SEMISIMPLE LIE GROUPS AND ALGEBRAS, REAL AND COMPLEX

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This is a compilation from several sources, in particular [2].

See also [1] for semisimple Lie algebras over other fields than $\mathbb{R}$ and $\mathbb{C}$.

1. List of groups and algebras

We assume below $n \geqslant 1$ and $p, q \geqslant 0$ with $p + q \geqslant 1$.

All groups are connected and semisimple, and all Lie algebras are semisimple unless otherwise said.

$I_{p,q}$, $J_{p,q}$ and $K_{p,q}$ denote $(p+q)\times(p+q)$-matrices with the block forms (with blocks of sizes p and q)

$$I_{p,q} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad J_{p,q} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad K_{p,q} = I_{p,q}J_{p,q} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The identifications of $n\times n$-matrices over $\mathbb{H}$ with $2n\times 2n$-matrices over $\mathbb{C}$, see $\mathfrak{gl}(n,\mathbb{H}) \cong \mathfrak{u}^*(2n)$ below, uses the identification of $\mathbb{H}$ with $\mathbb{C}^2$ where $(z,w) \in \mathbb{C}^2$ corresponds to $z + jw \in \mathbb{H}$.

- $\mathsf{GL}(n,\mathbb{R})$. General linear group in $\mathbb{H}^n$.
  Not semisimple.
  Center $= \{cI : c \in \mathbb{R}^*\}$. Commutator subgroup $\mathsf{SL}(n,\mathbb{R})$.
  Not connected; 2 components: determinant > 0 and < 0.
  $\mathfrak{gl}(n,\mathbb{R})$ = all real $n \times n$ matrices.
  Reductive, not semisimple.
  Center $= \mathbb{R}$. Commutator ideal $\mathfrak{sl}(n,\mathbb{R})$.
  Complexification $\mathfrak{gl}(n,\mathbb{C})$.
  Dimension n^2.
- $\mathsf{GL}(n,\mathbb{C})$. General linear group in $\mathbb{C}^n$.
  Not semisimple.
  Center $= \{cI : c \in \mathbb{C}^*\}$. Commutator subgroup $\mathsf{SL}(n,\mathbb{C})$.
  $\mathfrak{gl}(n,\mathbb{C})$ = all complex $n \times n$ matrices.
  Reductive, not semisimple.
  Center $= \mathbb{C}$. Commutator ideal $\mathfrak{sl}(n,\mathbb{C})$.
  Complex.
  Complex dimension n^2; real dimension $2n^2$.
- $\mathsf{GL}(n,\mathbb{H})$. General linear group in $\mathbb{H}^n$.
  $\cong \mathsf{U}^*(2n) := \Big\{T \in \mathsf{GL}(2n,\mathbb{C}) : T = \begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix}\Big\}$.
  Not semisimple.
  Center $= \{cI : c \in \mathbb{R}^*\}$. Commutator subgroup $\mathsf{SL}(n,\mathbb{H})$.
  $\mathfrak{gl}(n,\mathbb{H})$ = all quaternionic $n \times n$ matrices.
  $\cong \mathfrak{u}^*(2n) := \Big\{T \in \mathfrak{gl}(2n,\mathbb{C}) : T = \begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix}\Big\}$.

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Reductive, not semisimple.
Center $= \mathbb{R}$. Commutator ideal $\mathfrak{sl}(n,\mathbb{H})$.
Complexification $\mathfrak{gl}(2n,\mathbb{C})$.
Dimension $4n^2$.

- $\mathsf{SL}(n,\mathbb{R})$. Determinant $= 1$.
  Trivial for $n = 1$.
  $\pi_1\big(\mathsf{SL}(n,\mathbb{R})\big) = \pi_1\big(\mathsf{SO}(n)\big)$. Thus not simply connected if $n \geqslant 2$.
  $\mathfrak{sl}(n,\mathbb{R})$. Trace $= 0$.
  Trivial for $n = 1$.
  Simple if $n \geqslant 2$.
  Complexification $\mathfrak{sl}(n,\mathbb{C})$.
  Dimension $n^2 - 1$.

- $\mathsf{SL}(n,\mathbb{C})$. Determinant $= 1$.
  Trivial for $n = 1$.
  Simply connected.
  $\mathfrak{sl}(n,\mathbb{C})$. Trace $= 0$.
  Trivial for $n = 1$.
  Simple if $n \geqslant 2$.
  Complex.
  Complex dimension $n^2 - 1$; real dimension $2n^2 - 2$.

