EUCLIDEAN, SPHERICAL AND HYPERBOLIC TRIGONOMETRY

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Abstract. This is a collection of some standard formulae from Euclidean, spherical and hyperbolic trigonometry, including some standard models of the hyperbolic plane. Proofs are not given.

1. Introduction

We give here various formulae from Euclidean, spherical and hyperbolic trigonometry, denoting the three cases by E, S and H, respectively. The three cases are grouped together for easy comparison. Note that the hyperbolic formulae can be obtained from the corresponding spherical ones by multiplying all lengths by the imaginary unit i. Occasionally we also distinguish the cases by the sectional curvature K;

$$\mathsf{E}: \qquad K = 0, \tag{1.1}$$

$$\mathsf{S}: \qquad K = 1, \tag{1.2}$$

$$\mathsf{H}: \qquad K = -1. \tag{1.3}$$

The formulae are collected from various sources. For proofs and further formulae see e.g. [1], [2], [5] (in particular Chapter VI), [6]; see also [10, Spherical trigonometry, Triangle, Triangle properties] and the references given there. See also [7] for some related differential geometry.

Euclidean geometry is geometry in the usual Euclidean plane $\mathbb{R}^2$. Spherical geometry is geometry on the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. For hyperbolic geometry, there are several equivalent standard models, see the appendices for the most important ones. (We consider only plane geometry. See [5] for 3-dimensional hyperbolic geometry, and for striking uses of 3-dimensional geometry to prove results in plane hyperbolic geometry, sometimes providing also unified proofs of the hyperbolic and sperical cases.)

Remark 1.1. We consider only the standard cases with curvature $K = 0, \pm 1$ as in (1.1)–(1.3). More generally, formulae for $K = \pm 1/\rho^2$, for any $\rho > 0$, are obtained from the formulae for $K = \pm 1$ by dividing all lengths by ρ, but we leave this to the reader. (The curvature can thus be any real number K, taking $\rho = 1/\sqrt{|K|}$ when $K \neq 0$. In the spherical case $K > 0$, this means geometry on a sphere of radius $\rho = 1/\sqrt{K}$.) It can then be seen that the

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formulae for the Euclidean case $K = 0$ are the limits of the formulae for spherical or hyperbolic trigonometry as $K \to 0$.

1.1. **Points at infinity.** The sphere is bounded and compact, and the lines (geodesics) are closed circles. In the Euclidean and hyperbolic cases, it is often convenient to extend the planes by adding improper points at infinity. (From a topological point of view, we compactify the space.) The details are different, and we first recall the standard Euclidean case.

E: In the Euclidean case, we embed the Euclidean plane $\mathbb{R}^2$ in the projective plane $\mathbb{P}^2$, defined as the set of lines through the origin in $\mathbb{R}^3$, i.e., the set of equivalence classes $[x_1, x_2, x_3]$ with $(x_1, x_2, x_3) \in \mathbb{R}^3 \setminus \{0\}$ under the equivalence relation $[x_1, x_2, x_3] = [tx_1, tx_2, tx_3]$ for any $t \neq 0$. The plane $\mathbb{R}^2$ is embedded by $(x_1, x_2) \mapsto [x_1, x_2, 1]$ and the remaining points $\mathbb{P}^2 \setminus \mathbb{R}^2 = \{[x_1, x_2, 0]\}$ is the *line at infinity.* Each (proper) line is adjoined one point at infinity, which can be thought of as the limit point when going to infinity along the line; note that going along the line in any of the two directions yields the same point at infinity. Two lines have the same point at infinity if and only if they are parallel. Hence we can define the points at infinity intrinsically, as the set of equivalence classes of parallel lines. (If we include the improper points and the improper line at infinite, we obtain a model of *projective geometry*, where each pair of distinct points lie on a unique line, and each pair of distinct lines intersect in a unique point.)

H: In the hyperbolic case, we define an *end* as an equivalence class of parallel rays (i.e., half-lines)[1]; each line thus has two (distinct) ends. We extend the hyperbolic plane $\mathbb{H}$ by adding a *boundary* $\partial\mathbb{H}$ consisting of all ends, which are called *points at infinity* (or *infinite points* or *improper points*); see Appendices A–C for concrete models. Each (proper) line thus has two points at infinity. Conversely, for each (unordered) pair of distinct points at infinity, there is exactly one (proper) line with these ends. (In the hyperbolic case, the set of points at infinity should not be regarded as a line, for example because there already is a proper line connecting any pair of improper points. The set of infinite points can be seen as an improper *horocycle*, see Section 9, and may be called the *horocycle at infinity.*)

1.2. **Pairs of lines.** S: In spherical geometry, a pair of distinct lines always intersect in exactly *two* (antipodal) points.

E: In Euclidean geometry, there are two possibilities for a pair of distinct lines:

(i) The lines intersect in a unique point.
(ii) The lines are *parallel*; they do not not intersect in any proper point, but they intersect at infinity.

[1] We may for example define two rays to be parallel if a point moving to infinity on one of them has a distance to the other that tends to 0. (For rays that are not parallel, this distance tends to infinity.)

H: In hyperbolic geometry, there are three possibilities for a pair of distinct lines:

(i) The lines intersect in a unique point.
(ii) The lines are *parallel*; they do not not intersect in any proper point, but they have a common end, so they intersect at infinity. The distance between the lines (i.e., $\inf_{x,y} d(x,y)$ where x and y lie on each of the two lines) is 0.
(iii) The lines are *ultraparallel*; they do not intersect, not even at infinity. The distance between the lines is positive. In this case (and only in this case), the lines have a (unique) common normal. (The intersections with this normal are the unique points on the two lines of minimum distance to each other.)

We can assign a non-negative number $|\langle \ell, \ell' \rangle|$, which we call *gauge*[2], to any pair of lines by the following rules:

(i) If ℓ and ℓ' intersect, at an angle $\alpha \in (0, \pi/2]$, let
$$|\langle \ell, \ell' \rangle| := \cos \alpha \in [0, 1). \tag{1.4}$$
(ii) If ℓ and ℓ' are equal or parallel (Euclidean and hyperbolic cases), then
$$|\langle \ell, \ell' \rangle| := 1. \tag{1.5}$$
(iii) If ℓ and ℓ' are ultraparallel (hyperbolic case only), and have distance d, then
$$|\langle \ell, \ell' \rangle| := \cosh d \in (1, \infty). \tag{1.6}$$

In particular, note that ℓ and ℓ' are orthogonal if and only if $|\langle \ell, \ell' \rangle| = 0$.

We also define a signed version $\langle \ell, \ell' \rangle$ of the gauge for *directed* lines by the following modifications above: In (i), let $\alpha \in (0, \pi)$ be the angle between the positive directions of the lines; in (ii), let $\langle \ell, \ell' \rangle = 1$ if the lines are equal with the same orientation or parallel and have the same orientation at their common point at infinity; in the opposite case (antiparallel lines), let $\langle \ell, \ell' \rangle = -1$; in (iii), let $\langle \ell, \ell' \rangle = \cosh d$ if ℓ and ℓ' are directed towards the same side of their common normal; otherwise let $\langle \ell, \ell' \rangle = -\cosh d$. In all cases, the gauge changes sign if the orientation of one of the lines is reversed. (The absolute value $|\langle \ell, \ell' \rangle|$ is thus independent of the directions, and is well defined for undirected lines; this is the unsigned gauge defined above.) Note also the symmetry
$$\langle \ell, \ell' \rangle = \langle \ell', \ell \rangle. \tag{1.7}$$

2. Triangles

A triangle is defined by any three points (the *vertices*) not on a common line. (For infinite triangles in hyperbolic geometry, see Section 4.) In particular, the vertices are distinct; on the sphere, furthermore no two vertices are antipodal. Hence, there is a unique line through any pair of vertices; the

[2]This is not standard terminology, as far as I know.

part of this line that lies between the two vertices is a *side* of the triangle. The triangle itself can be defined as the convex hull of the three vertices; this is a convex set bounded by the three sides.

Consider a triangle ABC with (interior) angles A, B, C and sides a, b, c (with a opposite to A, etc.). (We follow the standard convention of using the same letter denote a vertex and the corresponding angle; this should not cause any confusion.) Further, let Δ be the area and let

$$s := \frac{a+b+c}{2}, \tag{2.1}$$

$$\sigma := \frac{A+B+C}{2}. \tag{2.2}$$

Thus, s is the *semiperimeter*. Note that in the Euclidean case E, $\sigma = \pi/2$ (see (2.51)).

Note that A, B, C always are interior angles, and thus have values in $(0, \pi)$. (For infinite hyperbolic trianges, 0 is also allowed, see Section 4.) In the spherical case, we further assume that each side is the shortest of the two great circle arcs (on the same great circle) connecting its endpoints; thus each side has length in $(0, \pi)$. (It follows that the entire triangle lies in some open half-sphere, see Section 10.) In the Euclidean and hyperbolic cases, the sidelengths are in $(0, \infty)$.

We give below a number of trigonometric formulae and other relations for these quantities. Note that (except the angle sum in the Euclidean case, (2.51) and (2.9)), all formulae usings sides and angles involve at least four of the three sides and three angles, since any three of these elements may be chosen more or less arbitrarily, see Section 7. Relations between four elements may be used to solve the triangle when three elements are given, but also relations with five or six elements are sometimes useful, see again Section 7.

We may obviously permute the vertices A, B, C in any order, i.e., make any simultaneous permutation of the angles A, B, C and sides a, b, c. Some formulae are symmetric, but not all, and in the latter case we often give only one form, leaving permutations to the reader.

2.1. **Fundamental formulae for sides and angles.**

The law of sines.

$$\mathsf{E}: \qquad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}, \tag{2.3}$$

$$\mathsf{S}: \qquad \frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}, \tag{2.4}$$

$$\mathsf{H}: \qquad \frac{\sin A}{\sinh a} = \frac{\sin B}{\sinh b} = \frac{\sin C}{\sinh c}. \tag{2.5}$$

The law of cosines.

$$\mathsf{E}: \quad a^2 = b^2 + c^2 - 2bc\cos A, \tag{2.6}$$

$$\mathsf{S}: \quad \cos a = \cos b \, \cos c + \sin b \, \sin c \, \cos A, \tag{2.7}$$

$$\mathsf{H}: \quad \cosh a = \cosh b \, \cosh c - \sinh b \, \sinh c \, \cos A. \tag{2.8}$$

The second law of cosines.

$$(\mathsf{E}: \quad \cos A = -\cos B \, \cos C + \sin B \, \sin C = -\cos(B+C),) \tag{2.9}$$

$$\mathsf{S}: \quad \cos A = -\cos B \, \cos C + \sin B \, \sin C \, \cos a, \tag{2.10}$$

$$\mathsf{H}: \quad \cos A = -\cos B \, \cos C + \sin B \, \sin C \, \cosh a, \tag{2.11}$$

2.2. **Further formulae for sides and angles.**

The law of tangents.

$$\mathsf{E}: \quad \frac{\tan\frac{A-B}{2}}{\tan\frac{A+B}{2}} = \frac{\sin A - \sin B}{\sin A + \sin B} = \frac{a-b}{a+b}, \tag{2.12}$$

$$\mathsf{S}: \quad \frac{\tan\frac{A-B}{2}}{\tan\frac{A+B}{2}} = \frac{\sin A - \sin B}{\sin A + \sin B} = \frac{\sin a - \sin b}{\sin a + \sin b} = \frac{\tan\frac{a-b}{2}}{\tan\frac{a+b}{2}}, \tag{2.13}$$

$$\mathsf{H}: \quad \frac{\tan\frac{A-B}{2}}{\tan\frac{A+B}{2}} = \frac{\sin A - \sin B}{\sin A + \sin B} = \frac{\sinh a - \sinh b}{\sinh a + \sinh b} = \frac{\tanh\frac{a-b}{2}}{\tanh\frac{a+b}{2}}. \tag{2.14}$$

The formulae hold for all triangles, but in the spherical case, (2.13) may be of the type $\infty = \infty$ (when $A + B = a + b = \pi$, cf. (2.87)). The law of tangents is equivalent to the law of sines, by simple manipulations and standard identities for trigonometric functions.

Napier's analogies. This is (in the spherical case) the traditional name for the following set of equations (some of which may be of the type $\infty = \infty$):

$$\mathsf{E}: \quad \frac{c}{a+b} = \frac{\cos\frac{A+B}{2}}{\cos\frac{A-B}{2}}, \tag{2.15}$$

$$\mathsf{E}: \quad \frac{c}{a-b} = \frac{\sin\frac{A+B}{2}}{\sin\frac{A-B}{2}}, \tag{2.16}$$

$$\mathsf{S}: \quad \frac{\tan\frac{c}{2}}{\tan\frac{a+b}{2}} = \frac{\cos\frac{A+B}{2}}{\cos\frac{A-B}{2}}, \tag{2.17}$$

$$\mathsf{S}: \quad \frac{\tan\frac{c}{2}}{\tan\frac{a-b}{2}} = \frac{\sin\frac{A+B}{2}}{\sin\frac{A-B}{2}}, \tag{2.18}$$

$$\mathsf{S}: \quad \frac{\cot\frac{C}{2}}{\tan\frac{A+B}{2}} = \frac{\cos\frac{a+b}{2}}{\cos\frac{a-b}{2}}, \tag{2.19}$$

$$\mathsf{S}: \quad \frac{\cot\frac{C}{2}}{\tan\frac{A-B}{2}} = \frac{\sin\frac{a+b}{2}}{\sin\frac{a-b}{2}}, \tag{2.20}$$

$$\mathsf{H}: \qquad \frac{\tanh \frac{c}{2}}{\tanh \frac{a+b}{2}} = \frac{\cos \frac{A+B}{2}}{\cos \frac{A-B}{2}}, \tag{2.21}$$

$$\mathsf{H}: \qquad \frac{\tanh \frac{c}{2}}{\tanh \frac{a-b}{2}} = \frac{\sin \frac{A+B}{2}}{\sin \frac{A-B}{2}}, \tag{2.22}$$

$$\mathsf{H}: \qquad \frac{\cot \frac{C}{2}}{\tan \frac{A+B}{2}} = \frac{\cosh \frac{a+b}{2}}{\cosh \frac{a-b}{2}}, \tag{2.23}$$

$$\mathsf{H}: \qquad \frac{\cot \frac{C}{2}}{\tan \frac{A-B}{2}} = \frac{\sinh \frac{a+b}{2}}{\sinh \frac{a-b}{2}}. \tag{2.24}$$

Note that Napier's analogies come in pairs, with the formulae in each pair equivalent by the law of tangents, (2.12)–(2.14). (The formulae can be derived from the law of cosines (2.6)–(2.8), or second law of cosines (2.10)–(2.11), by adding (or subtracting) the same equation with A and B interchanged, and then using the sine law (2.3)–(2.5) and standard identities for trigonometric functions.)

Napier's analogies are very useful for solving triangles in spherical and hyperbolic geometry, see Section 7.

Mollweide's formulae. In the Euclidean case, (2.15)–(2.16) are easily proved directly from the law of sines and the angle sum (2.51); using the latter they are further equivalent to the following formulae, involving all six sides and angles.

$$\mathsf{E}: \qquad \frac{a+b}{c} = \frac{\cos \frac{A-B}{2}}{\sin \frac{C}{2}}, \tag{2.25}$$

$$\mathsf{E}: \qquad \frac{a-b}{c} = \frac{\sin \frac{A-B}{2}}{\cos \frac{C}{2}}. \tag{2.26}$$

The formulae go Newton, at least, and are sometimes called *Newton's formulae*, see [8]. (In [10], (2.26) is called Mollweide's and (2.25) is called Newton's.)

The haversine formula.

$$\mathsf{E}: \qquad a^2 = (b-c)^2 + 4bc\sin^2 \frac{A}{2}, \tag{2.27}$$

$$\mathsf{S}: \qquad \sin^2 \frac{a}{2} = \sin^2 \frac{b-c}{2} + \sin b \, \sin c \, \sin^2 \frac{A}{2}, \tag{2.28}$$

$$\mathsf{H}: \qquad \sinh^2 \frac{a}{2} = \sinh^2 \frac{b-c}{2} + \sinh b \, \sinh c \, \sin^2 \frac{A}{2}. \tag{2.29}$$

This is a version of the law of cosines (by standard identities for trigonometric functions); it is (in the spherical case) called the *haversine formula*, because the function $\sin^2 \frac{x}{2}$ is called *haversine* [10, Haversine]. (The haversine formula is better than the law of cosines for numerical calculations of small distances in navigation.) There is (in the spherical and hyperbolic cases) a similar version of the second law of cosines, which we omit.

Another version of the law of cosines (by standard identities for trigonometric functions) is

$$\mathsf{E}: \quad \cos^2\frac{A}{2} = \frac{(b+c)^2-a^2}{4bc} = \frac{s(s-a)}{bc}, \tag{2.30}$$

$$\mathsf{S}: \quad \cos^2\frac{A}{2} = \frac{\cos a - \cos(b+c)}{2\sin b\,\sin c} = \frac{\sin s\,\sin(s-a)}{\sin b\,\sin c}, \tag{2.31}$$

$$\mathsf{H}: \quad \cos^2\frac{A}{2} = \frac{\cosh(b+c) - \cosh a}{2\sinh b\,\sinh c} = \frac{\sinh s\,\sinh(s-a)}{\sinh b\,\sinh c}. \tag{2.32}$$

Manipulation using trigonometric identities yields the equivalent

$$\mathsf{E}: \quad \sin^2\frac{A}{2} = \frac{a^2-(b-c)^2}{4bc} = \frac{(s-b)(s-c)}{bc}, \tag{2.33}$$

$$\mathsf{S}: \quad \sin^2\frac{A}{2} = \frac{\cos(b-c) - \cos a}{2\sin b\,\sin c} = \frac{\sin(s-b)\,\sin(s-c)}{\sin b\,\sin c}, \tag{2.34}$$

$$\mathsf{H}: \quad \sin^2\frac{A}{2} = \frac{\cosh a - \cosh(b-c)}{2\sinh b\,\sinh c} = \frac{\sinh(s-b)\,\sinh(s-c)}{\sinh b\,\sinh c}, \tag{2.35}$$

and

$$\mathsf{E}: \quad \tan^2\frac{A}{2} = \frac{a^2-(b-c)^2}{(b+c)^2-a^2} = \frac{(s-b)(s-c)}{s(s-a)}, \tag{2.36}$$

$$\mathsf{S}: \quad \tan^2\frac{A}{2} = \frac{\cos(b-c) - \cos a}{\cos a - \cos(b+c)} = \frac{\sin(s-b)\,\sin(s-c)}{\sin s\,\sin(s-a)}, \tag{2.37}$$

$$\mathsf{H}: \quad \tan^2\frac{A}{2} = \frac{\cosh a - \cosh(b-c)}{\cosh(b+c) - \cosh a} = \frac{\sinh(s-b)\,\sinh(s-c)}{\sinh s\,\sinh(s-a)}. \tag{2.38}$$

Similarly, the second law of cosines (2.9)–(2.11) yields

$$\mathsf{S}: \quad \sin^2\frac{a}{2} = -\frac{\cos A + \cos(B+C)}{2\sin B\,\sin C} = -\frac{\cos\sigma\,\cos(\sigma-A)}{\sin B\,\sin C}, \tag{2.39}$$

$$\mathsf{H}: \quad \sinh^2\frac{a}{2} = \frac{\cos A + \cos(B+C)}{2\sin B\,\sin C} = \frac{\cos\sigma\,\cos(\sigma-A)}{\sin B\,\sin C}, \tag{2.40}$$

and

$$\mathsf{S}: \quad \cos^2\frac{a}{2} = \frac{\cos A + \cos(B-C)}{2\sin B\,\sin C} = \frac{\cos(\sigma-B)\,\cos(\sigma-C)}{\sin B\,\sin C}, \tag{2.41}$$

$$\mathsf{H}: \quad \cosh^2\frac{a}{2} = \frac{\cos A + \cos(B-C)}{2\sin B\,\sin C} = \frac{\cos(\sigma-B)\,\cos(\sigma-C)}{\sin B\,\sin C}, \tag{2.42}$$

$$\mathsf{S}: \quad \tan^2\frac{a}{2} = -\frac{\cos A + \cos(B+C)}{\cos A + \cos(B-C)} = -\frac{\cos\sigma\,\cos(\sigma-A)}{\cos(\sigma-B)\,\cos(\sigma-C)}, \tag{2.43}$$

$$\mathsf{H}: \quad \tanh^2\frac{a}{2} = \frac{\cos A + \cos(B+C)}{\cos A + \cos(B-C)} = \frac{\cos\sigma\,\cos(\sigma-A)}{\cos(\sigma-B)\,\cos(\sigma-C)}. \tag{2.44}$$

Remark 2.1. In the days of calculation by hand, formulae involving only multiplications and divisions of trigonometric functions were much more convenient than formulae with both multiplications and additions or subtractions of such functions, since the former are converted into sums and differences by taking logarithms, and tables of logarithms of trigonometric functions were readily available.

