-2\bar\gamma). \tag{7.42}$$

We thus have (7.28) with

$$b_n = \log n + 1 - 2\bar\gamma + o(1). \tag{7.43}$$

□

7.3. **The case $\alpha = 2$.** If $\alpha = 2$, then a_n in (7.1) have to be chosen such that

$$\frac{n\mu(a_n)}{a_n^2} \to C \tag{7.44}$$

for some $C > 0$, see [3, (5.23)]; conversely any such sequence (a_n) will do.

Theorem 7.9. *If $\mu(x)$ is slowly varying with $\mu(x) \to \infty$ as $x \to \infty$ and (7.44) holds, then*

$$\frac{S_n - \mathbb{E}\,S_n}{a_n} \xrightarrow{\mathrm{d}} N(0, C). \tag{7.45}$$

Proof. Feller [3, Theorem XVII.5.3]. □

Example 7.10. Suppose that $\alpha = 2$ and that X is a random variable such that, as $x \to \infty$,

$$\mathbb{P}(X > x) \sim Cx^{-2}, \tag{7.46}$$

with $C > 0$, and $\mathbb{P}(X < -x) = o(x^{-2})$. Then (7.4) holds with $p_+ = 1$, and thus $p_- := 1 - p_+ = 0$. Furthermore, as $x \to \infty$,

$$\begin{aligned}
\mu(x) &= \mathbb{E}\Bigl(\int_0^{|X|} 2t\,\mathrm{d}t\,\mathbf{1}\{|X| \leqslant x\}\Bigr) = \mathbb{E}\int_0^x \mathbf{1}\{t \leqslant |X| \leqslant x\} 2t\,\mathrm{d}t \\
&= \int_0^x 2t\,\mathbb{P}(t \leqslant |X| \leqslant x)\,\mathrm{d}t = \int_0^x 2t\,\mathbb{P}(|X| > t)\,\mathrm{d}t - x^2\,\mathbb{P}(|X| > x) \\
&= \bigl(1 + o(1)\bigr)\int_1^x 2tCt^{-2}\,\mathrm{d}t + O(1) \sim 2C\log x.
\end{aligned} \tag{7.47}$$

Thus (7.3) holds with $L_1(x) = 2C \log x$.

We take $a_n := \sqrt{n \log n}$. Then $\mu(a_n) \sim 2C\frac{1}{2}\log n = C \log n$, so (7.44) holds and Theorem 7.9 yields

$$\frac{S_n - \mathbb{E}\, S_n}{\sqrt{n \log n}} \xrightarrow{\mathrm{d}} N(0, C). \tag{7.48}$$

□

8. Attraction and characteristic functions

We study the relation between the attraction property (7.1) and the characteristic function $\varphi_X(t)$ of X. For simplicity, we consider only the common case when $a_n = n^{1/\alpha}$. Moreover, for simplicity we state results for $\varphi_X(t)$, $t > 0$ only, recalling (3.18) and $\varphi_X(0) = 1$.

Theorem 8.1. *Let $0 < \alpha \leqslant 2$. The following are equivalent.*

(i) $\dfrac{S_n}{n^{1/\alpha}} \xrightarrow{\mathrm{d}} Z$ *for some non-degenerate random variable Z.*

(ii) *The characteristic function φ_X of X satisfies*

$$\varphi_X(t) = 1 - (\kappa - \mathrm{i}\tau)t^\alpha + o(t^\alpha) \qquad \text{as } t \searrow 0, \tag{8.1}$$

for some real $\kappa > 0$ and τ. In this case, Z is strictly α-stable and has the characteristic function (3.19). *(Hence, $|\tau| \leqslant \kappa \tan \frac{\pi\alpha}{2}$.)*

Proof. If (i) holds, then for every integer m,

$$\frac{S_{mn}}{(mn)^{1/\alpha}} = \frac{1}{m^{1/\alpha}} \sum_{k=1}^{m} \frac{1}{n^{1/\alpha}} \sum_{j=1}^{n} X_{(k-1)n+j} \xrightarrow{\mathrm{d}} \frac{1}{m^{1/\alpha}} \sum_{k=1}^{m} Z_k, \qquad \text{as } n \to \infty,$$

with $Z_k \overset{\mathrm{d}}{=} Z$ i.i.d. Since also $(mn)^{-1/\alpha} S_{mn} \xrightarrow{\mathrm{d}} Z$, we have $m^{-1/\alpha} \sum_{k=1}^{m} Z_k \overset{\mathrm{d}}{=} Z$, and thus Z is strictly α-stable.