- $\mathsf{SL}(n,\mathbb{H})$. Real determinant $= 1$.
  $\cong \mathsf{SU}^*(2n) := \Big\{T \in \mathsf{SL}(2n,\mathbb{C}) : T = \begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix}\Big\}$.
  For $n = 1$: $\mathsf{SL}(1,\mathbb{H}) = \mathsf{Sp}(1) \cong \mathsf{SU}^*(2) = \mathsf{SU}(2)$, the unit quaternions.
  Simply connected.
  $\mathfrak{sl}(n,\mathbb{H})$. Real part of trace $= 0$.
  $\cong \mathfrak{su}^*(2n) := \Big\{X \in \mathfrak{sl}(2n,\mathbb{C}) : X = \begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix}\Big\}$.
  Simple if $n \geqslant 1$.
  Complexification $\mathfrak{sl}(2n,\mathbb{C})$.
  Dimension $4n^2 - 1$.

- $\mathsf{O}(n)$. Preserves a positive definite bilinear form on $\mathbb{R}^n$, e.g. $x^t y$.
  $= \{A \in \mathsf{GL}(n,\mathbb{R}) : A^t A = I\}$.
  Not connected; 2 components: determinant 1 ($\mathsf{SO}(n)$) and -1.
  $\pi_1\big(\mathsf{O}(n)\big) = \pi_1(\mathsf{SO}(n))$.
  Compact.
  $\mathfrak{so}(n)$, see below.
  Dimension $n(n-1)/2$.

- $\mathsf{SO}(n) := \mathsf{O}(n) \cap \mathsf{SL}(n,\mathbb{R})$.
  $= \{A \in \mathsf{SL}(n,\mathbb{R}) : A^t A = I\}$.
  Connected component of identity in $\mathsf{O}(n)$.
  Semisimple for $n \geqslant 3$.
  Trivial if $n = 1$. Commutative and not semisimple if $n = 2$ ($\cong \mathbb{T}$).
  Compact.
  $\pi_1\big(\mathsf{SO}(n)\big) = \mathbb{Z}_1$ if $n = 1$, $\mathbb{Z}$ if $n = 2$, $\mathbb{Z}_2$ if $n \geqslant 3$.
  $\mathfrak{so}(n) = \{X \in \mathfrak{gl}(n,\mathbb{R}) : X + X^t = 0\}$.

Trivial if $n = 1$. Commutative and not semisimple if $n = 2$ ($\cong \mathbb{R}$).
Semisimple for $n \geqslant 3$.
Simple if $n = 3$ or $n \geqslant 5$. $\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$.
Complexification $\mathfrak{so}(n, \mathbb{C})$.
Dimension $n(n-1)/2$.