A cotangent formula.

$$\mathsf{E}: \qquad \cot B = \frac{c - b\cos A}{b\sin A}, \tag{2.45}$$

$$\mathsf{S}: \qquad \cot B = \frac{\cos b\,\sin c - \sin b\,\cos c\,\cos A}{\sin b\,\sin A}, \tag{2.46}$$

$$\mathsf{H}: \qquad \cot B = \frac{\cosh b\,\sinh c - \sinh b\,\cosh c\,\cos A}{\sinh b\,\sin A}. \tag{2.47}$$

By the law of sines (2.3)–(2.5), these are equivalent to

$$\mathsf{E}: \qquad a\cos B = c - b\cos A, \tag{2.48}$$

$$\mathsf{S}: \qquad \sin a\,\cos B = \cos b\,\sin c - \sin b\,\cos c\,\cos A, \tag{2.49}$$

$$\mathsf{H}: \qquad \sinh a\,\cos B = \cosh b\,\sinh c - \sinh b\,\cosh c\,\cos A, \tag{2.50}$$

which are easy consequences of the law of cosines (2.6)–(2.8).

2.3. **Area.**

Angle sum and area.

$$\mathsf{E}: \qquad 2\sigma := A + B + C = \pi, \tag{2.51}$$

$$\mathsf{S}: \qquad 2\sigma := A + B + C = \pi + \Delta, \tag{2.52}$$

$$\mathsf{H}: \qquad 2\sigma := A + B + C = \pi - \Delta. \tag{2.53}$$

We can combine these formulae to

$$\mathsf{E},\mathsf{S},\mathsf{H}: \qquad 2\sigma := A + B + C = \pi + K\Delta, \tag{2.54}$$

where K is the sectional curvature; this is an example of the Gauss–Bonnet theorem.

Area.

$$\mathsf{E}: \qquad \Delta = \frac{bc\sin A}{2}, \tag{2.55}$$

$$\mathsf{S}: \qquad \sin\frac{\Delta}{2} = \frac{\sin b\,\sin c\,\sin A}{4\cos\frac{a}{2}\,\cos\frac{b}{2}\,\cos\frac{c}{2}}, \tag{2.56}$$

$$\mathsf{H}: \qquad \sin\frac{\Delta}{2} = \frac{\sinh b\,\sinh c\,\sin A}{4\cosh\frac{a}{2}\,\cosh\frac{b}{2}\,\cosh\frac{c}{2}}. \tag{2.57}$$

(See (2.110)–(2.112) and (2.107)–(2.109) for another way to write this.) Combining the law of cosines (2.6)–(2.8) and (2.55)–(2.57) yields

$$\mathsf{E}: \qquad \cot A = \frac{b^2 + c^2 - a^2}{4\Delta}, \tag{2.58}$$

$$\mathsf{S}: \qquad \cot A = \frac{\cos a - \cos b\,\cos c}{4\sin\frac{\Delta}{2}\,\cos\frac{a}{2}\,\cos\frac{b}{2}\,\cos\frac{c}{2}}, \tag{2.59}$$

$$\mathsf{H}: \qquad \cot A = \frac{\cosh b\,\cosh c - \cosh a}{4\sin\frac{\Delta}{2}\,\cosh\frac{a}{2}\,\cosh\frac{b}{2}\,\cosh\frac{c}{2}}. \tag{2.60}$$

In the Euclidean case we thus have

$$\mathsf{E}: \qquad \cot A + \cot B + \cot C = \frac{a^2+b^2+c^2}{4\Delta}. \tag{2.61}$$

Heron's formula. Recall that $s := (a+b+c)/2$.

$$\mathsf{E}: \qquad \Delta^2 = s(s-a)(s-b)(s-c), \tag{2.62}$$

$$\mathsf{S}: \qquad 2\sin^2\frac{\Delta}{2} = 1-\cos\Delta = \frac{4\sin s\,\sin(s-a)\,\sin(s-b)\,\sin(s-c)}{(1+\cos a)(1+\cos b)(1+\cos c)}, \tag{2.63}$$

$$\mathsf{H}: \qquad 2\sin^2\frac{\Delta}{2} = 1-\cos\Delta = \frac{4\sinh s\,\sinh(s-a)\,\sinh(s-b)\,\sinh(s-c)}{(1+\cosh a)(1+\cosh b)(1+\cosh c)}. \tag{2.64}$$

Equivalently,

$$\mathsf{E}: \qquad \Delta = \sqrt{s(s-a)(s-b)(s-c)}, \tag{2.65}$$

$$\mathsf{S}: \qquad 2\sin\frac{\Delta}{2} = \frac{\sqrt{\sin s\,\sin(s-a)\,\sin(s-b)\,\sin(s-c)}}{\cos\frac{a}{2}\,\cos\frac{b}{2}\,\cos\frac{c}{2}}, \tag{2.66}$$

$$\mathsf{H}: \qquad 2\sin\frac{\Delta}{2} = \frac{\sqrt{\sinh s\,\sinh(s-a)\,\sinh(s-b)\,\sinh(s-c)}}{\cosh\frac{a}{2}\,\cosh\frac{b}{2}\,\cosh\frac{c}{2}}. \tag{2.67}$$

The Euclidean case can also be written

$$\mathsf{E}: \qquad \Delta = \tfrac14\sqrt{2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4} \tag{2.68}$$

$$= \tfrac14\sqrt{(a^2+b^2+c^2)^2-2(a^4+b^4+c^4)}. \tag{2.69}$$

The spherical and hyperbolic cases can also be written, using trigonometric identities (including (2.93)–(2.94) and (2.96)–(2.97)),

$$\mathsf{S}: \qquad \cos\frac{\Delta}{2} = \frac{1+\cos a+\cos b+\cos c}{4\cos\frac{a}{2}\cos\frac{b}{2}\cos\frac{c}{2}}, \tag{2.70}$$

$$\mathsf{H}: \qquad \cos\frac{\Delta}{2} = \frac{1+\cosh a+\cosh b+\cosh c}{4\cosh\frac{a}{2}\cosh\frac{b}{2}\cosh\frac{c}{2}}, \tag{2.71}$$

$$\mathsf{S}: \qquad \tan\frac{\Delta}{2} = \frac{2\sqrt{\sin s\,\sin(s-a)\,\sin(s-b)\,\sin(s-c)}}{1+\cos a+\cos b+\cos c}, \tag{2.72}$$

$$\mathsf{H}: \qquad \tan\frac{\Delta}{2} = \frac{2\sqrt{\sinh s\,\sinh(s-a)\,\sinh(s-b)\,\sinh(s-c)}}{1+\cosh a+\cosh b+\cosh c} \tag{2.73}$$

Remark 2.2 (S). In the spherical case, $0 < \Delta < 2\pi$ by (2.84); thus $\Delta/2 \in (0,\pi)$ and (2.70) and (2.72) determine Δ uniquely, while (2.66) yields two possibilities. We see from (2.70) that

$$\mathsf{S}: \qquad \left.\begin{matrix} \Delta < \pi \\ \Delta = \pi \\ \Delta > \pi \end{matrix}\right\} \iff \begin{cases} 1+\cos a+\cos b+\cos c > 0, \\ 1+\cos a+\cos b+\cos c = 0, \\ 1+\cos a+\cos b+\cos c < 0. \end{cases} \tag{2.74}$$

2.4. **Some inequalitites.** The *triangle inequality* says that in any triangle

$$\mathsf{E,S,H}: \qquad a < b+c, \qquad b < a+c, \qquad c < a+b. \tag{2.75}$$

(Strict inequalities hold since we assume that the three vertices do not lie on a line.)

Euclidean and hyperbolic triangles can have arbitrarily large perimeter, but in the spherical case, the perimeter is always less than the length of a line (great circle), i.e.

$$\mathsf{S}: \qquad 2s = a+b+c < 2\pi. \tag{2.76}$$

The angle sum $A+B+C$ is equal to 2π in the Euclidean case by (2.51); in the spherical and hyperbolic case we have by (2.52)–(2.53) inequalities:

$$\mathsf{S}: \qquad 2\sigma = A+B+C > \pi, \tag{2.77}$$

$$\mathsf{H}: \qquad 2\sigma = A+B+C < \pi. \tag{2.78}$$

There is no lower bound for the angle sum of a hyperbolic triangle; any value in $(0,\pi)$ may be attained. A trivial upper bound for the angle sum in the spherical case is provided by the assumptions $A,B,C < \pi$, yielding

$$\mathsf{S}: \qquad 2\sigma = A+B+C < 3\pi; \tag{2.79}$$

this is best possible and any angle sum between π and 3π may be attained by a spherical triangle (for example, by an equilateral triangle). (In the spherical case, see also the duality (5.7).)

Moreover, in the spherical case, the triangle inequality (2.75) is by duality (Section 5) equivalent to the inequalities

$$\mathsf{S}: \quad B+C < \pi+A, \qquad A+C < \pi+B, \qquad A+B < \pi+C. \tag{2.80}$$

By (2.52), this can also be written

$$\mathsf{S}: \qquad \Delta < 2\max\{A,B,C\}, \tag{2.81}$$

which is obvious geometrically, since $2A$ is the area of the lune (digon) between the lines AB and AC extended to the antipode $\bar{A}$ of A, and the triangle is a subset of this sector. (See also (6.3).)

In the spherical case we further have the inequalities

$$\mathsf{S}: \qquad -\tfrac{3}{2} < \cos a+\cos b+\cos c < 3 \tag{2.82}$$

and dually, cf. (5.4)–(5.6),

$$\mathsf{S}: \qquad -3 < \cos A+\cos B+\cos C < \tfrac{3}{2}. \tag{2.83}$$

(These are easily shown by considering the centroid, cf. Appendix E.3 and (E.26). The inequalities are best possible, as is shown by equilateral triangles. Note that no corresponding nontrivial inequalities hold for two angles or sides; (a,b) or (A,B) can be any pair in $(0,\pi)^2$.)

The area of a spherical triangle is obviously bounded (the sphere has total area 4π); note that (2.53) shows that the area of a triangle is bounded also in the hyperbolic case: by (2.52)–(2.53) and (2.79),

$$\mathsf{S}: \qquad \Delta < 2\pi, \tag{2.84}$$

$$\mathsf{H}: \qquad \Delta < \pi. \tag{2.85}$$

Any values in these ranges may be attained. (Note that (2.84) also follows from (2.81).)

In contrast, a Euclidean triangle can have arbitrarily large area.

The smallest side is opposite to the smallest angle, and the largest side is opposite to the largest angle; this also holds if there are ties, since two angles are equal if and only if the sides opposite them are equal (an *isosceles* triangle). In other words,

$$\mathsf{E,S,H}: \qquad \left.\begin{matrix} a<b \\ a=b \\ a>b \end{matrix}\right\} \iff \left\{\begin{matrix} A<B, \\ A=B, \\ A>B. \end{matrix}\right. \tag{2.86}$$

(In the hyperbolic case, this is an immediate consequence of the sine law (2.5); the other cases are somewhat less obvious since sine is not monotone.) In the spherical case, (2.86) applied to an adjacent triangle (Section 6) yields the additional relations

$$\mathsf{S}: \qquad \left.\begin{matrix} a+b<\pi \\ a+b=\pi \\ a+b>\pi \end{matrix}\right\} \iff \left\{\begin{matrix} A+B<\pi, \\ A+B=\pi, \\ A+B>\pi. \end{matrix}\right. \tag{2.87}$$

2.5. **Amplitudes.** Two quantities that appear in some formulae are called *amplitudes*. We denote them by am_s and am_v and define them as the square roots of

$$\mathsf{E,S,H}: \qquad \mathrm{am}_s^2 := \begin{vmatrix} 1 & -\cos A & -\cos B \\ -\cos A & 1 & -\cos C \\ -\cos B & -\cos C & 1 \end{vmatrix} \tag{2.88}$$

$$= 1-\cos^2 A-\cos^2 B-\cos^2 C-2\cos A\cos B\cos C \tag{2.89}$$

$$= -4\cos\sigma\cos(\sigma-A)\cos(\sigma-B)\cos(\sigma-C) \tag{2.90}$$

and

$$\mathsf{E}: \qquad \mathrm{am}_v^2 := 4s(s-a)(s-b)(s-c), \tag{2.91}$$

$$\mathsf{S}: \qquad \mathrm{am}_v^2 := \begin{vmatrix} 1 & \cos a & \cos b \\ \cos a & 1 & \cos c \\ \cos b & \cos c & 1 \end{vmatrix} \tag{2.92}$$

$$= 1 - \cos^2 a - \cos^2 b - \cos^2 c + 2\cos a \cos b \cos c \tag{2.93}$$
$$= 4 \sin s \sin(s-a)\sin(s-b)\sin(s-c), \tag{2.94}$$

$$\mathsf{H}: \quad \mathrm{am}_v^2 := \begin{vmatrix} 1 & \cosh a & \cosh b \\ \cosh a & 1 & \cosh c \\ \cosh b & \cosh c & 1 \end{vmatrix} \tag{2.95}$$
$$= 1 - \cosh^2 a - \cosh^2 b - \cosh^2 c + 2\cosh a \cosh b \cosh c \tag{2.96}$$
$$= 4 \sinh s \sinh(s-a)\sinh(s-b)\sinh(s-c). \tag{2.97}$$

Note that by (2.90) and (2.51)–(2.53) (and (5.4)–(5.6) in the spherical case which imply $\sigma - A < \pi/2$ etc. also in this case),

$$\mathsf{E}: \qquad \mathrm{am}_s^2 = 0 \tag{2.98}$$
$$\mathsf{S}: \qquad \mathrm{am}_s^2 > 0 \tag{2.99}$$
$$\mathsf{H}: \qquad \mathrm{am}_s^2 < 0. \tag{2.100}$$

We thus have no use for am_s in the Euclidean case, and in the hyperbolic case am_s is imaginary (so we use $|\mathrm{am}_s| > 0$); otherwise we define am_s and am_v as the *positive* square roots.[3] [4]

Remark 2.3. In the spherical case, am_v is the volume of the parallelepiped spanned by the vertices, regarded as unit vectors in $\mathbb{R}^3$. This holds also in the Euclidean and hyperbolic cases if we use suitable embeddings in $\mathbb{R}^3$, see Appendix E. This is one reason for considering (2.91)–(2.97) together, as versions for the three cases of the same quantity.

Similarly, in the spherical case, am_s is the volume of the parallelepiped spanned by the vertices of the dual triangle, i.e., of the unit vectors orthogonal to the sides, see Section 5. This too can be extended to the hyperbolic case by considering a suitable embedding in $\mathbb{R}^3$, see again Appendix E. (In the Euclidean case, am_s vanishes by (2.98), and no geometrical interpretation is needed.)

Remark 2.4. The notation am_s and am_v is taken from [5]; the subscripts stand for *sides* and *vertices*, respectively. It may seem more natural to use to opposite notations, since am_s is defined in terms of the angles at the vertices and am_v in terms of the lengths of the sides. The solution is to see am_s as defined using the angles between the sides and am_v using the distances between the vertices. See further Appendix E.2, where am_v is connected to the vertices and am_s to the sides in a more direct way.

The second law of cosines yields

$$\mathrm{am}_s^2 = \sin^2 B \sin^2 C - (\cos A + \cos B\,\cos C)^2$$

[3] [5] defines the signs differently, with $\mathrm{am}_s < 0$ and $\mathrm{am}_v < 0$ in the spherical case, and $-\mathrm{i}\mathrm{am}_s > 0$ and $\mathrm{am}_v > 0$ in the hyperbolic case.

[4] We could have defined am_s^2 with an absolute value, changing the sign in the hyperbolic case, in order to make the amplitudes real and non-negative in all cases.

$$\mathsf{E}: \qquad = 0, \tag{2.101}$$
$$\mathsf{S}: \qquad = \sin^2 B \, \sin^2 C \, \sin^2 a, \tag{2.102}$$
$$\mathsf{H}: \qquad = -\sin^2 B \, \sin^2 C \, \sinh^2 a \tag{2.103}$$

and thus

$$\mathsf{E}: \qquad \mathrm{am}_s = 0, \tag{2.104}$$
$$\mathsf{S}: \qquad \mathrm{am}_s = \sin B \, \sin C \, \sin a, \tag{2.105}$$
$$\mathsf{H}: \qquad |\mathrm{am}_s| = \sin B \, \sin C \, \sinh a. \tag{2.106}$$

Similarly (dually), using the law of cosines,

$$\mathsf{E}: \qquad \mathrm{am}_v = bc \sin A, \tag{2.107}$$
$$\mathsf{S}: \qquad \mathrm{am}_v = \sin b \, \sin c \, \sin A, \tag{2.108}$$
$$\mathsf{H}: \qquad \mathrm{am}_v = \sinh b \, \sinh c \, \sin A. \tag{2.109}$$

We can use amplitudes to write Heron's formula as

$$\mathsf{E}: \qquad \Delta = \frac{\mathrm{am}_v}{2} \tag{2.110}$$
$$\mathsf{S}: \qquad \sin \frac{\Delta}{2} = \frac{\mathrm{am}_v}{4 \cos \frac{a}{2} \cos \frac{b}{2} \cos \frac{c}{2}} \tag{2.111}$$
$$\mathsf{H}: \qquad \sin \frac{\Delta}{2} = \frac{\mathrm{am}_v}{4 \cosh \frac{a}{2} \cosh \frac{b}{2} \cosh \frac{c}{2}} \tag{2.112}$$

The spherical and hyperbolic cases can also be written

$$\mathsf{S}: \qquad \cos \frac{\Delta}{2} = \frac{1 + \cos a + \cos b + \cos c}{4 \cos \frac{a}{2} \cos \frac{b}{2} \cos \frac{c}{2}} \tag{2.113}$$
$$\mathsf{H}: \qquad \cos \frac{\Delta}{2} = \frac{1 + \cosh a + \cosh b + \cosh c}{4 \cosh \frac{a}{2} \cosh \frac{b}{2} \cosh \frac{c}{2}} \tag{2.114}$$
$$\mathsf{S}: \qquad \tan \frac{\Delta}{2} = \frac{\mathrm{am}_v}{1 + \cos a + \cos b + \cos c} \tag{2.115}$$
$$\mathsf{H}: \qquad \tan \frac{\Delta}{2} = \frac{\mathrm{am}_v}{1 + \cosh a + \cosh b + \cosh c} \tag{2.116}$$

By (2.105)–(2.106), the sine laws (2.4)–(2.5) may be written

$$\mathsf{S}: \qquad \frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} = \frac{\sin A \sin B \sin C}{\mathrm{am}_s} \tag{2.117}$$
$$\mathsf{H}: \qquad \frac{\sin A}{\sinh a} = \frac{\sin B}{\sinh b} = \frac{\sin C}{\sinh c} = \frac{\sin A \sin B \sin C}{|\mathrm{am}_s|}. \tag{2.118}$$

3. Special cases for a right triangle

Suppose that $C = \pi/2$. Then the following hold.

