ON MOMENTS OF THE LIMITING RANDOM VARIABLE FOR A SUPERCRITICAL GALTON–WATSON PROCESS

SVANTE JANSON

ABSTRACT. Consider a supercritical Galton–Watson process $(Z_n)_0^\infty$ with offspring distribution ξ and finite offspring mean $\lambda := \mathbb{E}\,\xi$. Assume the standard condition $\mathbb{E}\,\xi \log \xi < \infty$. It is well-known that then $Z_n/\lambda^n \overset{\text{a.s.}}{\longrightarrow} W$ for some non-trivial random variable W. We give a simple probabilistic proof of the result by Bingham and Doney (1974) that the r:th moment $\mathbb{E}\,W^r$ is finite if and only if $\mathbb{E}\,\xi^r$ is, for any real $r > 1$.

1. Introduction

Consider a Galton–Watson process $(Z_n)_0^\infty$ with $Z_0 = 1$ and offspring given by independent copies of a random variable ξ (Thus, $\xi \overset{\text{d}}{=} Z_1$.) We denote the mean number of children by $\lambda := \mathbb{E}\,\xi$. We assume that the process is supercritical, i.e., $\lambda > 1$; moreover, we assume that $\lambda < \infty$.

It is well-known that then $W_n := \lambda^{-n} Z_n$, $n \geqslant 0$, is a martingale, which converges a.s. to a limit W; furthermore, if $\mathbb{E}\,\xi \log \xi < \infty$ then $\mathbb{E}\,W = 1$ and $W_n \to W$ also in L^1, but if $\xi \log \xi = \infty$, then $W = 0$ a.s., see e.g. [1, Section I.10] or [5, Section 2.7]. We consider here only the first case.

The distribution of the limit W can usually not be found explicitly, but various properties of it can be shown. In particular, Bingham and Doney [3, Corollary to Theorem 5] proved the following result on existence of moments of W.

Theorem 1 (Bingham and Doney)**.** *Consider a Galton–Watson process with notation as above. Assume that* $1 < \lambda < \infty$ *and* $\mathbb{E}\,\xi \log \xi < \infty$. *Then, for any real* $r > 1$,

$$\mathbb{E}\,W^r < \infty \iff \mathbb{E}\,\xi^r < \infty. \tag{1.1}$$

The proof in [3] uses Laplace transforms. We give here a simple probabilistic proof.

Remark 2. Bingham and Doney [3] prove also more general results on the existence of $\mathbb{E}\big[W^r L(W)\big]$, where $L(x)$ is a slowly varying function; these results will not be discussed here. □

We split the proof of Theorem 1 into necessity and sufficiency of the condition $\mathbb{E}\,\xi^r < \infty$, and the latter into the two cases $r \leqslant 2$ and $r \geqslant 2$.

Let $\|X\|_r := (\mathbb{E}\,|X|^r)^{1/r}$ for $r > 0$ and a random variable X. Also, let $\xi' := \xi - \lambda$, so $\mathbb{E}\,\xi' = 0$. C denotes various finite constants, not necessarily the same each time; these may depend on the distribution of ξ, but not on n.

Date: 6 September, 2019.

Partly supported by the Knut and Alice Wallenberg Foundation.

Proof of Theorem 1, $\Longrightarrow$. This is easy. Since $(W_n)_0^\infty$ is a martingale which converges in L^1, $W_1 = \mathbb{E}\big(W \mid W_1\big)$, and thus (all variables are non-negative)

$$\mathbb{E}\, W_1^r \leqslant \mathbb{E}\, W^r < \infty. \tag{1.2}$$

Moreover, $\xi \overset{\mathrm{d}}{=} Z_1 = \lambda W_1$ and thus $\mathbb{E}\,\xi^r = \lambda^r\, \mathbb{E}\, W^r < \infty$. □

Proof of Theorem 1, $\Longleftarrow$*,* $1 < r \leqslant 2$. Conditioned on Z_n, we have that Z_{n+1} is a sum of Z_n independent copies of ξ. Hence,

$$Z_{n+1} - \lambda Z_n = \sum_{i=1}^{Z_n} (\xi_i - \lambda) = \sum_{i=1}^{Z_n} \xi_i' \tag{1.3}$$

where ξ_i' are conditionally independent copies of ξ'. Hence, by the von Bahr – Esseen inequality [2, Theorem 2],

$$\mathbb{E}\big(|Z_{n+1} - \lambda Z_n|^r \mid Z_n\big) \leqslant 2\, \mathbb{E}\, |\xi'|^r Z_n = C Z_n \tag{1.4}$$

and thus, by taking the expectation,

$$\mathbb{E}\, |Z_{n+1} - \lambda Z_n|^r \leqslant C\, \mathbb{E}\, Z_n = C \lambda^n \tag{1.5}$$

or

$$\|Z_{n+1} - \lambda Z_n\|_r \leqslant C \lambda^{n/r}. \tag{1.6}$$

Since $W_{n+1} - W_n = \lambda^{-n-1}(Z_{n+1} - \lambda Z_n)$, this yields

$$\|W_{n+1} - W_n\|_r = \lambda^{-n-1} \|Z_{n+1} - \lambda Z_n\|_r \leqslant C \lambda^{-n(1-1/r)}. \tag{1.7}$$