- $\mathsf{O}(p, q)$. Preserves a bilinear form of signature (p, q) on $\mathbb{R}^{p+q}$, e.g. $x^t I_{p,q} y$.
  $= \{A \in \mathsf{GL}(p+q, \mathbb{R}) : A^t I_{p,q} A = I_{p,q}\}$.
  $= \left\{T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathsf{GL}(p+q, \mathbb{R}) : T^{-1} = \begin{pmatrix} A^t & -C^t \\ -B^t & D^t \end{pmatrix}\right\}$.
  $\mathsf{O}(p, q) \cong \mathsf{O}(q, p)$. $\mathsf{O}(n, 0) = \mathsf{O}(n)$.
  Not connected: 4 components if $p, q \geqslant 1$, 2 components if $p = 0$ or $q = 0$.
  $\pi_1(\mathsf{O}(p, q)) = \pi_1(\mathsf{SO}(p, q)) \cong \pi_1(\mathsf{SO}(p)) \times \pi_1(\mathsf{SO}(q))$.
  $\mathfrak{so}(p, q)$, see below.
- $\mathsf{SO}(p, q) := \mathsf{O}(p, q) \cap \mathsf{SL}(p+q, \mathbb{R})$.
  $= \{A \in \mathsf{SL}(p+q, \mathbb{R}) : A^t I_{p,q} A = I_{p,q}\}$.
  $= \left\{T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathsf{SL}(p+q, \mathbb{R}) : T^{-1} = \begin{pmatrix} A^t & -C^t \\ -B^t & D^t \end{pmatrix}\right\}$.
  $\mathsf{SO}(p, q) \cong \mathsf{SO}(q, p)$. $\mathsf{SO}(n, 0) = \mathsf{SO}(n)$.
  Not connected if $p, q \geqslant 1$: 2 components.
  $\pi_1\big(\mathsf{SO}(p, q)\big) \cong \pi_1(\mathsf{SO}(p)) \times \pi_1(\mathsf{SO}(q))$.
  $\mathfrak{so}(p, q) = \{X \in \mathfrak{gl}(p+q, \mathbb{R}) : X^t I_{p,q} + I_{p,q} X = 0\}$.
  $= \left\{X \in \mathfrak{gl}(p+q, \mathbb{R}) : X = \begin{pmatrix} A & B \\ B^t & D \end{pmatrix}, \quad A^t = -A,\ D^t = -D\right\}$.
  $\mathfrak{so}(p, q) \cong \mathfrak{so}(q, p)$. $\mathfrak{so}(n, 0) = \mathfrak{so}(n)$.
  Trivial if $p + q = 1$. Commutative and not semisimple if $p + q = 2$ ($\cong \mathbb{R}$).
  Semisimple if $p + q \geqslant 3$.
  Simple if $p + q \geqslant 3$, except $\mathfrak{so}(4) \cong \mathfrak{so}(3) \times \mathfrak{so}(3)$ and $\mathfrak{so}(2, 2) \cong \mathfrak{sl}(2; \mathbb{R}) \times \mathfrak{sl}(2; \mathbb{R})$.
  Complexification $\mathfrak{so}(p+q, \mathbb{C})$.
  Dimension $n(n-1)/2$ with $n := p + q$.
- $\mathsf{O}(n, \mathbb{C})$. Preserves a non-degenerate symmetric bilinear form on $\mathbb{C}^n$, e.g. $x^t y$.
  $= \{A \in \mathsf{GL}(n, \mathbb{C}) : A^t A = I\}$.
  Not connected; 2 components: determinant 1 ($\mathsf{SO}(n, \mathbb{C})$) and -1.
  $\pi_1\big(\mathsf{O}(n, \mathbb{C})\big) = \pi_1\big(\mathsf{SO}(n)\big)$.
  $\mathfrak{so}(n, \mathbb{C})$, see below.
  Complex.
  Complex dimension $n(n-1)/2$; real dimension $n(n-1)$.
- $\mathsf{SO}(n, \mathbb{C}) := \mathsf{O}(n, \mathbb{C}) \cap \mathsf{SL}(n, \mathbb{C})$.
  $= \{A \in \mathsf{SL}(n, \mathbb{C}) : A^t A = I\}$.
  Connected component of identity in $\mathsf{O}(n, \mathbb{C})$.
  Trivial for $n = 1$. Commutative and not semisimple for $n = 2$ ($\cong \mathbb{C}^*$).
  Semisimple if $n \geqslant 3$.
  $\pi_1\big(\mathsf{SO}(n, \mathbb{C})\big) = \pi_1\big(\mathsf{SO}(n)\big)$.
  $\mathfrak{so}(n, \mathbb{C}) = \{X \in \mathfrak{gl}(n, \mathbb{C}) : X + X^t = 0\}$.
  Trivial if $n = 1$. Commutative and not semisimple if $n = 2$ ($\cong \mathbb{C}$).
  Semisimple for $n \geqslant 3$.
  Simple if $n = 3$ or $n \geqslant 5$; $\mathfrak{so}(4, \mathbb{C}) \cong \mathfrak{so}(3, \mathbb{C}) \oplus \mathfrak{so}(3, \mathbb{C})$.
  Complex.

Complex dimension $n(n-1)/2$; real dimension $n(n-1)$.

- $\mathsf{U}(n)$. Preserves a positive definite Hermitean form in $\mathbb{C}^n$, e.g. y^*x.
  $= \{A \in \mathsf{GL}(n,\mathbb{C}) : A^*A = I\}$.
  Not semisimple.
  Center $= \{cI : c \in \mathbb{T}\}$. Commutator subgroup $\mathsf{SU}(n)$.
  Compact.
  $\pi_1(\mathsf{U}(n)) \cong \mathbb{Z}$.
  $\mathfrak{u}(n) = \{X \in \mathfrak{gl}(n,\mathbb{C}) : X + X^* = 0\}$.
  Reductive, not semisimple.
  Center $= \{cI : c \in i\mathbb{R}\}$. Commutator ideal $\mathfrak{su}(n)$.
  Dimension n^2.