Pythagorean theorem.

$$\mathsf{E}: \quad c^2 = a^2 + b^2, \tag{3.1}$$

$$\mathsf{S}: \quad \cos c = \cos a \, \cos b, \tag{3.2}$$

$$\mathsf{H}: \quad \cosh c = \cosh a \, \cosh b. \tag{3.3}$$

Sines.

$$\mathsf{E}: \quad \sin A = \frac{a}{c}, \tag{3.4}$$

$$\mathsf{S}: \quad \sin A = \frac{\sin a}{\sin c}, \tag{3.5}$$

$$\mathsf{H}: \quad \sin A = \frac{\sinh a}{\sinh c}. \tag{3.6}$$

Cosines.

$$\mathsf{E}: \quad \cos A = \frac{b}{c}, \tag{3.7}$$

$$\mathsf{S}: \quad \cos A = \cos a \, \sin B = \frac{\cos a \, \sin b}{\sin c} = \frac{\tan b}{\tan c}, \tag{3.8}$$

$$\mathsf{H}: \quad \cos A = \cosh a \, \sin B = \frac{\cosh a \, \sinh b}{\sinh c} = \frac{\tanh b}{\tanh c}. \tag{3.9}$$

Tangents.

$$\mathsf{E}: \quad \tan A = a/b. \tag{3.10}$$

$$\mathsf{S}: \quad \tan A = \tan a / \sin b, \tag{3.11}$$

$$\mathsf{H}: \quad \tan A = \tanh a / \sinh b. \tag{3.12}$$

Cotangents.

$$(\mathsf{E}: \quad 1 = \cot A \, \cot B,) \tag{3.13}$$

$$\mathsf{S}: \quad \cos c = \cot A \, \cot B, \tag{3.14}$$

$$\mathsf{H}: \quad \cosh c = \cot A \, \cot B. \tag{3.15}$$

Area.

$$\mathsf{E}: \quad \Delta = \frac{ab}{2}, \tag{3.16}$$

$$\mathsf{S}: \quad \sin \Delta = \frac{\sin a \, \sin b}{1 + \cos c}, \tag{3.17}$$

$$\mathsf{H}: \quad \sin \Delta = \frac{\sinh a \, \sinh b}{1 + \cosh c}. \tag{3.18}$$

Remark 3.1. The main formulae for a right spherical triangle are described by *Napier's pentagon.* Write the angles $\bar{A}, b, a, \bar{B}, \bar{c}$, where $\bar{X} := \pi/2 - X$, in this order around a circle. Then the sine of each angle equals the product of the cosines of the opposite two angles, and equals also the product of the tangents of the two adjacent angles. (The latter relation follows easily from the first.)

The same holds for a hyperbolic triangle if we replace all trigonometric functions acting on a, b or c by the corresponding hyperbolic functions.

4. Infinite triangles (hyperbolic)

In hyperbolic geometry one might thus also consider *infinite* or *improper* or *asymptotic* triangles, where one, two or three vertices are infinite points; this means that the corresponding sides are infinite rays or lines that are parallel and do not intersect. Some sides thus have infinite length. The angle at an infinite vertex is defined to be 0.

We have the following cases:

(1) A singly infinite triangle ABC where C is infinite. The side AB is finite, while AC and BC are infinite rays. (Thus, $c < \infty$, while $a = b = \infty$. The angle $C = 0$, while $A, B > 0$.)
(2) A doubly infinite triangle ABC where B and C are infinite. The sides AB and AC are infinite rays, while BC is an infinite line. (Thus, $a = b = c = \infty$. The angles $B = C = 0$, while $A > 0$.)
(3) A triply infinite triangle ABC where A, B, C all are infinite. The three sides are infinite lines. (Thus, $a = b = c = \infty$. The angles $A = B = C = 0$.) All such triangles are congruent, and have area π.

The law of sines holds trivially (all ratios are 0).

The formula (2.53) for angle sum and area holds for all infinite triangles. In particuler, the area of a triangle is bounded; $\Delta \leqslant \pi$ with equality if and only if the triangle is triply infinite.

The second law of cosines (2.11) makes sense for a singly infinite triangle; with vertices A and B finite (so the angles $A, B > 0$ and $c < \infty$), it says

$$\mathsf{H}: \qquad 1 = -\cos A \, \cos B + \sin A \, \sin B \, \cosh c. \tag{4.1}$$

(All other combinations include $0 \cdot \infty$ and are thus meaningless.)

Similarly, Napier's analogy (2.21) holds for all infinite triangles; since $a + b = \infty$ in all cases (at least two sides are infinite), it can then be written

$$\mathsf{H}: \qquad \tanh \frac{c}{2} = \frac{\cos \frac{A+B}{2}}{\cos \frac{A-B}{2}}, \tag{4.2}$$

which is equivalent to (4.1) when $A, B > 0$, and is valid but trivial $(1 = 1)$ in all other cases. The relation (4.2) can also easily be transformed to

$$\mathsf{H}: \qquad e^{-c} = \tan \frac{A}{2} \, \tan \frac{B}{2}. \tag{4.3}$$

(Napier's analogy (2.22) is either trivial ($\mp 1 = \mp 1$ when a or b is finite), or meaningless (involving $\infty - \infty$). The other two analogies (2.23)–(2.24) are always meaningless (with $\infty - \infty$ or ∞/∞) for infinite triangles, although (4.3) (with vertices B and C interchanged) can be seen as the right interpretation when $B = 0$ (take the limit as $a \to \infty$).)

In particular, for an infinite right triangle with $B = \pi/2$ and $C = 0$, (4.1) yields

$$\mathsf{H}: \qquad \sin A = 1/\cosh c. \tag{4.4}$$

Equivalently,

$$\mathsf{H}: \qquad \cos A = \tanh c, \tag{4.5}$$

$$\mathsf{H}: \qquad \tan A = 1/\sinh c. \tag{4.6}$$

Remark 4.1 (E). In the Euclidean case, one can define an infinite triangle with one infinite vertex in the analoguous way; this "triangle" then has two parallel infinite sides (infinite rays lying on two parallel lines, and going in the same direction) and a third, finite, side connecting their finite endpoints (which are two arbitrary points on the infinite lines); if the infinite vertex is C, then $a = b = \infty$ and $A + B = \pi$, so (2.51) still holds with $C = 0$. We will not discuss such Euclidean infinite triangles, which in any case are simple and easily handled.

4.1. **Angle of parallelism.** We can interpret (4.4)–(4.6) in the following way (the *angle of parallelism*): Let A be a point and ℓ a line in the hyperbolic plane, with $A \notin \ell$. Then there are two lines through A parallel to ℓ, and the angle between them (which equals the angle under which ℓ is seen from A) equals $2A$, where A satisfies (4.4)–(4.6) with $c = d(A, \ell)$, the distance from A to ℓ.

5. Duality (spherical)

In spherical geometry, every triangle ABC has a dual triangle $A^*B^*C^*$, defined by regarding the points on the sphere as unit vectors in $\mathbb{R}^3$ and letting A^* be the unit vector orthogonal to the side BC and on the same side of it as A, and similarly for B^* and C^*, i.e.,

$$\mathsf{S}: \qquad \langle A^*, B\rangle = \langle A^*, C\rangle = 0, \qquad \langle A^*, A\rangle > 0, \tag{5.1}$$

$$\mathsf{S}: \qquad \langle B^*, A\rangle = \langle B^*, C\rangle = 0, \qquad \langle B^*, B\rangle > 0, \tag{5.2}$$

$$\mathsf{S}: \qquad \langle C^*, A\rangle = \langle C^*, B\rangle = 0, \qquad \langle C^*, C\rangle > 0. \tag{5.3}$$

using the standard inner product in $\mathbb{R}^3$.

Since (5.1)–(5.3) are symmetric in ABC and $A^*B^*C^*$, duality is a symmetric relation, i.e., the second dual of a triangle ABC is ABC again (with vertices in the same order).

We have

$$\mathsf{S}: \qquad A^* = \pi - a, \qquad a^* = \pi - A, \tag{5.4}$$

$$\mathsf{S}: \qquad B^* = \pi - b, \qquad b^* = \pi - B, \tag{5.5}$$

$$\mathsf{S}: \qquad C^* = \pi - c, \qquad c^* = \pi - C. \tag{5.6}$$

and thus

$$\mathsf{S}: \qquad s^* = \frac{3\pi}{2} - \sigma, \qquad \sigma^* = \frac{3\pi}{2} - s. \tag{5.7}$$

The amplitudes are interchanged:

$$\mathsf{S}: \qquad \mathrm{am}_s^* = \mathrm{am}_v \qquad \mathrm{am}_v^* = \mathrm{am}_s. \tag{5.8}$$

Thus, for example, the law of sines (2.4) is self-dual, while the two laws of cosines (2.7) and (2.10) are dual, in the sense that one formula for a triangle is the same as the other for the dual triangle. Furthermore, (2.76) and (2.77) are dual.

6. Adjacent triangles (spherical)

In spherical geometry, there is another useful symmetry besides duality (Section 5). Each point A has an antipode $\bar{A}$, and by replacing a subset of the vertices A, B, C by their antipodes, we obtain 8 different triangles (including the original ABC). These 8 triangles together cover the sphere, without overlaps (except at the boundaries); they are the 8 subsets obtained by cuttng the sphere along the three (Euclidean) planes through the sides of the triangle.

We call the three triangles $\bar{A}BC$, $A\bar{B}C$ and $AB\bar{C}$ obtained by replacing one vertex by an antipode the *adjacent* triangles of ABC.[5] The 8 triangles considered here are thus the original triangle, its 3 adjacent triangles, and the antipodes of these 4.

The adjacent triangles share one side each with the original triangle. The triangle $\bar{A}BC$ has sides

$$\mathsf{S}: \qquad a' = a, \qquad b' = \pi - b, \qquad c' = \pi - c \tag{6.1}$$

and angles

$$\mathsf{S}: \qquad A' = A, \qquad B' = \pi - B, \qquad C' = \pi - C. \tag{6.2}$$

The area of the adjacent triangle $\bar{A}BC$ is thus, by (2.52),

$$\mathsf{S}: \qquad \Delta' = 2A - \Delta. \tag{6.3}$$

This is obvious geometrically, since the union of the triangles ABC and $\bar{A}BC$ is the lune between the lines $AB\bar{A}$ and $AC\bar{A}$, which has area $2A$.

The amplitudes am_s and am_v are the same as for the original triangle. (They are thus the same for all 8 triangles.)

It follows immediately from (5.1)–(5.3) that taking the dual triangle commutes with replacing one or several vertices by their antipodes; thus the duals of two adjacent triangle are adjacent to each other.

[5] This is not standard terminology, as far as I know.

7. Solving triangles

A triangle is described by three angles and three sides. In many, but not all, cases, three of these six elements determine uniquely the triangle up to congruence; in particular they determine the remaining three elements.[6] To calculate the remaining sides and angles when some of them are given is called to solve the triangle. There are six different cases, which we discuss cases by case; we discuss both existence and uniqueness of the triangle and the calculations of the remaining sides and angles, and in many cases formulae for the area.

We assume tacitly that all given angles are in $(0, \pi)$ and that all given sides are positive and in the spherical case in $(0, \pi)$. In the hyperbolic case, we sometimes consider also infinite triangles, but only when we explicitly say so; otherwise all triangles are assumed to be finite. Note that cos is one-to-one on $[0, \pi]$; thus an angle is uniquely determined by its cosine; however, sin is not one-to-one on $[0, \pi]$, since $\sin A = \sin(\pi - A)$; thus the sine of an angle is in general not enough to determine the angle; there are typically two possibilities, although often (2.86) can be used to exclude one of them. In the spherical case, the same applies to sine and cosine of sides; in the Euclidean and hyperbolic cases there are no problems with sides, in the latter case because both cosh and sinh are one-to-one on $[0, \infty)$.

7.1. **Three sides given (SSS).** A triangle with sides a, b, c exists if and only if the (strict) triangle inequalities (2.75) hold, and in the spherical case furthermore (2.76) holds; i.e.,

$$\mathsf{E,S,H}: \qquad a < b + c, \qquad b < a + c, \qquad c < a + b, \tag{7.1}$$

$$\mathsf{S}: \qquad a + b + c < 2\pi. \tag{7.2}$$

The triangle then is unique. (In the hyperbolic case, there are also triangles with two or three sides infinite, e.g. $a = b = \infty$ and $c \in (0, \infty]$ arbitrary; these are not uniquely determined by the sides, since we may choose the angle A arbitrary, with $A \in (0, \pi)$ if $c < \infty$ and $A \in [0, \pi)$ if $a = b = c = \infty$, see the case SAS.)

With the sides a, b, c given, the angles can be found by the law of cosines (2.6)–(2.8).

The area is given by Heron's formula (2.65)–(2.67).

7.2. **Two sides and the included angle given (SAS).** There exists a unique triangle for any given sides b, c and included angle A. (In the hyperbolic case, this includes the case of an infinite triangle with b or c or both infinite.)

With the sides b, c and the angle A given, the remaining side a is given by the law of cosines (2.6)–(2.8). The remaining two angles then can be found, as in the case SSS, by by the law of cosines (2.6)–(2.8). (The law of sines (2.3)–(2.5) can also be used to find the angles, but give typically

[6] This goes back to Euclid, and perhaps to Thales of Miletus [8, Thales of Miletus].

two possibilities for each angle. In the Euclidean case, the third angle can be found by the angle sum (2.51) once the second has been determined. In the spherical and hyperbolic cases, another alternative is to use Napier's analogies (2.19)–(2.20) or (2.23)–(2.24), resp., to find $B+C$ and $B-C$.) The case of an infinite hyperbolic triangle is special; in this case the remaining side $a=\infty$, and if, say, $b=\infty$ and $c<\infty$, then $C=0$ (since vertex C is infinite) and B can be found by (4.3).

An alternative is to obtain the two remaining angles B and C directly by (2.45)–(2.47).

The area is, by (2.55)–(2.57) and the law of cosines, given by

$$\mathsf{E}: \qquad \Delta = \frac{bc\sin A}{2}, \tag{7.3}$$

$$\mathsf{S}: \quad \sin\frac{\Delta}{2} = \frac{\sin\frac{b}{2}\sin\frac{c}{2}\sin A}{\cos\frac{a}{2}} = \frac{\sqrt{2}\sin\frac{b}{2}\sin\frac{c}{2}\sin A}{\sqrt{1+\cos b\cos c+\sin b\sin c\cos A}}, \tag{7.4}$$

$$\mathsf{H}: \quad \sin\frac{\Delta}{2} = \frac{\sinh\frac{b}{2}\sinh\frac{c}{2}\sin A}{\cosh\frac{a}{2}} = \frac{\sqrt{2}\sinh\frac{b}{2}\sinh\frac{c}{2}\sin A}{\sqrt{1+\cosh b\cosh c-\sinh b\sinh c\cos A}}. \tag{7.5}$$

7.3. **Two sides and an opposite angle given (ASS = SSA).** This is the most complicated case. The conditions for existence are as follows, with several different cases for each of the three geometries. (Note the similarity between the Euclidean and hyperbolic cases, while the spherical case is somewhat different.) We assume that the sides a and b and the angle A opposite to a are given.

$$\mathsf{E}: \qquad A<\pi/2 \quad \text{and} \quad a\geqslant b\sin A \qquad \text{or} \tag{7.6}$$

$$A\geqslant\pi/2 \quad \text{and} \quad a>b; \tag{7.7}$$

$$\mathsf{S}: \qquad A,a<\pi/2 \quad \text{and} \quad \sin b\sin A\leqslant\sin a\leqslant\sin b \qquad \text{or} \tag{7.8}$$

$$A,a>\pi/2 \quad \text{and} \quad \sin b\sin A\leqslant\sin a\leqslant\sin b \qquad \text{or} \tag{7.9}$$

$$\sin a>\sin b \qquad \text{or} \tag{7.10}$$

$$A=a=b=\pi/2; \tag{7.11}$$

$$\mathsf{H}: \qquad A<\pi/2 \quad \text{and} \quad \sinh a\geqslant\sinh b\sin A \qquad \text{or} \tag{7.12}$$

$$A\geqslant\pi/2 \quad \text{and} \quad a>b. \tag{7.13}$$

Moreover, among these cases, there are *two* different triangles (with supplementary values for the angle B) in the cases

$$\mathsf{E}: \quad A<\pi/2 \quad \text{and} \quad b\sin A<a<b, \tag{7.14}$$

$$\mathsf{S}: \quad A,a<\pi/2 \quad \text{and} \quad \sin b\sin A<\sin a<\sin b \qquad \text{or} \tag{7.15}$$

$$A,a>\pi/2 \quad \text{and} \quad \sin b\sin A<\sin a<\sin b, \tag{7.16}$$

$$\mathsf{H}: \quad A<\pi/2 \quad \text{and} \quad \sinh b\sin A<\sinh a<\sinh b, \tag{7.17}$$

and *infinitely* many different triangles in the exceptional spherical case (7.11):

$$\mathsf{S}: \qquad A = a = b = \pi/2; \tag{7.18}$$

in this case $B = \pi/2$ while $c = C$ is arbitrary in $(0, \pi)$.

In all cases, B is given by the law of sines. This gives two possibilities (except when $B = \pi/2$); in many cases one solution can be excluded by (2.86); otherwise both values are possible, and give two different solutions (the cases (7.14)–(7.17)).

E: In the Euclidean case, the remaining angle C then can be found from the angle sum (2.51), and the remaining side c from the law of sines (2.3). This yields the formula

$$\mathsf{E}: \qquad c = b\cos A \pm \sqrt{a^2 - b^2 \sin^2 A}, \tag{7.19}$$

where of course only a sign that yields a positive result is acceptable. (It is easily verified that there are 0, 1 or 2 choices in agreement with (7.6)–(7.7) and (7.14).)

S,H: In the spherical and hyperbolic cases, the remaining angle and side can be found by Napier's analogies, for example by (2.17) and (2.19), or (2.21) and (2.23), except in the spherical case when $a + b = \pi$ and thus $A + B = \pi$ (cf. (2.87)) when we instead use (2.18) and (2.20), assuming we are not in the indeterminate case (7.18). (Alternatively, (2.18) and (2.20), or (2.22) and (2.24), can be used in all cases except the isosceles $a = b$ and thus $A = B$.) An infinite hyperbolic triangle with $a = \infty$, $b < \infty$ and $A > 0$ can be solved in the same way using (4.3); we have $B = 0$, $c = \infty$ and (4.3) (with vertices B and C interchanged) yields c.