We use Corollary 3.8 and suppose that Z has characteristic function (3.19). Then the continuity theorem yields

$$\varphi_X(t/n^{1/\alpha})^n \to \varphi_Z(t) = \exp\bigl(-(\kappa - \mathrm{i}\tau)t^\alpha\bigr), \qquad t \geqslant 0; \tag{8.2}$$

moreover, this holds uniformly for, e.g., $0 \leqslant t \leqslant 1$.

In some neighbourhood $(-t_0, t_0)$ of 0, $\varphi_X \neq 0$ and thus $\varphi_X(t) = e^{\psi(t)}$ for some continuous function $\psi : (-t_0, t_0) \to \mathbb{C}$ with $\psi(0) = 0$. Hence, (8.2) yields (for $n > 1/t_0$)

$$\exp\Bigl(n\psi\Bigl(\frac{t}{n^{1/\alpha}}\Bigr) + (\kappa - \mathrm{i}\tau)t^\alpha\Bigr) = 1 + o(1), \qquad \text{as } n \to \infty,$$

uniformly for $0 \leqslant t \leqslant 1$, which implies

$$n\psi\Bigl(\frac{t}{n^{1/\alpha}}\Bigr) + (\kappa - \mathrm{i}\tau)t^\alpha = o(1), \qquad \text{as } n \to \infty,$$

since the left-hand side is continuous and 0 for $t = 0$, and thus

$$\psi\left(\frac{t}{n^{1/\alpha}}\right) + (\kappa - \mathrm{i}\tau)\frac{t^\alpha}{n} = o(1/n), \qquad \text{as } n \to \infty, \tag{8.3}$$

uniformly for $0 \leqslant t \leqslant 1$.

For $s > 0$, define $n := \lfloor s^{-\alpha} \rfloor$ and $t := sn^{1/\alpha} \in (0,1]$. As $s \searrow 0$, we have $n \to \infty$ and (8.3) yields

$$\psi(s) = -(\kappa - \mathrm{i}\tau)s^\alpha + o(1/n) = -(\kappa - \mathrm{i}\tau)s^\alpha + o(s^\alpha). \tag{8.4}$$

Consequently, as $s \searrow 0$,

$$\varphi_X(s) = e^{\psi(s)} = 1 - (\kappa - \mathrm{i}\tau)s^\alpha + o(s^\alpha), \tag{8.5}$$

so (8.1) holds.

Conversely, if (8.1) holds, then, for $t > 0$,

$$\mathbb{E}\, e^{\mathrm{i}tS_n/n^{1/\alpha}} = \varphi_X\bigl(t/n^{1/\alpha}\bigr)^n = \Bigl(1 - (\kappa - \mathrm{i}\tau + o(1))\frac{t^\alpha}{n}\Bigr)^n \to \exp\bigl(-(\kappa - \mathrm{i}\tau)t^\alpha\bigr),$$

as $n \to \infty$, and thus by the continuity theorem $S_n/n^{1/\alpha} \xrightarrow{\mathrm{d}} Z$, where Z has the characteristic function (3.19). ☐

For $\alpha = 1$, it is not always possible to reduce to the case when $b_n = 0$ in (7.1) and the limit is strictly stable. The most common case is covered by the following theorem.

Theorem 8.2. *The following are equivalent, for any real b.*

(i) $\dfrac{S_n}{n} - b\log n \xrightarrow{\mathrm{d}} Z$ *for some non-degenerate random variable Z.*

(ii) *The characteristic function φ_X of X satisfies*

$$\varphi_X(t) = 1 - (\kappa - \mathrm{i}\tau)t - \mathrm{i}bt\log t + o(t) \qquad \text{as } t \searrow 0, \tag{8.6}$$

for some real $\kappa > 0$ and τ. In this case, Z is 1-stable and has the characteristic function (3.29). (Hence, $|b| \leqslant 2\kappa/\pi$.)