- $\mathsf{SU}(n) := \mathsf{U}(n) \cap \mathsf{SL}(n,\mathbb{C})$.
  $= \{A \in \mathsf{SL}(n,\mathbb{C}) : A^*A = I\}$.
  Trivial if $n = 1$.
  Compact.
  Simply connected.
  $\mathfrak{su}(n) = \{X \in \mathfrak{sl}(n,\mathbb{C}) : X + X^* = 0\}$.
  Trivial if $n = 1$.
  Simple if $n \geqslant 2$.
  Complexification $\mathfrak{sl}(n,\mathbb{C})$.
  Dimension $n^2 - 1$.

- $\mathsf{U}(p,q)$. Preserves a Hermitean form of signature (p,q) on $\mathbb{C}^{p+q}$, e.g.$x^*I_{p,q}y$.
  $= \{A \in \mathsf{GL}(p+q,\mathbb{C}) : A^*I_{p,q}A = I_{p,q}\}$.
  $= \left\{T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathsf{GL}(p+q,\mathbb{C}) : T^{-1} = \begin{pmatrix} A^* & -C^* \\ -B^* & D^* \end{pmatrix}\right\}$.
  $\mathsf{U}(p,q) \cong \mathsf{U}(q,p)$. $\mathsf{U}(n,0) = \mathsf{U}(n)$.
  Not semisimple.
  Center $= \{cI : c \in \mathbb{T}\}$. Commutator subgroup $\mathsf{SU}(p,q)$.
  $\pi_1\big(\mathsf{U}(p,q)\big) \cong \pi_1(\mathsf{U}(p)) \times \pi_1(\mathsf{U}(q)) \cong \mathbb{Z}^2$ if $p,q \geqslant 1$.
  $\mathfrak{u}(p,q) = \{X \in \mathfrak{gl}(p+q,\mathbb{C}) : X^*I_{p,q} + I_{p,q}X = 0\}$.
  $= \left\{X \in \mathfrak{gl}(p+q,\mathbb{C}) : X = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix}, \quad A^* = -A,\ D^* = -D\right\}$.
  $\mathfrak{u}(p,q) \cong \mathfrak{u}(q,p)$. $\mathfrak{u}(n,0) = \mathfrak{u}(n)$.
  Reductive, not semisimple.
  Center $= \{cI : c \in i\mathbb{R}\}$. Commutator ideal $\mathfrak{su}(p,q)$.
  Complexification $\mathfrak{gl}(p+q,\mathbb{C})$
  Dimension $(p+q)^2$.

- $\mathsf{SU}(p,q) := \mathsf{U}(p,q) \cap \mathsf{SL}(p+q,\mathbb{C})$.
  $= \{A \in \mathsf{SL}(p+q,\mathbb{C}) : A^*I_{p,q}A = I_{p,q}\}$.
  $= \left\{T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathsf{SL}(p+q,\mathbb{C}) : T^{-1} = \begin{pmatrix} A^* & -C^* \\ -B^* & D^* \end{pmatrix}\right\}$.
  $\mathsf{SU}(p,q) \cong \mathsf{SU}(q,p)$. $\mathsf{SU}(n,0) = \mathsf{SU}(n)$.
  $\pi_1\big(\mathsf{SU}(p,q)\big) \cong \mathbb{Z}$ if $p,q \geqslant 1$; thus not simply connected.
  $\mathfrak{su}(p,q) = \{X \in \mathfrak{sl}(p+q,\mathbb{C}) : X^*I_{p,q} + I_{p,q}X = 0\}$.
  $= \left\{X \in \mathfrak{sl}(p+q,\mathbb{C}) : X = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix}, \quad A^* = -A,\ D^* = -D\right\}$.
  $\mathfrak{su}(p,q) \cong \mathfrak{su}(q,p)$. $\mathfrak{su}(n,0) = \mathfrak{su}(n)$.

Trivial if $p+q=1$.
Simple if $p+q \geqslant 2$.
Complexification $\mathfrak{sl}(p+q,\mathbb{C})$.
Dimension $(p+q)^2-1$.