The area is in the Euclidean case, by (7.19),

$$\mathsf{E}: \qquad \Delta = \frac{b^2 \sin A \, \cos A \pm b \sin A \sqrt{a^2 - b^2 \sin^2 A}}{2}, \tag{7.20}$$

where as in (7.19), the sign has to be chosen such that the result is positive.

7.4. **One side and two adjacent angles given (ASA).** There exists a triangle with the side a and the adjacent angles B and C if and only if the following holds:

$$\mathsf{E}: \qquad B + C < \pi, \tag{7.21}$$

$$\mathsf{S}: \qquad \text{(no condition; a triangle always exists)}, \tag{7.22}$$

$$\mathsf{H}: \qquad B + C < \pi \quad \text{and} \quad \cosh a < \frac{1 + \cos B \, \cos C}{\sin B \, \sin C}. \tag{7.23}$$

The triangle then is unique. In the hyperbolic case, there is furthermore a (unique) infinite triangle with side a and angles B, C if

$$\mathsf{H}: \qquad B + C < \pi \quad \text{and} \quad \cosh a = \frac{1 + \cos B \, \cos C}{\sin B \, \sin C}. \tag{7.24}$$

In the hyperbolic case, (7.23) and (7.24) can be transformed to the equivalent, cf. (4.3),

$$\mathsf{H}: \qquad \tan\frac{B}{2}\tan\frac{C}{2} < e^{-a} \qquad \text{(finite)}, \tag{7.25}$$

$$\mathsf{H}: \qquad \tan\frac{B}{2}\tan\frac{C}{2} = e^{-a} \qquad \text{(infinite)}. \tag{7.26}$$

In all cases, the remaining angle A can be found by the second law of cosines (2.9)–(2.11), which in the Euclidean case simply amounts to using the angle sum (2.51). The remaining sides b and c can then be found by the law of sines in the Euclidean and hyperbolic cases, and by the second law of cosines in the spherical (and hyperbolic) cases. (The law of sines applies in the spherical case too, but gives usually two possible results each for b and c. Another alternative, in all three geometries, is to use Napier's analogies (2.15)–(2.16), (2.17)–(2.18) or (2.21)–(2.22), resp., to find $b+c$ and $b-c$, and thus b and c.)

The area is in the Euclidean case

$$\mathsf{E}: \qquad \Delta = \frac{a^2 \sin B \sin C}{2\sin(B+C)} = \frac{a^2}{2(\cot B + \cot C)}. \tag{7.27}$$

Remark 7.1. For any $a > 0$ and angles $B, C \in (0,\pi)$, one can construct, on an arbitrary line ℓ, two points B and C with distance a and lines ℓ_B and ℓ_C through B and C, respectively, making angles B and C with ℓ; more precisely, we let ℓ_B and ℓ_C be directed lines such that if w_B and w_C are the positive endpoints of ℓ_B and ℓ_C, then w_B and w_C lie on the same side of ℓ, and the angle CBw_B is B and the angle BCw_C is C. (This construction is obviously unique up to congruence.) Then (7.21)–(7.23) give conditions for the lines ℓ_B and ℓ_C to intersect on the positive side of ℓ, thus forming a triangle ABC with interior angles B and C, but in any case, the gauge $\langle \ell_B, \ell_C\rangle$ is given by

$$\mathsf{E}: \qquad \langle \ell_B, \ell_C\rangle = -\cos B\,\cos C + \sin B\,\sin C = -\cos(B+C), \tag{7.28}$$

$$\mathsf{S}: \qquad \langle \ell_B, \ell_C\rangle = -\cos B\,\cos C + \sin B\,\sin C\,\cos a, \tag{7.29}$$

$$\mathsf{H}: \qquad \langle \ell_B, \ell_C\rangle = -\cos B\,\cos C + \sin B\,\sin C\,\cosh a. \tag{7.30}$$

Note that in the case of a triangle, when the gauge is $\cos A$, this is just the second law of cosines (2.9)–(2.11).

7.5. **One side, one adjacent and the opposite angle given (AAS = SAA).** We assume that the side a and the two angles A and B are given. The conditions for existence of a triangle are the following. (In the spherical case, this follows by duality from the case ASS. The Euclidean and hyperbolic cases are much simpler.)

$$\mathsf{E}: \qquad A + B < \pi, \tag{7.31}$$

$$\mathsf{S}: \qquad A, a < \pi/2 \quad \text{and} \quad \sin B \sin a \leqslant \sin A \leqslant \sin B \qquad \text{or} \tag{7.32}$$

$$A, a > \pi/2 \quad \text{and} \quad \sin B \sin a \leqslant \sin A \leqslant \sin B \qquad \text{or} \tag{7.33}$$

$$\sin A > \sin B \qquad \text{or} \tag{7.34}$$

$$a = A = B = \pi/2; \tag{7.35}$$

$$\mathsf{H}: \qquad A + B < \pi. \tag{7.36}$$

Moreover, there are *two* such triangles in the spherical cases

$$\mathsf{S}: \qquad A, a < \pi/2 \qquad \text{and} \qquad \sin B \sin a < \sin A < \sin B \qquad \text{or} \tag{7.37}$$

$$A, a > \pi/2 \qquad \text{and} \qquad \sin B \sin a < \sin A < \sin B, \tag{7.38}$$

and *infinitely* many different triangles in the exceptional spherical case (7.35):

$$\mathsf{S}: \qquad a = A = B = \pi/2; \tag{7.39}$$

in this case $b = \pi/2$ while $c = C$ is arbitrary in $(0, \pi)$. In all other cases, including all Euclidean and hyperbolic cases, the triangle is unique.

In all cases, b is given by the law of sines. In the spherical case, this gives two possibilities (except when $b = \pi/2$); in many cases one solution can be excluded by (2.86); otherwise both values are possible, and give two different solutions (the cases (7.37)–(7.38)). We then know a, b, A, B and are in the same situation as in the case AAS, so we proceed as there:

E: In the Euclidean case, the remaining angle C then can be found as $\pi - A - B$ from the angle sum (2.51), and the remaining side can be found from the law of sines (2.3) as

$$\mathsf{E}: \qquad c = \frac{a}{\sin A} \sin(A + B) = a \sin B(\cot A + \cot B). \tag{7.40}$$

S,H: In the spherical and hyperbolic cases, the remaining angle and side can be found by Napier's analogies, for example by (2.17) and (2.19), or (2.21) and (2.23), except in the spherical case when $a + b = \pi$ and thus $A + B = \pi$ (cf. (2.87)) when we instead use (2.18) and (2.20), assuming we are not in the indeterminate case (7.18). (Alternatively, (2.18) and (2.20), or (2.22) and (2.24), can be used in all cases except the isosceles $a = b$ and thus $A = B$.) An infinite hyperbolic triangle with $a < \infty$, $A = 0$ and $B > 0$ can be solved similarly; we have $b = c = \infty$ and (4.3) (with vertices A and C interchanged) yields C.

The area is in the Euclidean case, by (7.40),

$$\mathsf{E}: \qquad \Delta = \frac{a^2 \sin B \sin(A + B)}{2 \sin A} = \frac{1}{2} a^2 \sin^2 B(\cot A + \cot B). \tag{7.41}$$

7.6. **Three angles given (AAA).** E: In the Euclidean case, the three angles have to satisfy (2.51), i.e.

$$\mathsf{E}: \qquad A + B + C = \pi. \tag{7.42}$$

If this holds, there are infinitely many triangles with these angles, since the angles determine a Euclidean triangle up to similarity, but not up to congruence. We may choose one of the sides arbitrarily, and then find the other two by the law of sines as in case ASA or AAS.

S: In the spherical case, the angle sum has to be greater than π by (2.77); furthermore, the angles have to satisfy the dual triangle inequality (2.80). We thus have the conditions

$$\mathsf{S}: \qquad A+B+C>\pi, \tag{7.43}$$

$$B+C<\pi+A, \qquad A+C<\pi+B, \qquad A+B<\pi+C. \tag{7.44}$$

These conditions are both necessary and sufficient; for such A, B, C, there is a unique triangle with these angles.

H: In the hyperbolic case, the angle sum has to be less than π by (2.78):

$$\mathsf{H}: \qquad A+B+C<\pi. \tag{7.45}$$

This is also sufficient; for any such A, B, C, there is a unique triangle with these angles. (This includes the case of infinite triangles, when one or several of the angles A, B, C is 0.)

S, H: In the spherical and hyperbolic cases, the sides are determined by the second law of cosines (2.10)–(2.11). (In the hyperbolic case, for an infinite triangle, this includes the case of the finite side in a singly infinite triangle, see (4.1); any side adjacent to a vertex with angle 0 is infinite.)

The area is given by (2.52)–(2.53).

8. Circles

A circle is the set of points with a given distance r (the radius) to some given point (the centre).

Remark 8.1 (S). In spherical geometry, a circle with centre O and radius R coincides with the circle with the antipodal centre $\bar{O}$ and radius $\pi - R$; the interior of one of these circles is the exterior of the other.

Note also that in spherical geometry, a circle with radius $\pi/2$ is a geodesic (and conversely).

A circle with radius r has circumference

$$\mathsf{E}: \qquad 2\pi r, \tag{8.1}$$

$$\mathsf{S}: \qquad 2\pi \sin r, \tag{8.2}$$

$$\mathsf{H}: \qquad 2\pi \sinh r \tag{8.3}$$

and area

$$\mathsf{E}: \qquad \pi r^2, \tag{8.4}$$

$$\mathsf{S}: \qquad 4\pi \sin^2(r/2) = 2\pi(1-\cos r), \tag{8.5}$$

$$\mathsf{H}: \qquad 4\pi \sinh^2(r/2) = 2\pi(\cosh r - 1). \tag{8.6}$$

The geodesic curvature is (see e.g. [7])

$$\mathsf{E}: \qquad 1/r, \tag{8.7}$$

$$\mathsf{S}: \qquad \cot r, \tag{8.8}$$

$$\mathsf{H}: \qquad \coth r, \tag{8.9}$$

directed towards the centre of the circle (in spherical geometry, for $r > \pi/2$ when $\cot r < 0$, the circle thus curves outwards, in accordance with Remark 8.1.)

In all cases, if a circle has circumference ℓ, area A and geodesic curvature κ, then

$$\mathsf{E}, \mathsf{S}, \mathsf{H}: \qquad 2\pi - \ell\kappa = AK = \begin{cases} 0, & \mathsf{E} \\ A, & \mathsf{S} \\ -A, & \mathsf{H} \end{cases} \tag{8.10}$$

where $K = 0, 1, -1$ is the sectional curvature; this is an example of the Gauss–Bonnet theorem.

9. Curves of constant curvature

E,S: The curves of constant (geodesic) curvature in the Euclidean plane or on the sphere are lines (geodesics) and circles.

H: In the hyperbolic plane, where all circles have curvature $\kappa > 1$ by (8.9), there are two further types of such curves: *horocycles* and *hypercycles*. A horocycle has curvature $\kappa = 1$ and a hypercycle curvature $\kappa \in (0, 1)$. Geometrically, a horocycle is orthogonal to a pencil of parallel lines; a hypercycle is the set of all points with a given distance $d > 0$ to some line ℓ, and lying on the same side of ℓ. (The line is uniquely determined by the hypercycle, and so is the distance d. We call the line the *axis* of the hypercycle.) See also Appendices A and B for descriptions in two standard models. (If we include points at infinity, a horocycle contains one infinite point, and is orthogonal to all lines having this point as one end; a hypercycle contains two infinite points which are the ends of its axis.)

The curvature of a hypercycle with distance d to its axis is

$$\mathsf{H}: \qquad \kappa = \tanh d. \tag{9.1}$$

The curvature of a curve with constant curvature is thus 0 for a line and otherwise, for a circle of radius r or (in the hyperbolic case H) a horocycle or a hypercycle of distance d from its axis, extending (8.7)–(8.9),

$$\mathsf{E}: \qquad \kappa = 1/r, \tag{9.2}$$

$$\mathsf{S}: \qquad \kappa = \cot r, \tag{9.3}$$

$$\mathsf{H}: \qquad \kappa = \begin{cases} \coth r, & \text{(circle)} \\ 1, & \text{(horocycle)} \\ \tanh d. & \text{(hypercycle)} \end{cases} \tag{9.4}$$

In all three cases (E, S, H), all curves of a given constant curvature $\kappa \in [0, \infty)$ are congruent.

Remark 9.1 (E,H). In the Euclidean and hyperbolic cases, the set of points at infinity may be regarded as the limit of a circle with radius r as $r \to \infty$ (and the centre is any fixed point); by (9.2) and (9.4), the curvature of this cycle tends to 0 (E) or 1 (H) as $r \to \infty$, which justifies regarding the set of points at infinity as being a line in the Euclidean case but a horocycle in the hyperbolic case, as claimed in Subsection 1.1. (In the hyperbolic case, the horocycle at infinity can furthermore be seen as the limit of a family of parallel horocycles; for example in the halfplane model in Appendix A the horocycles $\{z : \operatorname{Im} z = y\}$ with $y \to 0$.)

10. Circumcircle

Three distinct points lie on a unique curve of constant curvature (a line, a circle or, in the hyperbolic case, a horocycle or hypercycle, see Section 9). (In the hyperbolic case, this is easily seen using one of the models in Appendices A and B.)

Remark 10.1 (H). In the hyperbolic case (but not the Euclidean case), this includes the case when one, two or three of the points are infinite; more precisely we have the following possibilities for the curve and its curvature κ:

1 infinite point: $0 \leqslant \kappa \leqslant 1$, a line, hypercycle or horocycle;
2 infinite points: $0 \leqslant \kappa < 1$, a line or hypercycle;
3 infinite points: $\kappa = 1$, the horocycle at infinity, see Remark 9.1.

However, in the sequel we assume that the points are proper unless otherwise said.

Ignoring the case of three collinear points, this means that a (proper) triangle ABC has

E,S: a unique circumscribed circle;
H: a unique circumscribed circle, horocycle or hypercycle.

This can also be viewed in another way: The perpendicular bisector of a side, say AB, is the locus of all points with the same distance to the vertices A and B. The three perpendicular bisectors therefore intersect in the points that have the same distance to all three vertices, if any. If there is such a point O, there is a circle with centre O and some radius $R > 0$ that passes through all vertices; this is the *circumcircle*, and R is the *circumradius*. In the Euclidean and spherical cases, this is always the case; in the hyperbolic case, there are two further possibilities, a *circumhorocycle* or a *circumhypercycle*. More precisely, the following holds:

E: In the Euclidean case, the perpendicular bisectors intersect in a unique point, and there is a unique circumcircle.

S: In the spherical case, the perpendicular bisectors intersect in two antipodal points O and $\bar{O}$, that thus both can be taken as the centres of the circumcircle. They define the same circumcircle, with the radius R or $R' = \pi - R$ (see Remark 8.1). For definiteness, we define the circumcentre to be the point O such the circumradius $R < \pi/2$. ($R = \pi$ is excluded, since a circle with radius π is the same as a line (a great circle), and thus all three vertices lie on a line.)

H: In the hyperbolic case, there are three (mutually exclusive) possibilities (cf. Subsection 1.2 and Appendix C):

(i) All three perpendicular bisectors intersect in a common point, and there is a (unique) circumcircle.

(ii) The three perpendicular bisectors are parallel. There is a (unique) circumscribed horocycle (i.e., a horocycle going through the three vertices); this horocycle is orthogonal to the three perpendicular bisectors, and has the same point at infinity as they.

(iii) The three perpendicular bisectors are pairwise ultraparallel. There is a (unique) circumscribed hypercycle (i.e., a hypercycle going through the three vertices); the axis of this hypercycle is orthogonal to the three perpendicular bisectors (so it is the common normal of all three of them).

Remark 10.2. The disc bounded by a circle (with radius $\leqslant \pi$ in the spherical case) is convex. Hence, if a triangle has a circumcircle, then the sides and interior of the triangle lie inside the circumcircle (except for the vertices which by definition lie on the circle).

If the triangle ABC has a circumcircle, we let R be its radius. (In other words, A, B, C lie on a circle with radius R.) In the hyperbolic case, if there instead is a circumhypercycle, we let D be its distance from its axis; we can unify the formulae for the three hyperbolic case by using the curvature κ, see (9.4). We have, expressing R or D in the sides or the angles,

$$\mathsf{E}: \qquad R^2 = \kappa^{-2} = \frac{a^2b^2c^2}{4\mathrm{am}_v^2} = \frac{a^2b^2c^2}{16s(s-a)(s-b)(s-c)}, \tag{10.1}$$

$$\mathsf{S}: \qquad \tan^2 R = \kappa^{-2} = \frac{2(1-\cos a)(1-\cos b)(1-\cos c)}{\mathrm{am}_v^2} \tag{10.2}$$

$$= \frac{-\cos\sigma}{\cos(\sigma-A)\cos(\sigma-B)\cos(\sigma-C)} = \frac{4\cos^2\sigma}{\mathrm{am}_s^2} \tag{10.3}$$

$$\mathsf{H}: \qquad \left.\begin{matrix} \tanh^2 R \\ 1 \\ \coth^2 D \end{matrix}\right\} = \kappa^{-2} = \frac{2(\cosh a - 1)(\cosh b - 1)(\cosh c - 1)}{\mathrm{am}_v^2} \tag{10.4}$$

$$= \frac{\cos\sigma}{\cos(\sigma-A)\cos(\sigma-B)\cos(\sigma-C)} = \frac{4\cos^2\sigma}{-\mathrm{am}_s^2} \tag{10.5}$$

and thus

$$\mathsf{E}: \qquad R = \kappa^{-1} = \frac{abc}{2\mathrm{am}_v} = \frac{abc}{4\Delta}, \tag{10.6}$$

$$\mathsf{S}: \qquad \tan R = \kappa^{-1} = \frac{4\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{c}{2}}{\mathrm{am}_v} = \frac{\tan\frac{a}{2}\tan\frac{b}{2}\tan\frac{c}{2}}{\sin\frac{\Delta}{2}} \tag{10.7}$$

$$= \frac{2|\cos\sigma|}{\mathrm{am}_s}, \tag{10.8}$$

$$\mathsf{H}: \qquad \left.\begin{matrix}\tanh R\\ 1\\ \coth D\end{matrix}\right\} = \kappa^{-1} = \frac{4\sinh\frac{a}{2}\sinh\frac{b}{2}\sinh\frac{c}{2}}{\mathrm{am}_v} = \frac{\tanh\frac{a}{2}\tanh\frac{b}{2}\tanh\frac{c}{2}}{\sin\frac{\Delta}{2}} \tag{10.9}$$

$$= \frac{2\cos\sigma}{|\mathrm{am}_s|}. \tag{10.10}$$

In the hyperbolic case, (10.5) and (10.10) apply also to infinite triangles, see Remark 10.1. Of course, in this case, there is a circumhorocycle or circumhypercycle; this is proper for singly or doubly infinite triangles while a triply infinite triangle is circumscribed by the (improper) horocycle at infinity, see Remark 9.1.