We have assumed $r > 1$, and thus $1 - 1/r > 0$. Hence, by Minkowski's inequality, for any finite n,

$$\|W_n\| \leqslant \|W_0\| + \sum_{k=0}^{n-1} \|W_{k+1} - W_k\| \leqslant 1 + C \sum_{k=1}^{\infty} \lambda^{-k(1-1/r)} = C. \tag{1.8}$$

Since $W_n \xrightarrow{\text{a.s.}} W$, Fatou's lemma yields $\mathbb{E}\, W^r \leqslant C$. □

Proof of Theorem 1, $\Longleftarrow$*,* $r \geqslant 2$. We again condition on Z_n and have (1.3). This time we use Rosenthal's inequality [4, Theorem 3.9.1] and obtain

$$\begin{aligned} \mathbb{E}\big(|Z_{n+1} - \lambda Z_n|^r \mid Z_n\big) &\leqslant C Z_n\, \mathbb{E}\, |\xi'|^r + C\big(Z_n\, \mathbb{E}\, |\xi'|^2\big)^{r/2} = C Z_n + C Z_n^{r/2} \\ &\leqslant C Z_n^{r/2}. \end{aligned} \tag{1.9}$$

Hence, using also the Cauchy–Schwarz inequality,

$$\mathbb{E}\, |Z_{n+1} - \lambda Z_n|^r \leqslant C\, \mathbb{E}\, Z_n^{r/2} \leqslant C\big(\mathbb{E}\, Z_n^r\big)^{1/2}. \tag{1.10}$$

Thus,

$$\|Z_{n+1} - \lambda Z_n\|_r \leqslant C \|Z_n\|_r^{1/2}, \tag{1.11}$$

and Minkowski's inequality yields

$$\|Z_{n+1}\|_r \leqslant \lambda \|Z_n\|_r + C \|Z_n\|_r^{1/2}. \tag{1.12}$$

Let

$$A_n := \|W_n\|_r = \lambda^{-n} \|Z_n\|_r \tag{1.13}$$

and note that $A_n \geqslant \|W_n\|_1 = \mathbb{E}\, W_n = 1$. Then (1.12) and (1.13) yield

$$\begin{aligned} A_{n+1} &\leqslant \lambda^{-n}\|Z_n\|_r + C\lambda^{-n}\|Z_n\|_r^{1/2} = A_n + C\lambda^{-n/2}A_n^{1/2} \\ &\leqslant A_n + C\lambda^{-n/2}A_n = \bigl(1 + C\lambda^{-n/2}\bigr)A_n. \end{aligned} \tag{1.14}$$

Hence, by induction, noting $A_0 = \|Z_0\|_r = 1$,

$$A_n \leqslant \prod_{k=0}^{n-1}\bigl(1 + C\lambda^{-k/2}\bigr) \leqslant \exp\Bigl(\sum_{k=0}^{n-1} C\lambda^{-k/2}\Bigr) \leqslant C, \tag{1.15}$$

since $\sum_{k=0}^{\infty} \lambda^{-k/2} < \infty$.

We have shown $\mathbb{E}\, W_n^r = A_n^r \leqslant C$, and, as in the case $r \leqslant 2$, Fatou's lemma yields $\mathbb{E}\, W^r < \infty$. □

References

[1] Krishna B. Athreya & Peter E. Ney: *Branching Processes.* Springer-Verlag, Berlin, 1972.

[2] Bengt von Bahr & Carl-Gustav Esseen: Inequalities for the rth absolute moment of a sum of random variables, $1 \leqslant r \leqslant 2$. *Ann. Math. Statist.* **36** (1965), 299–303.

[3] N. H. Bingham & R. A. Doney: Asymptotic properties of supercritical branching processes I: The Galton–Watson process. *Advances in Appl. Probability* **6** (1974), 711–731.

[4] Allan Gut: *Probability: A Graduate Course*, 2nd ed., Springer, New York, 2013.

[5] Peter Jagers: *Branching Processes with Biological Applications.* Wiley, London, 1975.

Department of Mathematics, Uppsala University, PO Box 480, SE-751 06 Uppsala, Sweden

Email address: `svante.janson@math.uu.se`

URL: `http://www.math.uu.se/svante-janson`