- $\mathsf{Sp}(n)$. Preserves a positive definite Hermitean form in $\mathbb{H}^n$.
  $=\{A \in \mathsf{GL}(n,\mathbb{H}) : A^*A = I\} = \{A \in \mathsf{SL}(n,\mathbb{H}) : A^*A = I\}$.
  $\cong \mathsf{U}^*(2n) \cap \mathsf{U}(2n) = \mathsf{Sp}(n,\mathbb{C}) \cap \mathsf{U}(2n) = \mathsf{U}^*(2n) \cap \mathsf{Sp}(n,\mathbb{C}) = \mathsf{Sp}(n,\mathbb{C}) \cap \mathsf{SU}(2n)$.
  For $n=1$: $\mathsf{Sp}(1) = \mathsf{SL}(1,\mathbb{H}) \cong \mathsf{SU}^*(2) = \mathsf{SU}(2)$, the unit quaternions.
  Compact.
  Simply connected.
  $\mathfrak{sp}(n) = \{X \in \mathfrak{gl}(n,\mathbb{H}) : X + X^* = 0\}$.
  $\cong \mathfrak{u}^*(2n) \cap \mathfrak{u}(2n) = \mathfrak{sp}(n,\mathbb{C}) \cap \mathfrak{u}(2n) = \mathfrak{u}^*(2n) \cap \mathfrak{sp}(n,\mathbb{C}) = \mathfrak{sp}(n,\mathbb{C}) \cap \mathfrak{su}(2n)$.
  $= \left\{X \in \mathfrak{gl}(2n,\mathbb{C}) : X = \begin{pmatrix} A & B \\ -B^* & -A^t \end{pmatrix}, \quad A = -A^*,\ B = B^t\right\}$.
  Simple if $n \geqslant 1$.
  Complexification $\mathfrak{sp}(n,\mathbb{C})$.
  Dimension $n(2n+1)$.

- $\mathsf{Sp}(p,q)$. Preserves a Hermitean form of signature (p,q) on $\mathbb{H}^{p+q}$, e.g.$x^*I_{p,q}y$.
  $=\{A \in \mathsf{GL}(n,\mathbb{H}) : A^*I_{p,q}A = I_{p,q}\} = \{A \in \mathsf{SL}(n,\mathbb{H}) : A^*I_{p,q}A = I_{p,q}\}$.
  $\mathsf{Sp}(p,q) \cong \mathsf{Sp}(q,p)$. $\mathsf{Sp}(n,0) = \mathsf{Sp}(n)$.
  Simply connected.
  $\mathfrak{sp}(p,q) = \{X \in \mathfrak{gl}(p+q,\mathbb{H}) : X^*I_{p,q}+I_{p,q}X = 0\} = \{X \in \mathfrak{sl}(p+q,\mathbb{H}) : X^*I_{p,q}+I_{p,q}X = 0\}$.
  $= \left\{X \in \mathfrak{gl}(p+q,\mathbb{H}) : X = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix}, \quad A^* = -A,\ D^* = -D\right\}$.
  $\mathfrak{sp}(p,q) \cong \mathfrak{sp}(q,p)$. $\mathfrak{sp}(n,0) = \mathfrak{sp}(n)$.
  Complexification $\mathfrak{sp}(p+q,\mathbb{C})$.
  Dimension $n(2n+1)$ with $n := p+q$.

- $\mathsf{Sp}(n,\mathbb{R})$. Preserves an anti-symmetric non-degenerate bilinear form on $\mathbb{R}^{2n}$.
  $=\{A \in \mathsf{GL}(2n,\mathbb{R}) : A^tJ_{n,n}A = J_{n,n}\} = \{A \in \mathsf{SL}(2n,\mathbb{R}) : A^tJ_{n,n}A = J_{n,n}\}$.
  $\pi_1\big(\mathsf{Sp}(n,\mathbb{R})\big) = \mathbb{Z}$.
  $\mathfrak{sp}(n,\mathbb{R}) = \{X \in \mathfrak{gl}(2n,\mathbb{R}) : X^tJ_{n,n} + J_{n,n}X = 0\}$.
  $= \left\{X \in \mathfrak{gl}(2n,\mathbb{R}) : X = \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}, \quad B = B^t,\ C = C^t\right\}$.
  Simple if $n \geqslant 1$.
  Complexification $\mathfrak{sp}(n,\mathbb{C})$.
  Dimension $n(2n+1)$.