Returning to proper triangles, we have also the formulae

$$\mathsf{E}: \qquad R = \kappa^{-1} = \frac{a/2}{\sin A} = \frac{b/2}{\sin B} = \frac{c/2}{\sin C}, \tag{10.11}$$

$$\mathsf{S}: \qquad \tan R = \kappa^{-1} = \frac{\tan(a/2)}{\cos(\sigma - A)} = \frac{\tan(b/2)}{\cos(\sigma - B)} = \frac{\tan(c/2)}{\cos(\sigma - C)}, \tag{10.12}$$

$$\mathsf{H}: \qquad \left.\begin{matrix}\tanh R\\ 1\\ \coth D\end{matrix}\right\} = \kappa^{-1} = \frac{\tanh(a/2)}{\cos(\sigma - A)} = \frac{\tanh(b/2)}{\cos(\sigma - B)} = \frac{\tanh(c/2)}{\cos(\sigma - C)}. \tag{10.13}$$

In terms of two sides and the included angle, we have the formulas

$$\mathsf{E}: \qquad R^2 = \kappa^{-2} = \frac{b^2 + c^2 - 2bc\cos A}{4\sin^2 A}, \tag{10.14}$$

$$\mathsf{S}: \qquad \tan^2 R = \kappa^{-2} \tag{10.15}$$

$$= \frac{\tan^2(b/2) + \tan^2(c/2) - 2\cos A\,\tan(b/2)\,\tan(c/2)}{\sin^2 A}, \tag{10.16}$$

$$\mathsf{H}: \qquad \left.\begin{matrix}\tanh^2 R\\ 1\\ \coth^2 D\end{matrix}\right\} = \kappa^{-2} \tag{10.17}$$

$$= \frac{\tanh^2(b/2) + \tanh^2(c/2) - 2\cos A\,\tanh(b/2)\,\tanh(c/2)}{\sin^2 A} \tag{10.18}$$

$$= 1 + \frac{\big(\cos(\beta+\gamma) - \cos A\big)\big(\cos(\beta-\gamma) - \cos A\big)}{\sin^2 A}, \tag{10.19}$$

where in (10.19) (in the hyperbolic case) we define

$$\beta = \arcsin\Bigl(\frac{1}{\cosh(b/2)}\Bigr) = \arccos\bigl(\tanh(b/2)\bigr), \tag{10.20}$$

$$\gamma = \arcsin\Bigl(\frac{1}{\cosh(c/2)}\Bigr) = \arccos\bigl(\tanh(c/2)\bigr). \tag{10.21}$$

In the hyperbolic case, it can be shown, using (10.9), that a triangle has a circumscribed circle, horocycle or hypercycle according as the largest of $\sinh\frac{a}{2}$, $\sinh\frac{b}{2}$, $\sinh\frac{c}{2}$ is less than, equal to, or greater than the sum of the two others, i.e., if we assume that the triangle is labelled with $a \geqslant b \geqslant c$,

$$\mathsf{H}: \qquad \begin{cases} \kappa > 1 \text{ (circle)} & \iff \sinh\frac{a}{2} < \sinh\frac{b}{2} + \sinh\frac{c}{2} \\ \kappa = 1 \text{ (horocycle)} & \iff \sinh\frac{a}{2} = \sinh\frac{b}{2} + \sinh\frac{c}{2} \\ \kappa < 1 \text{ (hypercycle)} & \iff \sinh\frac{a}{2} > \sinh\frac{b}{2} + \sinh\frac{c}{2}. \end{cases} \tag{10.22}$$

In other words, the triangle has a circumcircle if and only if the three numbers $\sinh\frac{a}{2}$, $\sinh\frac{b}{2}$, $\sinh\frac{c}{2}$ satisfy the triangle inequality (strictly).

Similarly, still in the hyperbolic case, it follows from (10.19) that the triangle has a circumscribed circle if and only if A, β and γ (defined by (10.20)–(10.21)) satisfy the triangle inequalities

$$A < \beta + \gamma, \qquad \beta < A + \gamma, \qquad \gamma < A + \beta; \tag{10.23}$$

there is a circumscribed horocycle if there is equality in one of these inequalities, and a circumscribed hypercycle if one of these inequalities holds in the opposite direction.

If there is a circumcircle, then the circumcentre O and two vertices, say A and B, form an isosceles triangle, which has two equal angles at A and B. By considering the three isosceles triangles formed in this way, it follows that the angle

$$\mathsf{E},\mathsf{S},\mathsf{H}: \qquad OAB = OBA = \sigma - C, \tag{10.24}$$

where a negative value means that O and C lie on opposite sides of AB. It follows that if there is a circumcircle, then the circumcentre lies inside the triangle if and only if

$$\mathsf{E},\mathsf{S},\mathsf{H}: \qquad A, B, C < \sigma \tag{10.25}$$

or equivalently,

$$\mathsf{E},\mathsf{S},\mathsf{H}: \qquad A < B + C, \quad B < C + A, \quad C < A + B; \tag{10.26}$$

if there is equality in one relation then the circumcentre lies on the corresponding side.

Moreover, if there is a circumcircle, then the circumcentre O, a vertex, say A, and the midpoint M_c of the side AB form a triangle with a right angle at M_c and side lengths $|OA| = R$ and $|AM_c| = c/2$, and angle $|\sigma - C|$ at A by (10.24). The results of Section 3 then applies; in particular, (3.7)–(3.9) yield (10.11)–(10.13) (assuming there is a circumcircle).

11. Incircle

The bisector of an angle consists of points that have the same distance to the two sides. The three angle bisectors thus intersect in one point, the *incentre* I, which has the same distance r to all three sides; r is the radius of the *incircle* (*inscribed circle*), which has centre I and is tangent to each of the three sides; r is called the *inradius*. We can express r in the sides or the angles by

$$\mathsf{E}: \qquad r^2 = \frac{\mathrm{am}_v^2}{4s^2} = \frac{(s-a)(s-b)(s-c)}{s} \tag{11.1}$$

$$\mathsf{S}: \qquad \tan^2 r = \frac{\mathrm{am}_v^2}{4\sin^2 s} = \frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s} \tag{11.2}$$

$$= \frac{\mathrm{am}_s^2}{2(1+\cos A)(1+\cos B)(1+\cos C)} \tag{11.3}$$

$$\mathsf{H}: \qquad \tanh^2 r = \frac{\mathrm{am}_v^2}{4\sinh^2 s} = \frac{\sinh(s-a)\sinh(s-b)\sinh(s-c)}{\sinh s} \tag{11.4}$$

$$= \frac{-\mathrm{am}_s^2}{2(1+\cos A)(1+\cos B)(1+\cos C)} \tag{11.5}$$

and thus

$$\mathsf{E}: \qquad r = \frac{\mathrm{am}_v}{2s} = \frac{\Delta}{s} = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} \tag{11.6}$$

$$\mathsf{S}: \qquad \tan r = \frac{\mathrm{am}_v}{2\sin s} = \frac{2\cos\frac{a}{2}\cos\frac{b}{2}\cos\frac{c}{2}}{\sin s}\sin\frac{\Delta}{2} \tag{11.7}$$

$$= \frac{\mathrm{am}_s}{4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} \tag{11.8}$$

$$\mathsf{H}: \qquad \tanh r = \frac{\mathrm{am}_v}{2\sinh s} = \frac{2\cosh\frac{a}{2}\cosh\frac{b}{2}\cosh\frac{c}{2}}{\sinh s}\sin\frac{\Delta}{2} \tag{11.9}$$

$$= \frac{|\mathrm{am}_s|}{4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}. \tag{11.10}$$

In the hyperbolic case, (11.5) and (11.10) apply also to infinite triangles.

The incircle is tangent to the three sides at points $T_a \in BC$, $T_b \in CA$, $T_c \in AB$; the distances from the tangent points to the vertices are

$$\mathsf{E,S,H}: \qquad |AT_b| = |AT_c| = s-a \tag{11.11}$$

and so on.

The incentre I, a vertex A and an adjacent tangent point $T_c \in AB$ form a triangle with a right angle at T_c and the angle $A/2$ at A; furthermore, the sides $|IT_c| = r$ and $|AT_c| = s-a$. The results of Section 3 applies; in particular, (3.10)–(3.12) yield

$$\mathsf{E}: \qquad r = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2} \tag{11.12}$$

$$\mathsf{S}: \quad \tan r = \sin(s-a)\tan\frac{A}{2} = \sin(s-b)\tan\frac{B}{2} = \sin(s-c)\tan\frac{C}{2} \tag{11.13}$$

$$\mathsf{H}: \quad \tanh r = \sinh(s-a)\tan\frac{A}{2} = \sinh(s-b)\tan\frac{B}{2} = \sinh(s-c)\tan\frac{C}{2}. \tag{11.14}$$

These can also be written as (and similarly for B, C; these formulae are sometimes called the *law of cotangents*)

$$\mathsf{E}: \qquad \cot\frac{A}{2} = \frac{s-a}{r}, \tag{11.15}$$

$$\mathsf{S}: \qquad \cot\frac{A}{2} = \frac{\sin(s-a)}{\tan r}, \tag{11.16}$$

$$\mathsf{H}: \qquad \cot\frac{A}{2} = \frac{\sinh(s-a)}{\tanh r}. \tag{11.17}$$

Remark 11.1 (H). In the hyperbolic case, the inradius r is bounded, since the area of the triangle is bounded by (2.85). More precisely, it can be shown from the formulae above that $\tanh r \leqslant \frac{1}{2}$, i.e.,

$$\mathsf{H}: \qquad r \leqslant \tfrac{1}{2}\log 3, \tag{11.18}$$

with equality only for a triply infinite triangle. (The area of the incircle is thus at most $\pi(\sqrt{3}-1)^2/\sqrt{3}$ by (8.6).)

We have also relations between circumradius and inradius:

$$\mathsf{E}: \qquad r = 4R\sin\frac{A}{2}\,\sin\frac{B}{2}\,\sin\frac{C}{2}, \tag{11.19}$$

$$\mathsf{S}: \qquad \tan r = \frac{\cos(\sigma-A)\,\cos(\sigma-B)\,\cos(\sigma-C)}{2\cos\frac{A}{2}\,\cos\frac{B}{2}\,\cos\frac{C}{2}}\tan R, \tag{11.20}$$

$$\mathsf{H}: \qquad \tanh r = \frac{\cos(\sigma-A)\,\cos(\sigma-B)\,\cos(\sigma-C)}{2\cos\frac{A}{2}\,\cos\frac{B}{2}\,\cos\frac{C}{2}}\cdot\begin{cases}\tanh R\\ 1\\ \coth D,\end{cases} \tag{11.21}$$

which in the Euclidean case (11.19) follows from (10.11), (11.12), (2.51) and trigonometric formulae (in particular $\sin B + \sin C - \sin(B+C) = 4\sin\frac{B}{2}\sin\frac{C}{2}\sin\frac{B+C}{2}$) and in the spherical and hyperbolic cases (11.20)–(11.21) follows from (11.8), (11.10), (10.8), (10.10) and (2.90).

Remark 11.2 (S). In spherical geometry, the circumcentre O of ABC is the incentre I^* of the dual triangle $A^*B^*C^*$ and conversely, and

$$\mathsf{S}: \qquad r^* = \frac{\pi}{2} - R, \qquad\qquad R^* = \frac{\pi}{2} - r. \tag{11.22}$$

12. Excircles

The full locus of points that have the same distance to the two lines extending two sides, say AB and AC, is the union of the internal bisector of the angle at the vertex A where the sides intersect, and the exterior bisector, which passes through the same vertex and is perpendicular to the internal bisector.

In the Euclidean and spherical cases, and sometimes in the hyperbolic case, there are apart from the incentre three further points that have the same distance to all three lines extending the sides; these are the *excentres*, each of which is the centre of an *excircle* (or *escribed circle*) that is tangent to one side and the extensions of the two others. The radii of the excircles are called *exradii.* The excentre and the exradius of the excircle tangent to side $a = BC$ are denoted J_a and r_a, etc.

H: In the hyperbolic case, there are not always excircles. More precisely, considering for example the side $a = BC$, there are three (mutually exclusive) possibilities for the internal bisector at the vertex A and the external bisectors at the two other vertices B and C:

(i) The three bisectors intersect in a common point J_a, and there is a (unique) excircle as discussed above.
(ii) The three bisectors are parallel. There is a (unique) *exhorocycle* (*escribed horocycle*), i.e., a horocycle tangent to the side BC and the extensions of the two other sides; this horocycle is orthogonal to the three bisectors, and has the same point at infinity as they.
(iii) The three bisectors are pairwise ultraparallel. There is a (unique) *exhypercycle* (*escribed hypercycle*), i.e., a hypercycle tangent to the side BC and the extensions of the two other sides; the axis of this hypercycle is orthogonal to the three bisectors (so it is the common normal of all three of them). We denote the distance from this hypercycle to its axis by d_a.

As in Section 10, we may unify the formulae for the three hyperbolic cases by using the curvature κ_a of the excircle, exhorocycle or exhypercycle.

Remark 12.1 (S). In the spherical case, the excircle at a side BC is the incircle of the adjacent triangle $\bar{A}BC$, where $\bar{A}$ and A are antipodal, see Section 6; thus all formulae for incircles apply if we replace B, C, b, c by their supplements $\pi - B, \pi - C, \pi - b, \pi - c$ (note that this does not change the amplitudes am_s and am_v).

In total, we can obtain 8 triangles from ABC by replacing some subset of vertices by their antipodes; the corresponding 8 incircles are the incircle and the 3 excircles of ABC together with their antipodes (images under reflection in the centre).

We can express r_a or d_a in the sides or the angles by

$$\text{E}: \qquad r_a^2 = \kappa_a^{-2} = \frac{\mathrm{am}_v^2}{4(s-a)^2} = \frac{s(s-b)(s-c)}{s-a} \tag{12.1}$$

$$\text{S}: \qquad \tan^2 r_a = \kappa_a^{-2} = \frac{\mathrm{am}_v^2}{4\sin^2(s-a)} = \frac{\sin(s)\sin(s-b)\sin(s-c)}{\sin(s-a)} \tag{12.2}$$

$$= \frac{\mathrm{am}_s^2}{2(1+\cos A)(1-\cos B)(1-\cos C)} \tag{12.3}$$

$$\mathsf{H}: \quad \left.\begin{matrix} \tanh^2 r_a \\ 1 \\ \coth^2 d_a \end{matrix}\right\} = \kappa_a^{-2} = \frac{\mathrm{am}_v^2}{4\sinh^2(s-a)} = \frac{\sinh(s)\sinh(s-b)\sinh(s-c)}{\sinh(s-a)} \tag{12.4}$$

$$= \frac{-\mathrm{am}_s^2}{2(1+\cos A)(1-\cos B)(1-\cos C)} \tag{12.5}$$

and thus

$$\mathsf{E}: \quad r_a = \kappa_a^{-1} = \frac{s}{s-a} r = \frac{\Delta}{s-a} = \sqrt{\frac{s(s-b)(s-c)}{s-a}} \tag{12.6}$$

$$\mathsf{S}: \quad \tan r_a = \kappa_a^{-1} = \frac{\mathrm{am}_v}{2\sin(s-a)} = \frac{\sin s}{\sin(s-a)} \tan r \tag{12.7}$$

$$= \frac{\mathrm{am}_s}{4\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} \tag{12.8}$$

$$\mathsf{H}: \quad \left.\begin{matrix} \tanh r_a \\ 1 \\ \coth d_a \end{matrix}\right\} = \kappa_a^{-1} = \frac{\mathrm{am}_v}{2\sinh(s-a)} = \frac{\sinh s}{\sinh(s-a)} \tanh r \tag{12.9}$$

$$= \frac{|\mathrm{am}_s|}{4\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} \tag{12.10}$$

In the hyperbolic case, (12.5) and (12.10) apply also to infinite triangles.

We have further the formulae

$$\mathsf{E}: \quad r_a = s\,\tan\frac{A}{2} \tag{12.11}$$

$$\mathsf{S}: \quad \tan r_a = \sin s\,\tan\frac{A}{2} \tag{12.12}$$

$$\mathsf{H}: \quad \left.\begin{matrix} \tanh r_a \\ 1 \\ \coth d_a \end{matrix}\right\} = \sinh s\,\tan\frac{A}{2}, \tag{12.13}$$

which can be obtained from(12.6)–(12.9) and (11.12)–(11.14).

In the hyperbolic case, it can be shown, using (12.10), that a triangle has an excircle, exhorocycle or exhypercycle at the side $a = BC$ according as:

$$\mathsf{H}: \quad \begin{cases} \kappa_a > 1 \text{ (circle)} & \iff \cos\frac{A}{2} < \sin\frac{B}{2} + \sin\frac{C}{2}, \\ \kappa_a = 1 \text{ (horocycle)} & \iff \cos\frac{A}{2} = \sin\frac{B}{2} + \sin\frac{C}{2}, \\ \kappa_a < 1 \text{ (hypercycle)} & \iff \cos\frac{A}{2} > \sin\frac{B}{2} + \sin\frac{C}{2}. \end{cases} \tag{12.14}$$

The excircle (exhorocycle, exhypercycle) at the side $a = BC$ is tangent to $a = BC$ and the extensions of $b = CA$ and $c = AB$ at the at points

$T^a_a \in BC$, $T^a_b \in CA$, $T^a_c \in AB$; the distances from the tangent points to the vertices are

$$\mathsf{E,S,H}: \qquad |AT^a_b| = |AT^a_c| = s, \tag{12.15}$$

$$|BT^a_a| = |BT^a_c| = s - c, \tag{12.16}$$

$$|CT^a_a| = |CT^a_b| = s - b. \tag{12.17}$$

If there exists an excircle at $a = BC$ (in particular, always in the Euclidean and spherical cases), then the excentre J_a, the vertex A and the tangent point $T^a_c \in AB$ form a triangle with a right angle at T^a_c and the angle $A/2$ at A; furthermore, the sides $|IT^a_c| = r$ and $|AT^a_c| = s - a$. The results of Section 3 applies; in particular, (3.10)–(3.12) yield (12.11)–(12.13) (in this case).

Combining (11.6)–(11.9) and (12.6)–(12.9), we find

$$\mathsf{E}: \qquad rr_ar_br_c = \frac{\mathrm{am}_v^2}{4} = s(s-a)(s-b)(s-c) = \Delta^2, \tag{12.18}$$

$$\mathsf{S}: \quad \tan r \, \tan r_a \, \tan r_b \, \tan r_c = \frac{\mathrm{am}_v^2}{4} = \sin s \, \sin(s-a)\sin(s-b)\sin(s-c), \tag{12.19}$$

$$\mathsf{H}: \qquad \kappa^{-1}\kappa_a^{-1}\kappa_b^{-1}\kappa_c^{-1} = \frac{\mathrm{am}_v^2}{4} = \sinh s \, \sinh(s-a)\sinh(s-b)\sinh(s-c). \tag{12.20}$$

In the Euclidean case, we have further simple relations:

$$\mathsf{E}: \qquad \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{r}, \tag{12.21}$$

$$\mathsf{E}: \qquad r_a + r_b + r_c - r = 4R. \tag{12.22}$$

13. Medians

A *median* of a triangle is the line joining a vertex and the midpoint of the opposite side. There are thus three medians, and they are concurrent, i.e., they all meet at a common point G, which is called the *centroid* of the triangle.

Remark 13.1. In the Euclidean case, the centroid is easily seen to be the centre of mass of the triangle; regarding the points as vectors in $\mathbb{R}^2$, it is simply the average of the vertices:

$$\mathsf{E}: \qquad G = \frac{1}{3}(A + B + C). \tag{13.1}$$

Similarly, in the spherical case, if we regard the sphere as a subset of $\mathbb{R}^3$, and in the hyperbolic case, if we use the hyperboloid model in Appendix D to regard the hyperbolic plane as a subset of $\mathbb{R}^3$,

$$\mathsf{S,H}: \qquad G = k(A + B + C). \tag{13.2}$$

where $k > 0$ is a normalizing constant, chosen such that $\langle G, G\rangle = 1$. See Appendix E.