Proof. (ii) $\Longrightarrow$ (i). If (8.6) holds, for any $\kappa \in \mathbb{R}$, then, as $t \searrow 0$,

$$\log\varphi_X(t) = -(\kappa - \mathrm{i}\tau + o(1))t - \mathrm{i}bt\log t \tag{8.7}$$

and thus, as $n \to \infty$, for every fixed $t > 0$,

$$\begin{aligned}
\mathbb{E}\, e^{\mathrm{i}t(S_n/n - b\log n)} &= \varphi_X\bigl(t/n\bigr)^n e^{-\mathrm{i}bt\log n} \\
&= \exp\Bigl(n\Bigl(-(\kappa - \mathrm{i}\tau + o(1))\frac{t}{n} - \mathrm{i}b\frac{t}{n}\log\frac{t}{n}\Bigr) - \mathrm{i}bt\log n\Bigr) \\
&\to \exp\bigl(-(\kappa - \mathrm{i}\tau)t - \mathrm{i}bt\log t\bigr)
\end{aligned}$$

which shows (i), where Z has the characteristic function (3.29).

Furthermore, for use below, note that (3.29) implies $|\varphi_Z(t)| = e^{-\kappa t}$ for $t > 0$. Since $|\varphi_Z(t)| \leqslant 1$, this shows that $\kappa \geqslant 0$. Moreover, if $\kappa = 0$, then $|\varphi_Z(t)| = 1$ for $t > 0$, and thus for all t, which implies that $Z = c$ a.s. for some $c \in \mathbb{R}$, so Z is degenerate and $b = 0$. Hence, (8.6) implies $\kappa \geqslant 0$, and $\kappa = 0$ is possible only when $b = 0$ and $S_n/n \xrightarrow{\mathrm{p}} \tau$.

(i) $\Longrightarrow$ (ii). Let $\gamma_1 := |b|\pi/2$ and $\beta_1 := -\operatorname{sgn} b$. Let Y and Y_i be i.i.d., and independent of $(X_j)_1^\infty$ and Z, with distribution $\mathrm{S}_1(\gamma_1, \beta_1, 0)$. (If $b = 0$ we simply take $Y_i := 0$.) Then Y_i has, by (3.6), the characteristic function

$$\varphi_Y(t) = \exp\bigl(-\gamma_1 t + \mathrm{i}bt\log t\bigr), \qquad t > 0. \tag{8.8}$$

By Theorem 3.3(ii),

$$\sum_{i=1}^n Y_i \overset{\mathrm{d}}{=} nY - bn\log n. \tag{8.9}$$

Define $\widetilde{X}_i := X_i + Y_i$. Then,

$$\frac{1}{n}\sum_{i=1}^n \widetilde{X}_i = \frac{1}{n}\sum_{i=1}^n X_i + \frac{1}{n}\sum_{i=1}^n Y_i \overset{\mathrm{d}}{=} \frac{S_n}{n} + Y - b\log n \xrightarrow{\mathrm{d}} Z + Y. \tag{8.10}$$

Thus, by Theorem 8.1, for some $\kappa_2 > 0$ and τ_2,

$$\varphi_X(t)\varphi_Y(t) = \mathbb{E}\, e^{\mathrm{i}t\widetilde{X}_i} = 1 - (\kappa_2 - \mathrm{i}\tau_2)t + o(t) \qquad \text{as } t \searrow 0, \tag{8.11}$$

and hence, using (8.8),

$$\varphi_X(t) = \mathbb{E}\, e^{\mathrm{i}t\widetilde{X}_i}/\varphi_Y(t) = 1 - (\kappa_2 - \mathrm{i}\tau_2 - \gamma_1)t - \mathrm{i}bt\log t + o(t), \tag{8.12}$$

which shows (8.6), with $\kappa = \kappa_2 - \gamma_1 \in \mathbb{R}$.

Finally, we have shown in the first part of the proof that (8.6) implies $\kappa > 0$, because Z is non-degenerate. $\square$

We can use these theorems to show the following.

Theorem 8.3. *Let $0 < \alpha \leqslant 2$. Suppose that X is such that*

$$n^{-1/\alpha}\sum_{i=1}^n X_i \xrightarrow{\mathrm{d}} Z, \tag{8.13}$$

where Z is an α-stable random variable with characteristic function (3.19) *and that $Y \geqslant 0$ is a random variable with $\mathbb{E}\, Y^\alpha < \infty$. Let $(Y_i)_1^\infty$ be independent copies of Y that are independent of $(X_i)_1^\infty$. Then*

$$n^{-1/\alpha}\sum_{i=1}^n X_iY_i \xrightarrow{\mathrm{d}} Z' := \bigl(\mathbb{E}\, Y^\alpha\bigr)^{1/\alpha} Z, \tag{8.14}$$

where the limit Z' has the characteristic function

$$\varphi_{Z'}(t) = \exp\bigl(-(\mathbb{E}\, Y^\alpha\kappa - \mathrm{i}\,\mathbb{E}\, Y^\alpha\tau)t^\alpha\bigr), \qquad t \geqslant 0. \tag{8.15}$$

If $Z \sim \mathrm{S}_\alpha(\gamma, \beta, 0)$ (where $\beta = 0$ if $\alpha = 1$), then $Z' \sim \mathrm{S}_\alpha((\mathbb{E}\, Y^\alpha)^{1/\alpha}\gamma, \beta, 0)$.