- $\mathsf{Sp}(n,\mathbb{C})$. Preserves an anti-symmetric non-degenerate bilinear form on $\mathbb{C}^{2n}$.
  $=\{A \in \mathsf{GL}(2n,\mathbb{C}) : A^tJ_{n,n}A = J_{n,n}\} = \{A \in \mathsf{SL}(2n,\mathbb{C}) : A^tJ_{n,n}A = J_{n,n}\}$.
  Simply connected.
  $\mathfrak{sp}(n,\mathbb{C}) = \{X \in \mathfrak{gl}(2n,\mathbb{C}) : X^tJ_{n,n} + J_{n,n}X = 0\}$.
  $= \left\{X \in \mathfrak{gl}(2n,\mathbb{C}) : X = \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}, \quad B = B^t,\ C = C^t\right\}$.
  Simple if $n \geqslant 1$.
  Complex.
  Complex dimension $n(2n+1)$; real dimension $2n(2n+1)$.

- $\mathsf{SO}^*(2n)$. Preserves a skew-Hermitean form on $\mathbb{H}^n$, e.g. x^*iy, or its real part.

$= \{A \in \mathsf{GL}(n,\mathbb{H}) : A^* i A = iI\} = \{A \in \mathsf{SL}(n,\mathbb{H}) : A^* i A = iI\}.$
$\cong \mathsf{U}^*(2n) \cap \mathsf{U}(n,n) = \mathsf{SU}^*(2n) \cap \mathsf{SU}(n,n)$
$= \{A \in \mathsf{U}^*(2n) : A^t K_{n,n} A = K_{n,n}\}$
$= \{A \in \mathsf{U}(n,n) : A^t K_{n,n} A = K_{n,n}\}.$
$\pi_1\big(\mathsf{SO}^*(2n)\big) = \mathbb{Z}.$

$\mathfrak{so}^*(2n) = \{X \in \mathfrak{gl}(n,\mathbb{H}) : X^* i + iX = 0\} = \{X \in \mathfrak{sl}(n,\mathbb{H}) : X^* i + iX = 0\}.$
$\cong \mathfrak{u}^*(2n) \cap \mathfrak{u}(n,n) = \mathfrak{su}^*(2n) \cap \mathfrak{su}(n,n)$
$= \{X \in \mathfrak{u}^*(2n) : X^t K_{n,n} + K_{n,n} X = 0\}$
$= \{X \in \mathfrak{u}(n,n) : X^t K_{n,n} + K_{n,n} X = 0\}.$
$= \left\{X \in \mathfrak{gl}(2n,\mathbb{C}) : X = \begin{pmatrix} A & -\overline{C} \\ C & \overline{A} \end{pmatrix} = \begin{pmatrix} A & C^* \\ C & -A^t \end{pmatrix}, \quad A = -A^*,\ C = -C^t\right\}.$
Simple if $n \geqslant 1$.
Complexification $\cong \mathfrak{so}(2n,\mathbb{C})$.
Dimension $n(2n-1)$.

2. Complete lists of simple complex and real Lie algebras

2.1. $\mathbf{A}_\ell$, $\ell \geqslant 1$. Dimension $\ell(\ell+2)$.

2.1.1. *Complex.*

- $\mathfrak{sl}(\ell+1;\mathbb{C})$. $\big(\mathfrak{sl}_{\ell+1}\mathbb{C},\ P'_{\ell+1,L},\dots\big)$

2.1.2. *Real.*

- $\mathfrak{sl}(\ell+1;\mathbb{C})$. $\big(\mathfrak{sl}_{\ell+1}\mathbb{C},\ P'_{\ell+1,L},\dots\big)$
  Complex type. Double dimension.
- $\mathfrak{sl}(\ell+1;\mathbb{R})$. $\big(\mathfrak{sl}_{\ell+1}\mathbb{R},\ \Phi'_{\ell+1,L},\dots\big)$
  Split.
- $\mathfrak{sl}(\frac{\ell+1}{2};\mathbb{H})$, ℓ odd. $\big(\mathfrak{sl}_{\frac{\ell+1}{2}}\mathbb{H},\ \Delta'_{\frac{\ell+1}{2},L},\dots\big)$
  Subalgebra of $\mathfrak{gl}(\frac{\ell+1}{2};\mathbb{H})/\mathbb{R}$ with real trace 0.
  $\cong \mathfrak{gl}(\frac{\ell+1}{2};\mathbb{H})/\mathbb{R}$
- $\mathfrak{su}(p;q)$, $p+q=\ell+1$, $q \leqslant (\ell+1)/2$. $\big(\mathfrak{su}_{p,q},\ \mathfrak{S}(P,\ell+1,q),\dots\big)$
  $\mathfrak{su}(p;q) \cong \mathfrak{su}(q;p)$.
  $\mathfrak{su}(\ell+1) := \mathfrak{su}(\ell+1;0)$ is compact.