We denote the midpoints of the sides by A', B', C' (with A' on the side BC etc.); thus the medians are AA', BB', CC'. The lengths of the medians are given by the following formulae (which are easy consequences of the law of cosines, or of (E.21)–(E.24) and (E.1)–(E.2)):

$$\mathsf{E}: \qquad |AA'|^2 = \frac{2b^2+2c^2-a^2}{4} = \frac{1}{2}b^2 + \frac{1}{2}c^2 - \frac{1}{4}a^2, \tag{13.3}$$

$$\mathsf{S}: \qquad \cos|AA'| = \frac{\cos b + \cos c}{2\cos\frac{a}{2}}, \tag{13.4}$$

$$\mathsf{H}: \qquad \cosh|AA'| = \frac{\cosh b + \cosh c}{2\cosh\frac{a}{2}}. \tag{13.5}$$

Furthermore, the distances from the vertices to the centroid are given by

$$\mathsf{E}: \qquad |AG|^2 = \frac{2b^2+2c^2-a^2}{9} = \Bigl(\frac{2}{3}|AA'|\Bigr)^2 \tag{13.6}$$

$$\mathsf{S}: \qquad \cos|AG| = \frac{1+\cos b+\cos c}{\sqrt{3+2\cos a+2\cos b+2\cos c}}, \tag{13.7}$$

$$\mathsf{H}: \qquad \cosh|AG| = \frac{1+\cosh b+\cosh c}{\sqrt{3+2\cosh a+2\cosh b+2\cosh c}}. \tag{13.8}$$

Similarly, the length of the remaining part of the median is

$$\mathsf{E}: \qquad |A'G|^2 = \frac{2b^2+2c^2-a^2}{36} = \Bigl(\frac{1}{3}|AA'|\Bigr)^2 \tag{13.9}$$

$$\mathsf{S}: \qquad \cos|A'G| = \frac{2+2\cos a+\cos b+\cos c}{2\cos\frac{a}{2}\sqrt{3+2\cos a+2\cos b+2\cos c}}, \tag{13.10}$$

$$\mathsf{H}: \qquad \cosh|A'G| = \frac{2+2\cosh a+\cosh b+\cosh c}{2\cosh\frac{a}{2}\sqrt{3+2\cosh a+2\cosh b+2\cosh c}}. \tag{13.11}$$

The centroid divides the medians such that

$$\mathsf{E}: \qquad \frac{|AG|}{|A'G|} = 2, \tag{13.12}$$

$$\mathsf{S}: \qquad \frac{\sin|AG|}{\sin|A'G|} = 2\cos\frac{a}{2}, \tag{13.13}$$

$$\mathsf{H}: \qquad \frac{\sinh|AG|}{\sinh|A'G|} = 2\cosh\frac{a}{2}. \tag{13.14}$$

14. Altitudes

The *altitude* from a vertex, say A, in a triangle is the line through the vertex perpendicular to the opposite side BC, or more precisely, to the line through the opposite side. In other words, the altitude is the line from the vertex to the closest point H_A on the line through the opposite side.

Note that the point H_A may or may not lie on the side BC, i.e., between B and C. More precisely, it lies on the side BC if the angles B and C both are acute; it equals B $[C]$ if B $[C]$ is a right angle, and it lie outside the segment BC if B or C is obtuse; in the latter case, in Euclidean or hyperbolic geometry, H_A lies outside the obtuse vertex.

We have defined the altitude as an entire line; the name is also used for the segment AH_A, for example when talking about the length of the altitude; the precise meaning should hopefully be clear from the context.

In the Euclidean and hyperbolic cases, altitudes always exist uniquely. In the spherical case, the altitude intersects the line through the opposite side in two antipodal points; thus even if the altitude is unique as a line, there are two conceivable choices for H_A (antipodes of each other) and for the altitude segment AH_A (with complementary lenghts); to be precise we defined H_A to be the point closest to A and thus the altitude segment AH_A is be the shorter of the two line segments from the vertex A. Furthermore, in the spherical case, there is an exceptional case, viz. $b = c = \pi/2$, when the altitude is not unique; in this case, all points on the line through BC have the same distance $(\pi/2)$ to A. In the sequel, we sometimes exclude this case.

We denote the length of the altitude from A to the side through BC by $h_A := |AH_A|$. (Note that this is well-defined even in the exceptional case in spherical trigonometry; in that case $h_A = \pi/2$.)

Remark 14.1 (S). In the spherical case, the altitude through A is the line through A and the dual vertex A^*, see (5.1); the exceptional case above equals the case $A = A^*$. The altitudes, regarded as lines, are thus the same for a triangle and its dual, but they differ as segments.

The length of the altitude is, using (3.4)–(3.6), given by

$$\mathsf{E}: \qquad h_A = c \sin B = b \sin C, \tag{14.1}$$

$$\mathsf{S}: \qquad \sin h_A = \sin c \sin B = \sin b \sin C, \tag{14.2}$$

$$\mathsf{H}: \qquad \sinh h_A = \sinh c \sin B = \sinh b \sin C. \tag{14.3}$$

Using amplitudes, we have by (2.107)–(2.109) and (2.105)–(2.106),

$$\mathsf{E}: \qquad h_A = \frac{\mathrm{am}_v}{a}, \tag{14.4}$$

$$\mathsf{S}: \qquad \sin h_A = \frac{\mathrm{am}_v}{\sin a} = \frac{\mathrm{am}_s}{\sin A}, \tag{14.5}$$

$$\mathsf{H}: \qquad \sinh h_A = \frac{\mathrm{am}_v}{\sinh a} = \frac{|\mathrm{am}_s|}{\sin A}. \tag{14.6}$$

In the Euclidean case, we have also, by (2.110) and (10.11),

$$\mathsf{E}: \qquad h_A = \frac{2\Delta}{a} \tag{14.7}$$

$$= \frac{bc}{2R}. \tag{14.8}$$

The three altitudes of a triangle intersect in the Euclidean and spherical cases, but not always in the hyperbolic case. If they intersect, their intersection point is called the *orthocentre*, and denoted by H. [10, Orthocenter]. More precisely, we have the following possibilities:

E: In the Euclidean case, the altitudes intersect in a unique point, the orthocentre.

S: In the spherical case, the altitudes intersect in two antipodal points H and $\bar{H}$, and both can be taken as the orthocentre. (We do not know whether it is possible to define a canonical choice of one of them as H.) In the exceptional case of a triangle with two or three right angles, the orthocentre is undefined. (Otherwise at least two altitudes are uniquely defined and distinct, and thus the pair of their intersection points are uniquely defined. If there are exactly two right angles, say A and B, then the altitudes from A and B coincice, both equal the line AB, and the third altitude is undefined.)

H: In the hyperbolic case, there are three (mutually exclusive) possibilities (cf. Subsection 1.2):

(i) All three altitudes intersect in a common point, the orthocentre.
(ii) The three altitudes are parallel.
(iii) The three altitudes are pairwise ultraparallel. There is a (unique) line orthogonal to the three altitudes (i.e., a common normal).

In hyperbolic geometry, the three possibilities can be distinguished by

$$\mathsf{H}: \quad \sin^2 A \sin^2 B \sin^2 C + (2\cos A \cos B \cos C + 1)|\mathrm{am}_s|^2 \begin{cases} > 0, \\ = 0, \\ < 0, \end{cases}$$
$$\iff \begin{cases} \text{the altitudes intersect,} \\ \text{the altitudes are parallel,} \\ \text{the altitudes are ultraparallel.} \end{cases} \tag{14.9}$$

The distance $|HH_A|$ from the orthocentre H (provided it exists) to the side BC is given by

$$\mathsf{E}: \quad |HH_A| = \frac{a\,|\cos B \cos C|}{\sin A}, \tag{14.10}$$

$$\mathsf{S}: \quad \sin^2|HH_A| = \frac{\mathrm{am}_s^2 \cos^2 B \cos^2 C}{\sin^2 A \sin^2 B \sin^2 C - (2\cos A \cos B \cos C + 1)\mathrm{am}_s^2} \tag{14.11}$$

$$\mathsf{H}: \quad \sinh^2|HH_A| = \frac{|\mathrm{am}_s|^2 \cos^2 B \cos^2 C}{\sin^2 A \sin^2 B \sin^2 C + (2\cos A \cos B \cos C + 1)|\mathrm{am}_s|^2} \tag{14.12}$$

In the Euclidean case, (14.10) holds without absolute values if we measure the distance $|HH_A|$ as negative when H and A or on opposite sides of H_A (i.e., on opposite sides of BC), In the hyperbolic case, note that the denominator in (14.12) equals the left-hand side of (14.9). If this is negative, the right-hand side of (14.12) equals $\cosh^2 |H_A N|$, where $|H_A N|$ is the distance from H_A to the common normal N of the three altitudes.

Remark 14.2. The orthocentre (exists and) coincides with a vertex A if and only if A is a right angle. (Excluding the exceptional case of a spherical triangle with three right angles; note that a triangle cannot have exactly two right angles.)

Remark 14.3. If the orthocentre H exist and is distinct from the vertices (i.e., the triangle is not right, see Remark 14.2, then the four distinct points A, B, C, H have the property that the line through any pair of points is orthogonal to the line through the opposite pair. This is evidently a symmetric condition, so the triangle formed by any three of the points has the fourth as its orthocentre. (In spherical geometry, we might here consider pairs of antipodal points instead of points in order to have the orthocentre unique, i.e., to consider projective geometry instead of spherical.)

Such a set of four points is called an *orthocentric system*, see [10].

15. Example: equilateral triangles

For equilateral triangles we have $a = b = c$ and, equivalently, $A = B = C$. The law of cosines (2.6)–(2.8) yields

$$\mathsf{E}: \qquad \cos A = \frac{1}{2}, \tag{15.1}$$

$$\mathsf{S}: \qquad \cos A = \frac{\cos a}{1 + \cos a}, \tag{15.2}$$

$$\mathsf{H}: \qquad \cos A = \frac{\cosh a}{\cosh a + 1}; \tag{15.3}$$

in the spherical and hyperbolic cases, the second law of cosines (2.9)–(2.11) yields the equivalent

$$\mathsf{S}: \qquad \cos a = \frac{\cos A}{1 - \cos A}, \tag{15.4}$$

$$\mathsf{H}: \qquad \cosh a = \frac{\cos A}{1 - \cos A}. \tag{15.5}$$

Other equivalent versions are

$$\mathsf{E}: \qquad \sin A = \frac{\sqrt{3}}{2}, \tag{15.6}$$

$$\mathsf{S}: \qquad \sin A = \frac{\sqrt{1 + 2\cos a}}{1 + \cos a}, \tag{15.7}$$

$$\mathsf{H}: \qquad \sin A = \frac{\sqrt{1 + 2\cosh a}}{1 + \cosh a}; \tag{15.8}$$

$$\mathsf{E}: \qquad \tan \frac{A}{2} = \frac{1}{\sqrt{3}}, \tag{15.9}$$

$$\mathsf{S}: \qquad \tan\frac{A}{2} = \frac{1}{\sqrt{1+2\cos a}}, \tag{15.10}$$

$$\mathsf{H}: \qquad \tan\frac{A}{2} = \frac{1}{\sqrt{1+2\cosh a}}. \tag{15.11}$$

Note that for an equilateral triangle, by (2.51)–(2.53), see also (2.77)–(2.78),

$$\mathsf{E}: \qquad A = \frac{\pi}{3}, \tag{15.12}$$

$$\mathsf{S}: \qquad A > \frac{\pi}{3}, \tag{15.13}$$

$$\mathsf{H}: \qquad A < \frac{\pi}{3}; \tag{15.14}$$

moreover, in the spherical and hyperbolic cases, the area is given by

$$\mathsf{S}: \qquad \Delta = 3A - \pi, \tag{15.15}$$

$$\mathsf{H}: \qquad \Delta = \pi - 3A. \tag{15.16}$$

In the hyperbolic case, the formulae above include the infinite case, with $a = \infty$, $A = 0$, $\Delta = \pi$,

Heron's formula (2.65)–(2.73) yields, since $s = \frac{3}{2}a$,

$$\mathsf{E}: \qquad \Delta = \frac{\sqrt{3}}{4}a^2, \tag{15.17}$$

$$\mathsf{S}: \qquad \sin\frac{\Delta}{2} = \frac{\sqrt{\sin\frac{3a}{2}\,\sin^3\frac{a}{2}}}{2\cos^3\frac{a}{2}} = \frac{\sin^2\frac{a}{2}\,\sqrt{1+2\cos a}}{2\cos^3\frac{a}{2}}, \tag{15.18}$$

$$\mathsf{H}: \qquad \sin\frac{\Delta}{2} = \frac{\sqrt{\sinh\frac{3a}{2}\,\sinh^3\frac{a}{2}}}{2\cosh^3\frac{a}{2}} = \frac{\sinh^2\frac{a}{2}\,\sqrt{1+2\cosh a}}{2\cosh^3\frac{a}{2}}, \tag{15.19}$$

$$\mathsf{S}: \qquad \tan\frac{\Delta}{2} = \frac{2\sqrt{\sin\frac{3a}{2}\,\sin^3\frac{a}{2}}}{1+3\cos a} = \frac{(1-\cos a)\sqrt{1+2\cos a}}{1+3\cos a}, \tag{15.20}$$

$$\mathsf{H}: \qquad \tan\frac{\Delta}{2} = \frac{2\sqrt{\sinh\frac{3a}{2}\,\sinh^3\frac{a}{2}}}{1+3\cosh a} = \frac{(\cosh a-1)\sqrt{1+2\cosh a}}{1+3\cosh a}. \tag{15.21}$$

The amplitudes are, by (2.104)–(2.109) and (15.6)–(15.8), cf. (2.110)–(2.112) and (15.17)–(15.19),

$$\mathsf{E}: \qquad \mathrm{am}_s = 0, \tag{15.22}$$

$$\mathsf{S}: \qquad \mathrm{am}_s = \frac{\sin a(1+2\cos a)}{(1+\cos a)^2}, \tag{15.23}$$

$$\mathsf{H}: \qquad |\mathrm{am}_s| = \frac{\sinh a(1+2\cosh a)}{(1+\cosh a)^2} \tag{15.24}$$

and

$$\mathsf{E}: \qquad \mathrm{am}_v = \frac{\sqrt{3}}{2}a^2, \tag{15.25}$$

$$\mathsf{S}: \qquad \operatorname{am}_v = (1-\cos a)\sqrt{1+2\cos a} = 2\sqrt{\sin\frac{3a}{2}\,\sin^3\frac{a}{2}}, \tag{15.26}$$

$$\mathsf{H}: \qquad \operatorname{am}_v = (\cosh a-1)\sqrt{1+2\cosh a} = 2\sqrt{\sinh\frac{3a}{2}\,\sinh^3\frac{a}{2}}. \tag{15.27}$$

The circumradius is by (10.6)–(10.9) and (15.25)–(15.27)

$$\mathsf{E}: \qquad R = \frac{a}{\sqrt{3}}, \tag{15.28}$$

$$\mathsf{S}: \qquad \tan R = 2\sqrt{\frac{\sin^3\frac{a}{2}}{\sin\frac{3a}{2}}} = \frac{2\sin\frac{a}{2}}{\sqrt{1+2\cos a}}, \tag{15.29}$$

$$\mathsf{H}: \qquad \tanh R = 2\sqrt{\frac{\sinh^3\frac{a}{2}}{\sinh\frac{3a}{2}}} = \frac{2\sinh\frac{a}{2}}{\sqrt{1+2\cosh a}}, \tag{15.30}$$

Note that a circumcircle always exists, also in the hyperbolic case. (By symmetry, or because the right-hand side of (15.30) is less than 1, as is easily seen.)

The inradius is by (11.1)–(11.4), or by (11.19)–(11.21),

$$\mathsf{E}: \qquad r = \frac{a}{2\sqrt{3}} = \frac{1}{2}R, \tag{15.31}$$

$$\mathsf{S}: \qquad \tan r = \sqrt{\frac{\sin^3\frac{a}{2}}{\sin\frac{3a}{2}}} = \frac{\sin\frac{a}{2}}{\sqrt{1+2\cos a}} = \frac{1}{2}\tan R, \tag{15.32}$$

$$\mathsf{H}: \qquad \tanh r = \sqrt{\frac{\sinh^3\frac{a}{2}}{\sinh\frac{3a}{2}}} = \frac{\sinh\frac{a}{2}}{\sqrt{1+2\cosh a}} = \frac{1}{2}\tanh R. \tag{15.33}$$

Similarly, the exradius is by (12.1)–(12.4) and (12.6)–(12.9)

$$\mathsf{E}: \qquad r_a = \frac{\sqrt{3}}{2}a = 3r, \tag{15.34}$$

$$\mathsf{S}: \qquad \tan r_a = \sqrt{\sin\frac{a}{2}\,\sin\frac{3a}{2}} = \sin\frac{a}{2}\,\sqrt{1+2\cos a} = \frac{\sin\frac{3a}{2}}{\sin\frac{a}{2}}\tan r \tag{15.35}$$

$$\mathsf{H}: \qquad \left.\begin{matrix}\tanh r_a\\ 1\\ \coth d_a\end{matrix}\right\} = \sqrt{\sinh\frac{a}{2}\,\sinh\frac{3a}{2}} = \sinh\frac{a}{2}\,\sqrt{1+2\cosh a} = \frac{\sinh\frac{3a}{2}}{\sinh\frac{a}{2}}\tanh r. \tag{15.36}$$

It follows by some algebra that in the hyperbolic case, there exist excircles if and only if

$$\mathsf{H}: \quad \cosh a < \frac{3}{2} \iff \sinh\frac{a}{2} < \frac{1}{2} \iff \sin A < \frac{4}{5} \iff a < 2\ln\frac{\sqrt{5}+1}{2}; \tag{15.37}$$

if there is equality in the relations in (15.37) there exist exhorocycles on all sides, and if the inequalities are reversed there are exhypercycles.

16. Morley's triangle (Euclidean)

Morley's theorem says that, in Euclidean geometry, the (internal) trisectors of the angles A, B, C in a triangle intersect pairwise in three points A', B', C' (where A' is the intersection of the trisectors of B and C closest to the side BC, etc.), which form an *equilateral* triangle. (Many different proofs are known. For some very different proofs, see [3; 4; 9].)

The side of Morley's triangle is [9]

$$\mathsf{E}: \qquad 8R\sin\frac{A}{3}\,\sin\frac{B}{3}\,\sin\frac{C}{3}. \tag{16.1}$$

(Cf. the similar expression in (11.19).) The distance from A' to the side BC is

$$\mathsf{E}: \qquad 8R\sin\frac{A}{3}\,\sin\frac{B}{3}\,\sin\frac{C}{3}\,\sin\frac{A+\pi}{3}. \tag{16.2}$$

17. Quadrilaterals

Consider a quadrilateral (quadrangle) $ABCD$ with angles A, B, C, D and sides $a = |AB|$, $b = |BC|$, $c = |CD|$, $d = |DA|$, and area Δ. Let s be the semiperimeter

$$s := \frac{a+b+c+d}{2}. \tag{17.1}$$

We give only a few results.