Proof. By Theorem 8.1, for $t \geqslant 0$,

$$\varphi_X(t) = 1 - (\kappa - \mathrm{i}\tau)t^\alpha + t^\alpha r(t), \tag{8.16}$$

where $r(t) \to 0$ as $t \searrow 0$. Furthermore, (8.16) implies that $r(t) = O(1)$ as $t \to \infty$, and thus $r(t) = O(1)$ for $t \geqslant 0$.

Consequently, for $t>0$, assuming as we may that Y is independent of X,

$$\begin{aligned}\varphi_{XY}(t)&=\mathbb{E}\, e^{\mathrm{i}tXY}=\mathbb{E}\,\varphi_X(tY)=\mathbb{E}\big(1-(\kappa-\mathrm{i}\tau)t^\alpha Y^\alpha+t^\alpha Y^\alpha r(tY)\big)\\&=1-(\kappa-\mathrm{i}\tau)t^\alpha\,\mathbb{E}\,Y^\alpha+t^\alpha\,\mathbb{E}\big(Y^\alpha r(tY)\big),\end{aligned}\tag{8.17}$$

where $\mathbb{E}\big(Y^\alpha r(tY)\big)\to 0$ as $t\searrow 0$ by dominated convergence; hence

$$\varphi_{XY}(t)=1-(\kappa-\mathrm{i}\tau)t^\alpha\,\mathbb{E}\,Y^\alpha+o(t^\alpha)\qquad\text{as } t\searrow 0.\tag{8.18}$$

Theorem 8.1 applies and shows that $n^{-1/\alpha}\sum_{i=1}^n X_iY_i\overset{\mathrm{d}}{\longrightarrow}Z'$, where Z' has the characteristic function (8.15). Moreover, by (3.19), $(\mathbb{E}\,Y^\alpha)^{1/\alpha}$ has this characteristic function, so we may take $Z':=(\mathbb{E}\,Y^\alpha)^{1/\alpha}$.

The final claim follows by Remark 3.6. ☐

Theorem 8.4. *Suppose that X is such that, for some real b,*

$$n^{-1}\sum_{i=1}^n X_i-b\log n\overset{\mathrm{d}}{\longrightarrow}Z,\tag{8.19}$$

where Z is a 1-stable random variable, and that $Y\geqslant 0$ is a random variable with $\mathbb{E}\,Y\log Y<\infty$. Let $(Y_i)_1^\infty$ be independent copies of Y that are independent of $(X_i)_1^\infty$. Then, with $\mu:=\mathbb{E}\,Y$,

$$n^{-1}\sum_{i=1}^n X_iY_i-b\mu\log n\overset{\mathrm{d}}{\longrightarrow}Z':=\mu Z-b\big(\mathbb{E}(Y\log Y)-\mu\log\mu\big).\tag{8.20}$$

Z has the characteristic function (3.29) for some κ and τ, and then the limit Z' has the characteristic function, with $\nu:=\mathbb{E}(Y\log Y)$,

$$\varphi_{Z'}(t)=\exp\big(-\big(\mu\kappa+\mathrm{i}(b\nu-\mu\tau)t\big)-\mathrm{i}b\mu t\log t\big),\qquad t>0.\tag{8.21}$$

If $Z\sim \mathrm{S}_1(\gamma,\beta,\delta)$, then $Z'\sim\mathrm{S}_1(\mu\gamma,\beta,\mu\delta-b\nu)$.

Proof. By Theorem 8.2, for $t\geqslant 0$,

$$\varphi_X(t)=1-(\kappa-\mathrm{i}\tau)t-\mathrm{i}bt\log t+tr(t),\tag{8.22}$$

where $r(t)\to 0$ as $t\searrow 0$; moreover Z has the characteristic function (3.29). Furthermore, (8.22) implies that $r(t)=O(\log t)$ as $t\to\infty$, and thus $r(t)=O(1+\log_+ t)$ for $t\geqslant 0$.