2.1.3. *Nontrivial isomorphisms.*

$\mathbf{A}_1$:
$\mathfrak{su}(2) \cong \mathfrak{sl}(1;\mathbb{H})$. Compact.
$\mathfrak{su}(1,1) \cong \mathfrak{sl}(2;\mathbb{R})$. Split.

2.2. $\mathbf{B}_\ell$, $\ell \geqslant 2$. Dimension $\ell(2\ell+1)$.

$\mathbf{B}_1 = \mathbf{A}_1$.

2.2.1. *Complex.*

- $\mathfrak{so}(2\ell+1;\mathbb{C})$. $\big(\mathfrak{so}_{2\ell+1}\mathbb{C},\dots\big)$

2.2.2. *Real.*

- $\mathfrak{so}(2\ell+1;\mathbb{C})$. $\big(\mathfrak{so}_{2\ell+1}\mathbb{C}, \dots\big)$
  Complex type. Double dimension.
- $\mathfrak{so}(p;q)$, $p+q=2\ell+1$, $0\leqslant q\leqslant \ell$. $\big(\mathfrak{so}_{p,q}\mathbb{R},\ \mathfrak{S}(\Phi, 2\ell+1, q), \dots\big)$
  $\mathfrak{so}(p;q)\cong\mathfrak{so}(q;p)$.
  $\mathfrak{so}(2\ell+1) := \mathfrak{so}(2\ell+1;0)$ is compact.
  $\mathfrak{so}(\ell+1;\ell)$ is split.

2.2.3. *Nontrivial isomorphisms.*

$\mathbf{B}_1=\mathbf{A}_1$:
$\mathfrak{so}(3;\mathbb{C})\cong\mathfrak{sl}(2;\mathbb{C})$. Complex.
$\mathfrak{so}(3)\cong\mathfrak{su}(2)\cong\mathfrak{sl}(1;\mathbb{H})$. Compact.
$\mathfrak{so}(2;1)\cong\mathfrak{su}(1,1)\cong\mathfrak{sl}(2;\mathbb{R})$. Split.

2.3. **$\mathbf{C}_\ell$, $\ell\geqslant 3$.** Dimension $\ell(2\ell+1)$.
$\mathbf{C}_1=\mathbf{B}_1=\mathbf{A}_1$.
$\mathbf{C}_2=\mathbf{B}_2$.

2.3.1. *Complex.*

- $\mathfrak{sp}(\ell;\mathbb{C})$ $\big(\mathfrak{sp}_{2\ell}\mathbb{C}, \dots\big)$

2.3.2. *Real.*

- $\mathfrak{sp}(\ell;\mathbb{C})$ $\big(\mathfrak{sp}_{2\ell}\mathbb{C}, \dots\big)$
  Complex type. Double dimension.
- $\mathfrak{sp}(\ell;\mathbb{R})$ $\big(\mathfrak{sp}_{2\ell}\mathbb{R},\ \mathfrak{S}(\Phi, 2\ell, Q), \dots\big)$
  Split.
- $\mathfrak{sp}(p;q)$, $p+q=\ell$, $0\leqslant q\leqslant \ell/2$. $\big(\mathfrak{u}_{p,q}\mathbb{H},\ \mathfrak{S}(\Delta, \ell, q), \dots\big)$
  Linear operators (matrices) on $\mathbb{H}^\ell$ preserving a Hermitean form.
  $\mathfrak{sp}(p;q)\cong\mathfrak{sp}(q;p)$.
  $\mathfrak{sp}(\ell) := \mathfrak{sp}(\ell;0)$ is compact.

2.3.3. *Nontrivial isomorphisms.*

$\mathbf{C}_1=\mathbf{B}_1=\mathbf{A}_1$:
$\mathfrak{sp}(1;\mathbb{C})\cong\mathfrak{so}(3;\mathbb{C})\cong\mathfrak{sl}(2;\mathbb{C})$. Complex.
$\mathfrak{sp}(1)\cong\mathfrak{so}(3)\cong\mathfrak{su}(2)\cong\mathfrak{sl}(1;\mathbb{H})$. Compact.
$\mathfrak{sp}(1;\mathbb{R})\cong\mathfrak{so}(2;1)\cong\mathfrak{su}(1,1)\cong\mathfrak{sl}(2;\mathbb{R})$. Split.