17.1. Angle sum and area.

We have analogues of the triangle formulas (2.51)–(2.53):

$$\mathsf{E}: \qquad A+B+C+D = 2\pi, \tag{17.2}$$

$$\mathsf{S}: \qquad A+B+C+D = 2\pi+\Delta, \tag{17.3}$$

$$\mathsf{H}: \qquad A+B+C+D = 2\pi-\Delta. \tag{17.4}$$

As in the triangle case, these can be combined to

$$\mathsf{E,S,H}: \qquad A+B+C+D = 2\pi+K\Delta, \tag{17.5}$$

where K is the sectional curvature, another example of the Gauss–Bonnet theorem.

17.2. Area (Euclidean).

In the Euclidean case, there are several formulas for the area, which generalize Heron's formula (2.65) for triangles. (Regarding a triangle as the special case $d = 0$.) We do not know any analogue of any of these formulas in spherical or hyperbolic geometry.

Bretschneider's formula.

$$\mathsf{E}: \quad \Delta = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\,\cos^2\Big(\frac{A+C}{2}\Big)}\,. \tag{17.6}$$

Note that $\frac{A+C}{2} = \pi - \frac{B+D}{2}$ by (17.2), so $\cos^2\frac{A+C}{2} = \cos^2\frac{B+D}{2}$.

This is due to Bretschneider (1842) and Strehlke (1842). Some related formulas are, denoting the lengths of the diagonals by p and q,

$$\mathsf{E}: \quad \Delta = \sqrt{(s-a)(s-b)(s-c)(s-d) - \tfrac14(ac+bd+pq)(ac+bd-pq)} \tag{17.7}$$

$$= \sqrt{(s-a)(s-b)(s-c)(s-d) + \tfrac14 p^2q^2 - \tfrac14(ac+bd)^2} \tag{17.8}$$

$$= \tfrac14\sqrt{4p^2q^2 - (b^2+d^2-a^2-c^2)^2}. \tag{17.9}$$

See e.g. [10, Bretschneider's formula].

Brahmagupta's formula. A *cyclic quadrilateral* is a quadrilateral that can be inscribed in a circle, the *circumcircle.* In the Euclidean case considered here, a quadrilateral is cyclic if and only if $A+C=\pi$ (or, equivalently, $B+D=\pi$. In this case Bretschneider's formula simplifies to *Brahmagupta's formula.*

$$\mathsf{E}: \qquad \Delta = \sqrt{(s-a)(s-b)(s-c)(s-d)}\,. \tag{17.10}$$

Note that a triangle is a (degenerate) special case of a cyclic quadrilateral. See further e.g. [10, Brahmagupta's formula, Cyclic quadrilateral].

17.3. **Right-angled quadrilaterals.** For a quadrilateral $ABCD$ with $A = B = C = \pi/2$, also called *Lambert quadrangle*,

$$\mathsf{E}: \qquad |AB| = |BC|, \tag{17.11}$$

$$\mathsf{S}: \qquad \cos|AB| = \cos|CD|\,\sin D, \tag{17.12}$$

$$\mathsf{H}: \qquad \cosh|AB| = \cosh|CD|\,\sin D. \tag{17.13}$$

and

$$\mathsf{E}: \qquad \cos D = 0 \quad \text{and} \quad \Delta = |AB|\,|BC|, \tag{17.14}$$

$$\mathsf{S}: \qquad \cos D = -\sin\Delta = -\sin|AB|\,\sin|BC|. \tag{17.15}$$

$$\mathsf{H}: \qquad \cos D = \sin\Delta = \sinh|AB|\,\sinh|BC|. \tag{17.16}$$

In the hyperbolic case, further results for quadrilaterals with 2 or 3 right angles, pentagons with 4 or 5 right angles and hexagons with 6 right angles are given in Fenchel [5].

Appendix A. Poincaré halfplane (hyperbolic)

The Poincaré halfplane is a model of the hyperbolic plane, consisting of the upper halfplane $H := \{x + \mathrm{i}y : y > 0\}$ in the complex plane, with the Riemannian metric

$$\mathrm{d}s = \frac{|\mathrm{d}z|}{y}. \tag{A.1}$$

The metric is conformal, so angles equal the Euclidean angles. The (sectional) curvature is constant -1. See further [7], where more details are given.

The set of infinite points is $\mathbb{R}^* := \mathbb{R} \cup \{\infty\}$.

The lines are the Euclidean lines orthogonal to the x-axis and the circles with centres on the real axis. (I.e., the circles in the extended complex plane $\mathbb{C}^* := \mathbb{C} \cup \infty$ that are orthogonal to $\partial H = \mathbb{R}^*$.)

The circles are the Euclidean circles that lie completely in H. The horocycles are the Euclidean circles that lie in H and are tangent to $\partial H = \mathbb{R}^*$, and the horizontal Eucidean lines (which should be thought of as being tangent to ∂H at ∞). The hypercycles are the Euclidean circles and lines that intersect ∂H at an angle $\theta \in (0, \pi/2)$. (In this case, the curvature is $\cos\theta$.) Thus the set of curves of constant curvature, which is the collection of all lines, circles, horocycles and hypercycles, is the set of all Euclidean circles and lines that intersect H. (Of course, we only consider the part inside H.)

The distance d between two points is given by

$$\sinh^2\left(\frac{d(z,w)}{2}\right) = \frac{\cosh(d(z,w)) - 1}{2} = \frac{|z-w|^2}{4 \operatorname{Im} z \operatorname{Im} w}; \tag{A.2}$$

equivalently,

$$\cosh\bigl(d(x+\mathrm{i}y, u+\mathrm{i}v)\bigr) = \frac{(x-u)^2 + y^2 + v^2}{2yv}. \tag{A.3}$$

The proper isometries of H are the Möbius transformations preserving the upper halfplane H, i.e.,

$$PSL(2,\mathbb{R}) := \left\{\frac{az+b}{cz+d} : a,b,c,d \in \mathbb{R},\ ad - bc = 1\right\} \Big/ \{\pm 1\}. \tag{A.4}$$

These form a subgroup of index 2 in the group of all isometries, which thus equals

$$PSL(2,\mathbb{R}) \cup \bigl\{\sigma(-\bar z) : \sigma \in PSL(2,\mathbb{R})\bigr\}. \tag{A.5}$$

The signed version of the gauge (1.4)–(1.6) of two (directed) lines $\ell = xy$ and $\ell' = x'y'$ with endpoints $x, y, x', y' \in \partial H = \mathbb{R}^*$ is given by

$$\langle \ell, \ell' \rangle = \frac{1 + [x,y,x',y']}{1 - [x,y,x',y']} = 2[x,x',y',y] - 1, \tag{A.6}$$

where $[x,y,x',y']$ is the cross ratio defined by

$$[x,y,z,w] := \frac{(x-z)(y-w)}{(x-w)(y-z)} \tag{A.7}$$

(suitably interpreted if some point is ∞).

Appendix B. Poincaré disc (hyperbolic)

The Poincaré disc is a model of the hyperbolic plane, consisting of the unit disc $D := \{z : |z| < 1\}$ in the complex plane, with the Riemannian metric

$$\mathrm{d}s = \frac{2|\mathrm{d}z|}{1-|z|^2}. \tag{B.1}$$

The metric is conformal, so angles equal the Euclidean angles. The (sectional) curvature is constant -1. See further [7], where more details are given.

The set of infinite points is the unit circle ∂D.

The lines are the Euclidean lines through the origin and the circles orthogonal to the unit circle. (I.e., the circles in the extended complex plane $\mathbb{C}^* := \mathbb{C} \cup \infty$ that are orthogonal to the unit circle.)

The circles are the Euclidean circles that lie completely in D. The horocycles are the Euclidean circles that lie in D and are tangent to ∂D. The hypercycles are the Euclidean circles or lines that intersect ∂D at an angle $\theta \in (0, \pi/2)$. (In this case, the curvature is $\cos\theta$.) Thus the set of curves of constant curvature, which is the collection of all lines, circles, horocycles and hypercycles, is the set of all Euclidean circles and lines that intersect D. (Equivalently, the set of all circles in $\mathbb{C}^*$ that intersect D. Of course, we only consider the part inside D.)

The distance d between two points is given by

$$\sinh^2\left(\frac{d(z,w)}{2}\right) = \frac{\cosh(d(z,w))-1}{2} = \frac{|z-w|^2}{(1-|z|^2)(1-|w|^2)}; \tag{B.2}$$

equivalently,

$$\cosh d(z,w) = \frac{(1+|z|^2)(1+|w|^2) - 4\operatorname{Re}(z\bar{w})}{(1-|z|^2)(1-|w|^2)}. \tag{B.3}$$

In particular,

$$\cosh d(z,0) = \frac{1+|z|^2}{1-|z|^2}, \tag{B.4}$$

$$\sinh d(z,0) = \frac{2|z|}{1-|z|^2}, \tag{B.5}$$

$$\tanh d(z,0) = \frac{2|z|}{1+|z|^2}, \tag{B.6}$$

$$d(z,0) = \log\frac{1+|z|}{1-|z|} = \log(1+|z|) - \log(1-|z|), \tag{B.7}$$

$$|z| = \tanh\frac{d(z,0)}{2}. \tag{B.8}$$

The proper isometries of D are the Möbius transformations preserving the unit disc, i.e.,

$$\text{Möb}(D) := \left\{ \frac{az+b}{\bar{b}z+\bar{a}} : |a|^2 - |b|^2 = 1 \right\} / \{\pm 1\} \cong PSL(2,\mathbb{R}). \tag{B.9}$$

These form a subgroup of index 2 in the group of all isometries, which thus equals

$$\text{Möb}(D) \cup \bigl\{ \sigma(\bar{z}) : \sigma \in \text{Möb}(D) \bigr\}. \tag{B.10}$$

Similarly, D is isometric to the Poincareé halfplane H in Appendix A by any Möbius transformation mapping D onto H, for example the map

$$z \mapsto -\mathrm{i}\frac{z+1}{z-1}. \tag{B.11}$$

As for the Poincaré halfplane in Appendix A, the signed version of the gauge (1.4)–(1.6) of two (directed) lines $\ell = xy$ and $\ell' = x'y'$ with endpoints $x, y, x', y' \in \partial D$ is given by

$$\langle \ell, \ell' \rangle = \frac{1 + [x, y, x', y']}{1 - [x, y, x', y']} = 2[x, x', y', y] - 1, \tag{B.12}$$

where $[x, y, x', y']$ is the cross ratio defined by (A.7).

Appendix C. Klein disc (hyperbolic)

The Klein disc K is another model of the hyperbolic plane, as the Poincaré disc in Appendix B consisting of the unit disc $\{z : |z| < 1\}$ in the complex plane, but now equipped with the Riemannian metric

$$|\mathrm{d}s|^2 = \frac{|\mathrm{d}x|^2}{1-|x|^2} + \frac{|\langle \mathrm{d}x, x\rangle|^2}{(1-|x|^2)^2} = \frac{|\mathrm{d}x|^2 - |x \wedge \mathrm{d}x|^2}{(1-|x|^2)^2}, \tag{C.1}$$

(This metric is *not* conformal.) The (sectional) curvature is constant -1. See further [7], where more details are given.

The set of infinite points is the unit circle ∂K.

The lines are the Euclidean lines that intersect the disc. This is a great advantage of this model, which makes it useful for some purposes, although very often the Poincaré models are more convenient for other reasons.

The circles are in general not Euclidean circles.

The distance between two points $x, y \in K$ is given by, with $\langle x, y\rangle$ the Euclidean inner product,

$$\cosh d(x,y) = \frac{1 - \langle x, y\rangle}{\sqrt{(1-|x|^2)(1-|y|^2)}}. \tag{C.2}$$

In particular,

$$\cosh d(x,0) = \frac{1}{\sqrt{1-|x|^2}}, \tag{C.3}$$

$$\sinh d(x,0) = \frac{|x|}{\sqrt{1-|x|^2}}, \tag{C.4}$$

$$\tanh d(x, 0) = |x|. \tag{C.5}$$

If the line (= Euclidean line) through x and y has endpoints u and v on ∂D, then

$$d(x,y) = \frac{1}{2}\Bigl|\log \frac{|x-u|\,|y-v|}{|x-v|\,|y-u|}\Bigr| = \frac{1}{2}\left|\log[x,y,u,v]\right|. \tag{C.6}$$

The Klein disc K is isometric to the Poincaré models H and D in Appendices A–B. An isometry $D \to K$ is given by

$$x \mapsto \frac{2x}{1+|x|^2} \tag{C.7}$$

with inverse $K \to D$

$$x \mapsto \frac{x}{1+\sqrt{1-|x|^2}}; \tag{C.8}$$

these extend to the boundary $\partial K = \partial D$ (which is fixed by these maps).

As a consequence, just as for the Poincaré halfplane and disc in Appendices A and B, the signed version of the gauge (1.4)–(1.6) of two (directed) lines $\ell = xy$ and $\ell' = x'y'$ with endpoints $x, y, x', y' \in \partial K = \partial D$ is given by

$$\langle \ell, \ell' \rangle = \frac{1+[x,y,x',y']}{1-[x,y,x',y']} = 2[x,x',y',y] - 1, \tag{C.9}$$

where $[x, y, x', y']$ is the cross ratio defined by (A.7).

If ℓ is a line in the Klein disc K, let $\bar{\ell}$ denote its extension to a line in the Euclidean plane $\mathbb{R}^2$, or better (adjoining a point at infinity) in the projective plane $\mathbb{P}^2$. If ℓ_1 and ℓ_2 are two distinct (hyperbolic) lines in K, then the extensions $\bar{\ell}_1$ and $\bar{\ell}_2$ intersect in a point $Q \in \mathbb{P}^2$; the lines ℓ_1 and ℓ_2 intersect if and only if $Q \in K$ ($|Q| < 1$), they are parallel if and only if $Q \in \partial K$ ($|Q| = 1$), and they are ultraparallel if and only if $Q \in \mathbb{P}^2 \setminus \overline{K}$ ($|Q| > 1$).

Remark C.1. Given a point $Q \in \mathbb{P}^2$, the set of all hyperbolic lines ℓ such that $Q \in \bar{\ell}$ is called a *pencil* of lines. There are thus three cases:

(i) $Q \in K$: a pencil of intersecting lines, consisting of all lines through the (hyperbolic) point Q;

(ii) $Q \in \partial K$: a pencil of parallel lines, consisting of all lines with a common point at infinity $Q \in \partial K$;

(iii) $Q \in \mathbb{P}^2 \setminus \overline{K}$: a pencil of ultraparallel lines; this consists of all lines perpendicular to some line ℓ (the *axis* of the pencil); furthermore, $\bar{\ell}$ is the Euclidean polar $\{x \in \mathbb{R}^2 : \langle x, Q \rangle = 1\}$ of the point Q (if $Q \in \mathbb{P}^2 \setminus \mathbb{R}^2$, so Q is a point at infinity, this is interpreted as a line through the origin, perpendicular to the direction Q).

Note that each pair of hyperbolic lines defines a unique pencil containing both lines.

Given a triangle ABC in the Klein disc K, let ℓ_a, ℓ_b, ℓ_c be the three perpendicular bisectors of the sides. Then the Euclidean (or projective) extensions $\bar{\ell}_a, \bar{\ell}_b, \bar{\ell}_c$ intersect in a common point $Q \in \mathbb{P}^2$; in other words,

ℓ_a, ℓ_b, ℓ_c belong to a common pencil of lines. It follows that the three cases in Section 10 can be characterized by Q:

(i) If $Q \in K$ ($|Q| < 1$), then ABC has a (unique) circumcircle.
(ii) If $Q \in \partial K$ ($|Q| = 1$), then ABC has a (unique) circumscribed horocycle; this has Q as the point at infinity.
(iii) If $Q \in \mathbb{P}^2 \setminus \overline{K}$ ($|Q| > 1$), then ABC has a (unique) circumscribed hypercycle (i.e., a hypercycle going through the three vertices); the axis of this hypercycle is the Euclidean polar of Q.

Appendix D. Hyperboloid (hyperbolic)

Equip $\mathbb{R}^3$ with the Lorentzian indefinite inner product

$$\langle (x_1, x_2, x_3), (y_1, y_2, y_3) \rangle := -x_1 y_1 - x_2 y_2 + x_3 y_3. \tag{D.1}$$

We write the elements of $\mathbb{R}^3$ as (x, ξ) with $x \in \mathbb{R}^2$ and $\xi \in \mathbb{R}$, and thus

$$\langle (x, \xi), (y, \eta) \rangle = -\langle x, y \rangle + \xi \eta, \tag{D.2}$$

where $\langle x, y \rangle$ is the ordinary Euclidean inner product in $\mathbb{R}^2$.

The hyperboloid (or pseudosphere)

$$\tilde{H} := \bigl\{ (x, \xi) : \langle (x, \xi), (x, \xi) \rangle = 1 \bigr\} = \bigl\{ (x_1, x_2, x_3) : x_3^2 = x_1^2 + x_2^2 + 1 \bigr\} \tag{D.3}$$

has two connected components, with $\xi = x_3 > 0$ and < 0, respectively. Let H^+ be one of these, for definiteness

$$H^+ := \bigl\{ (x, \xi) : \langle (x, \xi), (x, \xi) \rangle = 1 \text{ and } \xi > 0 \bigr\} = \bigl\{ \bigl(x, \sqrt{|x|^2 + 1}\bigr) : x \in \mathbb{R}^2 \bigr\}. \tag{D.4}$$

H^+ is a model of the hyperbolic plane, with metric given by

$$\cosh d\bigl((x, \xi), (y, \eta)\bigr) = \langle (x, \xi), (y, \eta) \rangle \tag{D.5}$$

$$= \sqrt{(1 + |x|^2)(1 + |y|^2)} - \langle x, y \rangle. \tag{D.6}$$

(See further [7].)

The isometries are the Lorentz group, which is the subgroup of index 2 of maps preserving H^+ in the group $O(2, 1)$ of all linear maps of $\mathbb{R}^3$ onto itself that preserve the indefinite inner product (D.1).

The lines are the intersections of H^+ and planes in $\mathbb{R}^3$ (in vector space sense, i.e., containing 0). There is thus a one-to-one correspondence between lines and planes in $\mathbb{R}^3$ that contain a vector X with $\langle X, X \rangle > 0$. Each such plane has a normal vector N, determined up to a constant factor, with $\langle N, N \rangle < 0$, and conversely. Thus each line has a normal vector N, and if we normalize N by $\langle N, N \rangle = -1$, this yields is a 2–1 correspondence between vectors N with $\langle N, N \rangle = -1$ and lines. The correspondence is 2–1 since $\pm N$ are normal vectors to the same line, but we may make it into a 1–1 correspondence between normalized normal vectors and oriented lines, by giving each line a direction and requiring that if X is a point on the line, T a tangent vector in the positive direction, and N the corresponding ("positive") normal vector, then (X, T, N) is a positively oriented basis in $\mathbb{R}^3$.

The gauge (1.4)–(1.6) of two lines ℓ and ℓ' with normals N and N' (in the signed version for directed lines, with positively oriented N and N') is given by

$$\langle \ell, \ell' \rangle = -\langle N, N' \rangle. \tag{D.7}$$

The points at infinity can be identified by lines (through 0) in $\mathbb{R}^3$ containing only points with $\langle X, X \rangle = 0$ (the *light cone*), or (equivalently) by the points in the intersection of the light cone and the (affine) plane $\{(x, 1)\}$, which yields the identification

$$\partial H^+ \cong \{(x, 1) \in \mathbb{R}^3 : |x| = 1\}. \tag{D.8}$$

With this identification, the two ends of a line ℓ with normal vector N, i.e., the two points at infinity on ℓ, are the two points in (D.8) that are orthogonal to N.