Consequently, for $t>0$, assuming as we may that Y is independent of X,

$$\begin{aligned}\varphi_{XY}(t)&=\mathbb{E}\,\varphi_X(tY)\\&=1-(\kappa-\mathrm{i}\tau)t\,\mathbb{E}\,Y-\mathrm{i}bt\,\mathbb{E}\big(Y\log(tY)\big)+t\,\mathbb{E}\big(Yr(tY)\big),\\&=1-\big(\mu\kappa-\mathrm{i}\mu\tau+\mathrm{i}b\,\mathbb{E}(Y\log Y)\big)t-\mathrm{i}b\mu t\log t+t\,\mathbb{E}\big(Yr(tY)\big),\end{aligned}$$

where $\mathbb{E}\big(Yr(tY)\big)\to 0$ as $t\searrow 0$ by dominated convergence; hence

$$\varphi_{XY}(t)=1-\big(\mu\kappa-\mathrm{i}\mu\tau+\mathrm{i}b\nu\big)t-\mathrm{i}b\mu t\log t+o(t)\qquad\text{as } t\searrow 0.\tag{8.23}$$

Theorem 8.2 applies and shows that $n^{-1}\sum_{i=1}^n X_iY_i-b\mu\log n\overset{\mathrm{d}}{\longrightarrow}Z'$, where Z' has the characteristic function (8.21). Moreover, it follows easily from (3.29) that $\mu Z-b\big(\mathbb{E}(Y\log Y)-\mu\log\mu\big)$ has this characteristic function, and thus (8.20) follows.

Finally, if $Z \sim \mathrm{S}_1(\gamma, \beta, \delta)$, then $b = \frac{2}{\pi}\beta\gamma$ by Remark 3.10 and it follows easily from Remark 3.6 that $Z' \sim \mathrm{S}_1(\mu\gamma, \beta, \mu\delta - b\nu)$; alternatively, it follows directly from (8.20) and (3.6) that Z' has the characteristic function

$$\begin{aligned}\varphi_{Z'}(t) &= \varphi_Z(\mu t) \exp\bigl(-\mathrm{i}bt(\nu - \mu \log \mu)\bigr) \\ &= \exp\Bigl(-\gamma\mu|t|\Bigl(1 + \mathrm{i}\beta\frac{2}{\pi}\operatorname{sgn}(t)\log|t|\Bigr) + \mathrm{i}\delta\mu t - \mathrm{i}bt\nu\Bigr).\end{aligned} \tag{8.24}$$

□

Example 8.5. Let $X := U/U'$, where $U, U' \sim \mathrm{U}(0,1)$ are independent. By Example 7.7 and Theorem 8.4, with $Z \sim \mathrm{S}_1(\pi/2, 1, 1 - \bar\gamma)$, $b = 1$, $\mu := \mathbb{E}\, U = 1/2$ and

$$\nu := \mathbb{E}\, U \log U = \int_0^1 x \log x \,\mathrm{d}x = -\frac{1}{4}, \tag{8.25}$$

we obtain

$$\frac{S_n}{n} - \frac{1}{2}\log n \xrightarrow{\mathrm{d}} \frac{1}{2}Z - \nu + \frac{1}{2}\log\frac{1}{2} = \frac{1}{2}Z + \frac{1}{4} - \frac{1}{2}\log 2 \sim \mathrm{S}_1\Bigl(\frac{\pi}{4}, 1, \frac{3}{4} - \frac{\bar\gamma}{2}\Bigr). \tag{8.26}$$

□

Example 8.6. Let $X := Y/Y'$ where $Y, Y' \sim \mathrm{Exp}(1)$ are independent. (Thus X has the F-distribution $F_{2,2}$.) By Example 7.8 and Theorem 8.4, with $Z \sim \mathrm{S}_1(\pi/2, 1, 1 - 2\bar\gamma)$, $b = 1$, $\mu := \mathbb{E}\, Y = 1$ and

$$\nu := \mathbb{E}\, Y \log Y = \int_0^\infty x \log x \, e^{-x} \,\mathrm{d}x = \Gamma'(2) = 1 - \bar\gamma, \tag{8.27}$$

we obtain

$$\frac{S_n}{n} - \log n \xrightarrow{\mathrm{d}} Z - \nu = Z - 1 + \bar\gamma \sim \mathrm{S}_1(\pi/2, 1, -\bar\gamma). \tag{8.28}$$