$\mathbf{C}_2=\mathbf{B}_2$:
$\mathfrak{sp}(2;\mathbb{C})\cong\mathfrak{so}(5;\mathbb{C})$. Complex.
$\mathfrak{sp}(2)\cong\mathfrak{so}(5)$. Compact.
$\mathfrak{sp}(1;1)\cong\mathfrak{so}(4;1)$.
$\mathfrak{sp}(2;\mathbb{R})\cong\mathfrak{so}(3;2)$. Split.

2.4. **$\mathbf{D}_\ell$, $\ell\geqslant 4$.** Dimension $\ell(2\ell-1)$.
$\mathbf{D}_1$ does not exist. ($\mathfrak{so}(2)$ is commutative and not semisimple.)
$\mathbf{D}_2=\mathbf{A}_1\oplus\mathbf{A}_1$, semisimple but not simple.
$\mathbf{D}_3=\mathbf{A}_3$.

2.4.1. *Complex.*

- $\mathfrak{so}(2\ell;\mathbb{C})$. $\big(\mathfrak{so}_{2\ell}\mathbb{C},\dots\big)$

2.4.2. *Real.*

- $\mathfrak{so}(2\ell;\mathbb{C})$. $\big(\mathfrak{so}_{2\ell}\mathbb{C},\dots\big)$
  Complex type. Double dimension.
- $\mathfrak{so}(p;q)$, $p+q=2\ell$, $0\leqslant q\leqslant \ell$. $\big(\mathfrak{so}_{p,q}\mathbb{R},\ \mathfrak{S}(\Phi,2\ell,q),\dots\big)$
  $\mathfrak{so}(p;q)\cong\mathfrak{so}(q;p)$.
  $\mathfrak{so}(2\ell):=\mathfrak{so}(2\ell;0)$ is compact.
  $\mathfrak{so}(\ell;\ell)$ is split.
- $\mathfrak{so}^*(2\ell)$ $\big(\mathfrak{u}^*_\ell\mathbb{H},\ \mathfrak{S}(\Delta,\ell,Q),\dots\big)$
  Linear operators (matrices) on $\mathbb{H}^\ell$ preserving a skew-Hermitean form.

2.4.3. *Nontrivial isomorphisms.*

$\mathbf{D}_4$:
$\mathfrak{so}^*(8)\cong\mathfrak{so}(6,2)$.

$\mathbf{D}_2=\mathbf{A}_1\oplus\mathbf{A}_1$:
$\mathfrak{so}(4;\mathbb{C})\cong\mathfrak{sl}(2;\mathbb{C})\times\mathfrak{sl}(2;\mathbb{C})$. Complex.
$\mathfrak{so}(4)\cong\mathfrak{so}(3)\times\mathfrak{so}(3)\cong\mathfrak{su}(2)\times\mathfrak{su}(2)$. Compact.
$\mathfrak{so}(2,2)\cong\mathfrak{sl}(2;\mathbb{R})\times\mathfrak{sl}(2;\mathbb{R})\cong\mathfrak{su}(1,1)\times\mathfrak{su}(1,1)$. Split.
$\mathfrak{so}(3,1)\cong\mathfrak{sl}(2;\mathbb{C})$.
$\mathfrak{so}^*(4)\cong\mathfrak{su}(2)\times\mathfrak{sl}(2;\mathbb{R})\cong\mathfrak{su}(2)\times\mathfrak{su}(1,1)$.

$\mathbf{D}_3=\mathbf{A}_3$:
$\mathfrak{so}(6;\mathbb{C})\cong\mathfrak{sl}(4;\mathbb{C})$. Complex.
$\mathfrak{so}^*(6)\cong\mathfrak{su}(3;1)$.
$\mathfrak{so}(6)\cong\mathfrak{su}(4)$. Compact.
$\mathfrak{so}(5;1)\cong\mathfrak{sl}(2;\mathbb{H})$.
$\mathfrak{so}(4;2)\cong\mathfrak{su}(2;2)$.
$\mathfrak{so}(3;3)\cong\mathfrak{sl}(4;\mathbb{R})$. Split.

2.5. $\mathbf{E}_6$**,** $\mathbf{E}_7$**,** $\mathbf{E}_8$**,** $\mathbf{F}_4$**,** $\mathbf{G}_2$**.**
For the exceptional simple Lie algebras, see e.g. [2].

References

[1] Nathan Jacobson. *Lie algebras.* Interscience, New York–London, 1962.
[2] Anthony W. Knapp. *Lie groups beyond an introduction.* 2nd edition. Birkhuser, Boston, MA, 2002.

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