The hyperboloid H^+ is isometric to the hyperbolic models H, D, K in Appendices A–C. An isometry $H^+ \to D$ is given by

$$(x, \xi) \mapsto \frac{x}{1 + \xi} \tag{D.9}$$

with inverse

$$x \mapsto \frac{(2x, 1 + |x|^2)}{1 - |x|^2}; \tag{D.10}$$

geometrically these are stereographic projections with centre $(0, -1)$, with D^n seen as the subset $\{(x, 0) \in R^3 : |x| < 1\}$. Similarly, an isometry $H^+ \to K$ is given by

$$(x, \xi) \mapsto \frac{x}{\xi} \tag{D.11}$$

with inverse

$$x \mapsto \frac{(x, 1)}{\sqrt{1 - |x|^2}}; \tag{D.12}$$

geometrically these are stereographic projections with centre 0, with K seen as the subset $\{(x, 1) \in R^3 : |x| < 1\}$. These isometries $H^+ \to D$ and $H^+ \to K$ extend to the boundary (D.8) in the natural way, mapping $(x, 1) \mapsto x$.

D.1. **A matrix model.** Any three-dimensional real vector space V with an indefinite inner product of signature $(1, 2)$ is isometric to $\mathbb{R}^3$ with the Lorentzian inner product (D.1), and may thus be used to define a model of the hyperbolic plane as one sheet of the hyperboloid $\{X : \langle X, X \rangle = 1\}$. One useful version of the hyperboloid model is obtained by taking V as $\mathsf{sl}(2, \mathbb{R})$, the space of all 2×2-matrices with trace 0:

$$V := \mathsf{sl}(2, \mathbb{R}) := \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} : a, b, c \in \mathbb{R} \right\} \tag{D.13}$$

with the inner product

$$\langle A, B \rangle := -\tfrac{1}{2} \operatorname{Tr}(AB). \tag{D.14}$$

In particular, if $A = \left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right)$, then

$$\langle A, A\rangle = -a^2 - bc = \det(A). \tag{D.15}$$

Note that $\langle A, A\rangle = 1$ implies $bc < 0$, and thus $c \neq 0$; we may thus define H^+ as the sheet

$$H^+ := \left\{ A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathsf{sl}(2,\mathbb{R}) : \det(A) = 1 \text{ and } c > 0 \right\}. \tag{D.16}$$

Furthermore, a normalized normal vector N has

$$\det(N) = \langle N, N\rangle = -1. \tag{D.17}$$

This model is used by Iversen [6]; it is convenient because further tools from linear algebra are available, see [6, Chapter III]. In particular, the gauge (1.4)–(1.6) of two lines ℓ and ℓ' with normals N and N' (in the signed version for directed lines, with positively oriented N and N') is given by, cf. (D.7),

$$\langle \ell, \ell'\rangle = -\langle N, N'\rangle = \tfrac{1}{2}\operatorname{Tr}(NN'). \tag{D.18}$$

The light cone consists of all matrices A in $\mathsf{sl}(2,\mathbb{R})$ with $\det(A) = 0$, i.e., all singular matrices. These matrices are the matrices

$$\pm \begin{pmatrix} xy & -x^2 \\ y^2 & -xy \end{pmatrix}, \quad x, y \in \mathbb{R}. \tag{D.19}$$

The points at infinity, which as said above can be seen as lines in the light cone, can thus be identified with the projective line $\mathbb{R}^* = \mathbb{R} \cup \{\infty\}$; the matrix (D.19) (not identically 0) corresponds to the pair $[x, y]$, i.e., to the number $x/y \in \mathbb{R}$ if $y \neq 0$ and to ∞ if $y = 0$. The two endpoints of a line ℓ with normal vector N then correspond to the two pairs $[x_i, y_i]$, $i = 1, 2$, and thus to the extended real numbers $x_i/y_i \in \mathbb{R}^*$, where (x_i, y_i) are the eigenvectors of N (which are real and distinct; the eigenvalues are ± 1 if N is normalized).

Remark D.1. Fenchel [5] represents, more generally, lines in three-dimensional hyperbolic space H^3 by matrices in $\mathsf{sl}(2,\mathbb{C})$ (again up to multiplication by a constant factor); the ends of the lines are points in $\partial H^3 \cong \mathbb{S}^2$, which is seen as the Riemann sphere $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$. If we consider only lines with endpoints in the real projective line $\mathbb{R}^* \subset \mathbb{C}^*$, we obtain all lines in a projective plane embedded in H^3, and their defining matrices can be taken in $\mathsf{sl}(2,\mathbb{R})$; with the identification above of ∂H^+ and $\mathbb{R}^*$, this yields the same correspondence between lines and matrices as above, regarding the matrices as normal vectors in $\mathsf{sl}(2,\mathbb{R})$. (The normalization in [5] is slightly different, using $\langle N, N\rangle = \det N = +1$, which corresponds to multiplying the matrix by $\pm \mathrm{i}$.)

Appendix E. Canonical embeddings

The three geometries can be treated in a unified way as geometries on certain submanifolds of $\mathbb{R}^3$, which we call *canonical embeddings* in $\mathbb{R}^3$. We write vectors in $\mathbb{R}^3$ as $x = (x_1, x_2, x_3)$.

S: The standard model of spherical geometry is the unit sphere $\mathbb{S}^2 := \{x \in \mathbb{R}^3 : \langle x, x\rangle = 1\}$, where $\langle\cdot,\cdot\rangle$ is the standard Euclidean inner product in $\mathbb{R}^3$.

H: The model in Appendix D embeds the hyperbolic plane as the sheet $\{x_3 > 0\}$ of the hyperboloid $\{x \in \mathbb{R}^3 : \langle x, x\rangle = 1\}$, where $\langle\cdot,\cdot\rangle$ is the indefinite inner product of signature $(1, 2)$ defined in Appendix D.

E: We embed the Euclidean plane $\mathbb{R}^2$ in $\mathbb{R}^3$ as the affine plane $\{x \in \mathbb{R}^3 : x_3 = 1\} = \{(x_1, x_2, 1) : x_1, x_2 \in \mathbb{R}\}$. In analogy with the hyperbolic case, this can be seen as the sheet $\{x_3 > 0\}$ of the degenerate hyperboloid (really a pair of planes) $\{x \in \mathbb{R}^3 : \langle x, x\rangle = 1\}$, where $\langle\cdot,\cdot\rangle$ is the degenerate inner product $\langle x, y\rangle := x_3y_3$.

In all three cases, we denote the manifold by $\mathcal{S} \subset \mathbb{R}^3$, and let $\langle\cdot,\cdot\rangle$ be the inner product defined above (E: degenerate; S: positive definite; H: indefinite).

Remark E.1. In the Euclidean and hyperbolic cases, these embeddings can be seen as embeddings in the projective plane $\mathbb{P}^2$, since no two points in $\mathcal{S}$ are proportional. (Recall that $\mathbb{P}^2$ is defined as the set of equivalence classes $[x_1, x_2, x_3]$ with $(x_1, x_2, x_3) \in \mathbb{R}^3 \setminus \{0\}$ under the equivalence relation $[x_1, x_2, x_3] = [tx_1, tx_2, tx_3]$ for any $t \neq 0$.) This enables us to include also infinite points. In the Euclidean case, the embedding is just the standard embedding of $\mathbb{R}^2$ as a subset of $\mathbb{P}^2$, with $\mathbb{P}^2\setminus\mathbb{R}^2$ as the set of points at infinity (see Subsection 1.1).

In the hyperbolic case, the points at infinity are given by the points $[x_1, x_2, x_3] \in \mathbb{P}^2$ such that $-x_1^2 - x_2^2 + x_3^2 = 0$; these can uniquely be represented as $\{[x_1, x_2, 1] : x_1^2 + x_2^2 = 1\}$ (see Appendix D).

In the spherical and hyperbolic cases, the distance d between two points can be expressed using the inner products above:

$$\mathsf{S}: \qquad \cos d(x, y) = \langle x, y\rangle, \tag{E.1}$$

$$\mathsf{H}: \qquad \cosh d(x, y) = \langle x, y\rangle. \tag{E.2}$$

(In the Euclidean case, the inner product $\langle x, y\rangle = 1$ for all $x, y \in \mathcal{S}$.)

E.1. **Lines.** In all three cases, the lines are given by the intersections of $\mathcal{S}$ with planes (through 0) in $\mathbb{R}^3$. Hence, the line through two distinct points $x, y \in \mathcal{S}$ (in the spherical case we assume also $x \neq y$) is given by

$$\mathsf{E},\mathsf{S},\mathsf{H}: \qquad \{sx + ty : s, t \in \mathbb{R} \text{ and } sx + ty \in \mathcal{S}\}. \tag{E.3}$$

S,H: In the spherical and hyperbolic cases (when the inner product is nonsingular), a line can equivalently be described by a nonzero normal vector $N \in \mathbb{R}^3$ as $\{x \in \mathcal{S} : \langle x, N\rangle = 0\}$. Note that N is determined only up to a non-zero factor. In the hyperbolic case, we necessarily have $\langle N, N\rangle < 0$. Hence, we can normalize N be requiring $\langle N, N\rangle = 1$ in the spherical case and $\langle N, N\rangle = -1$ in the hyperbolic case, i.e.

$$\mathsf{S}, \mathsf{H}: \qquad \langle N, N\rangle = K, \tag{E.4}$$

the curvature. There is still a non-uniqueness, since $\pm N$ define the same line, and it follows that there is a 2–1 correspondence between vectors in $\mathbb{R}^3$ satisfying (E.4) and lines in the space $\mathcal{S}$. We can make this a 1–1 correspondence between vectors N satisfying (E.4) and directed lines; we may (for example) let T be a tangent vector in the positive (forward) direction at a point x on the line and say that N is positively oriented if the linearly independent vectors (x, T, N) form a poitively oriented basis of $\mathbb{R}^3$. With this convention, the gauge (1.4)–(1.6) of two lines ℓ and ℓ' with normals N and N' (in the signed version for directed lines, with positively oriented N and N') is given by (as is said in (D.7) for the hyperbolic case)

$$\mathsf{S}, \mathsf{H}: \qquad \langle \ell, \ell'\rangle = K\langle N, N'\rangle, \tag{E.5}$$

with $K = \pm 1$ the (sectional) curvature of the space, cf. (E.4).

A line (geodesic) parametrized as $\gamma(t)$ with unit speed satisfies

$$\mathsf{E}: \qquad \gamma(t) = x + tV, \tag{E.6}$$

$$\mathsf{S}: \qquad \gamma(t) = \cos t \cdot x + \sin t \cdot V, \tag{E.7}$$

$$\mathsf{H}: \qquad \gamma(t) = \cosh t \cdot x + \sinh t \cdot V, \tag{E.8}$$

where $x = \gamma(0) \in \mathcal{S}$ and $V = \dot\gamma(0) \in \mathbb{R}^3$ is such that $\langle x, V\rangle = 0$ and furthermore

$$\mathsf{E}: \qquad V = (v_1, v_2, 0) \text{ with } \|(v_1, v_2)\| = 1, \tag{E.9}$$

$$\mathsf{S}: \qquad \langle V, V\rangle = 1 \tag{E.10}$$

$$\mathsf{H}: \qquad \langle V, V\rangle = -1. \tag{E.11}$$

In all three cases, $\langle V, V\rangle = K$.

E.2. **Amplitudes.** The parallelepiped in $\mathbb{R}^3$ spanned by three vectors x, y, z is the set $\{sx + ty + uz : s, t, u \in [0, 1]\}$. (This is a polytope with vertices in $0, x, y, z, x + y, x + z, y + z, x + y + z$.) Denote its volume (in $\mathbb{R}^3$) by $V(x, y, z)$. (The tetrahedron with vertices $0, x, y, z$ has volume $V(x, y, z)/6$, e.g. by a linear map to the case of three orthonormal vectors.) We can express $V(x, y, z)$ using exterior algebra or coordinates as

$$V(x,y,z) = |x \wedge y \wedge z| = \pm \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}. \tag{E.12}$$

In the spherical case we thus get, by matrix algebra and (E.1),

$$\mathsf{S}: \quad V(x,y,z)^2 = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} \cdot \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = \begin{vmatrix} \langle x,x\rangle & \langle x,y\rangle & \langle x,z\rangle \\ \langle y,x\rangle & \langle y,y\rangle & \langle y,z\rangle \\ \langle z,x\rangle & \langle z,y\rangle & \langle z,z\rangle \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \cos d(x,y) & \cos d(x,z) \\ \cos d(y,x) & 1 & \cos d(y,z) \\ \cos d(z,x) & \cos d(z,y) & 1 \end{vmatrix}. \tag{E.13}$$

And in the hyperbolic case similarly, using (E.1),

$$\mathsf{H}: \qquad V(x,y,z)^2 = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} \cdot \begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

$$= \begin{vmatrix} \langle x,x\rangle & \langle x,y\rangle & \langle x,z\rangle \\ \langle y,x\rangle & \langle y,y\rangle & \langle y,z\rangle \\ \langle z,x\rangle & \langle z,y\rangle & \langle z,z\rangle \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \cosh d(x,y) & \cosh d(x,z) \\ \cosh d(y,x) & 1 & \cosh d(y,z) \\ \cosh d(z,x) & \cosh d(z,y) & 1 \end{vmatrix}. \tag{E.14}$$

In the Euclidean case, we obtain instead

$$\mathsf{E}: \qquad V(x,y,z) = \pm \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ 1 & 1 & 1 \end{vmatrix} = \pm \begin{vmatrix} x_1 - z_1 & y_1 - z_1 \\ x_2 - z_2 & y_2 - z_2 \end{vmatrix}. \tag{E.15}$$

Either from this, or geometrically by noting that $V(x,y,z)$ is 6 times the volume of the pyramid (tetrahedron) with base xyz and top vertex 0, and thus height (altitude) 1, it follows that $V(x,y,z)$ is 2 times the area Δ of the triangle xyz; thus by Heron's formula (2.65)

$$\mathsf{E}: \qquad V(x,y,z)^2 = 4\Delta^2 = 4s(s-a)(s-b)(s-c), \tag{E.16}$$

where a,b,c are the sides of the triangle xyz and $s = (a+b+c)/2$.

Comparing (E.13)–(E.16) and (2.91)–(2.95), we see that $V(x,y,z)$ equals the amplitude am_v defined in Subsection 2.5. In other words, as said in Remark 2.3, for a triangle ABC, using the canonical embedding,

$$\mathsf{E},\mathsf{S},\mathsf{H}: \qquad \mathrm{am}_v = V(A,B,C). \tag{E.17}$$

S,H: In the spherical and hyperbolic cases, we may similarly consider the volume of the parallelepiped spanned in $\mathbb{R}^3$ by the unit normals N_a, N_b, N_c to the sides of a triangle ABC. We obtain as in (E.13)–(E.14)

$$\mathsf{S},\mathsf{H}: \qquad V(N_a,N_b,N_c)^2 = \begin{vmatrix} \langle N_a,N_a\rangle & \langle N_a,N_b\rangle & \langle N_a,N_c\rangle \\ \langle N_b,N_a\rangle & \langle N_b,N_b\rangle & \langle N_b,N_c\rangle \\ \langle N_c,N_a\rangle & \langle N_c,N_b\rangle & \langle N_c,N_c\rangle \end{vmatrix}. \tag{E.18}$$

This is not affected if we change the sign of one or several of the normal vectors N_a, N_b, N_c. We may thus assume that the normal vectors correspond to the orientations AB, BC and CA of the sides (and the lines that extend them), and then, using (E.5), (1.4)–(1.5) (and comments after them on the directed version) and (2.88),

$$\mathsf{S,H}: \qquad V(N_a,N_b,N_c)^2 = K^3 \begin{vmatrix} 1 & \langle BC,CA\rangle & \langle BC,AB\rangle \\ \langle CA,BC\rangle & 1 & \langle CA,AB\rangle \\ \langle AB,BC\rangle & \langle AB,CA\rangle & 1 \end{vmatrix}$$

$$= K^3 \begin{vmatrix} 1 & -\cos C & -\cos B \\ -\cos C & 1 & -\cos A \\ -\cos B & -\cos A & 1 \end{vmatrix}$$

$$= K \mathrm{am}_s^2. \tag{E.19}$$

Consequently,

$$\mathsf{S,H}: \qquad |\mathrm{am}_s| = V(N_a,N_b,N_c). \tag{E.20}$$

We see also again that $\mathrm{am}_s^2 > 0$ in the spherical case but $\mathrm{am}_s^2 < 0$ in the hyperbolic case, see (2.99)–(2.100).

In the spherical case, the normals N_a, N_b, N_c may be taken as the vertices of the dual triangle, see Section 5 and (5.8). In the hyperbolic case, we may thus by analogy regard the normal vectors as forming a kind of dual, but note that the normal vectors do not belong to the hyperbolic plane, since $\langle N_a, N_a\rangle = -1$ etc., so they are some kind of "imaginary points".

E.3. **Midpoints and centroid.** It is easily verified (in all three cases) that for any two distinct points $x, y \in \mathcal{S}$ (with $x \neq y$ in the spherical case), the midpoint of the segment xy is

$$k(x+y) \tag{E.21}$$

where $k > 0$ is a normalizing constant, given by

$$k = \frac{1}{\sqrt{\langle x+y, x+y\rangle}} = \frac{1}{\sqrt{2+2\langle x,y\rangle}}$$

$$\mathsf{E}: \qquad = \frac{1}{2}, \tag{E.22}$$

$$\mathsf{S}: \qquad = \frac{1}{2\cos d(x,y)/2}, \tag{E.23}$$

$$\mathsf{H}: \qquad = \frac{1}{2\cosh d(x,y)/2}. \tag{E.24}$$

Similarly, and as an easy consequence, the centroid of a triangle ABC (see Section 13) is $k(A+B+C)$ for a normalizing constant $k > 0$, given by (using (E.1)–(E.2))

$$k = \frac{1}{\sqrt{\langle A+B+C, A+B+C\rangle}}$$

$$\mathsf{E}: \qquad = \frac{1}{3}, \tag{E.25}$$

$$\mathsf{S}: \qquad = \frac{1}{\sqrt{3+2\cos a+2\cos b+2\cos c}}, \tag{E.26}$$

$$\mathsf{H}: \qquad = \frac{1}{\sqrt{3 + 2\cosh a + 2\cosh b + 2\cosh c}}. \tag{E.27}$$

E.4. **Isometries.** An isometry of the space $\mathcal{S}$ can, in all three cases, be extended to a linear isomorphism of $\mathbb{R}^3$ onto itself that preserves the inner product. The group of isometries of the space $\mathcal{S}$ is

- E: The affine group $A(2;\mathbb{R})$ realized as matrices $\begin{pmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ 0 & 0 & 1 \end{pmatrix}$.
- S: The orthogonal group $O(3;\mathbb{R})$.
- H: The subgroup $O_+(2,1;\mathbb{R})$ of the Lorentz group $O(2,1;\mathbb{R})$, where $O(2,1;\mathbb{R})$ is the group of linear isomorphisms of $\mathbb{R}^3$ that preserve the indefinite inner product, and $O_+(2,1;\mathbb{R})$ is the subgroup of index 2 consisting of the maps that preserve the sheet H^+.

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