This is in accordance with Example 7.7, since, as is well-known, $U := Y'/(Y + Y') \sim \mathrm{U}(0,1)$, and thus we can write $X = (Y + Y')/Y' - 1 = 1/U - 1$. □

Example 8.7. Let $X := V^2/W$ where $V \sim \mathrm{U}(-\frac{1}{2}, \frac{1}{2})$ and $W \sim \mathrm{Exp}(1)$ are independent. By Example 7.8 and Theorem 8.4, with $Z \sim \mathrm{S}_1(\pi/2, 1, 1 - 2\bar\gamma)$, $b = 1$, $\mu := \mathbb{E}\, V^2 = 1/12$ and

$$\begin{aligned}\nu := 2\,\mathbb{E}\, V^2 \log|V| &= 4\int_0^{1/2} x^2 \log x \,\mathrm{d}x = 4\left[\frac{x^3}{3}\log x - \frac{x^3}{9}\right]_0^{1/2} \\ &= -\frac{3\log 2 + 1}{18},\end{aligned} \tag{8.29}$$

we obtain

$$\frac{S_n}{n} - \frac{1}{12}\log n \xrightarrow{\mathrm{d}} \frac{1}{12}Z - \nu + \frac{1}{12}\log\frac{1}{12} \sim \mathrm{S}_1\Bigl(\frac{\pi}{24}, 1, \frac{5 - 6\bar\gamma + 6\log 2}{36}\Bigr). \tag{8.30}$$

Equivalently, using Remark 3.6,

$$\frac{24 S_n}{\pi n} - \frac{2}{\pi}\log n \xrightarrow{\mathrm{d}} \frac{2}{\pi}Z - \frac{24\nu}{\pi} - \frac{2}{\pi}\log 12 \sim \mathrm{S}_1\Bigl(1, 1, \frac{2}{\pi}\Bigl(\frac{5}{3} - 2\bar\gamma + \log\frac{\pi}{6}\Bigr)\Bigr). \tag{8.31}$$

This is shown directly in Heinrich, Pukelsheim and Schwingenschlögl [5, Theorem 5.2 and its proof]. □

Example 8.8. More generally, let $X := V^2/W$ where $V \sim \mathrm{U}(q-1,1)$ and $W \sim \mathrm{Exp}(1)$ are independent, for some fixed real q. By Example 7.8 and Theorem 8.4, with $Z \sim \mathrm{S}_1(\pi/2, 1, 1-2\bar\gamma)$, $b = 1$,

$$\mu = \mathbb{E}\, V^2 = \Bigl(\mathbb{E}\, V\Bigr)^2 + \operatorname{Var} V = \Bigl(q - \frac{1}{2}\Bigr)^2 + \frac{1}{12} = \frac{3q^2 - 3q + 1}{3} \tag{8.32}$$

and

$$\begin{aligned}\nu := 2\,\mathbb{E}\, V^2 \log|V| &= 2\int_{q-1}^{q} x^2 \log|x| \,\mathrm{d}x = 2\left[\frac{x^3}{3}\log|x| - \frac{x^3}{9}\right]_{q-1}^{q} \\ &= 2\frac{q^3\log|q| + (1-q)^3\log|1-q|}{3} - 2\frac{3q^2-3q+1}{9},\end{aligned} \tag{8.33}$$

we obtain

$$\frac{S_n}{n} - \mu\log n \xrightarrow{\mathrm{d}} \mu Z - \nu + \mu\log\mu \sim \mathrm{S}_1\Bigl(\mu\frac{\pi}{2}, 1, (1-2\bar\gamma)\mu - \nu\Bigr). \tag{8.34}$$

Equivalently, using Remark 3.6,

$$\frac{S_n - n(\mathbb{E}\, V)^2}{\mu n} - \log n \xrightarrow{\mathrm{d}} Z - \frac{\nu}{\mu} + \log\mu - \frac{(\mathbb{E}\, V)^2}{\mu} \sim \mathrm{S}_1\Bigl(\frac{\pi}{2}, 1, b_q\Bigr), \tag{8.35}$$

with

$$b_q := \frac{2}{3} - 2\bar\gamma - 2\frac{q^3\log|q| + (1-q)^3\log|1-q|}{3q^2-3q+1} + \log\frac{3q^2-3q+1}{3} + \frac{1}{12\mu}. \tag{8.36}$$

This is shown (in the case $0 \leqslant q \leqslant 1$) directly in Heinrich, Pukelsheim and Schwingenschlögl [6, Theorem 4.2 and its proof]. □

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